8 Se considera

$$\int_{-1}^{1} f(x)(1-x^2)\,dx = \alpha_0 f(x_0) + \alpha_1 f(x_1) + R(f)$$

a) Determinar coeficientes y nodos para máximo grado de exactitud

El grado de exactitud máximo es, para $n+1=2 \Rightarrow n=1$ y por el teorema de existencia de la fórmula gaussiana $2n+1=3$ //

Calculamos la fórmula gaussiana para 2 nodos

$$\pi(x) = (x-x_0)(x-x_1) = x^2+bx+c$$

$$\int_{-1}^{1} (x^2+bx+c)(1-x^2)\,dx = \int_{-1}^{1} x^2+bx+c-x^4-bx^3-cx^2\,dx =$$

$$= \int_{-1}^{1} (1-c)x^2 - x^4 - bx^3 + bx + c\,dx = \left[-\frac{x^5}{5} - \cancel{b\frac{x^4}{4}}^{\,0} + (1-c)\frac{x^3}{3} + \cancel{b\frac{x^2}{2}}^{\,0} + cx\right]_{-1}^{1} =$$

$$= -\frac{2}{5} + \frac{2(1-c)}{3} + 2c = 0 \Leftrightarrow -\frac{1}{5}$$

$$\int_{-1}^{1} x(x^2+bx+c)(1-x^2)\,dx = \left[\cancel{\frac{-x^6}{6}}^{\,0} - b\frac{x^5}{5} + \cancel{(1-c)\frac{x^4}{4}}^{\,0} + b\frac{x^3}{3} + \cancel{c\frac{x^2}{2}}^{\,0}\right]_{-1}^{1} =$$

$$= -\frac{2b}{5} + \frac{2b}{3} = 0 \Leftrightarrow b = 0$$

Por tanto $\pi(x) = x^2 - \frac{1}{5} = (x + 1/\sqrt{5})(x - 1/\sqrt{5})$

y tenemos que los nodos son $x_0 = -\frac{1}{\sqrt{5}}$ y $x_1 = \frac{1}{\sqrt{5}}$

Calculamos ahora los coeficientes imponiendo exactitud

$$f(x)=1 \rightarrow \int_{-1}^{1} f(x)(1-x^2)\,dx = \int_{-1}^{1}(1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$

$$\Rightarrow \frac{4}{3} = \alpha_0 + \alpha_1$$

$$f(x)=x \rightarrow \int_{-1}^{1} f(x)(1-x^2)\,dx = \int_{-1}^{1}(x-x^4)\,dx = 0 \Rightarrow 0 = \frac{-\alpha_0}{\sqrt{5}} + \frac{\alpha_1}{\sqrt{5}}$$

$$\Leftrightarrow \alpha_0 = \alpha_1 \quad \Rightarrow \alpha_0 = \alpha_1 = \frac{2}{3}$$

Por lo que llegamos a

$$\int_{-1}^{1} f(x)(1-x^2)\,dx \approx \frac{2}{3} f\left(-\frac{1}{\sqrt{5}}\right) + \frac{2}{3} f\left(\frac{1}{\sqrt{5}}\right)$$

b) Obtener el error

Sabemos que el error de la fórmula gaussiana es

$$R(f) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi)\, L(\pi^2)$$

donde $n=1$ (del apartado anterior), $\xi \in [x_0, x_1] = [-1/\sqrt{5}, 1/\sqrt{5}]$

y $L(\pi^2) = \int_{-1}^{1} \pi^2(x)(1-x^2)\,dx = \int_{-1}^{1} (x^2 - \frac{1}{5})^2 (1-x^2)\,dx =$

$$= \int_{-1}^{1} \left(x^4 - \frac{2x^2}{5} + \frac{1}{25}\right)(1-x^2)\,dx = \int_{-1}^{1} x^4 - \frac{2x^2}{5} + \frac{1}{25} - x^6 + \frac{2x^4}{5} - \frac{x^2}{25}\,dx =$$

$$= \int_{-1}^{1} -x^6 + \frac{7}{5}x^4 - \frac{11}{25}x^2 + \frac{1}{25}\,dx = \frac{1}{25}\int_{-1}^{1} -25x^6 + 35x^4 - 11x^2 + 1\,dx =$$

$$= \frac{1}{25}\left[-25\frac{x^7}{7} + 35\frac{x^5}{5} - 11\frac{x^3}{3} + x\right]_{-1}^{1} = \frac{1}{25}\left(\frac{-50}{7} + \frac{70}{5} - \frac{22}{3} + 2\right) =$$

$$= \frac{32}{525} \Rightarrow R(f) = \frac{1}{4!} f^{(iv)}(\xi) \cdot \frac{32}{525} = \frac{4}{1575} f^{(iv)}(\xi) \quad \text{con } \xi \in [-1/\sqrt{5}, 1/\sqrt{5}]$$

Por lo que la fórmula completa será

$$\int_{-1}^{1} f(x)(1-x^2)\,dx = \frac{2}{3} f\left(-\frac{1}{\sqrt{5}}\right) + \frac{2}{3} f\left(\frac{1}{\sqrt{5}}\right) + \frac{4}{1575} f^{(iv)}(\xi)$$

con $\xi \in [-1/\sqrt{5}, 1/\sqrt{5}]$

c) Estimar

$$\int_{-1}^{1} \ln(x^2+1)(1-x^2)\,dx$$

Tenemos

$$\int_{-1}^{1} \ln(x^2+1)(1-x^2)\,dx \approx \frac{2}{3}\ln\left(\frac{1}{5}+1\right) + \frac{2}{3}\ln\left(\frac{1}{5}+1\right) =$$

$$= \frac{4}{3}\ln\left(\frac{6}{5}\right) \approx 0,2430954091$$