957633: Determination of Anisotropic Thermal Conductivity with Thermal Needle Probe Measurements

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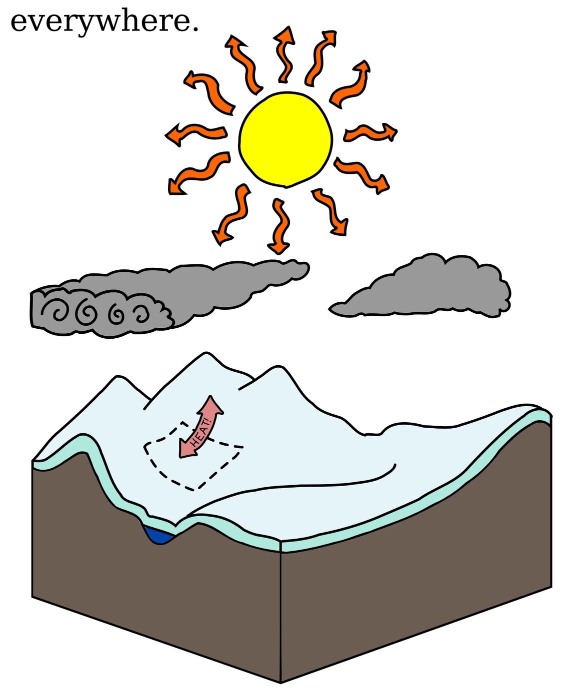
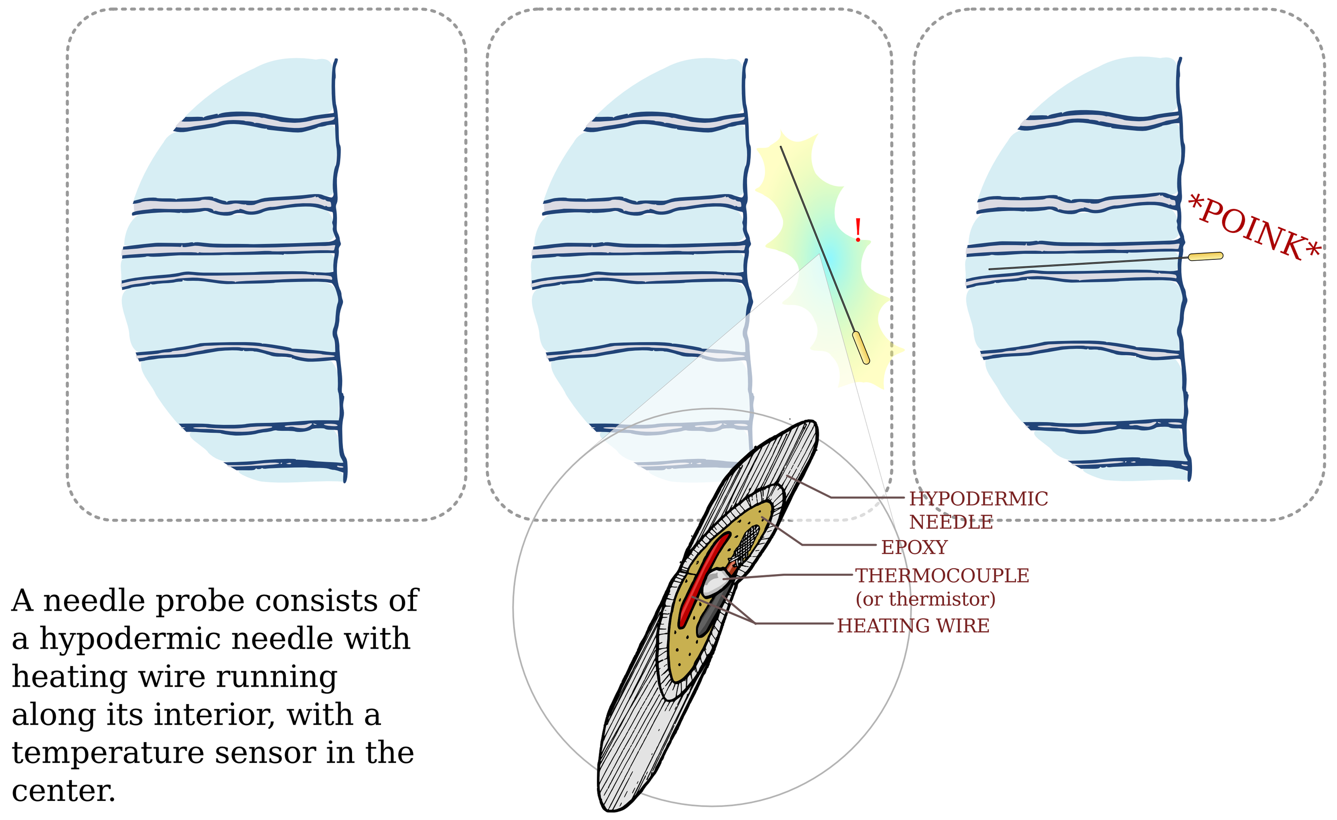
BACKGROUND:

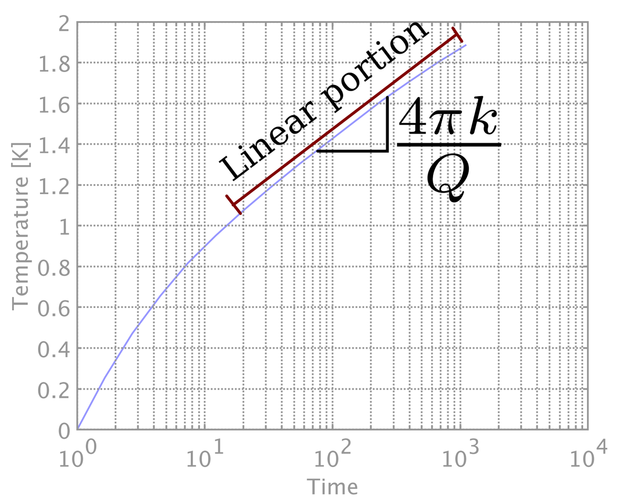
The thermal properties of snow are of interest to scientists because of the role they play in arctic climate models. After all, any heat transfer between the earth and the atmosphere must go through any snow first, and in the arctic, the stuff's everywhere.



One way to measure thermal conductivity---and possibly the most common way for snow---is to use what is called a needle probe.

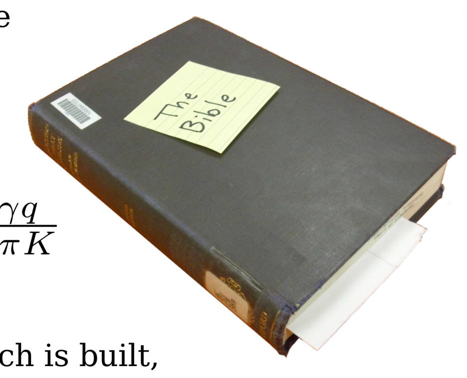


A needle probe consists of a hypodermic needle with heating wire running along its interior, with a temperature sensor in the center.



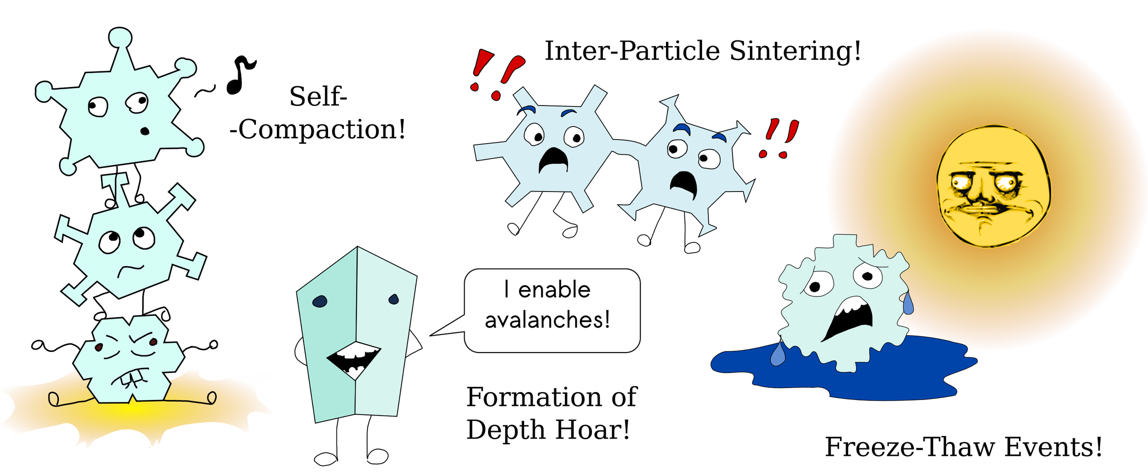
Using a needle probe for isotropic materials is straightforward in theory: A constant heat flux is generated along the needle and temperature is measured over time. By approximating the geometry as an infinite line source, it can be shown that temperature as a function of ln(t) approaches a linear asymptote whose slope is proportional to the thermal conductivity of the surrounding medium.

