Extracting maximum information from limited data

Jesús Rubio*, Paul Knott, and Jacob Dunningham
Department of Physics and Astronomy, University of Sussex, Brighton, UK
This work is funded by SEPnet and EPSRC through the NQIT Quantum Technologies Hub

Research problem

Empirical data constitute our primary source of information to construct theories that explain the world around us, and to develop the necessary technologies that help us to accomplish that task. However, the quality of this information is restricted in practice by factors such as the number of experiments that we can perform. This is particularly relevant for the study of fragile systems, or when only a few observations are possible before the system under study is out of reach. In this work we discuss a methodology to develop quantum-enhanced metrology protocols that are suitable for these scenarios.

Single-parameter methodology

Quantum metrology protocols

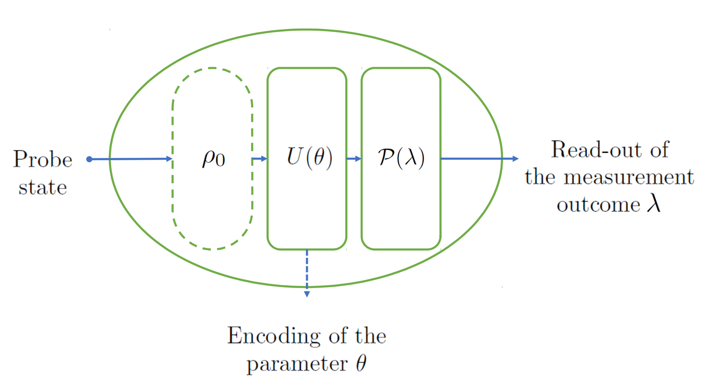

Figure 1: Quantum sensor model.

- *State preparation:*

$$\text{Experimental arrangement} \longrightarrow \rho_0$$

- *Unknown parameter encoding:*

$$\rho_0 \longrightarrow \rho(\theta) = U(\theta)\rho_0 U^{\dagger}(\theta)$$

- *Measurement scheme and data read-out:*

$$E(\lambda) \longrightarrow \text{outcome } \lambda,$$

with probability $p(\lambda|\theta) = \mathrm{Tr}\left[E(\lambda)\rho(\theta)\right]$

- *Parameter information summary:*

$$\text{prior } p(\theta), \text{ likelihood } p(\lambda|\theta) \longrightarrow p(\theta, \lambda)$$

- *Parameter estimation:*

$$p(\theta,\lambda) \longrightarrow \begin{cases} \text{estimate} : g(\lambda) \\ \text{uncertainty} : \sqrt{\bar{\epsilon}} \end{cases}$$

Our strategy

1. Optimal scheme for one shot, with
   (a) given ρ_0 and $U(\theta)$, and
   (b) a flat prior for $\theta \in [-W/2, W/2]$.
2. μ repetitions of the optimal one-shot strategy.

One-shot measure of uncertainty

If $W \lesssim 2$ (regime of moderate prior knowledge), then $\bar{\epsilon} \approx \bar{\epsilon}_{\mathrm{mse}}$, where

$$\bar{\epsilon}_{\mathrm{mse}} = \int d\lambda\, d\theta\, p(\theta) p(\lambda|\theta) [g(\lambda) - \theta]^2$$

is the mean square error.

One-shot optimal POVM

The optimal error for $\mu = 1$ is [1, 2]

$$\bar{\epsilon}_{\mathrm{mse}} \geqslant \int d\theta p(\theta)\theta^2 - \mathrm{Tr}\left(L^2 \rho\right),$$

where $L\rho + \rho L = 2\bar{\rho}$, $\rho = \int d\theta p(\theta)\rho(\theta)$ and $\bar{\rho} = \int d\theta p(\theta)\rho(\theta)\theta$, and the optimal strategy is given by

$$L = \int d\lambda\, \lambda |\lambda\rangle\langle\lambda|,$$

where λ are the estimates $E(\lambda) = |\lambda\rangle\langle\lambda|$ are the POVM elements.

