Question 2

The main prize at some TV show is a car. At the beginning the car is placed behind one door randomly chosen out of three (with equal probabilities). Participant has to guess the door with a car. If the guess is correct, the participant wins the car, otherwise she wins nothing. After the participant picks the door and announces her choice, host opens one of the two remaining doors and shows that there are no car there. Then the participant can either switch her decision and pick the remaining closed door or keep the door she picked initially. After the final decision is made, the chosen door opens and the participant get her prize (if any). Assume that the host never opens a door that is picked by the participant initially and never opens a door with a car. If the host can chose between several doors, the choice is random (with equal probabilities). Let us enumerate doors in such a way that the door initially picked by the participant has number 1.

Consider events H_1: *the car is behind door number 1*, H_2: *the car is behind door number 2* and H_3: *the car is behind door number 3*. Consider also event A: *the host opened door number 2*.

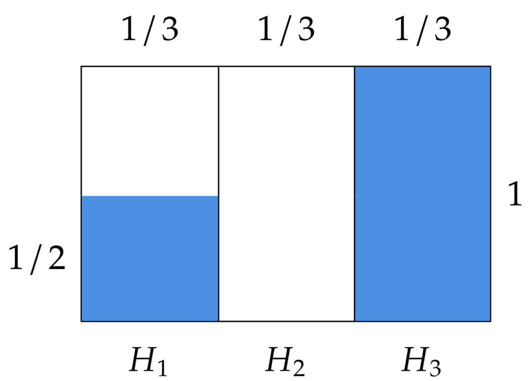
1. What can you say about probability of H_3 before any door is opened?
2. Assume that the host opened door number 2. What would you say about probability of H_3 after you observe that?
3. Are events A and H_1 independent? A and H_2? A and H_3?
4. Use Bayes' rule to find $P(H_1 \mid A)$ and $P(H_3 \mid A)$ (find all necessary probabilities that are used in Bayes' rule first). Compare with your previous answers.
5. Should the participant change the initial decision to increase probability of winning?

Please, provide full explanations of your answers, with proper references to the facts that were discussed in the videos and all necessary calculations and derivations.

1) Before any door is opened, the probability of H_3 is $P(H_3) = 1/3$. We know that the car is randomly placed behind one of the three doors with equal probability, therefore each door has an equal chance of having the car behind it so $P(H_1) = P(H_2) = P(H_3) = 1/3$.

2) The influence of event A on the probabiluty of event H_3 is rather counter-intuitive. Most common conclusion is that after A occured the probability of H_3 would increase from $1/3$ to $1/2$. However, this is not true and further we are showing it.

3) For next explanations let us introduce a mosaic plot reflecting the discussed probabilities. Event A is marked with blue color, and conditional probabilities that A occurs after either of events H_1, H_2 or H_3 are shown at the sides of the plot.



Formally we can write it the following way:
$P(A|H_1) = 1/2$ as after H_1 occurs the host has to choose between door 2 and door 3 randomly with eaqual probability, because he never opens the door picked by the participant and the door with the car, which is the same door in this case,
$P(A|H_2) = 0$ as after H_2 occurs the host cannot open door 2 because he never opens the door with the car,
$P(A|H_3) = 1$ as after H_3 occurs the host can open only door 2 because he never opens the door picked by the participant and the door with the car.

Now we can check for event independence by validating the equivalence for each pair of events:

$$P(A \cap H_i) = P(A) \cdot P(H_i), \text{ where}$$

$i = 1,\ 2,\ 3;$
$P(A) = 1/3 \cdot 1/2 + 1/3 \cdot 1 = 1/2.$

$P(A \cap H_1) = P(A) \cdot P(H_1)$ $1/2 \cdot 1/3 = 1/2 \cdot 1/3$ $1/6 = 1/6$ the equivalcence holds, the events are independent	$P(A \cap H_2) = P(A) \cdot P(H_2)$ $0 \cdot 1/3 = 1/2 \cdot 1/3$ $0 \neq 1/6$ the equivalcence doesn't hold, the events are not independent	$P(A \cap H_3) = P(A) \cdot P(H_3)$ $1 \cdot 1/3 = 1/2 \cdot 1/3$ $1/3 \neq 1/6$ the equivalcence doesn't hold, the events are not independent

As we can see only events A and H_1 are independent.

4) According to Bayes' rule $P(H|A) = \dfrac{P(A|H) \cdot P(H)}{P(A)}$.
Hence:

$$P(H_1|A) = \frac{P(A|H_1) \cdot P(H_1)}{P(A)} = \frac{1/2 \cdot 1/3}{1/2} = 1/3$$

$$P(H_3|A) = \frac{P(A|H_3) \cdot P(H_3)}{P(A)} = \frac{1 \cdot 1/3}{1/2} = 2/3$$

Now we can see that after event A accurs the probability of H_3 increases from $1/3$ to $2/3$, but not to $1/2$, accoring to the common conclusion in section 2) above.

5) The participant defenitely should change his initial decision and pick door 3 as it will double his chances to win the car.