$$T(r,t) = \frac{q}{4\pi K}\ln\left(\frac{4Kt}{r^2}\right) - \frac{\gamma q}{4\pi K}$$

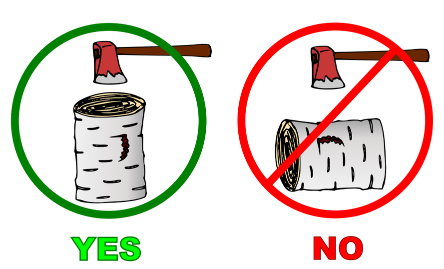


Details of the infinite line source solution, on which this research is built, may be found in "Conduction of Heat in Solids" by Carslaw & Jeager.

Unfortunately, snow is a complex material whose properties are heavily influenced by a number of inescapable metamorphic processes:



These make snow an anisotropic medium.



"Anisotropy" means that the material's properties depend on direction. An example of an anisotropic material is wood, which has lower yield shear stress with the grain than against the grain. Analogously, snow may conduct heat at different rates vertically, versus horizontally. As a consequence, the heat equation becomes more complex, and Carslaw & Jeager's solution no longer directly applies:

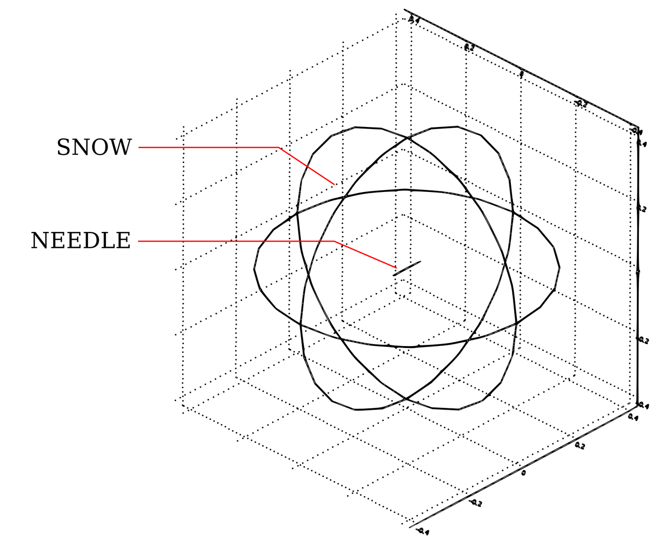
$$-k\nabla^2 T = \rho C\frac{\partial T}{\partial t} \xrightarrow{\text{BECOMES}} -\nabla\cdot\left(\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz}\end{bmatrix}\nabla T\right) = \rho C\frac{\partial T}{\partial t}$$

THIS   THIS.

We are researching methods for applying the standard needle probe method to anisotropic materials, particularly snow, with differing in-plane and out-of-plane conductivities, using both numerical finite element methods and analytical methods.

NUMERICAL SIMULATIONS:

COMSOL and MATLAB were used to run about 100 finite element models of a steel needle embedded in a sphere of snow with zero heat flux on the outside boundary. An infinite medium was simulated by using a sufficiently large sphere.



Parameter	Value
r_{snow}	0.4 m
r_{needle}	0.25 mm
l_{needle}	10 cm
ρ_{snow}	200 kg/m^3
ρ_{needle}	8000 kg/m^3
$C_{p,\text{ snow}}$	2050 $\text{J/kg}\cdot\text{K}$
$C_{p,\text{ needle}}$	460 $\text{J/kg}\cdot\text{K}$
q_{needle}	0.5 W/m
K_{needle}	160 $\text{W/m}\cdot\text{K}$

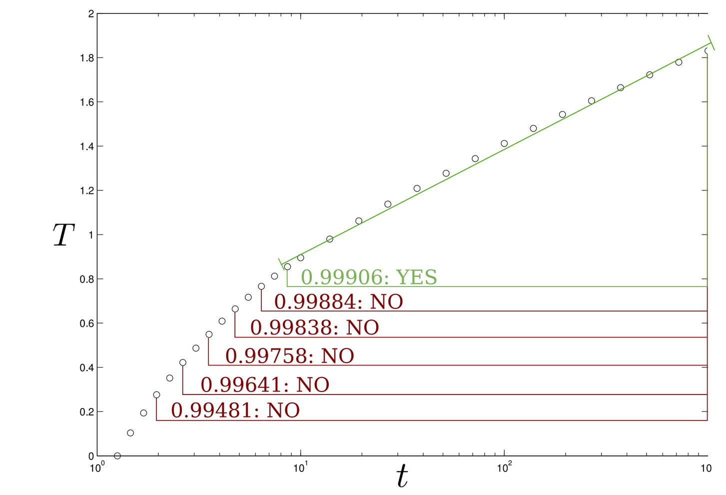
Instead of rotating the needle, the conductivity matrices were generated with rotation matrices around the y-axis, with the needle aligned with the x-axis.

