Clustering and Structural Balance in Graphs

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CARTWRIGHT & HARARY (1956) define a signed graph as balanced if all its cycles are 'positive' and state the following theorem:

> 'An *s*-graph is balanced if and only if its points can be separated into two mutually exclusive subsets such that each positive line joins two points of the same subset and each negative line joins points from different subsets.'

If the theory of structural balance is 'true' when applied to social relationships, it implies a tendency for groups to polarize—to split into exactly two cliques. Thus 'a "balanced" group consists of two highly cohesive cliques which dislike each other' (Harary, 1955–6, p. 144).

Because sociometric studies often suggest that groups may split into three, four, or more cliques, the following question may be asked; What conditions are necessary and sufficient for the points of a graph to be separated into two *or more* subsets such that each positive line joins two points of the same subset and each negative line joins points from different subsets?

We call the multiple clique phenomenon *clustering* to distinguish it from *balance*, although it will be shown that the two are closely related.

STATEMENT AND PROOF OF CLUSTER THEOREMS[2]

In the following discussion terms referring to signed graphs are defined as in Cartwright & Harary (1956). However, a few additional definitions are required. A *clustering* of a signed graph S is a partition of the point set $V(S)$ into subsets P_1, $P_2, \ldots, P_n$ (called *plus-sets*) such that each positive line joins two points in the same subset and each negative line joins two points from different subsets. If S has a unique clustering, its plus-sets are called *clusters*. A *cycle* consists of a path L, joining two points v_i and v_j, together with the line v_iv_j. If L has only positive lines, it is called an *all-positive path*.

THEOREM 1

Let S be any signed graph. Then S has a clustering if and only if S contains no cycle having exactly one negative line.

Proof. We first prove the necessity of this criterion. By hypothesis S has a clustering, and we assume S has a cycle with exactly one negative line, v_iv_j. Now v_i and v_j are

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2. The statement and proof presented here were kindly provided by Professors Dorwin Cartwright and Frank Harary as a substitute for the author's more lengthy and cumbersome one.

joined by an all-positive path, L. Clearly, all points of L are in the same plus-set since adjacent points of L are all joined by positive lines. But then there is a negative line joining two points in the same plus-set, which is a contradiction.

To prove the sufficiency, we are given that S contains no cycle having exactly one negative line, and we show that S has a partition that is a clustering. We form subsets $A_1, \ldots, A_k, \ldots, A_n$ by the rule that two points are in the same subset if and only if they are joined by an all-positive path. Clearly, the subsets so constructed are maximal and partition $V(S)$. Now every positive line of S joins two points in the same subset A_k. Let v_i and v_j be any two distinct points in A_k. Then v_i and v_j are joined by an all-positive path, and they are not joined by a negative line since S would then have a cycle with exactly one negative line. Hence, all negative lines of S join two points from different subsets, and our partition is a clustering.

