Quantum scale metrology

Highly-precise measurements beyond phase estimation

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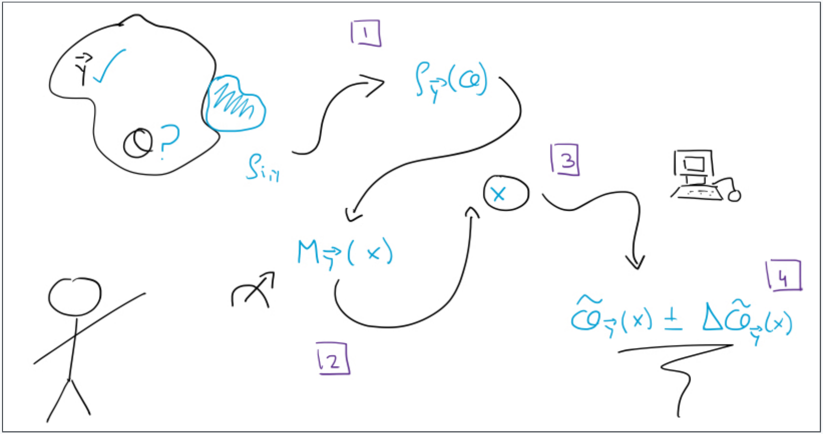
Our plan for today

1. Quantum metrology *beyond phase estimation*
2. The implications of scale invariance
3. Optimal strategies for scale estimation
4. A case study: atomic lifetimes



Quantum metrology *beyond phase estimation*



Quantum metrology: fundamental problem

Minimise the *error functional*

$$\bar{\epsilon}_{\boldsymbol{y}} = \int d\theta dx\, p(\theta) \mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta) M_{\boldsymbol{y}}(x)] \mathcal{D}[\tilde{\theta}_{\boldsymbol{y}}(x), \theta]$$

w.r.t. $\tilde{\theta}_{\boldsymbol{y}}(x)$, $M_{\boldsymbol{y}}(x)$ for given $p(\theta)$, $\rho_{\boldsymbol{y}}(\theta)$.

Setup	Estimation	Probabilities
$x \equiv$ measurand	$\theta \equiv$ hypothesis	$p(\theta) \equiv$ prior
$\boldsymbol{y} \equiv$ calibration	$\tilde{\theta}_{\boldsymbol{y}}(x) \equiv$ estimator	$\rho_{\boldsymbol{y}}(\theta) \equiv$ state
$\Theta \equiv$ unknown	$\mathcal{D}[\tilde{\theta}_{\boldsymbol{y}}(x), \theta] \equiv$ deviation	$M_{\boldsymbol{y}}(x) \equiv$ POM

Quantum metrology: ultimate precision limits

Let $\tilde{\vartheta}_{\boldsymbol{y}}(x), \mathcal{M}_{\boldsymbol{y}}(x)$ be the optimal strategy resulting from the minimisation problem above. Then,

$$\bar{\epsilon}_{\boldsymbol{y}} \geqslant \bar{\epsilon}_{\boldsymbol{y}}|_{\tilde{\vartheta}(x)} \geqslant \bar{\epsilon}_{\boldsymbol{y}}|_{\tilde{\vartheta}_{\boldsymbol{y}}(x),\, \mathcal{M}_{\boldsymbol{y}}(x)}.$$

Quantum metrology: optimal data processing

$$\tilde{\vartheta}_{\boldsymbol{y}}(x) \pm \Delta\tilde{\vartheta}_{\boldsymbol{y}}(x),$$

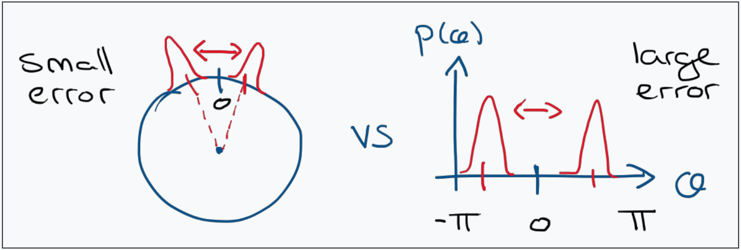
where $\Delta\tilde{\vartheta}_{\boldsymbol{y}}(x)$ is a suitable function of

$$\bar{\epsilon}_{\boldsymbol{y}}(x) = \int d\theta\, p(\theta|x, \boldsymbol{y})\mathcal{D}[\tilde{\theta}_{\boldsymbol{y}}(x), \theta],$$

with $p(\theta|x, \boldsymbol{y}) \propto p(\theta)\mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta)M_{\boldsymbol{y}}(x)]$ ($\equiv$ *Bayes theorem*).