Measure of uncertainty for μ observations

The experimental information of μ identical and independent experiments is encoded in

$$p(\boldsymbol{\lambda}|\theta) = p(\lambda_1|\theta) \dots p(\lambda_\mu|\theta),$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_\mu)$ are the outcomes. The estimation error is then

$$\bar{\epsilon}_{\mathrm{mse}} = \int d\boldsymbol{\lambda} d\theta p(\theta) p(\boldsymbol{\lambda}|\theta) [g(\boldsymbol{\lambda}) - \theta]^2,$$

which is the quantity of interest to study the low-μ regime.

Mach-Zehnder interferometer: results

Bounds for low μ and moderate prior

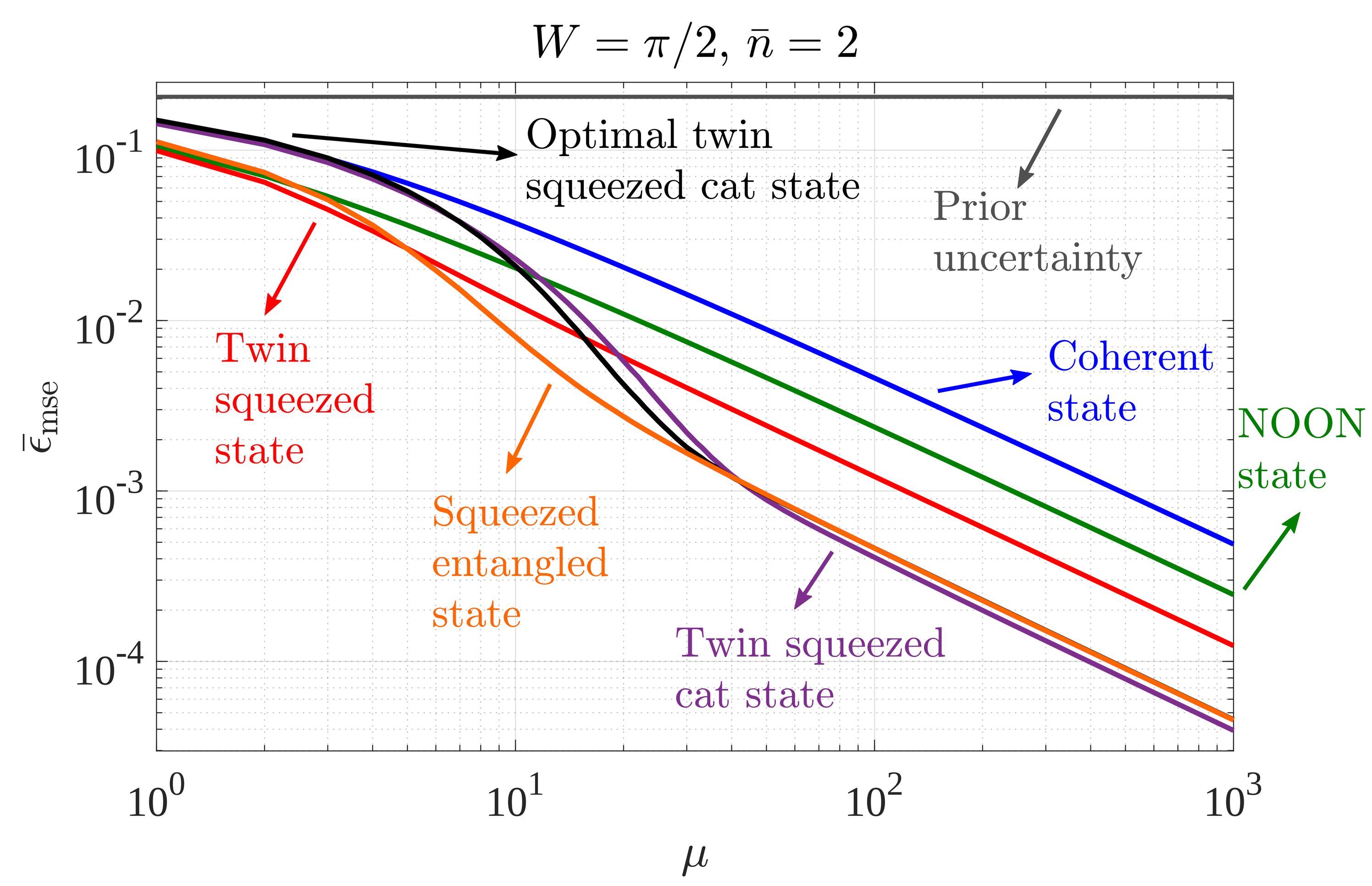

