# Equilibrium with Complete Markets

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# Introduction

- This course is an introduction to modern macroeconomic theory.
- Our main emphasis will be the analysis of resource allocations in dynamic stochastic environments.
- We will go though the analysis of:
  - Equilibrium with complete markets.
  - Dynamic Programming (DP)
  - Applications of DP (RBC models)
- We will start, however, with a simple environment: static exchange economy.

**References**: *Recursive Macroeconomic Theory* by Ljungqvist and Sargent and *The PhD Macro Book*

# Exchange Economy

- Simple environment: finite dimensional, static exchange economy.
- In an exchange economy, people interact in the market place.
- They buy and sell goods taking market prices as given in order to maximize their utility.
- Their choices are constrained by their endowments.

# Exchange Economy

- If we can find a set of selling and buying decision for all individuals and a set of prices such that:
  - Given these prices, people's selling and buying decision are optimal.
  - No excess demand or excess supply of any good.

$\Rightarrow$ Our economy is in equilibrium.

# Setup

- Consider an economy with $i = 1, \ldots, n$ consumers and $j = 1, \ldots, m$ commodities.
- Each individual $i$ is endowed with $w_i^j$ units of good $j$. $(w_i^1, w_i^2, \ldots, w_i^m)$
- Individuals have preferences over these goods and will trade with each other to maximize their well-being.

# Assumptions

1. Consumer's preferences are representable by a utility function $u.u_i : \boldsymbol{X} \equiv \mathbb{R}^m_+ \rightarrow \mathbb{R}$
2. $u$ is continuous and first and second derivatives exist.
3. Preferences are strictly monotonic (the more I consume, the better).
4. $u$ is strictly concave (no flat section in indifference curves).
5. Every agent is endowed with a positive amount of each good.
6. $\parallel Du_i(x_k) \parallel \rightarrow \infty$ as $x_k \rightarrow x$ where some component of $x$ is equal to zero.

# Problem

- Given a set of prices $\boldsymbol{p} = (p^1, \ldots, p^m)'$, consumers in this economy solve the following problem:

$$\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i)$$
$$\text{s.t.} \quad \boldsymbol{p}'(\boldsymbol{x}_i - \boldsymbol{w}_i) \leq 0$$

  Given that preferences are monotonic, individuals will be on their budget set: $\boldsymbol{p}'(\boldsymbol{x}_i - \boldsymbol{w}_i) = 0$

- Following is the Lagrangian of the consumer problem:

$$\mathcal{L} = u_i(\boldsymbol{x}_i) - \mu_i \boldsymbol{p}'(\boldsymbol{x}_i - \boldsymbol{w}_i)$$

# Problem
## FOCs

- FOCs are necessary and sufficient to characterize $\boldsymbol{x}_i$:

$$D_x u_i(\boldsymbol{x}_i) = \mu_i \boldsymbol{p} \quad (M \times 1)$$

- For each good we have:

$$\frac{\partial u_i(\boldsymbol{x}_i)}{\partial x_{i,j}} = \mu_i p_j$$

  - The MRS for any two goods must be equal to the ratio of prices
  - Any two agents hold the same MRS since they face the same prices.

# Definition

- **Competitive Equilibrium** is an allocation $\boldsymbol{x}^*$ and a price vector $\boldsymbol{p}^*$ such that:

    1. The allocation $\boldsymbol{x}_i^*$ solves agent $i$'s problem given $\boldsymbol{x}^*$, for all $i$'s.
    2. Market clears:

$$\sum_{i=1}^{n} x_{i,j}^* \leq \sum_{i=1}^{n} w_{i,j} \ \forall \ j$$

# Definition

- An allocation $\boldsymbol{x}$ is **Pareto optimal** if it is feasible and there is no other feasible allocation $\tilde{\boldsymbol{x}}$ such that $u_i(\tilde{\boldsymbol{x}}_i') \geq u_i(\boldsymbol{x}_i)$ for all $i \in \{1, ..., N\}$, and $u_j(\tilde{\boldsymbol{x}}_j) > u_j(\boldsymbol{x}_j)$ for at least one $j \in \{1, ..., N\}$.
- **First Welfare Theorem** Every competitive allocation is Pareto optimal.
- Sketch of the proof:
    1. Assume $\tilde{\boldsymbol{x}}$ is preferable by at least one agent $j$ and feasible.
    2. This allocation for agent $j$ was out of his budget set with prices p.
    3. All other agents $i$ cannot be consuming less and be as well off.
    4. Markets cannot clear $\Rightarrow$ allocation not feasible.

