Life-Cycle Models with Heterogeneous Agents

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Introduction

Life cycle is a very important dimension for many questions:

- Accounting for the wealth distribution.
  Castañeda, Díaz-Giménez, and Ríos-Rull (2009)
- Social security programs transfer resources from workers to retirees.
  Fuster, İmrohoroğlu, İmrohoroğlu (2007)
- Tax reforms
  Conesa, Kitao and Krueger (2009)
- Human capital accumulation and endogenous earnings inequality has a clear life-cycle component.
  Ben Porath (1967); Hugget, Ventura and Yaron (2011)
- Portfolio choice.
  Cocco, Gomes, and Maenhout (2005)

Huggett (1996)

- Extension of Diamond (1965) OLG model.
  - Multi-period.
  - Lifetime uncertainty.
  - Income uncertainty.

- It can also be seen as Aiyagari (1994) w/ life cycle.

- First serious attempt at accounting for the wealth distribution

- Results:
  - It matches the large Gini index of the US wealth distribution.
  - It does so through a counterfactually large share of people in zero wealth and too little concentration at the top.

Huggett (1996)

Setup

- Life-cycle dimension:
  - The average labor income changes with age.
  - Households retire at age J_R.
  - The probability of surviving to the next period is age-dependent
    In period J the probability of dying is 1

- Stationary age distribution:
  - Each period a continuum of households of size $\bar{N}_t$ are born.
  - New cohorts may grow in size at a constant rate $\bar{N}_{t+1} = (1+n)\bar{N}_t$.
  - The survival probabilities are time-independent.

- Stationary economy:
  - No aggregate uncertainty.
  - Wealth and income distribution identical across time for a given age.

- Standard production side.

Households

Setup

- Labor market income $e(z, j)w$
  - w is the market wage rate common to all agents.
  - $e(z, j)$ is the productivity of agents at j with idiosyncratic productivity z.
    (after retirement, age $j = J_R$, it will be zero)
  - $z \in \boldsymbol{Z} \equiv \{z_1, z_2, \ldots, z_M\}$ and follows a Markov process $\Gamma_{z,z'}$.

- There is a PAYG social security system,pays $b_j = b > 0$ for $j \geq J_R$.

- Agents can save and borrow through a risk free asset a:
  - to smooth out the life-cycle earnings profile.
  - to self-insure against earnings uncertainty.
  - to self-insure against excessive longevity risk.

There is a lower bound $\underline{a}$ on the holdings of this asset.

More generally, we establish $a \in \boldsymbol{a} \equiv [\underline{a}, \bar{a}]$

Households

Decision Problem

- Households have preferences over consumption at different points in time.

- At birth, expected utility is given by:

$$E\left[\sum_{j=1}^{J}\beta^{j-1}\left(\prod_{i=1}^{j}s_i\right)u(c_j)\right]$$

  where s_i are conditional survival probabilities.

- The budget constraints they face are of the type:

$$c_j + a_{j+1} = a_j R + (1-\theta)e(j,z)w + T + b_j$$

  T denotes accidental bequests, θ is the social security payroll tax and b_j the social security transfer.

- The feasibility and terminal constraints:
  $c_j \geq 0,\ a_j \geq \underline{a},\ a_1, z_1$ given, and $a_{j+1} \geq 0$ if $j = J$

A Note on Social Security

- It is important to introduce a public PAYG social security as in data:

  1. It helps generate the right incentives for retirement savings:
     - PAYG social security substitutes private savings
       (PAYG $\Rightarrow$ Lower aggregate capital in steady state)
     - Public pensions are paid out as life annuities
       (insurance against excessive longevity risk $\Rightarrow$ lower savings incentives)

  2. It helps produce a sizeable share of asset-poor households.

- In this formulation, the author does not link pensions to contributions. This implies that there is:
  - Lower uncertainty in the model economy.
  - Low incentives to save for income-poor households.
  - High incentives to save for income-rich households.
    (The model generates inequality through a wrong channel)

