Maximum Likelihood Estimation and Graph Matching in Errorfully Observed Networks

Jesús Arroyo

September 20th, 2020

Joint work with:

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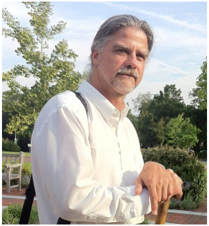
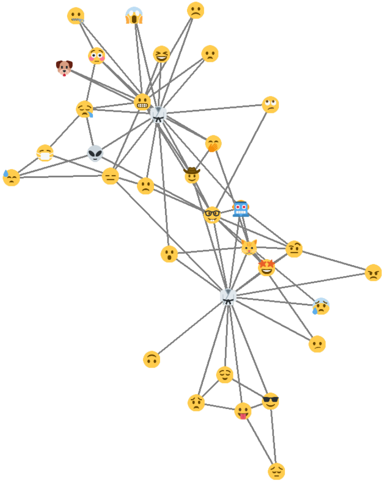
(Boston University)



Carey E. Priebe
(Johns Hopkins University)

Vince Lyzinski
(University of Maryland)

Graph matching



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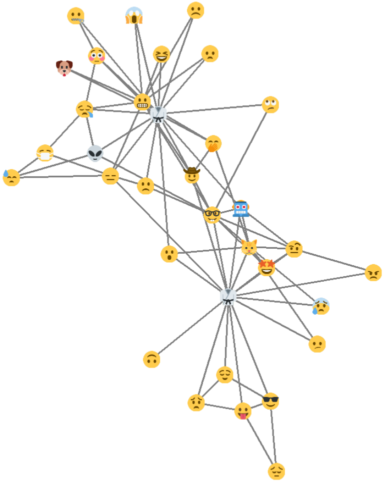

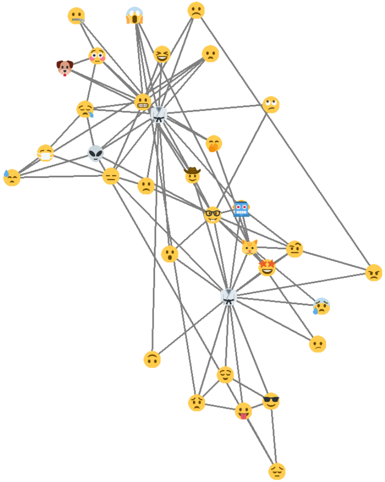

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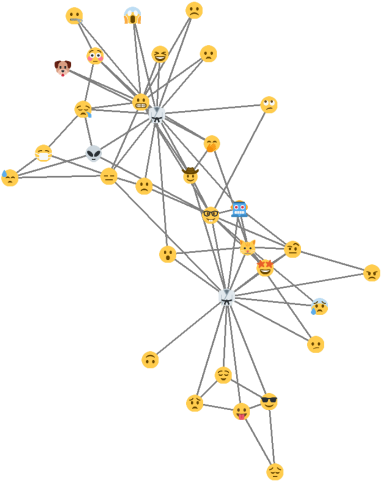

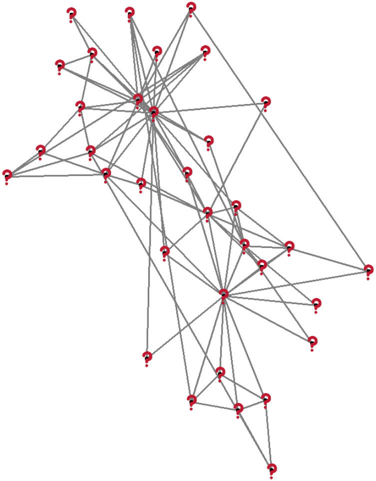

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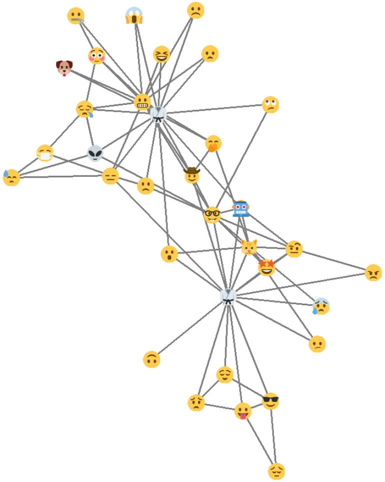

