Question 2

Find a decomposition $A = U\Sigma V^T$ of the matrix A = $\begin{bmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

, where Σ is a rectangular diagonal matrix of size 4×3, U and V are orthogonal matrices, and the upper right element of V is equal to $1/\sqrt{2}$.

1) At first let us find matrices $B = A^T A$ and $C = AA^T$.

$$B = \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} \tag{1}$$

$$C = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -9 & 18 & 9 \\ -9 & 9 & -18 & -9 \\ 18 & -18 & 36 & 18 \\ 9 & -9 & 18 & 9 \end{pmatrix} \tag{2}$$

2) Now we need to find eigenvalues λ of B (1), this means to find all λ such that $det(B \ - I\lambda) = 0$.

$$\begin{vmatrix} 28-\lambda & 14 & 28 \\ 14 & 7-\lambda & 14 \\ 28 & 14 & 28-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 28-\lambda & 14 & 28 \\ 14 & 7-\lambda & 14 \\ 28 & 14 & 28-\lambda \end{vmatrix} = (28-\lambda)\begin{vmatrix} 7-\lambda & 14 \\ 14 & 28-\lambda \end{vmatrix} - 14\begin{vmatrix} 14 & 14 \\ 28 & 28-\lambda \end{vmatrix} + 28\begin{vmatrix} 14 & 7-\lambda \\ 28 & 14 \end{vmatrix} =$$

$$= (28-\lambda)((7-\lambda)(28-\lambda) - 14\cdot 14) - 14(14(28-\lambda) - 14\cdot 28) + 28(14\cdot 14 - (7-\lambda)\cdot 28) =$$

$$= (28-\lambda)\left(\lambda^2 \ - 35\lambda\right) + 196\lambda + 784\lambda = 28\lambda^2 - 980\lambda - \lambda^3 + 35^2\lambda + 980\lambda = -\lambda^3 + 63\lambda^2$$

Let us find λ from the following equation:

$$-\lambda^3 + 63\lambda^2 = 0$$
$$\lambda^2(\lambda - 63) = 0$$

$$\lambda = 0, \ \lambda = 63 \tag{3}$$

3) In general after rearranging the eigenvalues in descending order so that $\lambda_1 \geq ... \geq \lambda_n$ we are able to find the singular values of A as $\sigma_1 = \sqrt{\lambda_1}, ..., \sigma_n = \sqrt{\lambda_n}$ and then construct the matrix Σ.

In our case we have only one non-zero eigenvalue of $\lambda = 63$ (3), so $\sigma = \sqrt{63} = 3\sqrt{7}$ and the matrix Σ has the following look:

$$\Sigma = \begin{pmatrix} 3\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

4) Having found eigenvalues of B (1) we can find a collection $b = \{b_1,..., b_n\}$ of eigenvectors of the matrix. For this purpose we should find all $\vec{v}$ such that $(B - I\lambda)v = 0$.

Considering the first value of $\lambda = 0$ (3):

$$\begin{pmatrix} 28-\lambda & 14 & 28 \\ 14 & 7-\lambda & 14 \\ 28 & 14 & 28-\lambda \end{pmatrix} \cdot v = 0$$

$$\begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\,; \qquad \begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$$v_1 + v_2/2 + v_3 = 0$$
$$v_1 = -v_2/2 - v_3$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_2/2 - v_3 \\ v_2 \\ v_3 \end{pmatrix}$$

Substituting $\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} -v_2/2 - v_3 \\ v_2 \\ v_3 \end{pmatrix}$ gives an independent set of eigenvectors for B associated with the eigenvalue of $\lambda = 0$:

$$b_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ b_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \tag{5}$$

Considering the second value of $\lambda = 63$ (3):

$$\begin{pmatrix} 28-\lambda & 14 & 28 \\ 14 & 7-\lambda & 14 \\ 28 & 14 & 28-\lambda \end{pmatrix} \cdot v = 0$$

$$\begin{pmatrix} -35 & 14 & 28 \\ 14 & -56 & 14 \\ 28 & 14 & -35 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\,; \qquad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

This gives a system of equations:

$$\begin{cases} v_1 & -v_3 & = 0 \\ v_2 & -v_3/2 & = 0 \end{cases} \Rightarrow \begin{cases} v_1 & = v_3 \\ v_2 & = v_3/2 \end{cases}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_3/2 \\ v_3 \end{pmatrix}$$

Letting $v_3 = 2$ in $\begin{pmatrix} v_3 \\ v_3/2 \\ v_3 \end{pmatrix}$, we find the eigenvector of the matrix B associated with the eigenvalue of $\lambda = 63$:

$$b_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{6}$$

5) Eigenvectors associated with different eigenvalues are orthogonal. That means b_3 (6) is orthogonal to b_1 and b_2 (5), but b_1 and b_2 are not orthogonal to each other. We can fix it by applying the Gram-Schmidt process to b_1. Vector b_2 is preserved to comply with the task requirements.

