

# Life-Cycle Models with Heterogeneous Agents

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# Introduction

Life cycle is a very important dimension for many questions:

- Accounting for the **wealth distribution**.  
Castañeda, Díaz-Giménez, and Ríos-Rull (2009)
- **Social security** programs transfer resources from workers to retirees.  
Fuster, İmrohoroglu, İmrohoroglu (2007)
- **Tax reforms**  
Conesa, Kitao and Krueger (2009)
- **Human capital** accumulation and endogenous earnings inequality has a clear life-cycle component.  
Ben Porath (1967); Hugget, Ventura and Yaron (2011)
- **Portfolio choice**.  
Cocco, Gomes, and Maenhout (2005)

## Huggett (1996)

- Extension of Diamond (1965) OLG model.
  - Multi-period.
  - Lifetime uncertainty.
  - Income uncertainty.
- It can also be seen as Aiyagari (1994) w/ life cycle.
- First serious attempt at accounting for the wealth distribution
- Results:
  - It matches the large Gini index of the US wealth distribution.
  - It does so through a counterfactually large share of people in zero wealth and too little concentration at the top.

# Huggett (1996)

## Setup

- Life-cycle dimension:
  - The average labor income changes with age.
  - Households retire at age  $J_R$ .
  - The probability of surviving to the next period is age-dependent  
In period  $J$  the probability of dying is 1
- Stationary age distribution:
  - Each period a continuum of households of size  $\bar{N}_t$  are born.
  - New cohorts may grow in size at a constant rate  $\bar{N}_{t+1} = (1 + n)\bar{N}_t$ .
  - The survival probabilities are time-independent.
- Stationary economy:
  - No aggregate uncertainty.
  - Wealth and income distribution identical across time for a given age.
- Standard production side.

# Households

## Setup

- Labor market income  $e(z, j)w$ 
  - $w$  is the market wage rate common to all agents.
  - $e(z, j)$  is the productivity of agents at  $j$  with idiosyncratic productivity  $z$ .  
(after retirement, age  $j = J_R$ , it will be zero)
  - $z \in \mathbf{Z} \equiv \{z_1, z_2, \dots, z_M\}$  and follows a Markov process  $\Gamma_{z, z'}$ .
- There is a PAYG social security system, pays  $b_j = b > 0$  for  $j \geq J_R$ .
- Agents can save and borrow through a risk free asset  $a$ :
  - to smooth out the life-cycle earnings profile.
  - to self-insure against earnings uncertainty.
  - to self-insure against excessive longevity risk.

There is a lower bound  $\underline{a}$  on the holdings of this asset.

More generally, we establish  $a \in \mathbf{a} \equiv [\underline{a}, \bar{a}]$

# Households

## Decision Problem

- Households have preferences over consumption at different points in time.
- At birth, expected utility is given by:

$$E \left[ \sum_{j=1}^J \beta^{j-1} \left( \prod_{i=1}^j s_i \right) u(c_j) \right]$$

where  $s_i$  are conditional survival probabilities.

- The budget constraints they face are of the type:

$$c_j + a_{j+1} = a_j R + (1 - \theta) e(j, z) w + T + b_j$$

$T$  denotes accidental bequests,  $\theta$  is the social security payroll tax and  $b_j$  the social security transfer.

- The feasibility and terminal constraints:

$$c_j \geq 0, \quad a_j \geq \underline{a}, \quad a_1, z_1 \text{ given, and } a_{j+1} \geq 0 \text{ if } j = J$$

## A Note on Social Security

- It is important to introduce a public PAYG social security as in data:
  1. It helps generate the right incentives for retirement savings:
    - PAYG social security substitutes private savings  
(PAYG  $\Rightarrow$  Lower aggregate capital in steady state)
    - Public pensions are paid out as life annuities  
(insurance against excessive longevity risk  $\Rightarrow$  lower savings incentives)
  2. It helps produce a sizeable share of asset-poor households.
- In this formulation, the author does not link pensions to contributions. This implies that there is:
  - Lower uncertainty in the model economy.
  - Low incentives to save for income-poor households.
  - High incentives to save for income-rich households.  
(The model generates inequality through a wrong channel)

