Question 3

Let $A = (1,1,1)$ be vector in R^3, and let L be a vector subspace of R^3 spanned by the vectors $(1,0,-1)$ and $(3,5,x)$ (where x is a real parameter). Find the values of x such that the distance between A and L is maximal. Find also the square of this maximal distance.

For the distance $d(A,L) = ||ort_L A|| = ||A - pr_L A||$ to be maximal we need $pr_L A$ to be minimal, which is equal to 0. That means that vector A should be orthogonal to L and therefore orthogonal to any vector of L. Moreover in this case we can state that $ort_L A$ and A itself coincide. We can check if this is true in our case by finding dot-product of vector A and the known vector of subspace L.

Let us denote vectors of subspace L as $v_1 = (1,0,-1)$ and $v_2 = (3,5,x)$.
We know that if vectors are orthogonal their dot-product is equal to 0.
Therefore if $\langle A, v_1 \rangle = 0$ and then A and L are orthogonal.

$$\langle A, v_1 \rangle = 0$$
$$1*1 + 1*0 + 1* - 1 = 0$$
$$0 = 0$$

The equivalence holds and we can state that A is orthogonal to L and the $d(A,L)$ is maximal notwithstanding value of x.

Now let's find x from v_2.

$$\langle A, v_2 \rangle = 0$$
$$1*3 \ + 1*5 + 1*x = 0$$
$$8 + x = 0$$
$$x = -8$$

The maximal distance can be expressed as $d(A,L) = ||ort_L A|| = ||A - pr_L A||$, where $pr_L A = 0,\ ||ort_L A|| = ||A||$. So the squared maximal distance $d^2(A,L) = 1^2 + 1^2 + 1^2 = 3$

OR we may use the following formula:

$$d(A,L) = ||A - V\left(V^T V\right)^{-1} VA||, \tag{1}$$

where V is the matrix of basis of L and $V\left(V^T V\right)^{-1} VA = pr_L A$. However, as we have stated above $pr_L A = 0$ and therefore the formula (1) can be simplified to $d(A,L) = ||A||$, which gives us the same result of $d(A,L) = 3$.