Figure 2: Bayesian uncertainty in the regime of low μ and moderate prior knowledge using optical probes. We can observe that the relative performance of each probe in the low-μ regime is significantly different from what happens in the asymptotic regime, and the relevance of the inter-mode correlations increases for low μ. In addition, the quantum Cramér-Rao bound is recovered asymptotically, which implies that our results are optimal for both one shot and a large number of them.

Experimentally accessible measurements

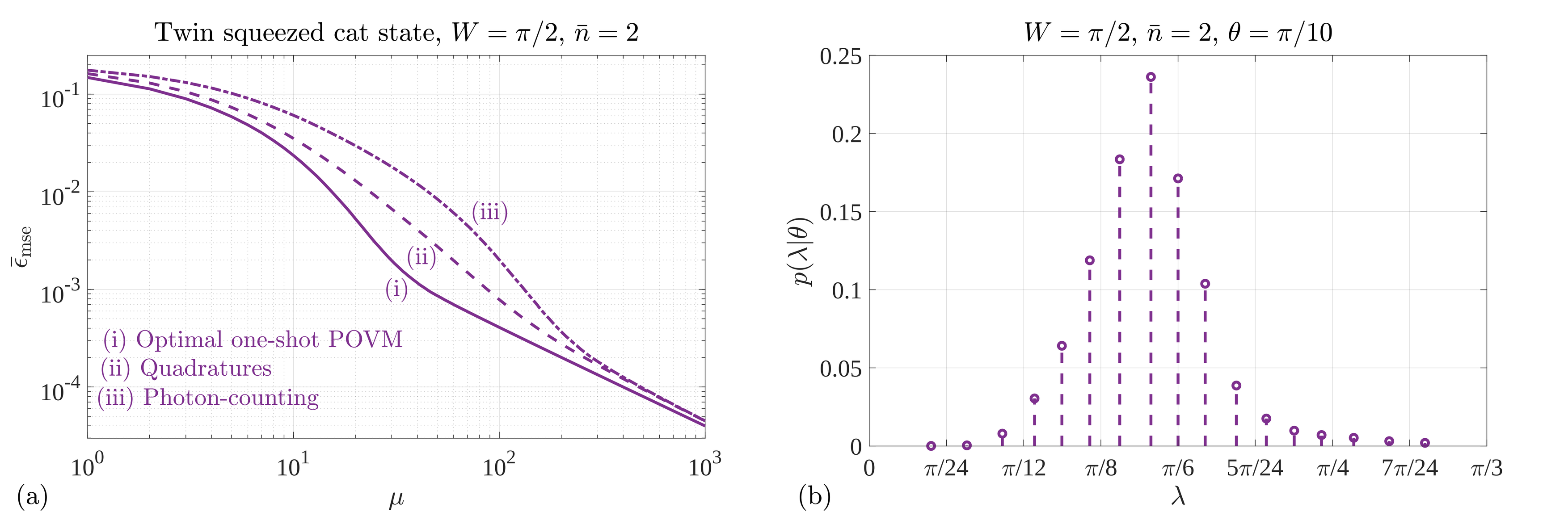

Figure 3: (a) Performance of energy and quadrature measurements for a twin squeezed cat state, and (b) likelihood function of the optimal POVM for one shot. We notice that these practical measurements approach both the one-shot bound and the quantum Cramér-Rao bound when $\mu \gg 1$.