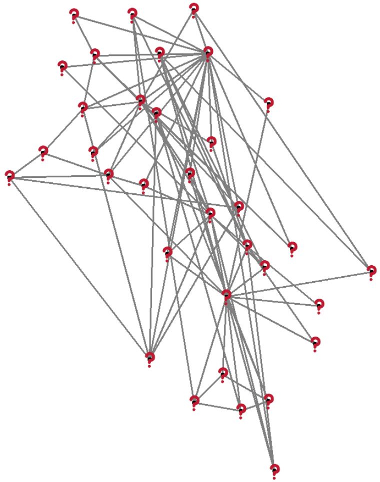

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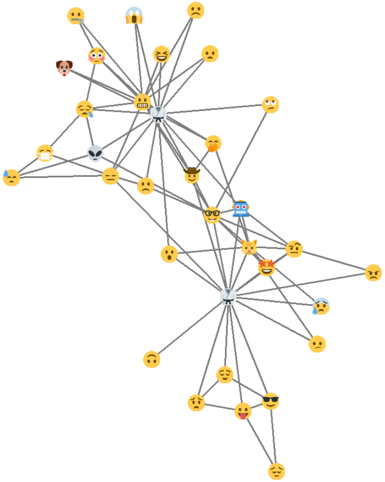

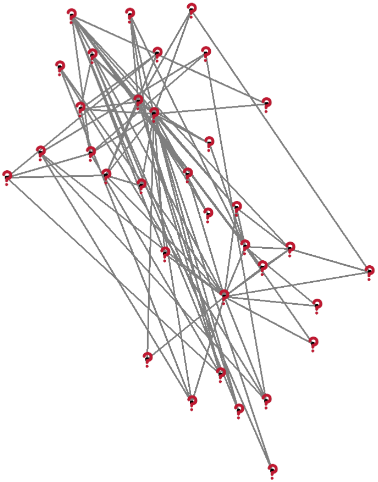

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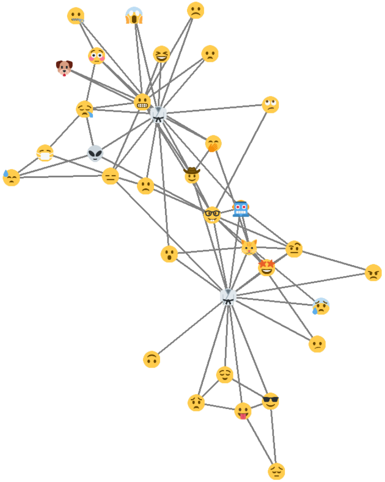

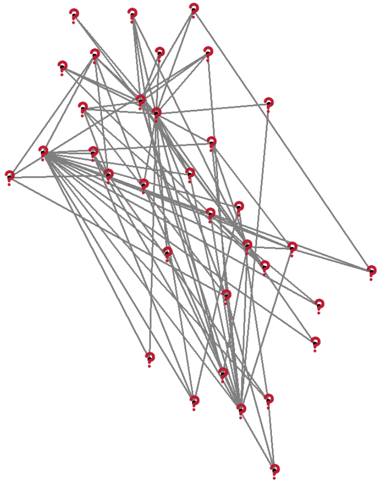

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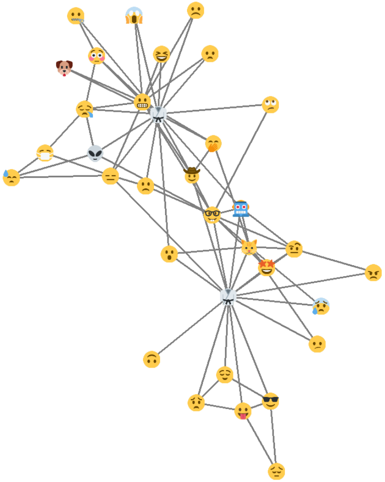

