Question 1

Find out if the rows of each of the matrices form $A=\begin{pmatrix} 4 & 8 & 14 & 0 \\ 2 & 4 & 7 & 0 \end{pmatrix}$, $B=\begin{pmatrix} -1 & -1 & 0 & 1 \\ 4 & 8 & 14 & 0 \end{pmatrix}$ fundamental system of solutions (i.e. basis in the solution space) for the system of linear equations $\begin{cases} 5x_1+x_2-2x_3+6x_4=0 \\ x_1+3x_2-2x_3+4x_4=0 \\ 3x_1+2x_2-2x_3+5x_4=0 \\ 4x_1+5x_2-4x_3+9x_4=0 \end{cases}$

1) At first let us solve the system of equations.
Below is the matrix of the system. We don't make it extended because a column with 0 values is never affected by elementary transformations of a matrix and always remains intact, so $rank(C|b) = rank(C)$.

$$\widetilde{C} = \begin{pmatrix} 5 & 1 & -2 & 6 \\ 1 & 3 & -2 & 4 \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix} \sim \left\| \begin{matrix} e_1(-1)+e_2 5 \rightarrow e_2 \\ e_1(-3)+e_3 5 \rightarrow e_3 \\ e_1(-4)+e_4 5 \rightarrow e_4 \end{matrix} \right\| \sim \begin{pmatrix} 5 & 1 & -2 & 6 \\ 0 & 14 & -8 & 14 \\ 0 & 7 & -4 & 7 \\ 0 & 21 & -12 & 21 \end{pmatrix} \sim \left\| \begin{matrix} e_1 7+e_2(-1/2) \rightarrow e_1 \\ \\ e_2(-1/2)+e_3 \rightarrow e_3 \\ e_2(-3/2)+e_4 \rightarrow e_4 \end{matrix} \right\| \sim$$

$$\sim \begin{pmatrix} 35 & 0 & -10 & 35 \\ 0 & 14 & -8 & 14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \left\| \begin{matrix} e_1(1/35) \rightarrow e_1 \\ e_2(1/14) \rightarrow e_2 \\ \\ \\ \end{matrix} \right\| \sim \begin{pmatrix} 1 & 0 & -2/7 & 1 \\ 0 & 1 & -4/7 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2/7 & 1 \\ 0 & 1 & -4/7 & 1 \end{pmatrix} \qquad (1)$$

Rank of resulting matrix (1) is 2 ($r = 2$), the total number of unknowns is 4 ($n = 4$), so the variables x_1 and x_2 are basic and x_3 and x_4 are free. Below is the system of equations corresponding to the transformed matrix.

$$\begin{cases} x_1 \qquad -2/7x_3 \quad +x_4 \quad = 0 \\ \qquad x_2 \; -4/7x_3 \quad +x_4 \quad = 0 \end{cases}, \qquad (2)$$

which gives the general solution of the system:

$$\begin{aligned} x_1 &= \; 2/7x_3 \; - x_4 \\ x_2 &= \; 4/7x_3 \; - x_4 \end{aligned} \qquad (3)$$

2) Now let's consider matrices A and B from the task.

Rows of some abstract matrix D may potentially form fundamental system of solutions, which is a basis of the set of solutions, for a homogenous SLAE $Cx = 0$ only if $rank(D)$ and $n - rank(D)$ are equal to $rank(C)$ and $n - rank(C)$ correspondingly.

In our case rank of matrix A is 1 because its rows are linearly dependent so it cannot form the fundamental system of solutions for the given system of linear equations.

Rows of matrix B are linearly independent therefore we'll continue considering only this matrix.
Let's transform if to the reduced row echelon form.

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 \\ 4 & 8 & 14 & 0 \end{pmatrix} \sim \left\| e_1 4 + e_2 \rightarrow e_2 \right\| \sim \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 4 & 14 & 4 \end{pmatrix} \sim \left\| e_1 4 + e_2 \rightarrow e_1 \right\| \sim$$

$$\sim \begin{pmatrix} -4 & 0 & 14 & 8 \\ 0 & 4 & 14 & 4 \end{pmatrix} \sim \left\| \begin{matrix} e_1(-1/4) \rightarrow e_1 \\ e_2(1/4) \rightarrow e_2 \end{matrix} \right\| \sim \begin{pmatrix} 1 & 0 & -7/2 & -2 \\ 0 & 1 & 7/2 & 1 \end{pmatrix}$$

$rank(B)$ and $n - rank(B)$ are equal to $rank(C)$ and $n - rank(C)$ correspondingly, that means we can further explore the option with matrix B.

Having the general solution of the system we can substitute values from matrix B for corresponding variables x_1, x_2, x_3 and x_4 of the SLAE.

$$\begin{matrix} x_1 = & 2/7x_3 & -x_4 \\ x_2 = & 4/7x_3 & -x_4 \end{matrix} \Rightarrow \begin{matrix} 1 = & 2/7(-7/2) & -(-2) \\ 1 = & 4/7(7/2) & -1 \end{matrix} \Rightarrow \begin{matrix} 1 = & -1 & +2 \\ 1 = & 2 & -1 \end{matrix} \Rightarrow \begin{matrix} 1 & = & 1 \\ 1 & = & 1 \end{matrix} \quad (4)$$

As the equivalense (4) holds, we can say that rows of matrix B form the fundamental system of solutions for the homogenous SLAE $Cx = 0$ given in the task.