Typical metrology frameworks

Parameter	phase	location
Support	$0 \leqslant \theta < 2\pi$	$-\infty < \theta < \infty$
Symmetry	$\theta \mapsto \theta' = \theta + 2\gamma\pi,\ \gamma \in \mathbb{Z}$	$\theta \mapsto \theta' = \theta + \gamma,\ \gamma \in \mathbb{R}$
Ignorance	$p(\theta) = 1/2\pi$	$p(\theta) \propto 1$
Error $\mathcal{D}(\tilde{\theta}, \theta)$	$4\sin^2[(\tilde{\theta} - \theta)/2]$	$(\tilde{\theta} - \theta)^2$



Remarks:

- The formulation of quantum metrology is greatly simplified by taking the notion of *error functional* as a primitive.
- The universality of this approach is apparent inasmuch as probability theory is an extension of propositional logic.
- **Different types of parameters demand different estimation-theoretic frameworks**.
- Phase estimation is just one of many.

The implications of scale invariance

Definition: scale parameter

Let $\boldsymbol{z} = (x, \boldsymbol{y})$. $\Theta \in (0, \infty)$ scales z_i if z_i is considered 'large' when $z_i/\Theta \gg 1$ and 'small' when $z_i/\Theta \ll 1$. This is **invariant under transformations**

$$z_i \mapsto z_i' = \gamma z_i, \;\; \Theta \mapsto \Theta' = \gamma\Theta,$$

with positive γ, since $z_i/\Theta = z_i'/\Theta'$.

Examples:

- Light speed: v/c
- Lifetime: t/τ
- Rate: kt
- Temperature: $E/(k_B T)$ or βE

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Maximum ignorance about scale parameters

- Alice and Bob wish to estimate Θ. They are told Θ is a *scale parameter*, but they are completely ignorant otherwise.
- Alice encodes her information in $p(\theta)d\theta$, while Bob does so in $p(\theta')d\theta'$, where $\theta' = \gamma\theta$.
- Since they hold the *same* information, $p(\theta)d\theta = p(\theta')d\theta'$, i.e.,

$$p(\theta) = \gamma p(\gamma\theta).$$

The solution is:

Jaynes's transformation groups: Jeffreys's prior

$$p(\theta) \propto 1/\theta$$

uniquely represents maximum ignorance about scales.

IEEE Trans Cybern **4**, 227–241 (1968)

The logarithmic error family

- Let $\phi \in (-\infty, \infty)$ be a location parameter.
- By virtue of translation invariance, $\mathcal{D}(\tilde{\phi}, \phi) = |\tilde{\phi} - \phi|^k$ and maximum ignorance is represented by $p(\phi) \propto 1$.
- This scenario can be mapped to scale estimation by setting $\phi = \alpha \log(\theta/\theta_u)$, where α, θ_u are free parameters.
- That is, $p(\phi)d\phi = p(\theta)d\theta$ implies $p(\phi) \propto 1 \mapsto p(\theta) \propto 1/\theta$.

Therefore:

Deviation function: the logarithmic family

$$\mathcal{D}(\tilde{\theta}, \theta) = |\alpha \log(\tilde{\theta}/\theta)|^k$$

Phys. Rev. Lett. **127**, 190402 (2021)

Deviation function: properties of the logarithmic family

- Scale invariant, i.e., $\mathcal{D}(\gamma\tilde{\theta}, \gamma\theta) = \mathcal{D}(\tilde{\theta}, \theta)$.
- Symmetric, i.e., $\mathcal{D}(\tilde{\theta}, \theta) = \mathcal{D}(\theta, \tilde{\theta})$.
- Reaches its absolute minimum at $\tilde{\theta} = \theta$, where it vanishes.
- Grows (decreases) monotonically from (towards) that minimum when $\tilde{\theta} > \theta$ ($\tilde{\theta} < \theta$).
- Can be interpreted as a generalised **noise-to-signal ratio** when $\alpha = 1$, $k = 2$, i.e.,

$$\mathcal{D}(\tilde{\theta}, \theta) = \log^2(\tilde{\theta}/\theta).$$

Why does the logarithmic error family matter?