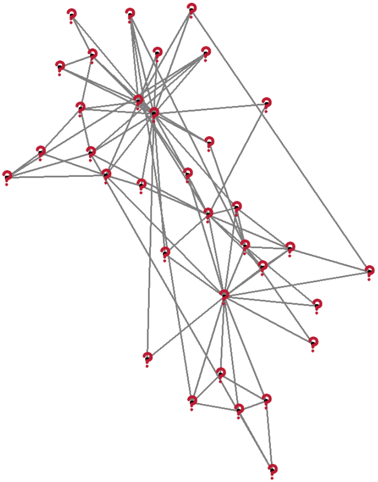

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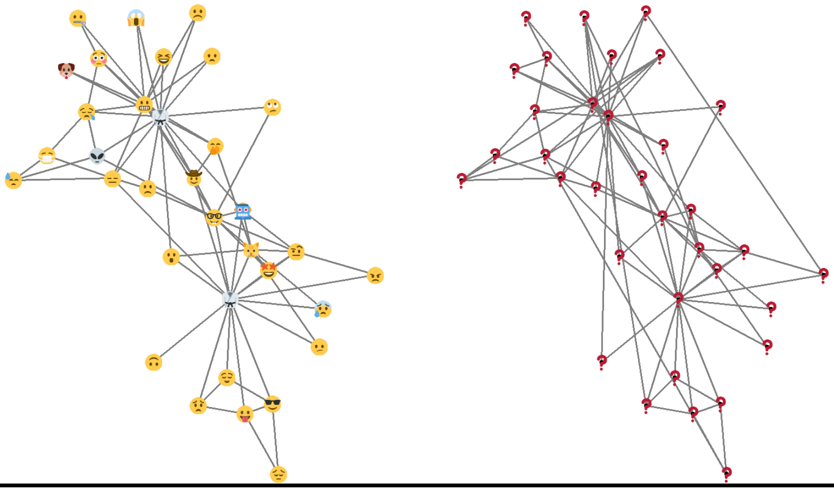

- How can we formulate a statistical model for graph matching?
- When is statistically feasible to solve the problem?

Graph matching problem

Goal: find a meaningful correspondence between the vertices of the graphs

- Typically defined as the permutation that minimizes the number of edge disagreements:

$$\begin{aligned}\widehat{Q}_{\mathsf{GM}} &= \operatorname*{argmin}_{Q \in \Pi_n} \|A - QBQ^T\|_F^2 \\ &= \operatorname*{argmin}_{Q \in \Pi_n} \sum_{i \neq j} \left(A_{ij} - B_{\sigma(i),\sigma(j)}\right)^2\end{aligned}$$

  - $A, B \in \{0,1\}^{n \times n}$ adjacency matrices
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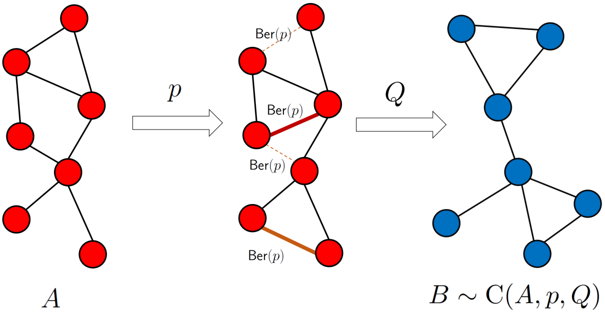
Some applications:

- Aligning biological networks
- De-anonymization in social networks
- Unsupervised word translation

Corrupting channel model

Model: B is an edge and vertex-label corrupted version of A

1. Edges and non-edges of A are independently flipped with probability p.
2. Vertices are shuffled with a permutation Q.



Maximum likelihood estimation

- Consider the maximum likelihood estimator for the parameters

$$(\widehat{p}_{\mathsf{MLE}}, \widehat{Q}_{\mathsf{MLE}}) := \underset{p,Q}{\operatorname{argmax}} \sum_{u>v} \log \mathbb{P}_p \left(A_{uv} = (QBQ^T)_{uv} \right).$$

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- Extensions of the model to non-uniform corrupting probabilities
  - This result also holds with non-uniform corrupting probabilities.

When is the MLE correct?

