Question 1

A linear operator $f : R_2 \to R_2$ maps the vectors $a = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ to the vectors $p = f(a) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $q = f(b) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$. Find the matrix of f in the basis $\{a, q\}$.

1) Let us denote the matrix of vectors $\vec{a}$ and $\vec{b}$ as M_1, and the matrix of vectors $\vec{p}$ and $\vec{q}$ as M_2. Then we can write down application of the linear operator in the following way $AM_1 = M_2$, where A is the matrix of linear operator f. Hence $A = M_2 M_1^{-1}$, or

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \tag{1}$$

2) Let us denote matrix A (1) in the basis of $\{a, q\}$ as matrix A'. Then the formula for change of the basis of linear operator is $A' = T^{-1} A\, T$, where T is the transformation matrix from standars basis to $\{a, q\}$, which can be denoted as $T_{e \to \{a,q\}}$.

Knowing vectors $\vec{a}$ and $\vec{q}$ we can compose the thransition matrix: $T_{e \to \{a,q\}} = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$, and calculate the matrix A':

$$\begin{aligned} A' &= \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} = \\ &= \begin{pmatrix} 0.4 & -0.2 \\ -0.2 & -0.4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 11 \end{pmatrix} \end{aligned} \tag{2}$$

ADDITIONALLY we can make sure that our solution is correct by checking if the following equivalence holds:

$$AT = TA'$$

$$\begin{aligned} \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} &= \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ -2 & 11 \end{pmatrix} \\ \begin{pmatrix} 2 & -11 \\ 4 & -22 \end{pmatrix} &= \begin{pmatrix} 2 & -11 \\ 4 & -22 \end{pmatrix} \end{aligned} \tag{3}$$

The equivalence (3) holds so we have found correct matrices A (1) and A' (2).