$$[K] = \begin{bmatrix}\cos(x) & 0 & \sin(x)\\ 0 & 1 & 0\\ -\sin(x) & 0 & \cos(x)\end{bmatrix}\begin{bmatrix}k_{xy} & 0 & 0\\ 0 & k_{xy} & 0\\ 0 & 0 & k_z\end{bmatrix}\begin{bmatrix}\cos(x) & 0 & -\sin(x)\\ 0 & 1 & 0\\ \sin(x) & 0 & \cos(x)\end{bmatrix}$$

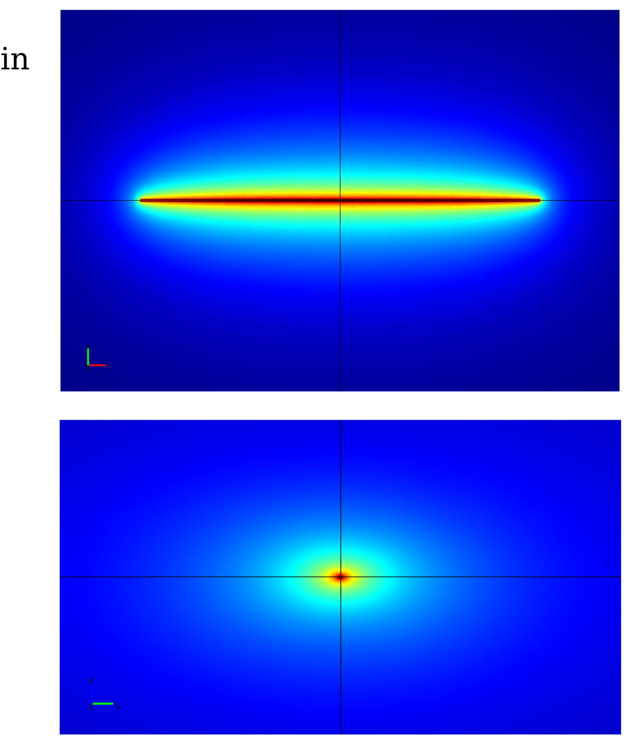
The piece of data used in the original method is the temperature at the very center of the needle, or at (0, 0, 0) in the model. This time-series data was exported from COMSOL and post-processed with MATLAB.

We found the linear section of the graph by calculating correlation coefficients, and dropping points from the beginning of the time-series data until the correlation coefficients were above a certain threshold.

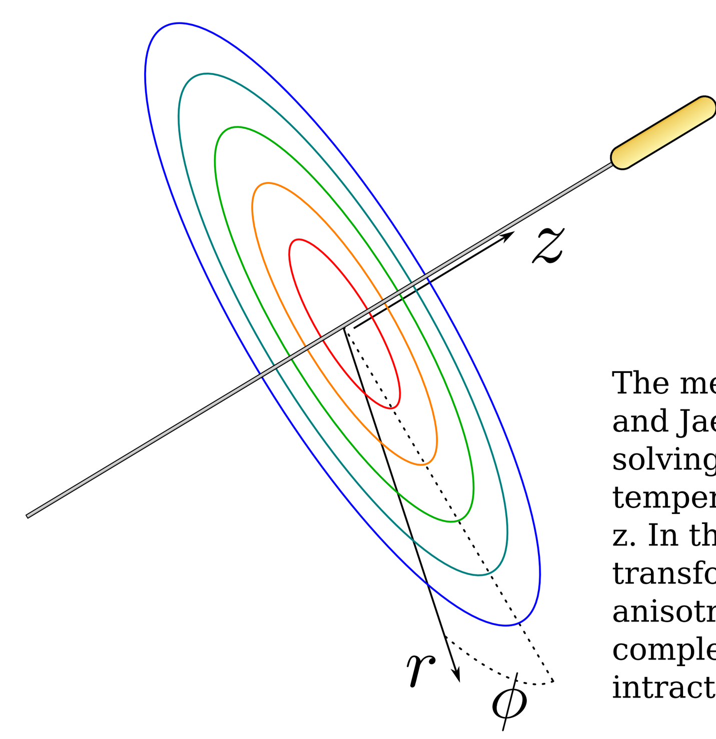
Curve fits of the linear data were used to find a measured k-value, which we called k_{meas}.



