Question 3

Consider arbitrary sample $x = (x_1, \dots, x_n)$. Let us find such value a that $\sum_{i=1}^{n}(x_i - a)^2$ is minimized. Prove that a is sample average $\bar{x}$ of x.

In order to find such value a that $\sum_{i=1}^{n}(x_i - a)^2$ is minimized, first we should take the derivative of this expression with respect to a. Then we set the result equal to 0 ans solve for a, to find the value of a which minimizes this sum of squares.

If would be helpful to express the initial sum the following way:

$$\sum_{i=1}^{n}(x_i - a)^2 = \sum_{i=1}^{n}\left(x_i^2 - 2x_i a + a^2\right) \tag{1}$$

Using (1) and differentiating with respect to a, we get:

$$\frac{d}{da}\sum_{i=1}^{n}(x_i - a)^2 = \frac{d}{da}\left(\sum_{i=1}^{n} x_i^2 - 2a\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} a^2\right) = \frac{d}{da}\left(\sum_{i=1}^{n} x_i^2 - 2a\sum_{i=1}^{n} x_i + na^2\right) =$$
$$= -2\sum_{i=1}^{n} x_i + 2na$$

Setting this equal to 0 and solving for a gives us:

$$2na = 2\sum_{i=1}^{n} x_i$$
$$a = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Therefore, a is the sample average $\overline{x}$ of x, which minimizes the sum of squares $\sum_{i=1}^{n}(x_i - a)^2$.