Question 2

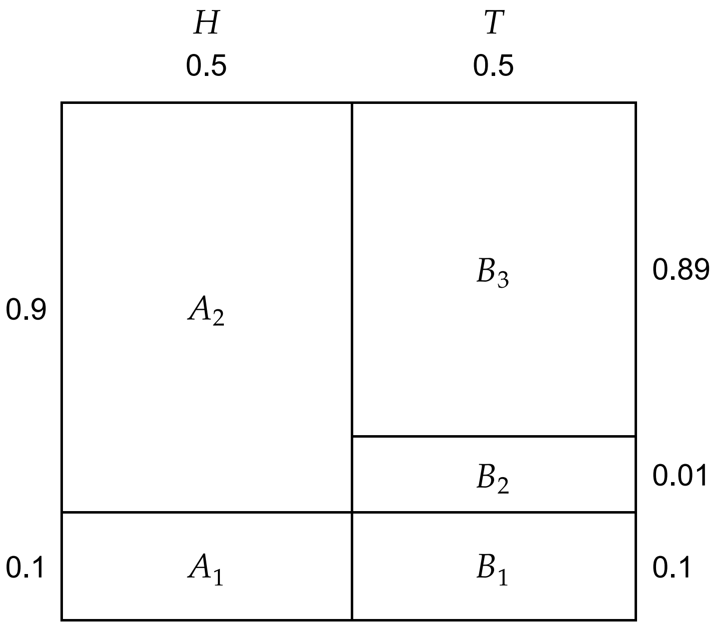
> I can buy one lottery ticket out of two available. In the first lottery I can win \$100 with probability 0.1, and the price of ticket is \$10. In the second lottery, I can win \$50 with probability 0.1 and \$500 with probability 0.01. The price of ticket is \$20. To decide which ticket to buy I toss a fair coin once. I chose first ticket in case of head and second otherwise. Let X be random variable that denotes my net payout (taking into account price of a ticket). Find probability mass function of X (hint: use law of total probability). Show that expected value of X is an average of expected values of net payouts for each of two lotteries. Explain, why. Will it still hold if lotteries has different payouts or probabilities? Prove it.

1) Let us introduce a table structure of given data and further calculations.

We do not distinguish events of coin tossing outcomes (H and T for head and tail correspondingly) and consequent buying of lottery ticket (A for lottery 1 and B for lottery 2), where conditional probabilities would be $P(A|H) = P(B|T) = 1$ and $P(A \cap H) = P(B \cap T) = 0.5$. This is done because no other options of interdependence for these events were supposed by the task, such as non-fair coin toss, for instance. Lines (1) to (4) in the table below represent the data from the task. Additionally we have denoted each considered event and corresponding conditional probabilities.

		LOTTERY 1		LOTTERY 2		
(*) coin tossing nad lottery choosing events		H		T		
		$H \cap T = \varnothing$ $H \cup T = \Omega$				
(1) probability of coin tossing event and buying a ticket of this lottery		0.5		0.5		
(**) events of lottery outcomes		A_1	A_2	B_1	B_2	B_3
		$A_1 \cap A_2 = \varnothing$ $A_1 \cup A_2 = \Omega H$		$B_1 \cap B_2 \cap B_3 = \varnothing$ $B_1 \cap B_2 \cup B_3 = \Omega T$		
(2) win amount		100	0	50	500	0
(3) ticket price		-10	-10	-20	-20	-20
(4) lottery event probability		0.1	0.9	0.1	0.01	0.89
(***) notation correspondence of lottery events probabilities	lottery-wise probabilities	$P(A_1)$	$P(A_2)$	$P(B_1)$	$P(B_2)$	$P(B_3)$
	conditional probabilities provided coin tossing events	$P(A_1\|H)$	$P(A_2\|H)$	$P(B_1\|T)$	$P(B_2\|T)$	$P(B_3\|T)$

The mosaic plot below illustrates the total sample space and gives an idea of "shares" taken by events and their corresponding probabilities.



2) Next we calculate values x of random variable X that denotes net payout, then we move to probability distribution of X, and expected value of X.

(5) net payout, values x of random variable X	90	-10	30	480	-20
(6) probability, $P(X = x)$ $0 \leq p(x_k) \leq 1, \sum_{k=1}^{5} p(x_k) = 1$	0.05	0.45	0.05	0.005	0.445
(7) expected net payout per outcome $(X = x) \cdot P(X = x)$	4.5	-4.5	1.5	2.4	-8.9
(8) overall expected value of net payout $E(X)$	-5.0				

EXPLANATIONS:
Line (5) is a summation result of lines (2) and (3) above.
Line (6) shows probability distribution and together with line (5) composes the probability mass function; it was calculated basing on the law of total probability as follows.

