Question 1

Use definition of variance and properties of expected value to prove that for any discrete random variable $X$, $\mathrm{Var}\, X = \mathbb{E}(X^2) - (\mathbb{E}X)^2$.

We can use the following properties of expected value:
(1) $E[c] \;=\; c$ for any constant c,
(2) $E[X+Y] = E[X] + E[Y]$.

Let us start with the definition of variance:

$$Var(X) = E\left[(X - E[X])^2\right].$$

Expanding the square inside the expectation, we get:

$$Var(X) = E\left[X^2 - 2XE[X] + E[X]^2\right].$$

Using the linearity of expectation, i.e. property (2) above, we can split this into three terms:

$$Var(X) = E\left[X^2\right] - 2E[X]E[E[X]] + E[E[X]]^2.$$

Since $E[X]$ is just a constant, we can write using propery (1):

$$2E[X]E[E[X]] = 2E[X]E[X] = 2E[X]^2 \text{ and}$$
$$E[E[X]]^2 = E[E[X]]E[E[X]] = E[X]E[X] = E[X]^2.$$

SimplifyingP this further we get:

$$Var(X) = E\left[X^2\right] - 2E[X]^2 + E[X]^2.$$

And finally:

$$Var(X) = E\left[X^2\right] - E[X]^2.$$

This shows that for any discrete random variable $X$, the variance can be expressed as the dif-ference between the expected value of the square of $X$ and the square of the expected value of $X$.