These screenshots from COMSOL, showing temperature distributions around the needle, demonstrate both the elliptical nature of the temperature distribution around the needle, and the edge effects that come with a finitely-long needle.

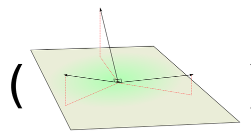


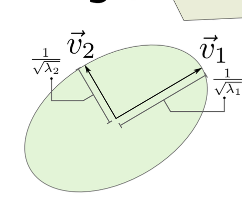
ANALYTICAL METHODS:



The method based on Carslaw and Jaeger's solution depends on solving a 2-D problem, where temperature is not a function of z. In the isotropic case this transformation is simple, but the anisotropic case is more complex---but not completely intractible.

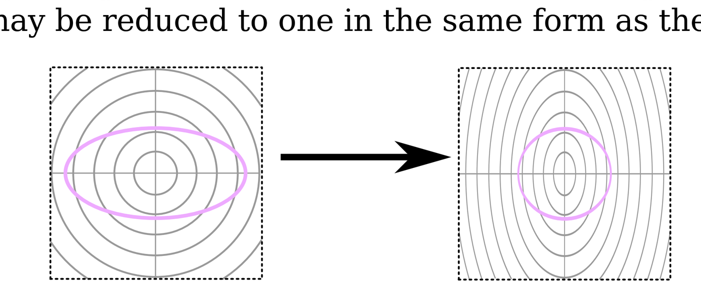
By finding the components of the conductivity in-plane and finding their eigenvalues and eigenvectors, we may reduce the problem to the form

In [7]: eig()

Out[7]: 

$$-\nabla\cdot\left(\begin{bmatrix}k_x & 0\\ 0 & k_y\end{bmatrix}\nabla T\right) = \rho C\frac{\partial T}{\partial t}$$

By doing a coordinate transformation, the governing equation may be reduced to one in the same form as the isotropic case.



$$x' = x$$

$$y' = \sqrt{\frac{k_x}{k_y}}\,y$$

However, the long-time approximation is no longer a function of r, but of the transformed distance r'.

$$T(r',t) = \frac{q}{4\pi k_x}\ln\left(\frac{4k_x t}{r'^2}\right) - \frac{\gamma q}{4\pi k_x}$$

$$r' = \sqrt{\cos^2(\phi) + \frac{k_x}{k_y}\sin^2(\phi)}$$

Assuming that the measurement is equivalent to the average temperature along a constant radius of the non-transformed domain, we can find predict the measured temperature as a function of time by evaluating the quotient of two elliptical integrals:

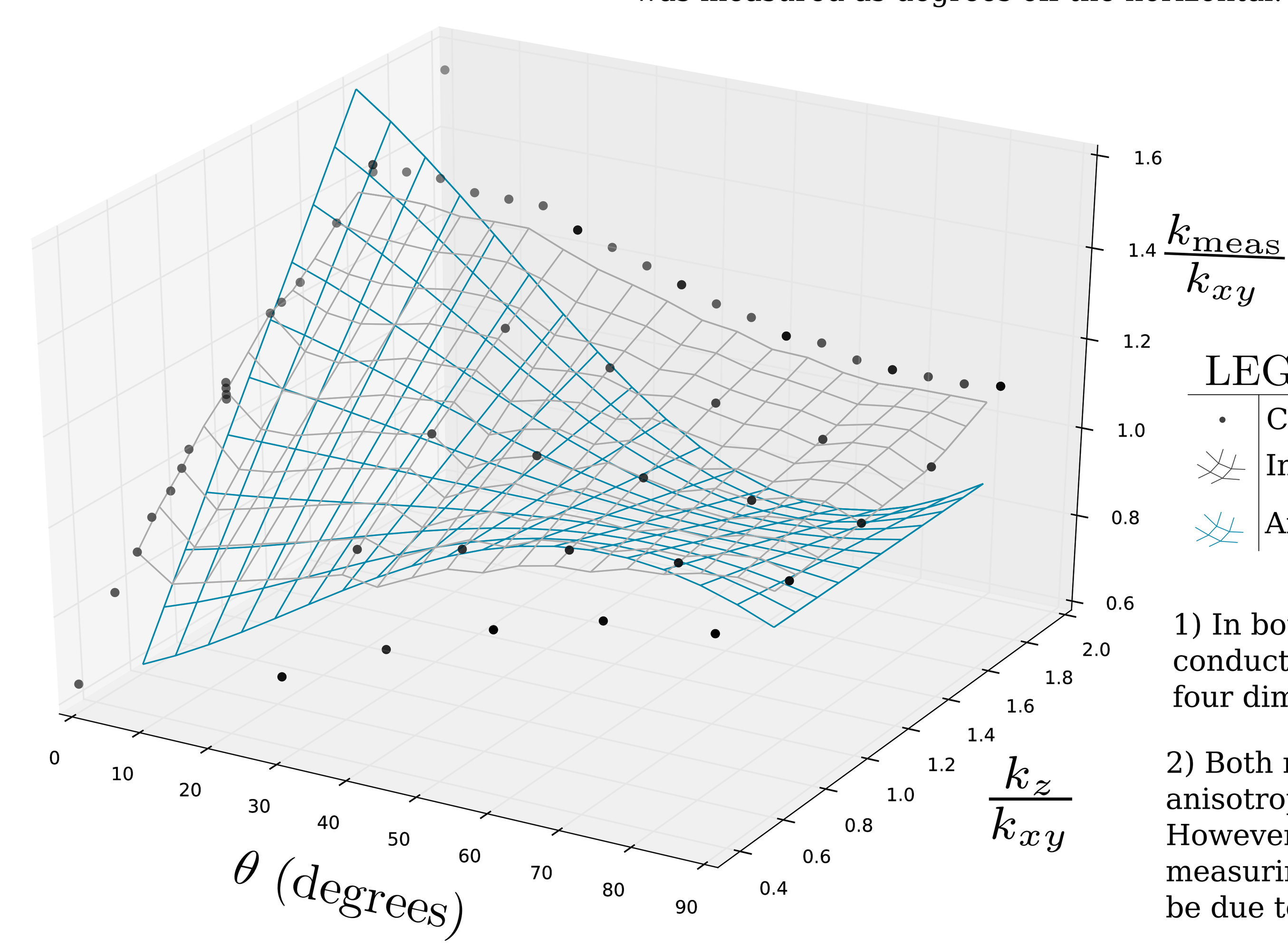
$$T_{\text{avg}} = \frac{4\pi k_x}{q}\frac{\mathcal{E}(\ln(t), \frac{k_x}{k_y})}{\mathcal{E}(1, \frac{k_x}{k_y})}$$

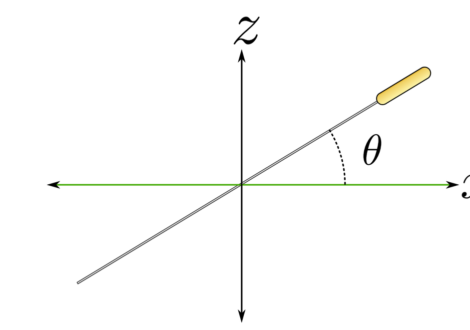
$$\mathcal{E}(f(\phi,\alpha),\alpha) = \int_0^{2\pi} f\sqrt{\cos^2(\phi) + \alpha\sin^2(\phi)}\,d\phi$$

Using this predicted function of temperature over time, we may predict the measured thermal conductivity by using a linear curve fit against T(ln(t)).

RESULTS:

The input conductivities and predicted measurements from both analyses were non-dimensionalized using k_{xy} , the in-plane conductivity. The orientation of the needle was measured as degrees off the horizontal.





LEGEND
- COMSOL simulation sample
- Interpolating Surface for samples
- Analytical Results

1) In both cases, non-dimensionalizing thermal conductivities successfully collapsed the results from four dimensions into three.

2) Both methods show measurements near one for anisotropic cases regardless of angle, as expected. However, numerical simulations contained an error, measuring about 110% of the expected value. This may be due to edge effects.

3) Both simulations yield reasonable values for the cases of 0 degrees and 90 degrees, and display non-monotonic behavior. However, the analytical case shows much stronger non-monotonic behavior, and the numerical simulations show an unexpected slope as a function of conductivity ratio for 90 degrees. This may also be due to edge effects.

4) This analysis indicates that measuring anisotropic thermal conductivity with needle probes should be a viable technique, requiring as few as two measurements given prior knowledge of the snow's geometry.

ACKNOWLEDGEMENTS:

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