Multi-parameter methodology

Uncertainty for μ observations and d parameters

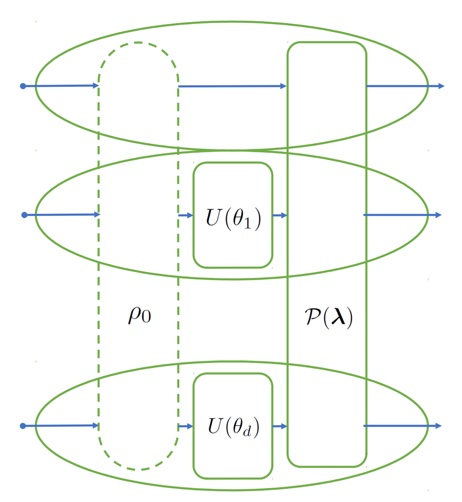

Figure 4: Quantum network model, where $U(\theta_i)$ encodes the i-th parameter.

- *Information summary for d parameters:*

$$p(\boldsymbol{\theta}, \boldsymbol{\lambda}) = p(\boldsymbol{\theta})\mathrm{Tr}[E(\boldsymbol{\lambda})\rho(\boldsymbol{\theta})]$$

- *Measure of uncertainty:*

$$\bar{\epsilon}_{\mathrm{mse}} = \int d\boldsymbol{\lambda} d\boldsymbol{\theta} p(\boldsymbol{\theta}, \boldsymbol{\lambda}) \sum_{i=1}^{d} \frac{\left(g_i(\boldsymbol{\lambda}) - \theta_i\right)^2}{d},$$

where the prior widths W_i are small enough to assume that $\bar{\epsilon} \approx \bar{\epsilon}_{\mathrm{mse}}$.

Simultaneous and independent strategies

Given $\bar{n}$ quanta, we can [3]

(a) prepare a single generalised NOON state using all the resources, or

(b) estimate r parameters using independent NOON states with $n+1$ quanta and $d - r$ parameters using NOON states with n quanta, where $\bar{n} = nd + r$.

Quantum Ziv-Zakai bound

When the probe is symmetric with respect to the exchange of sensors, we have that [4]

$$\bar{\epsilon}_{zz} = \frac{d}{2}\int_0^W d\theta\, h(\theta) \left\{1 - \sqrt{1 - |f(\theta)|^{2\mu}}\right\},$$

where $\rho_0 = |\psi_0\rangle\langle\psi_0|$, $h(\theta) = \theta(1 - \theta/W)$, $f(\theta) = \langle\psi_0|U(\theta)|\psi_0\rangle$ and $\bar{\epsilon}_{\mathrm{mse}} \geqslant \bar{\epsilon}_{zz}$. Our goal is to study the advantage of the simultaneous scheme as a function of μ and d.