# Household Problem

## Recursive Problem

- The HH problem in recursive form:

$$\begin{aligned}
 v_j(a, z) = \max_{a', c} & \left\{ u(c) + s_j \beta \sum_{z'} \Gamma_{z', z} v_{j+1}(a', z') \right\} \\
 \text{s.t. } & c + a' = aR + (1 - \theta)e(j, z)w + b_j + T \\
 & a' \geq \underline{a} \text{ and } c \geq 0
 \end{aligned}$$

- The standard Euler equation:

$$\begin{aligned}
 & u_c(aR + (1 - \theta)e(j, z)w + b_j + T - a') \\
 & = s_j \beta R \sum_{z'} \Gamma_{z, z'} u_c(a'R + (1 - \theta)e(j + 1, z')w + b_{j+1} + T - a'')
 \end{aligned}$$

- We are looking for policy function  $g_j^a(a, z)$  and  $g_j^c(a, z)$



# Solving the Household Problem

## Backwards Induction

- Analogous to value function iteration.
- In the life-cycle problem, the Bellman equation is not stationary:  $v_{j+1}(a, z)$  is a different function than  $v_j(a, z)$ .
- Hence, we do not look for a fixed point exploiting the Contraction Mapping Theorem.
- Instead, we solve by backwards induction:

- Period  $J$  is the last one. Hence we know that:

$$g_J^a(a, z) = 0 \text{ and } g_J^c(a, z) = aR + (1 - \theta)e(J, z)w + b_J + T$$

- Hence the value at  $J$ :

$$v_J(a, z) = u(g_J^c(a, z))$$

- From here on, we can solve backwards for every period  $j$  because we know  $v_{j+1}$

# Solving the Household Problem

## Backwards Induction

In period  $j$  do as follows:

- Solve:

$$\begin{aligned}
 v_j &= \max_{a', c} \{u(c) + s_j \beta \sum_{z'} \Gamma_{z, z'} v_{j+1}(a', z')\} \\
 \text{s.t. } c + a' &= aR + (1 - \theta)e(j, z)w + b + T \\
 a' &\geq \underline{a} \text{ and } c \geq 0
 \end{aligned}$$

where  $v_{j+1}(a', z')$  is known from  $j + 1$  period solution

- Obtain  $g_j^a(a, z)$  and  $g_j^c(a, z)$ .
- Obtain the value function:

$$v_j(a, z) = u(g_j^c(a, z)) + s_j \beta \sum_{z'} \Gamma_{z, z'} v_{j+1}(a', z')$$

- Move on and solve for period  $j - 1$ .

# Solving the Household Problem

## Using the Euler Equation

- The same idea of backwards induction can be applied in the Euler equation when looking for the policy function.
- Let's discretize the space  $a$  of our endogenous state variable into a dimension- $I$  real-valued vector  $\tilde{a} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_I\}$ .
- Let's define  $J \times M \times I$  matrices  $\tilde{g}_j^a$ , where  $M$  is the number of elements of the earnings space  $Z$  and  $I$  is the number of elements of  $\tilde{a}$ .
- Every element  $\{m, i\}$  of the matrix  $\tilde{g}_j^a$  states the choice  $a'$  for an individual of type  $\{z_m, \tilde{a}_i\}$  at age  $j$ .
- Our approximation  $\hat{g}_j^a$  to the true policy function  $g_j^a$  is constructed by linear interpolation of  $\tilde{g}_j^a$

# Solving the Household Problem

## Using the Euler Equation

- Let's define the exogenous income state for a given value  $m$  of the  $z$ -shock as:

$$d_j(w, z_m) = (1 - \theta)e(j, z_m)w + b_j + T$$

as the non-financial income for individual of age  $j$  with shock  $z_m$  and assets level  $\tilde{a}_i$

- Then, we can write the Euler equation as,

$$0 = u_c[d_j(w, z_m) + R\tilde{a}_i - \tilde{g}_j^a(z_m, \tilde{a}_i)] -$$

$$s_j \beta R \sum_{z'} \Gamma(z_m, z') u_c[d_{j+1}(w, z') + R\tilde{g}_j^a(z_m, \tilde{a}_i) - \hat{g}_{j+1}^a(z', \hat{g}_j^a(z, a; \tilde{g}_j^a); \tilde{g}_{j+1}^a)]$$