# Social Planner's Problem

- Next, we would like to know whether every Pareto optimal allocation can be sustained by a competitive equilibrium.

- The set of Pareto optimal allocation can be characterized by the solution to the following planner's problem:

$$\max_{\boldsymbol{x}} \sum_{i=1}^{n} \alpha_i u_i(\boldsymbol{x}_i) \text{ with } \sum_{i}^{n} \alpha_i = 1$$

$$\text{s.t. } \sum_{i=1}^{n} \boldsymbol{w}_i = \sum_{i=1}^{n} \boldsymbol{x}_i$$

with $\alpha_i$ representing the weights of the different agents in the planner's objective.

# Social Planner's Problem

- The solutions to the planner's problem is characterized by:

$$\alpha_i Du_i(\boldsymbol{x}_i) = \boldsymbol{\pi}$$
$$\sum_{i=1}^{n} \boldsymbol{w}_i = \sum_{i=1}^{n} \boldsymbol{x}_i^*$$

- The competitive allocation instead was characterized by:

$$Du_i(\boldsymbol{x}_i) = \mu_i \boldsymbol{p}$$
$$\boldsymbol{p}'(\boldsymbol{w}_i - \boldsymbol{x}_i) = 0$$
$$\sum_{i=1}^{n} \boldsymbol{w}_i = \sum_{i=1}^{n} \boldsymbol{x}_i$$

- Therefore if $\alpha_i = 1/\mu_i$ and $\boldsymbol{p} = \boldsymbol{\pi}$, the social planner and the competitive equilibrium coincide.

# Social Planner's Problem

- Then, whether a Pareto optimal allocation can be decentralized boils down to whether at prices $\boldsymbol{\pi}$, the allocation $\boldsymbol{x}$ is feasible for each consumer.

- In order the allocation to be affordable to every agent, the planner has to redistribute income across agents:

$$\tau_i(\boldsymbol{\alpha}) = \boldsymbol{\pi}'(\boldsymbol{x}_i - \boldsymbol{w})$$

- Note that such redistribution comes at zero cost:

$$\sum_{i=1}^{n} \tau_i(\boldsymbol{\alpha}) = 0$$

# Social Planner's Problem

- **Second Welfare Theorem** Every Pareto optimal allocation can be decentralized as a competitive equilibrium with transfers, i.e. given Pareto optimal allocation $\boldsymbol{x}$, we can find a price vector $\boldsymbol{p}$ and transfers $\boldsymbol{\tau}_i$ such that given the initial endowments and transfers, $\boldsymbol{x}$ is a competitive equilibrium.

# Exchange Economy with Infinitively-Lived Agents

- In our static exchange economy agents live for a single period.
- In this section we will analyze model economies where they live forever.
- Time discrete, infinite, finite number of agents $N$, only one consumption good.
- The consumption good is not storable.
- Agents have deterministic endowments $w^i = \{w_t^i\}_{t=0}^{\infty}$

# Exchange Economy with Infinitively-Lived Agents

- Let $c_t^i$ be consumption of agent $i$ at time $t$, and let $c^i = \{c_t^i\}_{t=0}^{\infty}$ be a consumption sequence.
- Agents preferences are given by,

$$U(c^i) = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i),$$

where $\beta$ is the discount factor.

- $U(c^i)$ is time separable.

# Exchange Economy with Infinitively-Lived Agents

## Market Structures

- We are going to study two system of markets:
    1. *Arrow-Debreu* structure with complete markets all trade takes place at time 0.
    2. Sequential trading structure with one period securities.
- These two structures will entail different assets and timing of trades but have identical consumption allocations.

# Arrow-Debreu Markets

- There is a market at time 0 where agents can buy and sell goods of different time periods.
- There is a price for every period's good.
- We assume that all contracts that are agreed at time 0 are honored.
- The consumer therefore faces a single budget constraint:

$$\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t w_t^i$$

- We call this market arrangement, Arrow-Debreu markets.
- We normalize $p_0 = 1$ (goods in period 1 are the *numeraire*)

# Arrow-Debreu Equilibrium

- **Definition**: sequence of allocation $c^i = \{c_t^i\}_{t=0}^{\infty}$ for each $i$, and a sequence of prices $p = \{p_t\}_{t=0}^{\infty}$ such that:

  1. Given $p$, $c^i$ solves the agent $i$'s maximization problem for each $i$:

$$\max_{c^i} \sum_{t=0}^{\infty} \beta^t u_i(c_t^i),$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t w_t^i$$

  2. Markets clear for each $t$:

$$\sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} w_t^i$$

# Arrow-Debreu Equilibrium

- The equilibrium allocations are characterized by:

  1. Consumer's FOCs:

  $$\beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i} = \mu^i p_t, \text{ for each } i \text{ and each } t$$

  2. Individual's budget constraints

  $$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t w_t^i$$

  3. Aggregate resource constraint:

  $$\sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} w_t^i$$

# Arrow-Debreu Equilibrium

## Intertemporal optimization

- From FOCs:

$$\frac{\beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i}}{\beta^{t+1} \frac{\partial u_i(c_{t+1}^i)}{\partial c_{t+1}^i}} = \frac{p_t}{p_{t+1}}$$

Intertemporal optimization conditions:

$$\frac{\partial u_i(c_t^i)}{\partial c_t^i} = \beta \frac{p_t}{p_{t+1}} \frac{\partial u_i(c_{t+1}^i)}{\partial c_{t+1}^i} \tag{1}$$

- The consumer allocates her resources optimally such that the marginal cost of reducing time-t consumption today equals the marginal benefit of increasing time-t+1 consumption tomorrow taking into account the discount factor and price dynamics.

# Arrow-Debreu Equilibrium

- From FOCs:

$$\frac{\dfrac{\partial u_i(c_t^i)}{\partial c_t^i}}{\dfrac{\partial u_j(c_t^j)}{\partial c_t^j}} = \frac{\mu^i}{\mu^j}$$

- Therefore the ratio of marginal utilities across two agents is constant across time.

# Arrow-Debreu Equilibrium

## Example: Aggregate Time-Invariant Endowment

- Imagine that $\sum_{i=1}^{n} w_{it} = W$ constant through time.
- Then the aggregate ressource constraint can be written as:

$$\sum_{i=1}^{N} (u_c^i)^{-1}\left(\frac{\lambda_i}{\lambda_j} u_c^j(c_t^j)\right) = W$$

- As $W$ is invariant, $c_{jt}$ must be invariant too and therefore equation 1 becomes :

$$\begin{aligned} p_{t+1} &= \beta p_t \\ p_t &= \beta^t p_0 \\ p_t &= \beta^t \;\; \text{w.l.o.g.} \end{aligned}$$

- Prices completely offset individuals impatience to induce them to maintain a constant consumption level.

# Arrow-Debreu Equilibrium

## Pareto Optimality of the Equilibrium

- **Proposition**: Any Arrow-Debreu equilibrium is Pareto optimal.
- Sketch of the proof:
  - Assume, it is not pareto optimal; there exists another feasible allocation $\tilde{c}$ such that

$$u(\tilde{c}^i) \geq u(c^i) \ \forall i$$
$$u(\tilde{c}^j) > u(c^j) \text{ for at least one } j$$

  - This implies that

$$\sum_{t=0}^{\infty} p_t \tilde{c}_t^j > \sum_{t=0}^{\infty} p_t c_t^j$$

  - Given that other individuals are on their budget set, adding across individuals and time:

$$\sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} \tilde{c}_t^i > \sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} c_t^i$$

# Pareto Optimal Allocation

- As before, we can characterize characterize the set of Pareto optimal allocations as solutions to the following planner's problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{i=0}^{n} \alpha_i \beta^t u_i(c_t^i)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} w_t^i, \text{ for all t}$$

# Pareto Optimal Allocation

- The solution to this problem is characterized by the following FOCs:

$$\alpha_i \beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i} = \pi_t, \text{ for all } i \text{ and } t,$$

where $\pi_t$ is the Lagrange multiplier on the time-t constraint.

- Given $\alpha$, allocations that solves the planner's problem are Pareto optimal.

# Pareto Optimal Allocation

- In order to decentralize the Pareto optimal allocation, we use Lagrange multiplier as prices and transfer resources among consumers according to:

$$\tau_i(\alpha) = \sum_{t=0}^{\infty} \pi_t(\alpha)[c_t^i(\alpha) - w_t^i],$$

  where $c_t^i(\alpha)$ is the pareto optimal allocation of goods.

- We can use this framework to compute the Arrow-Debreu equilibrium by finding $\alpha^*$, such that $\tau_i(\alpha^*) = 0$ for all $i$
  - The $\pi_t(\alpha^*)$ are the Arrow-Debreu prices and allocation $c_t^i$ are the Arrow-Debreu allocations.