- Alice and Bob wish to estimate Θ, with $\theta \in [0.01, 100]$.
- Alice is not sure about scale estimation, so she continues to use $p(\theta) \propto 1$ and minimises $\int d\theta\, p(\theta)(\tilde{\theta} - \theta)^2$, finding

$$\tilde{\theta} = \int d\theta p(\theta)\theta \simeq 50.$$

- Bob, on the other hand, uses $p(\theta) \propto 1/\theta$ and minimises $\int d\theta\, p(\theta) \log^2(\tilde{\theta}/\theta)$, finding

$$\tilde{\theta} = \theta_u \exp\left[\int d\theta p(\theta) \log\left(\frac{\theta}{\theta_u}\right)\right] = 1.$$

- $\tilde{\theta} = 1$ is the *middle point w.r.t. the orders of magnitude within the prior range*, and so the correct answer.

In summary:

Parameter	scale
Support	$0 < \theta < \infty$
Symmetry	$\theta \mapsto \theta' = \gamma\theta,\ \gamma \in \mathbb{R}_{++}$
Ignorance	$p(\theta) \propto 1/\theta$
Error $\mathcal{D}(\tilde{\theta}, \theta)$	$\log^2(\tilde{\theta}/\theta)$

Optimal strategies for scale estimation

Quantum scale metrology: fundamental problem

Minimise the *mean logarithmic error*

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}} = \int d\theta dx\, p(\theta)\mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta)M_{\boldsymbol{y}}(x)]\log^2\left[\frac{\tilde{\theta}_{\boldsymbol{y}}(x)}{\theta}\right]$$

w.r.t. $\tilde{\theta}_{\boldsymbol{y}}(x)$, $M_{\boldsymbol{y}}(x)$ for given $p(\theta)$, $\rho_{\boldsymbol{y}}(\theta)$. Here, it is assumed that: (i) $p(\theta|\boldsymbol{y}) \mapsto p(\theta)$, and (ii) Θ scales $\boldsymbol{y}$, but not x, which is dimensionless.

This can be solved *analytically* via **Jensen's operator inequality and** an operator version of **the calculus of variations**.

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Result 1: optimal strategy

Let $\mathcal{S}_{\boldsymbol{y}} = \int ds\, \mathcal{P}_{\boldsymbol{y}}(s)\, s$ solve the *Lyaponuv equation*

$$\mathcal{S}_{\boldsymbol{y}} \rho_{\boldsymbol{y},0} + \rho_{\boldsymbol{y},0} \mathcal{S}_{\boldsymbol{y}} = 2\rho_{\boldsymbol{y},1},$$

where

$$\rho_{\boldsymbol{y},k} = \int d\theta\, p(\theta) \rho_{\boldsymbol{y}}(\theta) \log^k \left(\frac{\theta}{\theta_u}\right);$$

then, the **optimal estimator** is

$$\tilde{\theta}_{\boldsymbol{y}}(x) \mapsto \tilde{\vartheta}_{\boldsymbol{y}}(s) = \theta_u \exp(s),$$

and the **optimal POM** is

$$M_{\boldsymbol{y}}(x) \mapsto \mathcal{M}_{\boldsymbol{y}}(s) = \mathcal{P}_{\boldsymbol{y}}(s).$$

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Result 2: ultimate precision limits

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}} \geqslant \bar{\epsilon}_p - \mathcal{K}_{\boldsymbol{y}} \geqslant \bar{\epsilon}_p - \mathcal{J}_{\boldsymbol{y}}$$

$\bar{\epsilon}_p$	$\int d\theta\, p(\theta) \log^2(\theta/\tilde{\vartheta}_p)$	prior error
$\tilde{\vartheta}_p$	$\theta_u \exp[\int d\theta\, p(\theta) \log(\theta/\theta_u)]$	prior estimate
$\mathcal{K}_{\boldsymbol{y}}$	$\int dx\, \{\mathrm{Tr}[M_{\boldsymbol{y}}(x)\rho_{\boldsymbol{y},1}]^2/\mathrm{Tr}[M_{\boldsymbol{y}}(x)\rho_{\boldsymbol{y},0}]\}$	classical IG*
$\mathcal{J}_{\boldsymbol{y}}$	$\mathrm{Tr}(\rho_{\boldsymbol{y},0}\mathcal{S}_{\boldsymbol{y}}^2) = \mathrm{Tr}(\rho_{\boldsymbol{y},1}\mathcal{S}_{\boldsymbol{y}})$	quantum IG*