Network of interferometers: results

Simultaneous versus independent strategies in the low-μ regime

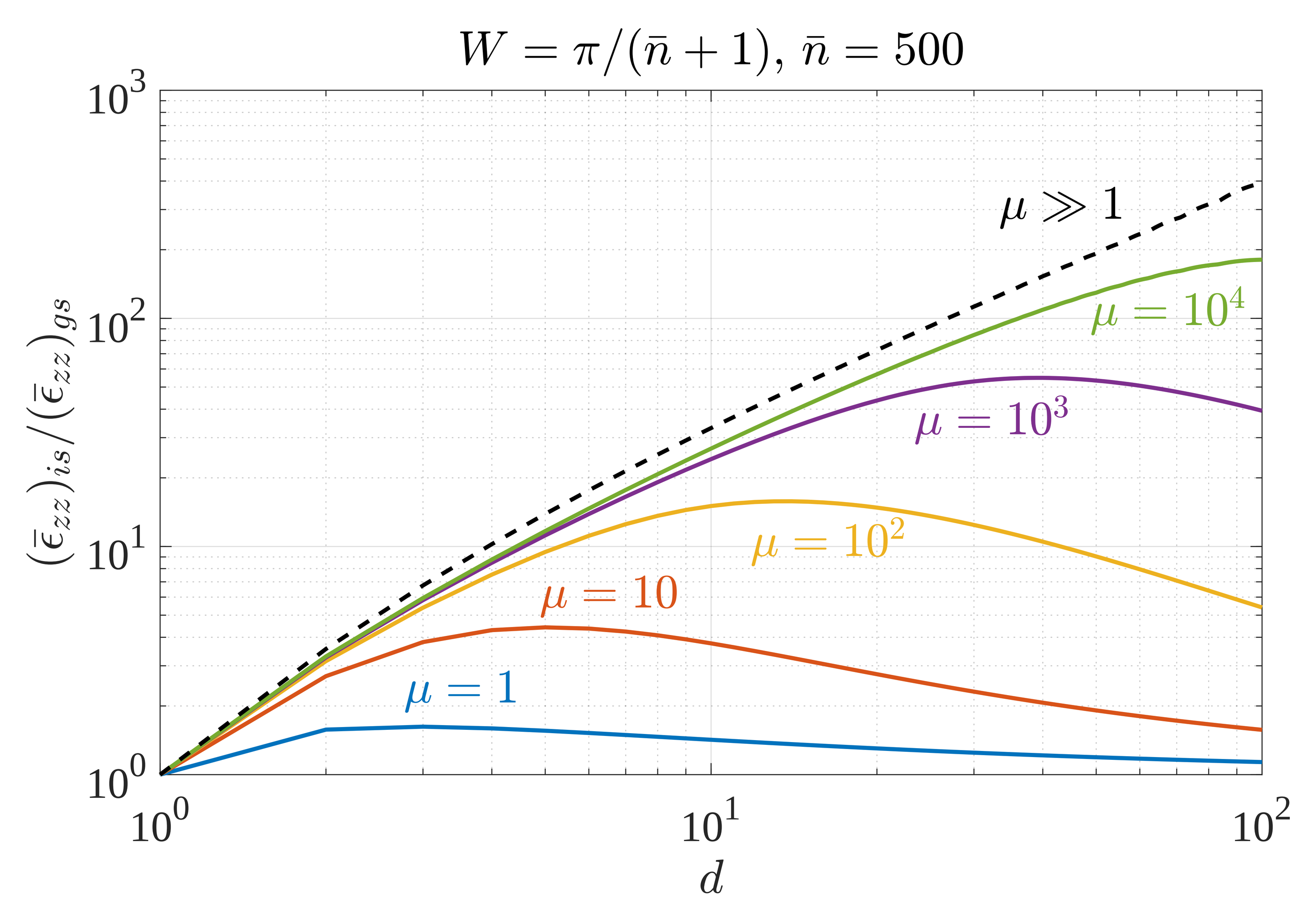

Figure 5: Ratio of the uncertainty for an independent strategy to the error of a simultaneous scheme in terms of the quantum Ziv-Zakai bound. Our result provides the transition between the $\mathcal{O}(d)$ advantage of the asymptotic regime and the small advantage observed in the one-shot regime. Moreover, it can be seen that the asymptotic prediction is reasonably accurate when $d \lesssim \sqrt{\mu}$.

References

* J.Rubio-Jimenez@sussex.ac.uk

[1] S. Personick. Application of quantum estimation theory to analog communication over quantum channels. *IEEE Transactions on Information Theory*, 17(3):240–246, 1971.

[2] C. W. Helstrom. *Quantum Detection and Estimation Theory*. Academic Press, New York, 1976.

[3] P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley. Quantum Enhanced Multiple Phase Estimation. *Phys. Rev. Lett.*, 111:070403, 2013.

[4] Yu-Ran Zhang and Heng Fan. Quantum metrological bounds for vector parameters. *Phys. Rev. A*, 90:043818, 2014.

[5] Jesús Rubio and Jacob Dunningham. Quantum metrology in the presence of limited data. *In preparation*.

[6] Jesús Rubio, Paul Knott, and Jacob Dunningham. Non-asymptotic analysis of quantum metrology protocols beyond the Cramér-Rao bound. *Journal of Physics Communications*, 2(1):015027, 2018.

[7] E. T. Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, 2003.

[8] R. Demkowicz-Dobrzanski, M. Jarzyna, and J. Kolodynski. Quantum Limits in Optical Interferometry. *Progress in Optics*, 60:345–435, 2015.

[9] P. A. Knott, T. J. Proctor, A. J. Hayes, J. P. Cooling, and J. A. Dunningham. Practical quantum metrology with large precision gains in the low-photon-number regime. *Phys. Rev. A*, 93:033859, 2016.