Household Problem

Recursive Problem

- The HH problem in recursive form:

$$
\begin{aligned}
v_j(a,z) = &\max_{a',c} \left\{ u(c) + s_j \beta \sum_{z'} \Gamma_{z',z} v_{j+1}(a',z') \right\} \\
\text{s.t. } & c + a' = aR + (1-\theta)e(j,z)w + b_j + T \\
& a' \geq \underline{a} \text{ and } c \geq 0
\end{aligned}
$$

- The standard Euler equation:

$$
\begin{aligned}
&u_c(aR + (1-\theta)e(j,z)w + b_j + T - a') \\
&\quad = s_j \beta R \sum_{z'} \Gamma_{z,z'} u_c(a'R + (1-\theta)e(j+1,z')w + b_{j+1} + T - a'')
\end{aligned}
$$

- We are looking for policy function $g_j^a(a,z)$ and $g_j^c(a,z)$

Solving the Household Problem

Backwards Induction

- Analogous to value function iteration.
- In the life-cycle problem, the Bellman equation is not stationary: $v_{j+1}(a, z)$ is a different function than $v_j(a, z)$.
- Hence, we do not look for a fixed point exploiting the Contraction Mapping Theorem.
- Instead, we solve by backwards induction:
  - Period J is the last one. Hence we know that:

$$g_J^a(a, z) = 0 \text{ and } g_J^c(a, z) = aR + (1 - \theta)e(J, z)w + b_J + T$$

  - Hence the value at J:

$$v_j(a, z) = u(g_J^c(a, z))$$

  - From here on, we can solve backwards for every period j because we know v_{j+1}

Solving the Household Problem

Backwards Induction

In period j do as follows:

- Solve:

$$v_j = \max_{a',c} \{u(c) + s_j \beta \sum_{z'} \Gamma_{z,z'} v_{j+1}(a', z')\}$$

$$\text{s.t. } c + a' = aR + (1-\theta)e(j,z)w + b + T$$

$$a' \geq \underline{a} \text{ and } c \geq 0$$

where $v_{j+1}(a', z')$ is known from $j+1$ period solution

- Obtain $g_j^a(a,z)$ and $g_j^c(a,z)$.
- Obtain the value function:

$$v_j(a,z) = u(g_j^c(a,z)) + s_j \beta \sum_{z'} \Gamma_{z,z'} v_{j+1}(a', z')$$

- Move on and solve for period $j-1$.

Solving the Household Problem

Using the Euler Equation

- The same idea of backwards induction can be applied in the Euler equation when looking for the policy function.
- Let's discretize the space a of our endogenous state variable into a dimension-I real-valued vector $\tilde{a} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_I\}$.
- Let's define $J \quad M \times I$ matrices $\tilde{g}_j^a$, where M is the number of elements of the earnings space Z and I is the number of elements of $\tilde{a}$.
- Every element $\{m, i\}$ of the matrix $\tilde{g}_j^a$ states the choice a' for an individual of type $\{z_m, \tilde{a}_i\}$ at age j.
- Our approximation $\hat{g}_j^a$ to the true policy function g_j^a is constructed by linear interpolation of $\tilde{g}_j^a$

Solving the Household Problem

Using the Euler Equation

- Let's define the exogenous income state for a given value m of the z-shock as:

$$d_j(w, z_m) = (1-\theta)e(j, z_m)w + b_j + T$$

  as the non-financial income for individual of age j with shock z_m and assets level $\tilde{a}_i$

- Then, we can write the Euler equation as,

$$0 = u_c[d_j(w, z_m) + R\tilde{a}_i - \tilde{g}^a_j(z_m, \tilde{a}_i)] -$$

$$s_j \beta R \sum_{z'} \Gamma(z_m, z') u_c[d_{j+1}(w, z') + R\tilde{g}^a_j(z_m, \tilde{a}_i) - \hat{g}^a_{j+1}(z', \hat{g}^a_j(z, a; \tilde{g}^a_j); \tilde{g}^a_{j+1})]$$

