# Time Series Econometrics

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EUI

# The Course in a Nutshell

- This course introduces students to the analysis, modeling and estimation of time series processes.
- Part I (Jesús):
  - Difference equations.
  - ARMA processes.
  - Estimation and inference: Maximum likelihood with serially dependent observations.
  - Vector Autoregressions.
- Part II (Barbara):
  - Non-stationary time series processes
  - State-space representation
  - Identification of structural VAR

# References

- The **main** references of the course is *Time Series Analysis* by James D. Hamilton.
- Other useful references:
  - *New Introduction to Multiple Time Series Analysis* by Helmut Lütkepohl.
  - *Applied Econometric Time Series* by Walter Enders.
  - *Econometric Modelling with Time Series* by V. L. Martin, A. S. Hurn and D. Harris.

# Chapter 1: Difference Equations

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# First-Order Difference Equation

- This part follows Hamilton's Chapter 1.
- The theory of difference equations underlies all the time-series process that we will see in the course.
- Suppose we are studying a variable (scalar) whose value at time $t$ is denoted $y_t$.
- Suppose we also know the dynamic equation relating the value $y$ at date $t$ to another variable $w_t$ and to the value of $y$ in the previous period:
$$y_t = \phi y_{t-1} + w_t, \tag{1}$$
where $\{w_t\}$ is exogenously given and bounded.
- This is a first-order (one lag) linear difference equation.

# First-Order Difference Equation

## Solving by Recursive Substitution

- The presumption is that equation 1 governs the behavior of $y$ for all dates $t$.
- If we knew the value of $y$ for date $t = -1$ and the value of $w$ for all dates, then, it is possible to simulate the dynamic system to recover $y$.

$$
\begin{aligned}
y_0 &= \phi y_{-1} + w_0 \\
y_1 &= \phi^2 y_{-1} + \phi w_0 + w_1 \\
&\vdots \\
y_t &= \phi^{t+1} y_{-1} + \sum_{j=0}^{t} \phi^j w_{t-j}
\end{aligned}
$$

# First-Order Difference Equation

## Impulse Response

- Note that we have expressed $y_t$ as a linear function of the initial value $y_0$ and the historical values of $w$.
- Therefore the effect of an increase of $w_0$ on $y_t$ would be given by:

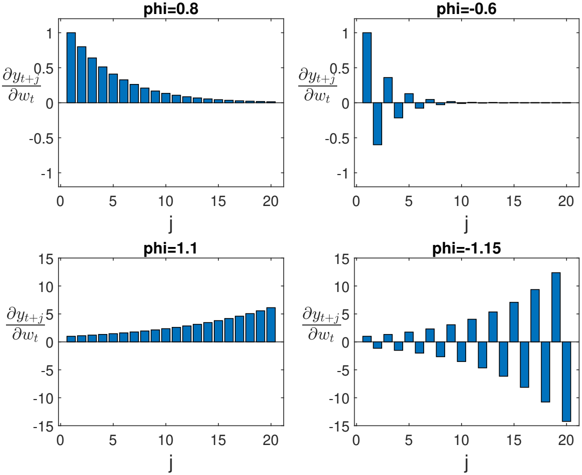
$$\frac{\partial y_t}{\partial w_0} = \phi^t = \frac{\partial y_{t+j}}{\partial w_j}$$

- The impulse response depends only on $j$ not on time.
- The impulse response function is also referred as the dynamic multiplier.
- We say that the system is stable if $|\phi| < 1$; the consequences of a given change in $w_t$ will eventually die out.

# First-Order Difference Equation

## Impulse Response

- Different values of $\phi$ can produce a variety of dynamic responses of $y_{t+j}$ to $w_t$.



# p*th* Order Difference Equation

- We can generalize the dynamic system in equation (1) by allowing the value of $y$ at date $t$ to depend on $p$ of its own lags:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t \tag{2}$$

- Since we already know how to compute the solution to a first-order difference equation, we can rewrite equation (2) as a first-order equation in a vector .

