Dynamic Optimization

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Dynamic Optimization

- In this chapter we are going to characterize solutions to dynamic optimization problems
- In order to solve them, we are going to introduce discrete dynamic programming.
- Along our way, we are going to revise some mathematical concepts covered by Villanacci.
- Reference: *The PhD Macro Book* (Chapter 4).

Motivating the Recursive Formulation

A Cake Eating Problem

- We will go over a very simple dynamic optimization problem.
- Suppose that you are presented with a cake of size W_1.
- At each point in time $t = 1, 2, \ldots, T$, you can eat some of the cake but must save the rest.
- Let c_t be your consumption at time t and $u(c_t)$ represent the flow of utility.
- u twice differentiable, strictly increasing, strictly concave, $\lim\limits_{c \to 0} u'(c) = \infty$.
- Discount factor: $0 < \beta < 1$

The Sequential Formulation

A Cake Eating Problem

- The agent is solving:

$$\max_{\{c_t, W_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$
$$\text{s.t. } c_t + W_{t+1} = W_t \ \forall t$$
$$W_{T+1} \geq 0$$

- The Lagrangian associated to this problem is given by:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t (W_t - c_t - W_{t+1}) + \phi W_{T+1}$$

The Sequential Formulation

A Cake Eating Problem

- FOCs:

$$
\begin{aligned}
\beta^t u_c(c_t) &= \lambda_t \\
\lambda_t &= \lambda_{t+1} \\
\lambda_T &= \phi \\
\phi &\geq 0 \;\text{ with }\; \phi W_{T+1} = 0 \Rightarrow \beta^T u_c(c_t) W_{T+1} = 0
\end{aligned}
$$

$$
\begin{aligned}
&u'(c_t) = \beta u'(c_{t+1}) \; \forall t \in [0, T-1] \\
&W_{T+1} = 0
\end{aligned}
$$

- With the set of T intertemporal equations (euler equations), an initial condition and a terminal condition

The Recursive Formulation

A Cake Eating Problem

- In order to solve finite-horizon dynamic programming problems, we are going to proceed by backwards induction.
- For $t = T$, given the properties of u and the constraint, the optimal solution is given by:

$$c_T = W_T$$
$$u(c_T) = u(W_T)$$

The Recursive Formulation

A Cake Eating Problem

- We define the **value function** at time T for the problem at time T as:

$$V_T(W_T) = \max_{c_T} u(c_T)$$
$$c_T + W_{T+1} = W_T$$

- The optimal cake-saving decision is thus:

$$g_T(W_T) = 0$$

and the value function is given by:

$$V_T(W_T) = u(W_T)$$

The Recursive Formulation

A Cake Eating Problem

- Now let's go to $t = T - 1$ given that we have solved the problem for $t = T$ and define V_{T-1}.

$$V_{T-1} = \max_{c_{T-1}, c_T, W_T, W_{T+1}} u(c_{T-1}) + \beta u(c_T)$$
$$s.t. \;\; c_{T-1} + W_T = W_{T-1}$$
$$c_T + W_{T+1} = W_T$$

- Given that we already we know what is optimal to do in the next period, we can simplify the problem at $T - 1$ as:

$$V_{T-1} = \max_{c_{T-1}, W_T} u(c_{T-1}) + \beta V_T(W_T)$$
$$s.t. \;\; c_{T-1} + W_T = W_{T-1}$$

The Recursive Formulation

A Cake Eating Problem

- Le's write the optimality conditions as:

$$u'(c_{T-1}) = \beta V_T'(W_T)$$
$$u'(c_{T-1}) = \beta u_T'(W_T)$$

- The solution coincides with the sequential formulation in the last period.
- We are in good track but what about previous periods?

The Recursive Formulation

A Cake Eating Problem

- Since it's going to be useful let's first derive the value of $V'_{T-1}(W_{T-1})$ given the optimal cake saving decision $g_{T-1}(W_{T-1})$ obtained from the previous FOC.

$$
\begin{aligned}
V_{T-1}(W_{T-1}) =& u(W_{T-1} - g_{T-1}(W_{T-1})) + \beta V_T(g_{T-1}(W_{T-1})) \\
V'_{T-1}(W_{T-1}) =& u'(c_{T-1}) - u'(c_{T-1}) g'_{T-1}(W_{T-1}) + \\
& \beta g'_{T-1}(W_{T-1}) V'_T(W_T) \\
V'_{T-1}(W_{T-1}) =& u'(c_{T-1}) + \\
& g'_{T-1}(W_{T-1})(-u'(c_{T-1}) + \beta V'_T(W_T)) \\
V'_{T-1}(W_{T-1}) =& u'(c_{T-1})
\end{aligned}
$$

- Now we can go to $T-2$

Infinite Horizon

A Cake Eating Problem

- Suppose for the cake-eating problem, we allow the horizon to go to infinity.
- The main advantage of an infinite horizon is that the agent problem becomes stationary: the maximization problem at date t is exactly the same as in period $t+1$
- In general, the infinite horizon maximization problems involve the same mathematical techniques as the finite horizon ones.
- Unlike in finite horizon case, we don't have a terminal condition in the cake eating problem we will thus need to impose a transversality condition:

$$\lim_{t\to\infty} \beta^t u_c(c_t) W_{t+1} = 0$$

if discounted marginal utility is positive, the amount of cake needs to go to zero to rule out over-accumulation

Infinite Horizon

A Cake Eating Problem

- One can consider solving the infinite horizon sequence given by:

$$\max_{\{c_t, W_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + W_{t+1} = W_t \ \forall \ t$$
$$\lim_{t\to\infty} \beta^t u_c(c_t) W_{t+1} = 0$$

- Written in recursive form:

$$V(W_t) = \max_{\{c_t, W_{t+1}\}} u(c_t) + \beta V(W_{t+1}) \tag{1}$$
$$\text{s.t. } c_t + W_{t+1} = W_t$$
$$\lim_{t\to\infty} \beta^t V(W_t) = 0 \tag{2}$$

The transversality condition (2) is frequently avoided because with V being bounded and $\beta < 1$, it will automatically be satisfied.

Infinite Horizon

A Cake Eating Problem

- Equation (1) is referred as the Bellman equation.
- It is a functional equation: the unknown represents as function.
- By FOCs:

$$u_c(c_t) = \beta \frac{\partial V(W_{t+1})}{\partial W_{t+1}} \tag{3}$$

- Let's define $g(W)$ the optimal savings function associated with equation (1):

$$\begin{aligned} g(W_t) =& arg \max_{W_{t+1}} u(W_t - W_{t+1}) + \beta V(W_{t+1}) \\ V(W_t) =& u(W_t - g(W_t)) + \beta V(g(W_t)) \end{aligned}$$

Infinite Horizon

A Cake Eating Problem

- Provided that g is differentiable we can now compute:

$$
\begin{aligned}
V'(W_t) &= u_c(c_t) + g'(W_t)\Big(\beta V'(W_{t+1}) - u_c(c_t)\Big) \\
V'(W_t) &= u_c(c_t) \\
V'(W_{t+1}) &= u_c(c_{t+1})
\end{aligned}
$$

- Then we can write equation (3) as:

$$
u_c(c_t) = \beta u_c(c_{t+1})
$$

Infinite Horizon

A Cake Eating Problem

- If the functional equation was a contraction mapping, then Bellman equation would have a unique solution.
- Moreover, this solution could be found by iterating on the Bellman equation, irrespectively of any initial guess for the value function $V^g(W)$ for $V(W)$.