- Knowing the matrix $\tilde{g}^a_{j+1}$ the Euler equation delivers a matrix $\tilde{g}^a_j$:
  - At J, agents are constrained so they are not on their Euler equation: we know that $\tilde{g}^a_J = 0$
  - Then at $j = J-1$, knowing $\tilde{g}^a_J$ we can solve for $\tilde{g}^a_{J-1}$
  - Iterating backwards, we can solve by all $\tilde{g}^a_j$ j with knowledge of $\tilde{g}^a_{j+1}$

Solving the Household Problem

Inverting the Euler Equation: EGM

- The idea of endogenous grid methods is to solve for the current level of asset fixing the choice of savings

  What level of assets (a) should an agent with income shock (z_m) have so that it would be optimal for him to save ($\tilde{a}'_i$)?

- Then, we can write the Euler equation as,

$$0 = u_c[d_j(w, z_m) + Ra - \tilde{a}'_i] -$$
$$s_j \beta R \sum_{z'} \Gamma(z_m, z') u_c[d_{j+1}(w, z') + R\tilde{a}'_i - \hat{g}^a_{j+1}(z', \hat{g}^a_j(z, a; \tilde{g}^a_j); \tilde{g}^a_{j+1})]$$

  $\Rightarrow$ Closed form solution for a!

- Clever idea by Carroll (2006)
  - Avoids root finding (faster)
  - More precise: euler equation error equal to zero at $\tilde{a}'_i$
  - Be careful with corners

Firm's Problem

- The firm's problem is very standard.
- We assume Cobb-Douglas production function.
- Firm's maximize:

$$\max_{L,K} K^{\alpha} L^{1-\alpha} - (r+\delta)K - wL$$

- FOC:

$$\alpha K^{\alpha-1} L^{1-\alpha} = r + \delta$$
$$(1-\alpha) K^{\alpha} L^{-\alpha} = w$$

- The wage is a function of the interest rate and L which is given because of inelastic labor supply.

Steady State Equilibrium

Definition

A steady state equilibrium for this economy is:

- a set of functions $\{v_j, g_j^a, g_j^c\}_{j=1}^{J}$
- a pair of aggregate allocations K and L (in per capita terms)
- an amount of transfers T (in per capita terms)
- a series of probability measures $\{\mu_j\}_{j=1}^{J}$
- a series of transition functions $\{Q_j\}_{j=1}^{J}$
- a pair of prices $\{w, r\}$
- a pair of social security parameters $\{\theta, b\}$

such that

Steady State Equilibrium

Definition

- Households solve their optimization problem. Thus, given a pair of prices $\{w, r\}$ and social security parameters $\{\theta, b\}$, the functions $\{v_j, g_j^a, g_j^c\}_{j=1}^J$ solve the hh pb.
- Firms solve their optimization problem. Factor prices are thus given by the first order conditions of the firm:

$$R = 1 + F_K(K/L) - \delta \text{ and } w = F_L(K/L)$$

- Labor market clears

$$\sum_{j=1}^{J_R-1} \psi_j \int_{\boldsymbol{Z} \times a} e(z, j) d\mu_j = L$$

- Capital market clears

$$\sum_{j=1}^{J} \psi_j \int_{\boldsymbol{Z} \times a} g_j^a(z, a) d\mu_j = K' = K$$

Steady State Equilibrium

Definition

- The social security administration is in balance
$$\theta w L = b \sum_{j=J_R}^{J} \psi_j$$
- Accidental bequests are given back as transfers,
$$\sum_{j=1}^{J} \psi_j (1 - s_j) \int_{\boldsymbol{Z} \times a} R g_j^a(z,a) d\mu_j = T' = T$$
- The measures of households at each age is given by,
$$\mu_{j+1}(B) = \int_{\boldsymbol{Z} \times a} Q_j(b, B) d\mu_j \text{ and } \mu_1, \text{ given}$$
- The transition functions Q_j arise from the optimal behavior of households and the markov chain Γ.
- Goods market clears:
$$F(K, L) + (1 - \delta) K = \sum_{j=1}^{J} \psi_j \int_{\boldsymbol{Z} \times a} (g_j^a(z,a) + g_j^c(z,a)) d\mu_j$$

Calibration

- Demographics
  Life tables to obtain s_j , average population growth to obtain n
- Income process
  Estimate from panel data: deterministic age component and residual
- Social security b and θ
  Match average replacement rate in the data and budget balance
- Technology parameters δ, α
  I/Y and capital share
- Preferences parameters σ, β
  Standard values off the shelves
- Borrowing limit, $\underline{a}$
- Initial conditions: μ_1
  Zero wealth and earnings dispersion of young households.