If $h_1 = b_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ then $h_2 = b_2 - \dfrac{\langle b_2, h_1 \rangle}{\langle h_1, h_1 \rangle} h_1 - \dfrac{\langle b_2, h_3 \rangle}{\langle h_3, h_3 \rangle} h_3 = b_2 - h_1/2 = \begin{pmatrix} -1/2 \\ 2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$,
where h_3 is just the vector b_3 (6).

Now we have three orthogonal vectors:

$$h_1 = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}, \; h_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \; h_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{7}$$

6) Once we get the vectors (7) we need to convert them into unit vectors V^1, V^2 and V^3 and compose the matrix V.

$||h_1|| = \sqrt{16+1+1} = 3\sqrt{2}, \qquad ||h_2|| = \sqrt{1+1} = \sqrt{2}, \qquad ||b_3|| = \sqrt{4+1+4} = 3$

$$V^1 = \begin{pmatrix} -1/3\sqrt{2} \\ 4/3\sqrt{5} \\ -1/3\sqrt{2} \end{pmatrix}, \qquad V^2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \qquad V^3 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

Below is the matrix V with vectors arranged in a decreasing order according to associated λ. In case of equal λ (5), in order to follow conditions of the task let's order the vectors in an increasing order according to values of v_3 and v_2 above:

$$V = \begin{pmatrix} 2/3 & -1/3\sqrt{2} & -1/\sqrt{2} \\ 1/3 & 4/3\sqrt{2} & 0 \\ 2/3 & -1/3\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{8}$$

7) In order to compose the matrix U we need to find its vectors one by one.

As a matter of fact, having non-zero singular values we can find vectors of U one by one through the following calculation: $U^i = \sigma_i A V^i$. In this case we shall need only to adjust the very right vector of U either by solving a SLAE of vector pair zero dot-products or by orthogonalazing an arbitrary vector through applying the Gram-Schmidt process.
However, in our case we can find only the first vector of U using this approach as we have only one non-zero σ.

$$U^1 = \sigma_1 A V^1 = \frac{1}{3\sqrt{7}} \cdot \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \frac{1}{3\sqrt{7}} \cdot \begin{pmatrix} 3 \\ -3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{7} \\ -1/\sqrt{7} \\ 2/\sqrt{7} \\ 1/\sqrt{7} \end{pmatrix}$$

Generally, the other 3 vectors of U may be taken arbitrarily and then orthogonalized with regard to U^1 and each other through the Gram-Schmidt process.
Or we may even make up and manually compose a couple of vectors by adjusting them to get a zero dot-product with U^1 and each other and then solve a system of linear algebraic equations derived from such zero dot-products of vector pairs to obtain the last vector.

But let us compose U by going, probably, the longest way and finding eigenvectors of the matrix C (2) from above. Looking ahead we should note that our left-most vector of the matrix U, which we are going to find below, will be the same as U^1 above, and it will be denoted as U^1 as well.

At first, for that we need to find eigenvalues λ, this means to find all λ such that $det(C - I\lambda) = 0$.

$$\begin{vmatrix} 9-\lambda & -9 & 18 & 9 \\ -9 & 9-\lambda & -18 & -9 \\ 18 & -18 & 36-\lambda & 18 \\ 9 & -9 & 18 & 9-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 9-\lambda & -9 & 18 & 9 \\ -9 & 9-\lambda & -18 & -9 \\ 18 & -18 & 36-\lambda & 18 \\ 9 & -9 & 18 & 9-\lambda \end{vmatrix} = \lambda^4 - 63\lambda^3$$

$$\lambda = 0, \ \lambda = 63 \tag{9}$$

Considering the first value of $\lambda = 0$ (9):

$$\begin{pmatrix} 9-\lambda & -9 & 18 & 9 \\ -9 & 9-\lambda & -18 & -9 \\ 18 & -18 & 36-\lambda & 18 \\ 9 & -9 & 18 & 9-\lambda \end{pmatrix} \cdot v = 0$$

$$\begin{pmatrix} 9 & -9 & 18 & 9 \\ -9 & 9 & -18 & -9 \\ 18 & -18 & 36 & 18 \\ 9 & -9 & 18 & 9 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0\,; \qquad \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} v_2 - 2v_3 - v_4 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

Substituting $\begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} v_2 - 2v_3 - v_4 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ gives an independent set of eigenvectors for C associated with the eigenvalue of $\lambda = 0$:

$$c_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ c_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} and\ c_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Considering the second value of $\lambda = 63$ (9):

$$\begin{pmatrix} 9-\lambda & -9 & 18 & 9 \\ -9 & 9-\lambda & -18 & -9 \\ 18 & -18 & 36-\lambda & 18 \\ 9 & -9 & 18 & 9-\lambda \end{pmatrix} \cdot v = 0$$

$$\begin{pmatrix} -54 & -9 & 18 & 9 \\ -9 & -54 & -18 & -9 \\ 18 & -18 & -27 & 18 \\ 9 & -9 & 18 & -54 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0\,; \qquad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_\$ \end{pmatrix} = 0$$