*IG $\equiv$ information gain

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Result 3: optimal data processing

$$\tilde{\vartheta}_{\boldsymbol{y}}(s) \pm \Delta\tilde{\vartheta}_{\boldsymbol{y}}(s) = \tilde{\vartheta}(s)[1 \pm \bar{\epsilon}^{1/2}_{\boldsymbol{y},\mathrm{mle}}(s)],$$

where

$$\tilde{\vartheta}_{\boldsymbol{y}}(s) = \theta_u \exp\left[\int d\theta\, p(\theta|s,\boldsymbol{y}) \log\left(\frac{\theta}{\theta_u}\right)\right]$$

and

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}}(s) = \int d\theta\, p(\theta|s,\boldsymbol{y}) \log^2\left(\frac{\theta}{\theta_u}\right) - \tilde{\vartheta}^2_{\boldsymbol{y}}(s),$$

with $p(\theta|s,\boldsymbol{y}) \propto p(\theta)\mathrm{Tr}[\rho_{\boldsymbol{y}}(\theta)\mathcal{P}_{\boldsymbol{y}}(s)]$ ($\equiv$ *Bayes theorem*).

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PRX Quantum **3**, 040330 (2022)

Remarks:

- **We can now calculate *universally optimal* estimators and POMs** for any given prior and state in scale metrology.
- **This enables the search for ultimate precision limits and optimal protocols** for data analysis in scale metrology.
- It may be argued that local estimation theory, while valid and useful in its regime of applicability, is not essential.

A case study: atomic lifetimes

- Let a two-level atom prepared as $|\psi\rangle = \sqrt{1-a}\,|g\rangle + \sqrt{a}\,|e\rangle$ undergo *spontaneous photon emission*.
- Such a process may be described as

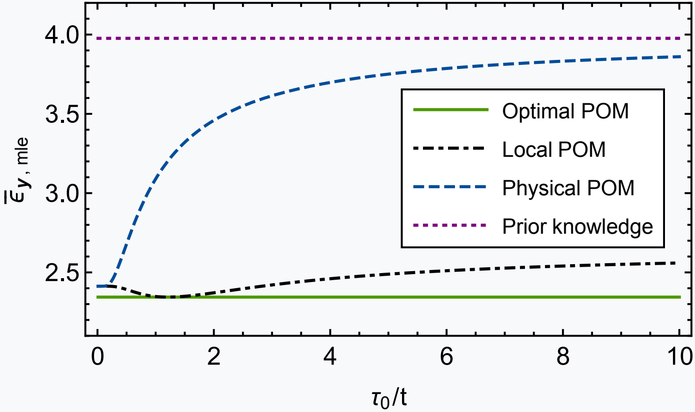
$$\rho_t(\tau) = \begin{pmatrix} [1 - a\,\eta_t(\tau)] & [a(1-a)\,\eta_t(\tau)]^{\frac{1}{2}} \\ [a(1-a)\,\eta_t(\tau)]^{\frac{1}{2}} & a\,\eta_t(\tau) \end{pmatrix},$$

with $\eta_t(\tau) = \exp(-t/\tau)$, **lifetime** τ and elapsed time t.

Problem: quantum estimation of a time scale

Unknown parameter: $\Theta = \tau$; prior information: $\theta/t \in [0.01, 10]$, $a = 0.9$.

S. M. Barnett, *Quantum Information*, Oxford: Oxford University Press (2009)



- **'Yes'/'No' measurement**: $M^{Y}_{t,\tau_0} = [1 - \eta_t(\tau_0)]\,|e\rangle\langle e|$ ('Yes'), $M^{N}_{t,\tau_0} = |g\rangle\langle g| + \eta_t(\tau_0)\,|e\rangle\langle e|$ ('No').
- Informative (reduces $\bar{\epsilon}_p$), but τ easier to estimate when decay likely to have already happened ($\tau_0/t \ll 1$).
- Initial 'hint' τ_0 needed, and generally suboptimal.