# Sequential Equilibrium

## Setup

- Our previous analysis was built on Arrow-Debreu markets where all trade takes place at time-0 market.
- Suppose now that trades takes place in *spot markets* that open every period.
- Hence, at time $t$; agents only trade time-t goods in a spot market.
- If agents can only trade time-t good at time t; and there are no credit arrangements, then this economy would look like a sequence of static exchange economies.

# Sequential Equilibrium

## Setup

- With spot markets we need a credit mechanism that will allow agents to move their resources between periods.
- Therefore, we will assume that there is a one period credit market that works as follows:
  - Each period, agents can borrow or lend in this one period credit market.
  - Let $r_t$ be the interest rate on time-$t$ borrowing/lending.

# Sequential Equilibrium

## Individual Problem

- Given a sequence of prices $\{r_t\}_{t=0}^{\infty}$, the agent $i$'s problem can be written as:

$$
\begin{aligned}
\max_{\{c_t^i, l_t^i\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) \\
\text{s.t.} \quad & c_0^i + l_1^i = w_0^i \\
& c_1^i + l_2^i = w_1^i + (1 + r_1) l_1^i \\
& \ldots \\
& c_t^i + l_{t+1}^i = w_1^i + (1 + r_t) l_t^i
\end{aligned}
$$

# Sequential Equilibrium

## No-Ponzi Condition

- How can we make sure that agents don't borrow more than what they can honor?
- We are interested in specifying a borrowing limit that prevents Ponzi schemes, yet is high enough so that household are never constrained in the amount they can borrow.
- We need to impose an extra condition:

In t=0: $l_1^i = w_0^i - c_0^i$

In t=1: $l_2^i = w_2^i + (1+r_1)w_0^i - c_1^i - (1+r_1)c_0$

$\vdots$

In t: $l_{t+1}^i = w_t^i + \sum_{s=0}^{t-1} \prod_{j=s+1}^{t} (1+r_j)w_s^i - c_t^i - \sum_{s=0}^{t-1} \prod_{j=s+1}^{t} (1+r_j)c_s$

# Sequential Equilibrium

## No-Ponzi Condition

- Dividing both sides by $\prod_{j=1}^{t}(1+r_j)$:

$$\frac{l_{t+1}^{i}}{\prod_{j=1}^{t}(1+r_j)} = \sum_{s=1}^{t} \frac{w_s}{\prod_{j=1}^{s}(1+r_j)} + w_0 - \sum_{s=1}^{t} \frac{c_s}{\prod_{j=1}^{s}(1+r_j)} - c_0$$

- Which is simply the time-0 present value of agent's resources minus consumption.

- We need to impose a condition on it such that agents don't run a game where they keep borrowing and never pay back:

$$\lim_{t\to\infty} \frac{l_{t+1}^{i}}{\prod_{s=1}^{t}(1+r_s)} \geq 0$$

# Sequential Equilibrium

## No-Ponzi Condition

- The weakest possible debt limit would be to impose *the natural debt limit*:
  - It has to be feasible for the consumer to repay her debt at every time $t$.

$$l^i_{t+1} \geq - \sum_{s=t+1}^{\infty} \frac{w_s}{\prod_{j=t+1}^{s}(1+r_j)} + w_t$$

  - At every time $t$ the value of her debt cannot exceed the discounted value of present and future endowments.

# Sequential Equilibrium

## FOCs

- From FOCs we get:

$$\beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i} = \lambda_t$$

$$\lambda_t = (1 + r_{t+1})\lambda_{t+1}$$

- Combining them,

$$\frac{\partial u_i(c_t^i)}{\partial c_t^i} = (1 + r_{t+1})\beta \frac{\partial u_i(c_{t+1}^i)}{\partial c_{t+1}^i}$$

# Sequential Equilibrium

- **Definition** A sequential market equilibrium is a sequence of allocations $c^i = \{c^i_t\}_{t=0}^{\infty}$ and a sequence of lending/borrowing decisions $l^i = \{l^i_t\}_{t=0}^{\infty}$ for each $i$, and sequence of prices $r = \{r_t\}_{t=0}^{\infty}$ such that