# p*th* Order Difference Equation

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\boldsymbol{\xi}_t = \boldsymbol{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{v}_t$$

# p*th* Order Difference Equation

## Recursive Substitution

- Exactly as we did before, if we knew the dynamics of $\boldsymbol{v}_t$ and an initial condition on the state of $\boldsymbol{\xi}_{-1}$, we could back-out the state of $\boldsymbol{\xi}_t$ for any $t$

$$
\begin{aligned}
\boldsymbol{\xi}_0 =& \boldsymbol{F}\boldsymbol{\xi}_{-1} + \boldsymbol{v}_0 \\
\boldsymbol{\xi}_1 =& \boldsymbol{F}^2\boldsymbol{\xi}_{-1} + \boldsymbol{F}\boldsymbol{v}_0 + \boldsymbol{v}_1 \\
&\vdots \\
\boldsymbol{\xi}_t =& \boldsymbol{F}^{t+1}\boldsymbol{\xi}_{-1} + \sum_{j=0}^{t} \boldsymbol{F}^j \boldsymbol{v}_{t-j}
\end{aligned}
$$

- Consider the first equation of this system:

$$
\begin{aligned}
y_t =& f^{t+1}_{(1,1)} y_{-1} + f^{t+1}_{(1,2)} y_{-2} + \cdots + f^{t+1}_{(1,p)} y_{-p} + \\
& f^{t}_{(1,1)} w_0 + f^{t-1}_{(1,1)} w_1 + \cdots + f_{(1,1)} w_{t-1} + w_t
\end{aligned}
$$

# p*th* Order Difference Equation

## Impulse-Response Function

- The effect of an increase of $w_1$ on $y_t$ is given by:

$$\frac{\partial y_t}{\partial w_0} = f^{t}_{(1,1)}$$

or more equivalently:

$$\frac{\partial y_{t+j}}{\partial w_t} = f^{j}_{(1,1)}$$

- For $j = 1$: $\phi_1$
- For $j = 2$: $\phi_1^2 + \phi_2$
- For larger values of $j$: simulate (set $y_{-1} = y_{-1} = \cdots = y_{-p} = 0$ and $w_0 = 1$ and iterate on equation 2)

# p*th* Order Difference Equation

- The system is stable whenever the eigenvalues of **F** are within the unit circle (smaller than one if real, modulus smaller than one if imaginary).
- Example 2nd order difference: equations:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

in this case the **F** matrix is given by:

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}$$

$$|\mathbf{F} - \lambda \mathbf{I}| = 0$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

then,

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}; \lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

# p*th* Order Difference Equation

- Real root if $\phi_2 \geq -\phi_1^2/4$
  - Stable if $\lambda_1 < 1 \Leftrightarrow \phi_2 < 1 - \phi_1$ and $\lambda_2 > -1 \Leftrightarrow \phi_2 < 1 + \phi_1$
- Imaginary root if $\phi_2 < -\phi_1^2/4$
  - Stable if $|\,\phi_1/2 \pm i\sqrt{-\phi_1^2 - 4\phi_2}/2\,| < 1 \Leftrightarrow \phi_2 > -1$

# Lag Operators

- Operation represented by the symbol $L$: $Lx_t = x_{t-1}$.
- We could rewrite a first-order difference equation using the lag operator:

$$y_t = \phi y_{t-1} + w_t \Leftrightarrow y_t - \phi y_{t-1} = w_t \Leftrightarrow (1 - \phi L) y_t = w_t$$

To find the solution, multiply by $(1 + \phi L + \phi^2 L^2 + \cdots + \phi^t L^t)$:

$$(1 + \phi L + \phi^2 L^2 + \cdots + \phi^t L^t)(1 - \phi L) y_t = (1 + \phi L + \phi^2 L^2 + \cdots + \phi^t L^t) w_t$$

$$(1 - \phi^{t+1} L^{t+1}) y_t = (1 + \phi L + \phi^2 L^2 + \cdots \phi^t L^t) w_t$$

$$y_t = \phi^{t+1} y_{-1} + \sum_{i=0}^{t} \phi^i w_{t-i}$$

# Lag Operators

- For $|\phi| < 1$ and $t$ large,$(1 - \phi^{t+1} L^{t+1}) y_t \simeq y_t$

  Thus, $(1 - \phi L)^{-1} \simeq (1 + \phi L + \phi^2 L^2 + \cdots \phi^t L^t)$ (equal in the $t$ limit)

  Then: $y_t = \sum_{i=0}^{\infty} \phi^i w_{t-i}$

# Lag Operators

- We can also re-write a p-th order difference equation as:

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + w_t \\
(1 - \phi_1 L - \cdots - \phi_p L^p) y_t &= w_t
\end{aligned}
$$

  Now you can factor a p-th order polynomial as:

$$
(1 - \phi_1 L - \cdots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_p L)
$$

- Again the system is stable when $\lambda$'s are within the unit circle.
- It is equivalent as looking for the eigenvalues of matrix $\boldsymbol{F}$.