Question 2

Prove that the following system of linear equations is inconsistent. Find a least squares solution of the system. $\begin{cases} x+2y+3z=8, \\ x+3z=-2, \\ 2x+4z=0, \\ x-y+2z=16. \end{cases}$

1) According to Rouche-Capelli Theorem, a system of linear equations $Ax=b$ has solutions only if $rank(A)=rank(A|b)$.

Let us find ranks of the two matrices by elementary transformations. Transforming to reduced row echelon form is not necessary for this purpose, as a regular row echelon form would be sufficient. However let's make it reduced as this does now require significantly more calculations.

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix} \sim \left\| \begin{matrix} \\ e_1(-1)+e_2\to e_2 \\ e_1(-2)+e_3\to e_3 \\ e_1(-1)+e_4\to e_4 \end{matrix} \right\| \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & -2 & -2 \\ 0 & -3 & -1 \end{pmatrix} \sim \left\| \begin{matrix} e_1+e_2\to e_1 \\ \\ e_2(-1)+e_3\to e_3 \\ e_2(-3)+e_4 2\to e_4 \end{matrix} \right\| \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \sim \left\| \begin{matrix} e_1 2+e_3 3\to e_1 \\ \\ \\ e_3(-1)+e_4\to e_4 \end{matrix} \right\| \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \left\| \begin{matrix} e_1(1/2)\to e_1 \\ e_2(-1/2)\to e_2 \\ e_3(-1/2)\to e_3 \\ \\ \end{matrix} \right\| \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

This implies $rank(A)$ = 3.

$$A|b=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 1 & 0 & 3 & -2 \\ 2 & 0 & 4 & 0 \\ 1 & -1 & 2 & 16 \end{array}\right) \sim \left\| \begin{matrix} \\ e_1(-1)+e_2\to e_2 \\ e_1(-2)+e_3\to e_3 \\ e_1(-1)+e_4\to e_4 \end{matrix} \right\| \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & -2 & 0 & -10 \\ 0 & -2 & -2 & -8 \\ 0 & -3 & -1 & 8 \end{array}\right) \sim \left\| \begin{matrix} e_1+e_2\to e_1 \\ \\ e_2(-1)+e_3\to e_3 \\ e_2(-3)+e_4 2\to e_4 \end{matrix} \right\| \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & -2 & 0 & -10 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & 14 \end{array}\right) \sim \left\| \begin{matrix} e_1 2+e_3 3\to e_1 \\ \\ \\ e_3(-1)+e_4\to e_4 \end{matrix} \right\| \sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & -10 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 12 \end{array}\right) \sim \left\| \begin{matrix} e_1+e_4(-1/6)\to e_1 \\ e_2+e_4(5/6)\to e_2 \\ e_3+e_4(-1/6)\to e_3 \\ e_3(-1)+e_4\to e_4 \end{matrix} \right\|$$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 12 \end{array}\right) \sim \left\| \begin{matrix} e_1(1/2)\to e_1 \\ e_2(-1/2)\to e_2 \\ e_3(-1/2)\to e_3 \\ e_4(1/12)\to e_4 \end{matrix} \right\| \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{2}$$

This shows that $rank(A|b=4)$.

Having $rank(A)\ \neq rank(A|b)$ we can state that the given system of linear equations is inconsistent.

2) We know that if we suppose that A is an $m\times n$ matrix with $rank\ A\ =\ n$, and $b\ \in\ \mathbb{R}^m$, then the least

square solution of the system $Ax = b$ is $\widetilde{x} = \left(A^T A\right)^{-1} A^T b$.
Let us make step-by-step calculations:

$$A^T A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 3 & 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 16 \\ 1 & 5 & 4 \\ 16 & 4 & 38 \end{pmatrix} \tag{3}$$

$$\left(A^T A\right)^{-1} A^T b = \begin{pmatrix} \frac{87}{14} & \frac{13}{14} & \frac{-19}{7} \\ \frac{13}{14} & \frac{5}{14} & \frac{-3}{7} \\ \frac{-19}{7} & \frac{-3}{7} & \frac{17}{14} \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 3 & 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ -2 \\ 0 \\ 16 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{-1}{14} & \frac{-27}{14} & \frac{11}{7} & \frac{-1}{7} \\ \frac{5}{14} & \frac{-5}{14} & \frac{1}{7} & \frac{-2}{7} \\ \frac{1}{14} & \frac{13}{14} & \frac{-4}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 8 \\ -2 \\ 0 \\ 16 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4}$$

The least square solution is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$