    1. Given $r, c^i$ and $l^i$ solves agent's maximization problem
    2. Markets clear.

$$\sum_{i=1}^{n} w^i_t = \sum_{i=1}^{n} c^i_t \text{ for all } t$$

$$\sum_{i=1}^{n} l^i_t = 0 \text{ for all } t$$

# Sequential Equilibrium

- **Proposition** *If* $\{c_t, p_t\}_{t=0}^{\infty}$ *is a competitive Arrow-Debreu equilibrium allocation, then letting:*

$$r_{t+1} = \frac{p_t}{p_{t+1}} - 1$$

$\{c_t, r_t\}_{t=0}^{\infty}$ *is a competitive equilibrium with sequential markets.*

- Sketch of the proof: If $r_{t+1} = \frac{p_t}{p_{t+1}} - 1$, $c_t$ satisfies FOCs, markets clear and the no-ponzi condition is satisfied.

# Stochastic Endowments

- So far we have analyzed economies where everything was certain.
- However, uncertainty is an important element in many economic activities.
- We are going to extend the previous analysis to a stochastic environment.

# Stochastic Endowments

## Setup

- Time discrete, infinite, $N$ agents, one consumption good.
- Endowments depend on the history of states in the economy $(s^t)$ which is uncertain: $w^i(s^t)$
- We will assume that state of the economy at a given time $t$ $(s_t)$ can take values from a given finite set $S$.

# Arrow-Debreu Market

## Setup

- We assume there is a time-0 Arrow-Debreu market where agents can buy and sell goods of different histories ($s^t = \{s_1, \ldots, s_t\}$).
    - Agents at time 0 choose a contingent plan where they decide her consumption for every date and every possible realization of the history.

$$c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$$

# Arrow-Debreu Market

## Agent Problem

- Agents maximize

$$U(c^i) = \max_{c^i} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) u(c_t^i(s^t))$$

$$\text{s.t} \quad \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) w_t^i(s^t)$$

# Arrow-Debreu Equilibrium

- **Definition**: An Arrow-Debreu equilibrium in this economy is a sequence of consumption plans $c^i$ for each $i$, and a sequence of history dependent prices $p$ such that given $s_0$,

  1. Given $p$, $c^i$ solves agent's $i$ maximization problem.
  2. Market clears

$$\sum_{i=1}^{n} c_t^i(s^t) \leq \sum_{i=1}^{n} w_t^i(s^t), \text{ for each } t \text{ and } s^t$$

# Arrow-Debreu Equilibrium

## FOCs

- By FOCs we get:

$$\beta^t \pi(s^t|s_0) \frac{\partial u(c_t^i(s^t))}{\partial c_t^i(s^t)} = \lambda p_t(s^t)$$

- Therefore the intertemporal FOC becomes:

$$\frac{\partial u(c_t^i(s^t))}{\partial c_t^i(s^t)} = \beta \frac{p_t(s^t)}{p_{t+1}(s^{t+1})} \pi(s^{t+1}|s^t) \frac{\partial u(c_{t+1}^i(s^{t+1}))}{\partial c_{t+1}^i(s^{t+1})}$$

# Pareto Optimal Allocations

- As in the case without uncertainty, we can can characterize the set of Pareto optimal allocations as solutions to the following planner's problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \sum_{i=0}^{n} \alpha_i \pi(s^t|s_0) \beta^t u_i(c_t^i)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_t^i(s^t) = \sum_{i=1}^{n} w_t^i(s^t), \text{ for all t}$$

- Then, we could compute the competitive equilibrium by finding the set of $\alpha$'s such that the transfer function that you would need to sustain this equilibrium is 0 for all individuals.

# Pareto Optimal Allocations

## Perfect Insurance

- Note also that at time-t, history $s^t$ consumption of any two agents is related by:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\alpha_j}{\alpha_i}$$

- **Definition**: An allocation has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time (independent of the state of the world).

# Pareto Optimal Allocations

## Irrelevance of History

- From previous equation,

$$c_t^i(s^t) = u'^{-1}\left(\frac{\alpha_j}{\alpha_i} u'(c_t^j(s^t))\right)$$

- Summing across individuals and using aggregate resources constraint:

$$\sum_{i=1}^{I} w_t^i(s^t) = \sum_{i=1}^{I} u'^{-1}\left(\frac{\alpha_j}{\alpha_i} u'(c_t^j(s^t))\right)$$

which is one equation on one unknown $c_t^j(s^t)$

- The Pareto optimal allocations $\{c_t^i\}_{t=0}^{\infty}$ only depends on the aggregate state of the economy and not on the whole history.