- Knowing the matrix  $\tilde{g}_{j+1}^a$  the Euler equation delivers a matrix  $\tilde{g}_j^a$ :
  - At  $J$ , agents are constrained so they are not on their Euler equation: we know that  $\tilde{g}_J^a = 0$
  - Then at  $j = J - 1$ , knowing  $\tilde{g}_J^a$  we can solve for  $\tilde{g}_{J-1}^a$
  - Iterating backwards, we can solve by all  $\tilde{g}_j^a$   $j$  with knowledge of  $\tilde{g}_{j+1}^a$

# Solving the Household Problem

## Inverting the Euler Equation: EGM

- The idea of endogenous grid methods is to solve for the current level of asset fixing the choice of savings

*What level of assets ( $a$ ) should an agent with income shock ( $z_m$ ) have so that it would be optimal for him to save ( $\tilde{a}'_i$ )?*

- Then, we can write the Euler equation as,

$$0 = u_c[d_j(w, z_m) + Ra - \tilde{a}'_i] - s_j \beta R \sum_{z'} \Gamma(z_m, z') u_c[d_{j+1}(w, z') + R\tilde{a}'_i - \hat{g}_{j+1}^a(z', \hat{g}_j^a(z, a; \tilde{g}_j^a); \tilde{g}_{j+1}^a)]$$

$\Rightarrow$  Closed form solution for  $a$ !

- Clever idea by Carroll (2006)
  - Avoids root finding (faster)
  - More precise: euler equation error equal to zero at  $\tilde{a}'_i$
  - Be careful with corners

## Firm's Problem

- The firm's problem is very standard.
- We assume Cobb-Douglas production function.
- Firm's maximize:

$$\max_{L,K} K^{\alpha} L^{1-\alpha} - (r + \delta)K - wL$$

- FOC:

$$\begin{aligned}\alpha K^{\alpha-1} L^{1-\alpha} &= r + \delta \\ (1 - \alpha) K^{\alpha} L^{-\alpha} &= w\end{aligned}$$

- The wage is a function of the interest rate and L which is given because of inelastic labor supply.

# Steady State Equilibrium

## Definition

A steady state equilibrium for this economy is:

- a set of functions  $\{v_j, g_j^a, g_j^c\}_{j=1}^J$
- a pair of aggregate allocations  $K$  and  $L$  (in per capita terms)
- an amount of transfers  $T$  (in per capita terms)
- a series of probability measures  $\{\mu_j\}_{j=1}^J$
- a series of transition functions  $\{Q_j\}_{j=1}^J$
- a pair of prices  $\{w, r\}$
- a pair of social security parameters  $\{\theta, b\}$

such that

# Steady State Equilibrium

## Definition

- Households solve their optimization problem. Thus, given a pair of prices  $\{w, r\}$  and social security parameters  $\{\theta, b\}$ , the functions  $\{v_j, g_j^a, g_j^c\}_{j=1}^J$  solve the hh pb.
- Firms solve their optimization problem. Factor prices are thus given by the first order conditions of the firm:

$$R = 1 + F_K(K/L) - \delta \text{ and } w = F_L(K/L)$$

- Labor market clears

$$\sum_{j=1}^{J_R-1} \psi_j \int_{\mathbf{Z} \times a} e(z, j) d\mu_j = L$$

- Capital market clears

$$\sum_{j=1}^J \psi_j \int_{\mathbf{Z} \times a} g_j^a(z, a) d\mu_j = K' = K$$



# Steady State Equilibrium

## Definition

- The social security administration is in balance

$$\theta wL = b \sum_{j=J_R}^J \psi_j$$

- Accidental bequests are given back as transfers,

$$\sum_{j=1}^J \psi_j (1 - s_j) \int_{\mathbf{Z} \times a} R g_j^a(z, a) d\mu_j = T' = T$$

- The measures of households at each age is given by,

$$\mu_{j+1}(B) = \int_{\mathbf{Z} \times a} Q_j(b, B) d\mu_j \text{ and } \mu_1, \text{ given}$$

- The transition functions  $Q_j$  arise from the optimal behavior of households and the markov chain  $\Gamma$ .