- The difficulty of the problem depends on how different is A from QAQ^T for any $Q \neq I$, measured by

$$\|A - QAQ^T\|_F^2 = \sum_{i \neq j} (A_{ij} - A_{\sigma(i),\sigma(j)})\big)^2.$$

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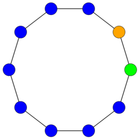
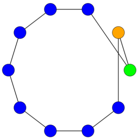
Examples:

- $k = 2$
- $\|A - QAQ^T\|_F^2 = 4.$



A



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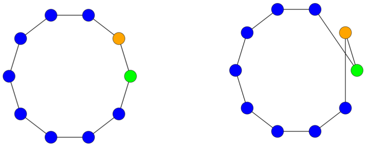
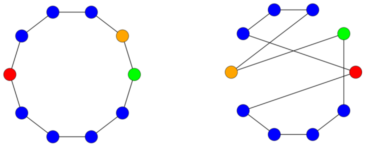


A QAQ^T

- $k = 3$
- $\|A - QAQ^T\|_F^2 = 10.$



A QAQ^T

Consistency of the MLE

Consider a sequence of networks $\{A_n\}$ with n vertices, and parameters $\{p_n, Q_n\}$

Definition

$\{\widehat{Q}_n\}$ is **consistent** if $\|\widehat{Q}_n - Q_n\|_F \xrightarrow{a.s.} 0$ as $n \to \infty$.

Consistency: all vertices are correctly matched in the limit.

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- $\widehat{Q}_{MLE}$ *is* **consistent** *if*

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- $\widehat{Q}_{MLE}$ *is* **not consistent** *if there exists* $m = \Omega(n)$ *disjoint permutations* $Q_1, \ldots, Q_m$

$$\max_{i \in [m]} \|A_n - Q_iA_nQ_i^T\|_F^2 = o\left(\frac{\log n}{(1/2 - p_n)^2}\right).$$

Relaxed consistency of the MLE

Matching all vertices correctly can be hard or impossible, for example:

- low-degree vertices
- automorphisms in the graph

Definition (Relaxed consistency)

$\{\widehat{Q}_n\}_{n=n_0}^{\infty}$ is $\{k_n\}$-**consistent** if $\frac{\|\widehat{Q}_n - Q_n\|_F}{k_n} \xrightarrow{a.s.} 0.$ as $n \to \infty$

Relaxed consistency: at most $o(k_n)$ misaligned vertices.

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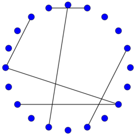
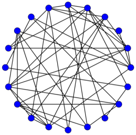
Consistency in random graph models

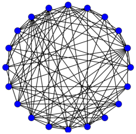
Erdös-Rényi graph model: $A_n \sim \mathrm{G}(n, \alpha_n)$



$\alpha = 0.05$



$\alpha = 0.2$



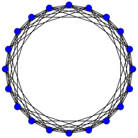
$\alpha = 0.4$

Corollary

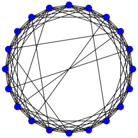
$\widehat{Q}_{\mathrm{MLE}}$ is **consistent** if

$$\alpha_n \geq \frac{c_1 \log n}{n(1/2 - p_n)^2}.$$

Newman-Watts small world graph model: $A_n \sim \mathrm{NW}(n, d_n, \beta_n)$,



$d = 4$,
$\beta = 0$



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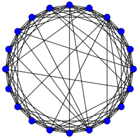

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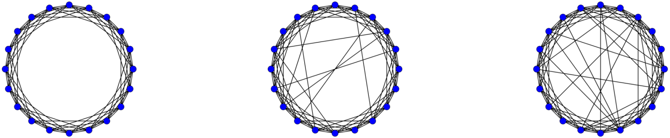

$d = 4$, $\beta = 0$ | $d = 4$, $\beta = 0.05$ | $d = 4$, $\beta = 0.2$

Corollary

- For any $\{d_n\}$, $\widehat{Q}_{\mathsf{MLE}}$ is **not consistent** if $(1/2 - p_n)^2 = o(\sqrt{\log n / n})$ and

$$\beta_n = o\left(\frac{\log n}{(1/2 - p_n)^2 n}\right).$$

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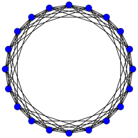

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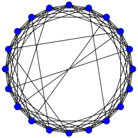

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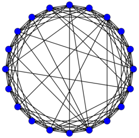