# Pareto Optimal Allocations

## Irrelevance of History

- Assume $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$, then, we have:

$$c_t^i(s^t) = c_t^j(s^t)\left(\frac{\alpha_i}{\alpha_j}\right)^{1/\sigma}$$

- Given the feasibility constraint:

$$\sum_{i=1}^{I} c_t^j(s^t)\left(\frac{\alpha_i}{\alpha_j}\right)^{1/\sigma} = c_t^j(s^t)\left(\frac{1}{\alpha_j}\right)^{1/\sigma}\sum_{i=1}^{I}\alpha_i^{1/\sigma} = W(s^t)$$

which allows us to find

$$c_t^j(s^t) = \frac{\alpha_j^{1/\sigma}}{\sum_{i=1}^{I}\alpha_i^{1/\sigma}} W_t(s^t)$$

Agent $j$ consumes a constant fraction of total endowment in every period.

# Pareto Optimal Allocations

## Irrelevance of History

- We can write the last expression in logs as:

$$\log c_t^j(s^t) = \log \theta_j + \log W_t(s^t)$$

or in first-differences, we could estimate using CEX data:

$$\Delta \log c_t^j(s^t) = \alpha_1 \Delta \log W_t(s^t) + \alpha_2 \Delta \log w_t^j(s^t) + \epsilon_{j,t}$$

- We get $\alpha_2 > 0$: excess sensitivity of consumption

# Sequential Markets

## Setup

- Suppose now that trade takes place sequentially in spot markets each period.
- Agents can buy and sell one period contingent claims or **Arrow securities** each period.
    - Securities that pay 1 unit of good at time $t+1$ for a particular realization of $s_{t+1}$ tomorrow.
    - Let $Q(s_{t+1}|s^t)$ be the price of such contract at time $t$.
    - Let $a^i_{t+1}(s_{t+1}, s^t)$ be the purchase of agent $i$ of such contract.
- Period $t$ budget constraint is given by:

$$c^i(s^t) + \sum_{s_{t+1}} a^i_{t+1}(s^t, s_{t+1}) Q(s^t, s_{t+1}) = w^i_t(s^t) + a_t(s^t)$$

- Note that although the agent buys a portfolio of Arrow securities at time $t$, at $t+1$ only one of these securities will deliver returns.

# Sequential Equilibrium

## No-Ponzi Condition

- With a sequential market structure we again need to put a debt limit to rule out Ponzi schemes.
- A natural debt limit $A_t^i(s^t)$ for an agent can be calculated as

$$p_t(s^t)A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) w_t^i(s^\tau),$$

$$\text{Debt limit: } -A_t^i(s^t) \leq a_t(s^t)$$

which means that the current value of your future endowments cannot be larger than the value of your debt using Arrow-Debreu prices.

# Sequential Equilibrium

- **Definition**: A sequential market equilibrium in this economy is prices for Arrow securities $Q(s^t, s_{t+1})$ for all $t$ and for all $s^t$, allocations $c_t^i(s^t)$ and $a_{t+1}^i(s^t, s_{t+1})$ for all agents, all $t$ and all $s^t$ such that

  1. For each $i$, given $Q(s^t, s_{t+1}), c_t^i(s^t)$ and $a_{t+1}^i(s_{t+1}, s^t)$ solve

$$\max_{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) u(c^i(s^t))$$

$$\text{s.t.} \quad c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s^t, s_{t+1}) Q(s^t, s_{t+1}) = w_t^i(s^t) + a_t(s^t)$$

$$a_{t+1}^i(s^t, s_{t+1}) \geq -A_{t+1}^i(s^{t+1})$$

  2. Markets clear:

Agg. ressource constraint: $\sum_{i=1}^{n} w_t^i(s^t) = \sum_{i=1}^{n} c_t^i(s^t)$ for all $s^t$

Securities are in zero net supply: $\sum_{i}^{n} a_{t+1}^i(s^t, s_{t+1}) = 0$ for all $s^t$ and $s_{t+1}$

# Sequential Equilibrium
## FOCs

- By FOCs we get:

$$Q(s^t, s_{t+1})u'(c_t^i(s^t)) = \beta\pi(s_{t+1}|s_t)u'(c_{t+1}^i(s^{t+1}))$$

from which we can see that if we let:

$$Q(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}$$

the allocations under the Arrow-Debreu and Sequential market structure coincide as the natural debt limit will not bind, the FOCs hold, and the aggregate ressource countraint holds.