- Goods market clears:

$$F(K, L) + (1 - \delta)K = \sum_{j=1}^J \psi_j \int_{\mathbf{Z} \times a} (g_j^a(z, a) + g_j^c(z, a)) d\mu_j$$

# Calibration

- **Demographics**

Life tables to obtain  $s_j$  , average population growth to obtain  $n$

- **Income process**

Estimate from panel data: deterministic age component and residual

- **Social security**  $b$  and  $\theta$

Match average replacement rate in the data and budget balance

- **Technology** parameters  $\delta, \alpha$

I/Y and capital share

- **Preferences** parameters  $\sigma, \beta$

Standard values off the shelves

- **Borrowing limit**,  $\underline{a}$

- **Initial conditions**:  $\mu_1$

Zero wealth and earnings dispersion of young households.

# Calibration

## Social Security

- The social security payroll tax  $\theta$  is calibrated analytically.
  - Let's call  $\omega$  the average replacement ratio in the data.
  - Then, we want the model to satisfy

$$\omega = \frac{b}{wL} \sum_{j=1}^{J_R-1} \psi_j \text{ and } \theta wL = b \sum_{j=J_R}^J \psi_j$$

- Both expressions together give

$$\theta = \omega \frac{\sum_{j=J_R}^J \psi_j}{\sum_{j=1}^{J_R-1} \psi_j}$$

- So, with  $\omega$  from the data we recover analytically the payroll tax  $\theta$
- The pension  $b$  is calibrated together with the equilibrium algorithm

# Steady State Equilibrium

How to find it?

1. Algorithm starts at iteration  $k$  with a guess on  $r_k$
2. Obtain the wage rate  $w_k$  and the social security parameter  $b_k$

$$R_k = 1 + F_K(K_k^d/L) - \delta \text{ and } w_k = F_L(K_k/L) \text{ and } \theta w_k L = b_k \sum_{j=J_R}^J \psi_j$$

3. Iterate to find accidental bequests

3.1 Guess transfers  $T_k^g$

3.2 Solve hh problem with  $T_k^g$

3.3 Aggregate and compute accidental bequests  $T_k^{g+1}$

3.4 If they are equal go on. Otherwise set  $T_k^{g+1} = T_k^g$  and come back to (3.2)

4. Aggregate household savings  $K_k^s = \sum_{j=1}^J \psi_j \int_{\mathbf{Z} \times \mathbf{a}} g_j^a(z, a) d\mu_j$
5. If  $|K_k^s - K_k^d| < \epsilon$ , stop. Otherwise set  $R_{k+1} = 1 + F_K(K_k^s/L) - \delta$  and back to 2

# Aggregating

## In Theory

- We keep track of the population in the economy by means of
  - $\psi_j$ , the fraction of individuals with age  $j$  (exogenous).
  - $\mu_j(B)$ , the probability measure that tells us the density of individuals of age  $j$  in any subset  $B \subset \mathbf{Z} \times \mathbf{a}$  of the state space.
  - The law of motion for  $\mu_j$  is given by,

$$\mu_{j+1}(B) = \int_{\mathbf{Z} \times \mathbf{a}} Q_j(b, B) d\mu_j$$

- Hence, note that there are  $J$  distributions  $\mu_j$ , one for every age group.
- Notice that we need to give an initial condition  $\mu_1$ , which describes the joint distribution of assets and labor earnings of every cohort that enters the labor market.

# Aggregating

## In Practice: Monte-Carlo Simulation

- Take an initial finite sample  $\hat{\mu}_1$   
(This should be a calibration sample)
- At any period  $j$ , take  $\hat{\mu}_j$ , use the  $\hat{g}_j^a$ , the  $\Gamma_{z',z}$ , and a random number generator to compute  $\hat{\mu}_{j+1}$ .
- In this manner, you end up with  $J$  distributions  $\hat{\mu}_j$ .
- Then, the  $\psi_j$  can be computed deterministically  
(there is no need to kill anybody)
  - Compute the cross-sectional age distribution at period  $t$ :

$$\tilde{\psi}_{t,j+1} = s_j \tilde{\psi}_{t,j} (1+n)^{-1} \text{ and } \tilde{\psi}_{t,1} = \bar{N}_t$$

- And then normalize by population size such that the  $\psi_j$  sum up to one:

$$\psi_{j=a} = \frac{\tilde{\psi}_{j=a}}{\sum_{j=1}^J \tilde{\psi}_j}$$