Question 2

Let X be uniform random variable on a segment $[0, 2]$. Consider random variable $Y = X^2$. Find CDF and PDF of Y. Is PDF a bounded function?

PDF of the random variable X is given by:

$$pdf_X(x) = \begin{cases} 1/2 & : \quad 0 \le x \le 2 \\ 0 & : \quad otherwise \end{cases}$$

To find the CDF of $Y = X^2$, we use the definition of CDF:

$$F_Y(y) = P(Y \le y) = P\left(X^2 \le y\right) = P\left(-\sqrt{y} \le X \le \sqrt{y}\right).$$

Since X is distributed and has non-zero PDF only on the segment $[0, 2]$ we can assume that $P\left(-\sqrt{y} \le X < 0\right) = 0$. Thus, we shall consider the following CDF:

$$F_Y(y) = P(Y \le y) = P\left(X^2 \le y\right) = P\left(0 \le X \le \sqrt{y}\right).$$

Now, in terms of $Y \le y$:

If $y < 0$, then $Y = X^2 < 0$. As $X \ge 0$ and, of course, $X^2 \ge 0$, $F_Y(y) = 0$ for $y < 0$.

If $y > 4$, then $Y = X^2 > 4$, so $F_Y(y) = 1$ for $y > 4$.

If $0 \le y \le 4$, then we can find CDF of $Y = X^2$ by following integration on the segment $\left[0, \sqrt{y}\right]$ where PDF of X is not equal to 0:

$$F_Y(y) = P\left(0 \le X \le \sqrt{y}\right) = \int_0^{\sqrt{y}} pdf_X(x)\, dx = \int_0^{\sqrt{y}} \frac{1}{2}\, dx =$$

$$= \frac{x}{2}\bigg|_0^{\sqrt{y}} = \frac{\sqrt{y}}{2}$$

Therefore, the CDF of Y is given by:

$$F_Y(y) = \begin{cases} 0 & : \quad y < 0 \\ \dfrac{\sqrt{y}}{2} & : \quad 0 \le y \le 4 \\ 1 & : \quad y > 4 \end{cases}$$

To find the PDF of Y, we differentiate the CDF.

For $0 \le y \le 4$ we have:

$$pdf_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\frac{\sqrt{y}}{2} = \frac{1}{4\sqrt{y}}$$

For $y < 0$ and $y > 4$ we have $\dfrac{d}{dy} F_Y(y) = 0$.

Therefore, the PDF of Y is:

$$pdf_Y(y) = \begin{cases} \dfrac{1}{4\sqrt{y}} & : \quad 0 \le y \le 4 \\ 0 & : \quad otherwise \end{cases}$$

Boundedness of the PDF :

The PDF is not defined at $y = 0$ and it approaches infinity as y approaches 0, so we cannot say that the PDF is bounded on the interval $[0, 4]$.