This gives a system of equations:

$$\begin{cases} v_1 & -v_4 & = 0 \\ v_2 & +v_4 & = 0 \\ v_3 & -2v_4 & = 0 \end{cases} \Rightarrow \begin{cases} v_1 & = v_4 \\ v_2 & = -v_4 \\ v_3 & = 2v_4 \end{cases}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} v_4 \\ -v_4 \\ 2v_4 \\ v_4 \end{pmatrix}$$

Letting $v_4 = 1$ in $\begin{pmatrix} v_4 \\ -v_4 \\ 2v_4 \\ v_4 \end{pmatrix}$, we find the eigenvector of the matrix C associated with the eigenvalue of $\lambda = 63$:

$$c_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} \tag{11}$$

8) And again eigenvectors associated with different eigenvalues are orthogonal. That means c_4 (11) is orthogonal to c_1, c_2 and c_3 (10), but c_1, c_2 and c_3 are not orthogonal to each other. Fixing that by applying the Gram-Schmidt process. To save some space we won't write the summand representing orthogonal projection of c_4 as it is equal to 0 for all of c_1, c_2 and c_3.

If $g_1 = c_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ then $g_2 = c_2 - \dfrac{\langle c_2, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 = c_2 - g_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, and

$$g_3 = c_3 - \frac{\langle c_3, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 - \frac{\langle c_3, g_2 \rangle}{\langle g_2, g_2 \rangle} g_2 = c_3 - (-1/2) g_1 - (-1/3) g_2 =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 0 \\ 1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 1 \\ 1/3 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

Now we have four orthogonal vectors, where g_4 is just the vector c_4 (11):

$$g_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \; g_3 = \begin{pmatrix} 1 \\ 6 \\ 2 \\ 1 \end{pmatrix}, \; g_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} \tag{12}$$

9) Once we get the vectors (12) we need to convert them into unit vectors $U^1, \; U^2, \; U^3$ and U^4 and compose the matrix U

$||g_1|| = \sqrt{2}$, $||g_2|| = \sqrt{3}$, $||g_3|| = \sqrt{42}$, $||g_4|| = \sqrt{7}$

$$U^1 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad U^2 = \begin{pmatrix} -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}, \quad U^3 = \begin{pmatrix} 1/\sqrt{42} \\ 6/\sqrt{42} \\ 2/\sqrt{42} \\ 1/\sqrt{42} \end{pmatrix}, \quad U^3 = \begin{pmatrix} 1/\sqrt{7} \\ -1/\sqrt{7} \\ 2/\sqrt{7} \\ 1/\sqrt{7} \end{pmatrix}$$

The matrix U with vectors ordered in a decreasing order according to associated λ. In case of equal λ (10), let's again order vectors in an increasing order according to values of v_4, v_3 and v_2 above:

$$
U = \begin{pmatrix} 1/\sqrt{7} & -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{42} \\ -1/\sqrt{7} & 0 & 0 & 6/\sqrt{42} \\ 2/\sqrt{7} & 0 & 1/\sqrt{3} & 2/\sqrt{42} \\ 1/\sqrt{7} & 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{42} \end{pmatrix} \tag{13}
$$

CONCLUSION.

Having found matrices U (13), Σ (4) and V (8) we can make sure that the decomposition is correct by checking if the following equivalence holds. Additionally we multiply both U and V by -1 to meet the conditions of the task that the upper right element of V should be equal to $1/\sqrt{2}$.

$$
A = (-1)U\Sigma((-1)V)^T
$$

$$
A = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} =
$$

$$
= \begin{pmatrix} -1/\sqrt{7} & 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{7} & 0 & 0 & -6/\sqrt{42} \\ -2/\sqrt{7} & 0 & -1/\sqrt{3} & -2/\sqrt{42} \\ -1/\sqrt{7} & -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{42} \end{pmatrix} \cdot \begin{pmatrix} 3\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ -1/3 & -4/3\sqrt{2} & 0 \\ -2/3 & 1/3\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^T =
$$

$$
= \begin{pmatrix} -3 & 0 & 0 \\ 3 & 0 & 0 \\ -6 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2/3 & -1/3 & -2/3 \\ 1/3\sqrt{2} & -4/3\sqrt{2} & 1/3\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} = A
$$

Now we can say that the SVD decomposition is correct and the matrices U, Σ and V are:

$$
U = \begin{pmatrix} -1/\sqrt{7} & 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{7} & 0 & 0 & -6/\sqrt{42} \\ -2/\sqrt{7} & 0 & -1/\sqrt{3} & -2/\sqrt{42} \\ -1/\sqrt{7} & -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{42} \end{pmatrix}
$$

$$
\Sigma = \begin{pmatrix} 3\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} -2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ -1/3 & -4/3\sqrt{2} & 0 \\ -2/3 & 1/3\sqrt{2} & -1/\sqrt{2} \end{pmatrix}
$$