Calibration

Social Security

- The social security payroll tax θ is calibrated analytically.
  - Let's call ω the average replacement ratio in the data.
  - Then, we want the model to satisfy

$$\omega = \frac{b}{wL} \sum_{j=1}^{J_R - 1} \psi_j \text{ and } \theta w L = b \sum_{j=J_R}^{J} \psi_j$$

  - Both expressions together give

$$\theta = \omega \frac{\sum_{j=J_R}^{J} \psi_j}{\sum_{j=1}^{J_R - 1} \psi_j}$$

  - ► So, with ω from the data we recover analytically the payroll tax θ
- The pension b is calibrated together with the equilibrium algorithm

Steady State Equilibrium

How to find it?

1. Algorithm starts at iteration k with a guess on r_k
2. Obtain prices K_k^d, w_k and the social security parameter b_k

$$R_k = 1 + F_K(K_k^d/L) - \delta \text{ and } w_k = F_L(K_k/L) \text{ and } \theta w_k L = b_k \sum_{j=J_R}^{J} \psi_j$$

3. Iterate to find accidental bequests
   - 3.1 Guess transfers T_k^g
   - 3.2 Solve hh problem with T_k^g
   - 3.3 Aggregate and compute accidental bequests T_k^{g+1}
   - 3.4 If they are equal go on. Otherwise set $T_k^{g+1} = T_k^g$ and come back to (3.2)
4. Aggregate household savings $K_k^s = \sum_{j=1}^{J} \psi_j \int_{\boldsymbol{Z}\times\boldsymbol{a}} g_j^a(z,a) d\mu_j$
5. If $|K_k^s - K_k^d| < \epsilon$, stop. Otherwise set $R_{k+1} = 1 + F_K(K_k^s/L) - \delta$ and back to 2

Aggregating

In Theory

- We keep track of the population in the economy by means of
  - ψ_j, the fraction of individuals with age j (exogenous).
  - $\mu_j(B)$, the probability measure that tells us the density of individuals of age j in any subset $B \subset \boldsymbol{Z} \times \boldsymbol{a}$ of the state space.
  - The law of motion for μ_j is given by,

$$\mu_{j+1}(B) = \int_{\boldsymbol{Z} \times a} Q_j(b, B) d\mu_j$$

  - Hence, note that there are J distributions μ_j , one for every age group.
  - Notice that we need to give an initial condition μ_1, which describes the joint distribution of assets and labor earnings of every cohort that enters the labor market.

Aggregating

In Practice: Monte-Carlo Simulation

- Take an initial finite sample $\hat{\mu}_1$
  (This should be a calibration sample)

- At any period j, take $\hat{\mu}_j$, use the $\hat{g}^a_j$, the $\Gamma_{z',z}$, and a random number generator to compute $\hat{\mu}_{j+1}$.

- In this manner, you end up with J distributions $\hat{\mu}_j$.

- Then, the ψ_j can be computed deterministically
  (there is no need to kill anybody)
  - Compute the cross-sectional age distribution at period t:
  $$\tilde{\psi}_{t,j+1} = s_j \tilde{\psi}_{t,j} (1+n)^{-1} \text{ and } \tilde{\psi}_{t,1} = \bar{N}_t$$
  - And then normalize by population size such that the ψ_j sum up to one:
  $$\psi_{j=a} = \frac{\tilde{\psi}_{j=a}}{\sum_{j=1}^{J} \tilde{\psi}_j}$$