Question 1

There is a connection between confidence intervals and hypothesis testing. Assume that we have an i.i.d. sample $x = (x_1, \ldots, x_n)$ from some random variable X with finite variance. Consider one-sample t-test with null hypothesis $\mathbb{E}X = \mu_0$ and symmetric alternative. For simplicity, let us assume that n is large enough and replace T-distribution with standard normal distribution. Assume that one found confidence interval I for $\mathbb{E}X$ with confidence level 95%. Prove that standard decision-making procedure of t-test is equivalent to the following: reject null hypothesis if and only if μ_0 does not belong to I. Follow the plan:

1. Assume that null hypothesis holds. We believe that t-statistics in this case is distributed according to standard normal law (due to assumption that n is large). Recall that t-statistics for sample x is defined as $t \approx \frac{\bar{x}-\mu_0}{\mathrm{SD}(x)}\sqrt{n}$.
2. If μ_0 does not lie in I, either μ_0 is larger than the right endpoint of I or μ_0 is smaller than the left endpoint of I. Let us consider the latter case.
3. Consider event "μ_0 is smaller than the left endpoint of I". Write this condition as an inequality using μ_0, $\bar{x}$, $\mathrm{SD}(x)$, n and a constant 1.96. (Recall that we assume that null hypothesis holds.)
4. Transform this inequality such that it becomes $(\ldots) > 1.96$. Does the left-hand part look similar to something?
5. Recall why we use number 1.96, how it is connected to standard normal distribution.
6. Find probability that μ_0 is smaller than the left endpoint of I provided that null hypothesis holds.
7. Find probability that μ_0 does not lie in I provided that null hypothesis holds.
8. Assume we are following rule "reject null hypothesis if and only if μ_0 does not belong to I." Find probability that we falsely reject null hypothesis provided that it is true.
9. Explain in what cases (in terms of $\bar{x}$) will we reject null hypothesis if we follow mentioned rule.
10. Explain that this rule is equivalent to the rule used in ordinary two-sided one-sample t-test.

Considering the event "μ_0 is smaller than the left endpoint of I" and assuming that the null hypothesis holds we can write this condition as an inequality using μ_0, $\bar{x}$, $SD(x)$, n, and the constant 1.96:

$$\mu_0 < \bar{x} - 1.96 \cdot \frac{SD(x)}{\sqrt{n}}$$

Transforming the inequality we get:

$$\frac{\bar{x} - \mu_0}{SD(x)}\sqrt{n} > 1.96\ ,$$

where the left part is exactly the t-statistic for sample x.

For the t-test, the critical value depends on the sample size and the desired confidence level. When the sample size is large enough (as assumed in this case), the t-distribution approaches the standard normal distribution. Therefore, the critical value for the t-test is approximately 1.96 for a 95% confidence level, which is equivalent to the critical region (-1.96, 1.96) for the standard normal distribution.
The cumulative probability of the standard normal distribution (the area under the curve) between -1.96 and 1.96 is approximately 0.95, which means that there is a 95% probability that a randomly selected value from a standard normal distribution falls within this range.
Multiplying the standard error by 1.96 provides the scale or width of the confidence interval, expressed in the units of standard deviation, which helps us estimate the range around the sample mean where the true population mean is likely to be.

The probability that μ_0 is smaller than the left endpoint of I, given that the null hypothesis holds, is equivalent to the probability that the t-statistic is greater than 1.96, which is 0.025.
Generally, in a symmetric alternative, where H_1 is that the true mean is not equal to μ_0, rejecting the null hypothesis corresponds to extreme values of the t-statistic in either tail of the distribution.
Totally, this probability is equal to the significance level α, which is 0.05 for a 95% confidence level.
Therefore, in a two-tailed test with a 95% confidence level, each tail of the normal distribution contains an area of (1 - 0.95) / 2 = 0.025. The critical value of 1.96 corresponds to the point on the standard normal distribution where the cumulative probability in each tail is 0.025.

The probability that μ_0 does not lie in I, given that the null hypothesis holds, is equal to the sum of the probabilities in both tails of the standard normal distribution. This probability is also equal to the significance level α, which is chosen as 0.05 for a 95% confidence level.

Type I error, or the probability that we falsely reject the null hypothesis, provided that it is true, is equal to the significance level α (0.05 for a 95% confidence level).

We will reject the null hypothesis if the observed sample mean $\overline{x}$ forms such confidence interval I

$$\left(\overline{x} - 1.96 \cdot \frac{SD(x)}{\sqrt{n}},\ \overline{x} + 1.96 \cdot \frac{SD(x)}{\sqrt{n}}\right),$$

that μ_0 falls out of it, i.e. if $\mu_0 < \overline{x} - 1.96 \cdot \dfrac{SD(x)}{\sqrt{n}}$ or $\mu_0 > \overline{x} + 1.96 \cdot \dfrac{SD(x)}{\sqrt{n}}$.

This rule is equivalent to the rule used in an ordinary two-sided one-sample t-test. In the t-test, we reject the null hypothesis if the test statistic falls outside the critical region (-1.96, 1.96), which corresponds to the confidence interval of 95% for normal distribution.

The condition that t-statistic > 1.96 or the t-statistic < -1.96 can be expressed as:

$$\frac{\overline{x} - \mu_0}{SD(x)}\sqrt{n} > 1.96 \text{ or } \frac{\overline{x} - \mu_0}{SD(x)}\sqrt{n} < -1.96,$$

and transformed into

$$\mu_0 < \overline{x} - 1.96 \cdot \frac{SD(x)}{\sqrt{n}} \text{ or } \mu_0 > \overline{x} + 1.96 \cdot \frac{SD(x)}{\sqrt{n}}$$

This statement accurately captures the equivalence between the decision-making rule in the t-test and the rule based on the confidence interval. When the true population mean falls outside the confidence interval, it indicates that the test statistic (t-statistic) will also fall outside the critical region. This alignment leads to the rejection of the null hypothesis in both cases.