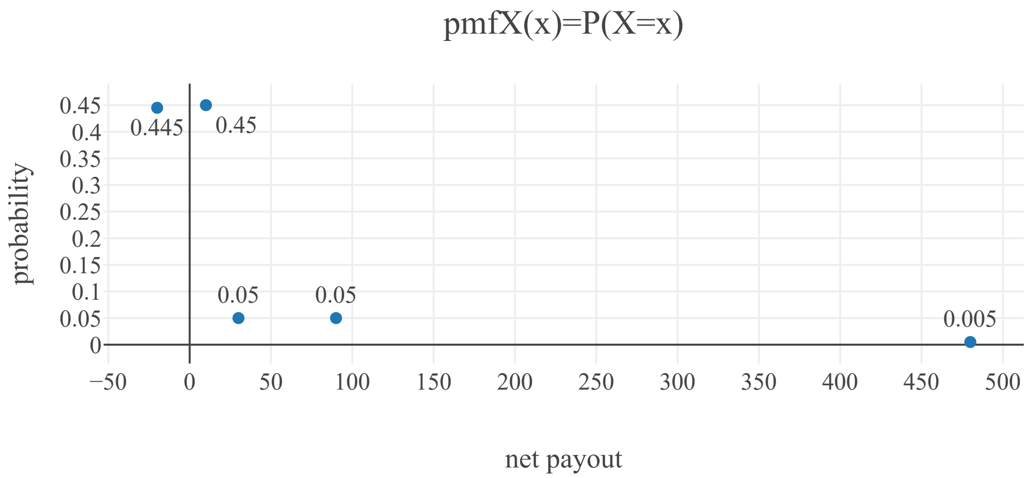
Having event properties as described in lines (*) and (**) above we can write

$$P(H) = P((A_1 \cap H) \cup (A_2 \cap H) = P(A_1 \cap H) + P(A_2 \cap H) =$$
$$P(H) \cdot P(A_1|H) + P(H) \cdot (A_2|H) = \sum_{i=1}^{2} P(H) \cdot P(A_i|H)$$

$$P(T) = (B_1 \cap T) \cup (B_2 \cap T) \cup (B_3 \cap T)) = P(B_1 \cap T) + P(B_2 \cap T) + P(B_3 \cap T) =$$
$$= P(T) \cdot (B_1|T) + P(T) \cdot P(B_2|T) + P(T) \cdot P(B_3|T) = \sum_{i=1}^{3} P(T) \cdot P(B_i|T)$$

This clearly shows that each value of $P(X = x)$ for a certain event is calculated through mutiplication of corresponding conditional probability of the event and the probability that this certain lottery was picked by coin-tossing.

Below is a plot of probability mass function for the probability distribution of X.



Line (7) represents results of multiplication of corresponding values of lines (6) and (7).

Line (8) contains a sum of all values of line (7): $E(X) = \sum_{k=1}^{5} x_k \cdot p(x_k)$.

3) Now we calculate expected values of net payout per lottery, find teir average and compare it to the overall expected values of net payout obtained in line (8) above.

	LOTTERY 1		LOTTERY 2		
(9) lotterywise expected net payout, X_H and X_T for lottery 1 and lottery 2 correspondingly	9.0	-9.0	3.0	4.8	-17.8
(10) lottery-wise expected value of net payout, $E(X_H)$ and $E(X_T)$	0.0		-10.0		
(11) average of expected values of lottery-wise net payouts	-5.0				

EXPLANATIONS:
Line (9) was calculated as product of lines (5) and (4) above.
Line (10) contains sum of line (9) values for each lottery.
Line (11) is an average of the values of line (10),

$$E_{avg} = \frac{E(X_H) + E(X_T)}{2} = \\ = \frac{x_{A_1} \cdot P(A_1) + x_{A_2} \cdot P(A_2) + x_{B_1} \cdot P(B_1) + x_{B_2} \cdot P(B_2) + x_{B_3} \cdot P(B_3)}{2}. \qquad (1)$$

3) As we can see the average value on line (11) is equal to overall expected value of net payout $E(X)$ on

line (8).

To explain this let us decompose expected value of X the following way:

$$E(X) = \sum_{k=1}^{5} x_k \cdot p(x_k) = \sum_{i=1}^{2} x_{A_i} \cdot P(H) \cdot P(A_i|H) + \sum_{i=1}^{3} x_{B_i} \cdot P(T) \cdot P(B_i|H) =$$
$$= P(H)\sum_{i=1}^{2} x_{A_i} \cdot P(A_i|H) + P(T)\sum_{i=1}^{3} x_{B_i} \cdot P(B_i|H). \quad (2)$$

Having $P(H) = P(T) = 1/2$ we can rewrite formula (2) as

$$\frac{x_{A_1} \cdot P(A_1|H) + x_{A_2} \cdot P(A_2|H) + x_{B_1} \cdot P(B_1|T) + x_{B_2} \cdot P(B_2|T) + x_{B_3} \cdot P(B_3|T)}{2}, \quad (3)$$

which is exactly the formula of average E_{avg} (1), taking into account notation correspondence in (***) above.

This decomposition proves that notwithstanding values of lottery payouts and their probabilities, which jointly form expected values of net lottery-wise payouts and compose denominators in (1) and (3), such equivalence will always hold as long as for each and every lottery a probability of buying its ticket is equal to the coefficient (weight), with which expected value of net payouts of this lottery is accounted for in an average.

In other words, there is no difference, if we first multiply by 1/2 to account for fair coin toss outcomes, then do the calculations and sum everything up, or we first perform calculations lottery-wise and only then multiply each outcome by 1/2 and sum up.

That means, in our case with lotteries and simple average we can obtain different results for coin tossing and afterward averaging, for example, in case of non-fair coin toss.