Question 4

Find a rank of a matrix A as a function of α. $A = \begin{pmatrix} \alpha & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & \alpha & 2 \end{pmatrix}$. For $\alpha = 7$ find a rank p of matrix A and a decomposition of A as a product of two matrices B and C, such that B has p columns and C has p rows.

1) Let us find rank of matrix A as a function of α. In order to make a deeper investigation we shall transform the matrix to a reduced row echelon form by applying the Gaussian elimination process. This will allow us to use the resulting matrix for further rank decomposition of the initially given matrix A.

$$A = \begin{pmatrix} \alpha & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & \alpha & 2 \end{pmatrix} = \begin{pmatrix} \alpha & 1 & 2 \\ 1 & \alpha & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \end{pmatrix} \sim \left\| \begin{array}{c} e_2 3 + e_4(-1) \to e_2 \\ e_3 3 + e_4(-5) \to e_3 \\ e_4(5) + e_3(-3) \to e_4 \end{array} \right\| \sim \begin{pmatrix} \alpha & 1 & 2 \\ 0 & 3\alpha - 3 & 0 \\ 0 & 0 & -36 \\ 0 & 0 & 36 \end{pmatrix} \sim$$

$$\sim \left\| \begin{array}{c} e_1 + e_2(-1/(3\alpha - 3)) \to e_1 \\ \\ \\ e_4 + e_3 \to e_4 \end{array} \right\| \sim \begin{pmatrix} \alpha & 0 & 2 \\ 0 & 3\alpha - 3 & 0 \\ 0 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} \sim \left\| \begin{array}{c} e_1 + e_3(1/18) \to e_1 \\ \\ \\ \\ \end{array} \right\| \sim$$

$$\sim \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 3\alpha - 3 & 0 \\ 0 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} \sim \left\| \begin{array}{c} e_1(1/\alpha) \to e_1 \\ e_2(1/(3\alpha - 3)) \to e_2 \\ e_3(-1/36) \to e_3 \end{array} \right\| \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^* \qquad (1)$$

$*$ For this elementary operations we should assume that $\alpha \neq 0,\ \alpha \neq 1$

The reduced row echelon form (1) of matrix A that we have achieved implies that notwithstanding the value of α matrix A has a rank of 3, i.e. $p = 3$.

2) Now we need to find two matrices B and C, such that B has p columns and C has p rows.
In our case matrix C with p rows is the reduced row echelon form matrix (1) we have got above.
As matrix B we should take p columns of the initially given matrix A. Having $p = 3$ we have to take the entire matrix A as matrix if the rank decomposition.
Therefore a decomposition of A as a product of two matrices B and C with $\alpha = 7$ would look as follows:

$$BC = \begin{pmatrix} 7 & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & 7 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$$