Quantum Mechanics 2

The statistical interpretation of quantum mechanics

(Year 2, core module)

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A brief recap

Postulates of quantum mechanics:

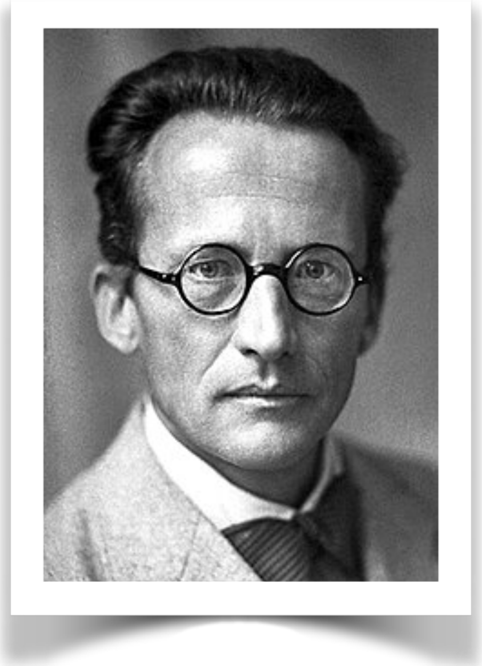
(i) State described by a **wavefunction** $\psi(x,t)$

(ii) **Born rule**:

$$\boxed{P_{ab|t} = \int_a^b dx\, |\psi(x,t)|^2}$$

(iii) **Schrödinger equation:**

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\,\psi(x,y)$$



Our plan for today

By the end of this lecture, you will have learnt…

…to make **statistical predictions using the Born** rule

…to **normalise a wavefunction**

…to use wave functions to **calculate expectation values and uncertainties**

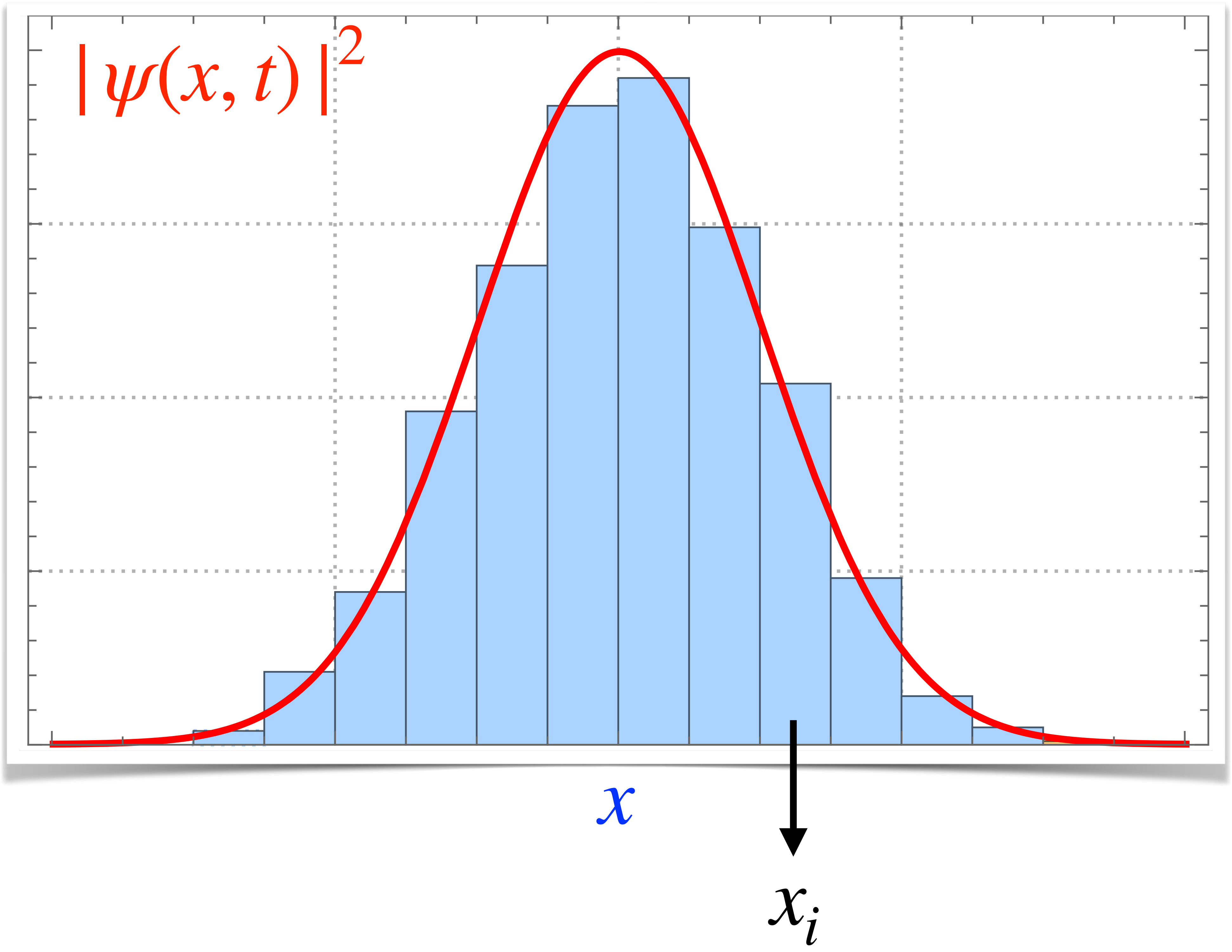
Working with probabilities

- $0 \leq P \leq 1$ (from impossible to certain)
- **Normalisation:** $\sum_i P_i = 1$
- **Average value:** $\langle x \rangle = \sum_i P_i x_i$
- **Standard deviation:** $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

Exercise
Four people attend a lecture. One of them is 18 years old, while the other three are are 21 years old. Calculate the **average age** and the **width of the distribution** (standard deviation).

Working with probabilities

In continuous variables:



mnemonic rule

$$\sum_i \mapsto \int dx$$

$$\downarrow$$

$$\sum_i P_i = 1$$

$$\boxed{\int_{-\infty}^{\infty} dx\,|\psi(x,t)|^2 = 1}$$

$$\downarrow$$

$$\langle x \rangle = \sum_i P_i\, x_i$$

$$\boxed{\langle x \rangle_t = \int_{-\infty}^{\infty} dx\,|\psi(x,t)|^2\, x}$$

Statistical predictions from a wave function

For the sake of concreteness, consider the following wavefunction:

$$\psi(x,0) = \mathcal{N} e^{-x^2/\lambda^2} .$$

- **Normalisation:**

1 $$\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 = \mathcal{N}^2 \int_{-\infty}^{\infty} dx\, e^{-2x^2/\lambda^2}$$

2 $$= \frac{\lambda \mathcal{N}^2}{\sqrt{2}} \int_{-\infty}^{\infty} dy\, e^{-y^2}$$

3 $$= \lambda \mathcal{N}^2 \sqrt{\frac{\pi}{2}} = 1 \longrightarrow \boxed{\psi(x,0) = \frac{e^{-x^2/\lambda^2}}{\sqrt{\lambda\sqrt{\pi/2}}}}$$

Statistical predictions from a wave function

For the sake of concreteness, consider the following wavefunction:

$$\psi(x,0) = \frac{e^{-x^2/\lambda^2}}{\sqrt{\lambda\sqrt{\pi/2}}}.$$

- **Average value:**

1 $$\langle x \rangle = \int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 x$$

2 $$\propto \int_{-\infty}^{\infty} dx\, \underbrace{e^{-2x^2/\lambda^2}}_{\text{even}}\, \underbrace{x}_{\text{odd}} = 0 \longrightarrow \boxed{\langle x \rangle = 0}$$

Statistical predictions from a wave function

For the sake of concreteness, consider the following wavefunction:

$$\psi(x,0) = \frac{e^{-x^2/\lambda^2}}{\sqrt{\lambda\sqrt{\pi/2}}}.$$

- **Standard deviation:**

① $$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \, |\psi(x,t)|^2 x^2$$

② $$= \frac{1}{\lambda}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \, e^{-2x^2/\lambda^2} x^2 = 0$$

③ $$= \frac{\lambda^2}{4}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dy \, e^{-y^2} y^2 = \frac{\lambda^2}{4} \longrightarrow \boxed{\sigma = \sqrt{\langle x^2 \rangle} = \frac{\lambda}{2}}$$

Next time

- Conservation of probability
- **Heisenberg's uncertainty principle**

Challenge of the week

Given the definition:

$$\langle x \rangle = \sum_i P_i x_i,$$

show that

$$\boxed{\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle = \langle x^2 \rangle - \langle x \rangle^2}.$$

Hint: $\langle x \rangle$ and $\langle x^2 \rangle$ are constant with respect to the summation symbol $\sum_i$.