Question 4

A dangerous virus spreads on an island with a population of 10,000. Every day, island authorities collect statistics and want to understand if they have enough health system resources. Doctors report that every day 15% of healthy people become infected and have a mild illness (which does not require hospitalization), 12% of healthy people become infected and have a difficult illness (they need to get to the hospital). At the same time, 12% of people with a mild form of the disease recover completely, and 15% go into the category of seriously ill patients. In the category of seriously ill patients, the situation is as follows: 20% go into the category of patients with a mild form of the disease and 10% completely recover. Recovered patients may become infected again. At the initial time on the island, 500 patients were identified in a mild form of the disease and 100 patients in a severe form. Luckily, the virus is not lethal. How will the number of patients behave with increasing time? From a mathematical point of view, find the limits of the number of patients in mild and severe forms, if they exist. Please do not forget that with real viruses everything is not so simple.

1) Generally speaking we are considering three different states, viz. healthy (h), medium (m), severe (s), and probabilities at which inter-state transition may happen. This can by described by a Markov transition matrix:

$$P = \begin{array}{c} \\ h \\ m \\ s \end{array}\!\!\begin{array}{c} \begin{array}{ccc} h & m & s \end{array} \\ \begin{bmatrix} 0.73 & 0.12 & 0.1 \\ 0.15 & 0.73 & 0.2 \\ 0.12 & 0.15 & 0.7 \end{bmatrix} \end{array} \tag{1}$$

By normalizing the eigenvector v corresponding to eigenvalue $\lambda = 1$ we can get a vector g which represents upper bounds of each state probability, or a rate of inter-state limit-distribution.

In order to find the limits of patients number in each state we need to multiply the total number of population by g.

$$v = \begin{pmatrix} 0.50067394 \\ 0.67738239 \\ 0.53896077 \end{pmatrix}, \; g = \begin{pmatrix} 0.2916 \\ 0.3945 \\ 0.3139 \end{pmatrix}, \; 10\,000 \cdot g = \begin{array}{c} h \\ m \\ s \end{array}\!\begin{pmatrix} 2916 \\ 3945 \\ 3139 \end{pmatrix}$$

2) To describe the behaviour of figures representing the inter-state distribution over time we may use following Python code:

```
import numpy as np

A = np.array([
    [0.73, 0.12, 0.1],
    [0.15, 0.73, 0.2],
    [0.12, 0.15, 0.7]])

num_iterations = 20
x = np.array([500, 100, 9400])
for i in range(num_iterations):
    x = A @ x
```

```
    print(x)
```

This gives the following printout:

```
[1317. 2028. 6655.]
[1870.27 3008.99 5120.74]
[2238.4499 3501.2512 4260.2989]
[2480.248461 3743.740641 3776.010898]
[2637.43134325 3860.17011668 3502.39854007]
[2738.78514858 3914.01859468 3347.19625674]
[2803.7150155  3937.49059775 3258.79438675]
[2845.09027172 3946.68426603 3208.22546225]
[2871.3405565  3949.48814741 3179.17129608]
[2887.93431355 3949.6616903  3162.40399615]
[2898.39185134 3948.92398018 3152.68416848]
[2904.96534595 3948.01011693 3147.02453712]
[2909.08837029 3947.19709468 3143.71453504]
[2911.66961517 3946.56004166 3141.77034316]
[2913.28305839 3946.09334132 3140.62360028]
[2914.29019361 3945.76531798 3139.9444884 ]
[2914.91812834 3945.54110885 3139.54076281]
[2915.30924303 3945.39088127 3139.2998757 ]
[2915.55264073 3945.29170492 3139.15565434]
[2915.70399776 3945.22697157 3139.06903067]
```

CONCLUSION
We have a perfect match to our 10 000g figures.
Moreover this iteration experiment shows that initial interstate distribution between h, m, and s does not matter. We may start with even all 10 000 of m or s and shall inevitably get the same
10 000g distribution within 20 iterations.

This conclusion is supported by the following theorem taking into account that restrictons for irreducible and aperiodic Markov chain are met:

> (1.9) THEOREM [BASIC LIMIT THEOREM]. *Let $X_0, X_1, \ldots$ be an irreducible, aperiodic Markov chain having a stationary distribution $\pi(\cdot)$. Let X_0 have the distribution π_0, an arbitrary initial distribution. Then $\lim_{n\to\infty} \pi_n(i) = \pi(i)$ for all states i.*

*We say i **communicates with** j if j is accessible from i and i is accessible from j. We say that the Markov chain is **irreducible** if all pairs of states communicate.*

(1.21) DEFINITION. *An irreducible Markov chain is said to be **aperiodic** if its period is 1, and **periodic** otherwise.*