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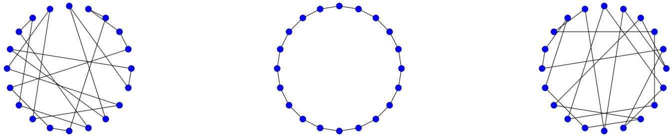
- $\widehat{Q}_{\mathrm{MLE}}$ is **consistent** if $d_n = o(\beta_n^2 n)$ and

$$\beta_n \geq C\sqrt{\frac{\log n}{n\,(1/2 - p_n)^2}}.$$

Relaxed consistency

Random regular graphs: $A \sim \mathrm{G}_{n,d_n}$



$d = 2$

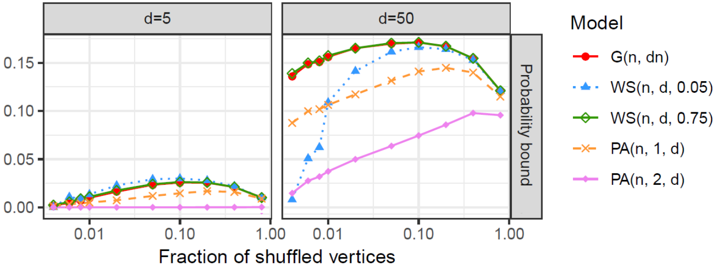
Corollary

If $d_n = c_1 n^{2/3+\varepsilon}$ and $(\frac{1}{2} - p_n)^2 \geq c_2 \frac{\log n}{d_n}$, then $\widehat{Q}_{\mathsf{MLE}}$ is $\{n^{2/3+\varepsilon}\}$-consistent.

A negligible fraction of the vertices ($o(n^{-(1/3-\epsilon)})$) might be incorrectly matched.

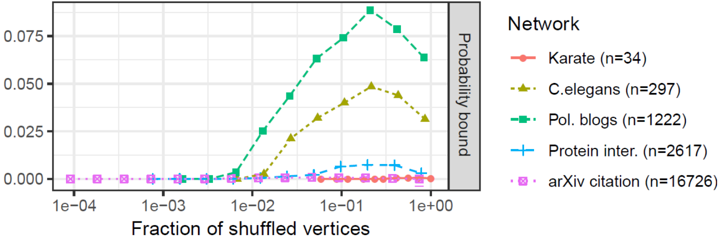
Matchability on random graphs

Measure matching feasibility: Upper bound for the noise probability tolerated by a graph based on the theory



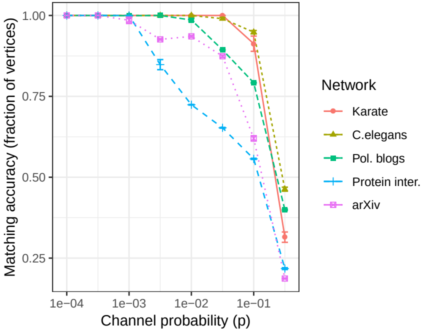
- Models considered
  - $G(n, dn)$ Erdös-Rényi
  - $WS(n, d, \beta)$: Watts-Strogatz
  - $PA(n, \gamma, d)$: preferential attachment

Matchability on real networks



Matchability on real networks

- Sample B from corrupting channel, and approximately compute the MLE
- Evaluate percentage of correctly matched vertices (left) and large degree vertices that are correctly matched (right).



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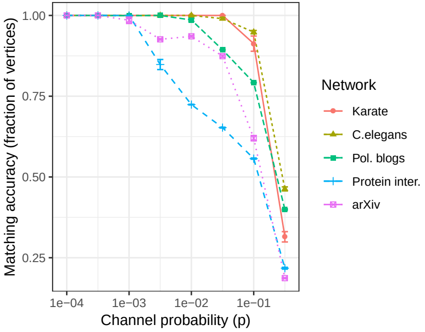

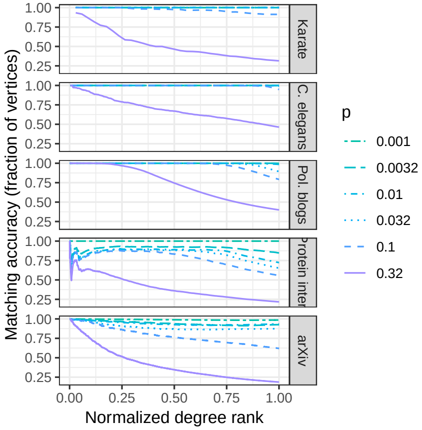

Conclusion

- Graph matching on errorfully observed networks is *statistically* feasible when:
    - The topology of the network is sufficiently irregular
    - The noise is small enough
- What are the computational limits?
- Matching on other relational data structures? multilayer networks, vertex or edge attributes, bipartite graphs, etc.
    - *Graph matching between bipartite and unipartite networks:* arXiv:2002.01648.

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