Question 1

> A fair coin is tossed 400 times. Let X be number of heads. Prove that $P(X > 240) \leq 1/32$.
> (Hint: use Chebyshev's inequality and symmetry considerations.)

Having $n = 400$ and $p = 0.5$ we can calculate expected value and variance of X:

$$E(X) = np = 400 \cdot 0.5 = 200$$
$$VarX = np(1-p) = 400 \cdot 0.5 \cdot 0.5$$

Generally, Chebyshev's inequality expression is:

$$P(|X - E(X)| > \alpha) \leq \frac{VarX}{\alpha^2}$$

We can rewrite it using previously calculated values:

$$P(|X - 200| > \alpha) \leq \frac{100}{\alpha^2}$$

Or, considering symmetry of $pmf_X(x)$:

$$P(200 - X > \alpha) + P(X - 200 > \alpha) \leq \frac{100}{\alpha^2}$$
$$P(X < 200 - \alpha) + P(X > 200 + \alpha) \leq \frac{100}{\alpha^2}$$

Having the target probability $P(X > 240)$ we substitute 40 for α:

$$P(X < 200 - 40) + P(X > 200 + 40) \leq \frac{100}{40^2}$$
$$P(X < 160) + P(X > 240) \leq \frac{1}{16} \tag{1}$$

Again, on the basis of symmetry of $pmf_X(x)$ we can state that $P(X < 160) = P(X > 240)$, which turns the inequality (1) above into

$$2 \cdot P(X > 240) \leq \frac{1}{16}$$
$$P(X > 240) \leq \frac{1}{32}$$