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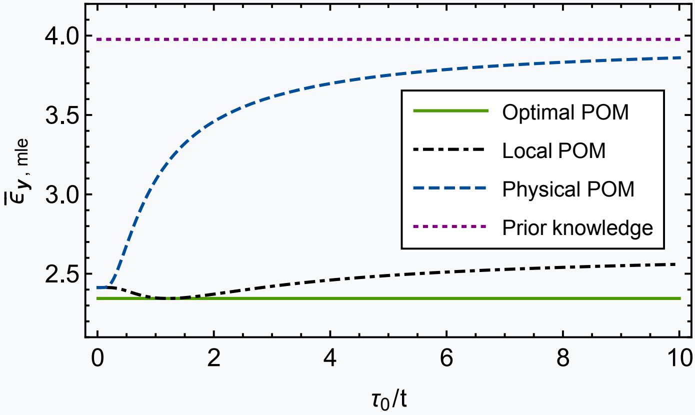

- **SLD measurement**: $M^i_{t,\tau_\circ} = |\lambda^i_{t,\tau_\circ}\rangle\langle\lambda^i_{t,\tau_\circ}|$, with $L_t(\tau_\circ)|\lambda^i_{t,\tau_\circ}\rangle = \lambda^i_{t,\tau_\circ}|\lambda^i_{t,\tau_\circ}\rangle$ and $L_t(\tau)\rho_t(\tau) + \rho_t(\tau)L_t(\tau) = 2\partial_\tau\rho_t(\tau)$.
- More informative than 'Yes'/'No' measurement.
- Initial 'hint' τ_o still needed, and suboptimal for $\tau_\circ/t \gg 1$.

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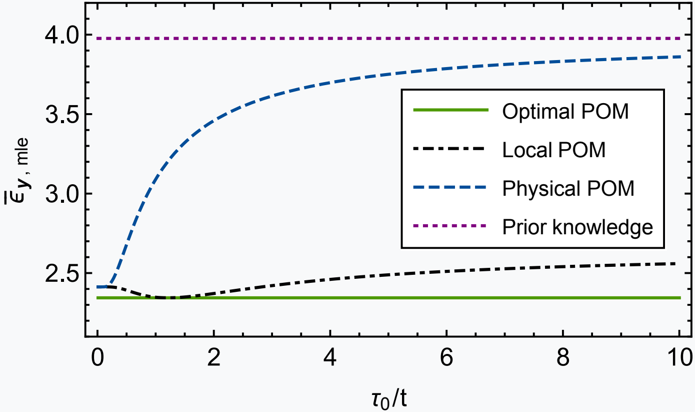

- **Optimal measurement**: $|\psi_+\rangle = 0.094\,|g\rangle + 0.996\,|e\rangle$, $|\psi_-\rangle = 0.996\,|g\rangle - 0.094\,|e\rangle$.
- Globally optimal (τ_0-independent).
- Establishes the fundamental precision limit for the estimation of τ.

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Remarks:

- Scale metrology enables the possibility of **exploiting quantum resources to estimate time and other scales**.
- Moreover, it can establish fundamental precision limits **in the presence of finite prior information**.

Epilogue: multiparameter estimation à la Bayes

Multiparameter metrology: scales

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mle}} \geqslant \sum_i w_i \left[\int d\theta \, p(\theta) \log^2\left(\frac{\theta_i}{\theta_{u,i}}\right) - \mathrm{Tr}(\rho^{\mathrm{mle}}_{\boldsymbol{y},1,i} \mathcal{S}^{\mathrm{mle}}_{\boldsymbol{y},i}) \right]$$

Multiparameter metrology: locations

$$\bar{\epsilon}_{\boldsymbol{y},\mathrm{mse}} \geqslant \sum_i w_i \left[\int d\theta \, p(\theta) \theta_i^2 - \mathrm{Tr}(\rho^{\mathrm{mse}}_{\boldsymbol{y},1,i} \mathcal{S}^{\mathrm{mse}}_{\boldsymbol{y},i}) \right]$$

- $w_i \equiv$ importance weight for the *i*th parameter
- Not saturable when $[\mathcal{S}_{\boldsymbol{y},i}, \mathcal{S}_{\boldsymbol{y},j}] \neq 0$ for $i \neq j$
- Starting point to study an **uncertainty- and prior-dependent notion of quantum incompatibility**

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Phys. Rev. A **101**, 032114 (2020)

Conclusions:

- Scale metrology **enables the most precise estimation of scale parameters that is allowed by quantum mechanics**.
- It provides a more fundamental picture of metrology, while also being practical and easy to use.
- It **opens the door to constructing new quantum estimation theories for all kinds of parameters**.

Key works:

Quantum Sci. Technol. **8**, 015009 (2022)
Phys. Rev. Lett. **127**, 190402 (2021)
Phys. Rev. A **101**, 032114 (2020)