FIGURE 1

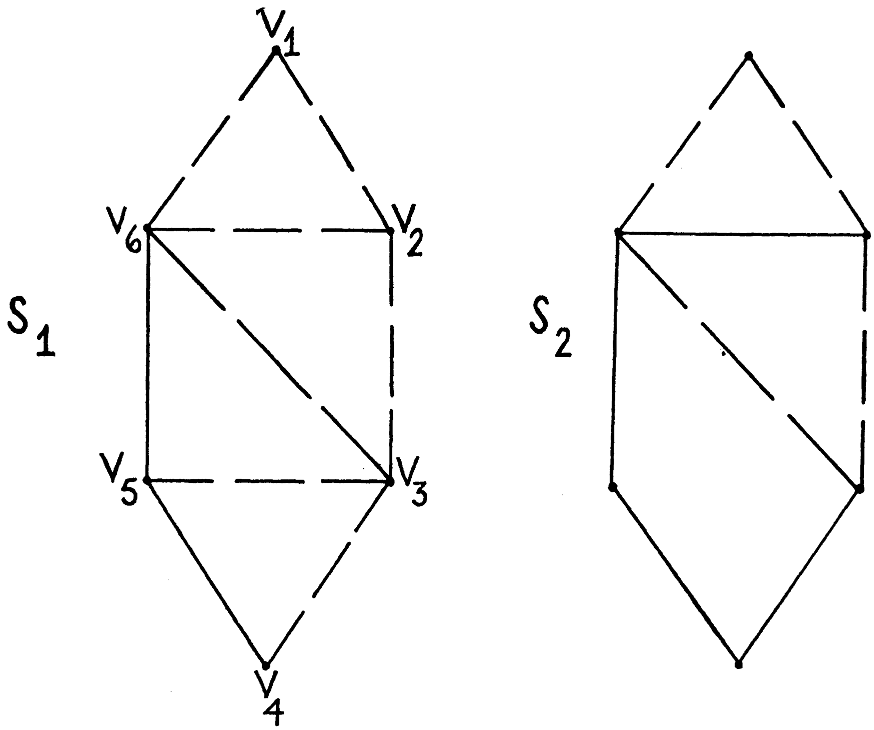

Figure 1 shows a signed graph, S_1, that has a clustering and another, S_2, that does not. The procedure given in the proof of *Theorem 1*, when applied to S_1, yields four plus-sets: $P_1=\{v_4, v_5, v_6\}$, $P_2=\{v_1\}$, $P_3=\{v_2\}$, $P_4=\{v_3\}$. It should be noted, however, that S_1 has another clustering: $P_1=\{v_4, v_5, v_6\}$, $P_2=\{v_1, v_3\}$, $P_3=\{v_2\}$.

The results of *Theorem 1* are applicable, of course, to any appropriate realization of a signed graph. We might, for example, represent the structure of interpersonal sentiments among a group of people by a signed graph S by letting each point of S correspond to a person. And if two people, v_i and v_j, like each other, we join the corresponding points with a positive line; if they dislike each other, we join them with a negative line. Now let us assume a tendency for these interpersonal sentiments to form a structure whose signed graph has a clustering. In other words, we assume a tendency for the group to divide into subgroups that correspond to the plus-sets of a clustering. In this case, all people who like each other are in the same subgroup and all who dislike each other are in different subgroups. *Theorem 1* gives a necessary and sufficient condition for the existence of such a structure.

A graph in which every two points are joined by a line is called *complete*. Clearly, if every two people in a group either like or dislike each other, the corresponding signed graph is complete. The next theorem gives three criteria for such a structure to be clusterable. Its statement and proof refer to an *n-cycle*, by which we mean a cycle of length n.

THEOREM 2

The following statements are equivalent for any complete signed graph S.

(1) S has a clustering.
(2) S has a unique clustering.
(3) S has no cycle with exactly one negative line.
(4) S has no 3-cycle (triangle) with exactly one negative line.

Proof. The equivalence of (1) and (2) is immediate. The equivalence of (1) and (3) was established by *Theorem 1*. Since (4) follows directly from (3), we need prove only that (4) implies (3). By hypothesis, S has no 3-cycle with exactly one negative line. We assume that S has a k-cycle with exactly one negative line, whose points are labeled $v_1, v_2, \ldots, v_k$, such that the single negative line joins v_1 and v_2. Now the line v_1v_2 is negative and v_2v_3 is positive. Since S has no 3-cycle with exactly one negative line, the line v_1v_3 must be negative. Similarly, v_1v_4 must be negative, and in general, all remaining points must be joined to v_1 by a negative line. In particular, the line v_1v_k must be negative. But then the k-cycle has two negative lines, which is a contradiction.

It is important to note two differences between *Theorem 1 and Theorem 2*, that is, differences between clustering in a complete graph and in an incomplete graph.

First, in a complete clusterable graph, the plus-sets are unique. There is only one way to form them. In some incomplete graphs (for example, a large graph with only a handful of lines 'missing') the plus-sets will be unique. However, in many incomplete graphs there is more than one acceptable way to form plus-sets. Thus we distinguish between *clusters*, unique plus-sets, and plus-sets in general.

Putting it another way, a graph may lack clusters for two different reasons: because it contains internal contradictions in the form of cycles with a single negative line and/or because it is incomplete in such a way as to make the plus-sets ambiguous.

This notion leads to some thoughts on the dynamics of a social group. Substantive structural theories usually have dynamic implications, hypotheses to the effect that structures will change toward greater conformity with the model (Flament, 1963, pp. 107–24). One obvious sociological hypothesis is that relationships which lie on a cycle with a single negative line will tend to change in such a fashion as to make the graph clusterable. If we think of such a process as 'reducing structural strain', we can posit an analogous process in incomplete clusterable graphs: 'structural growth'.

Now consider a group of people who are initially all strangers to one another. Let us assume that sentiments will form over time in such a way that the structure always has a clustering. We note in passing that the initial structure satisfies this requirement since a signed graph with no lines has a clustering. Let us assume, further, that sentiments once established remain unchanged. If we know the structure at a given time, can we predict the signs of new sentiments that subsequently develop? In other words, if S is the signed graph of the group at a given time and T is the signed graph of the same group at a later time, can we predict, on the basis of S, the signs of lines that are in T but not in S? The following corollary to *Theorem 1* gives conditions under which definite predictions can be made.

COROLLARY

Let S and T be clusterable signed graphs with the same set of points such that every line of S is in T, and let $v_i v_j$ be a line in T which is not in S.

(1) If v_i and v_j are joined by an all-positive path in S, then $v_i v_j$ is positive.
(2) If v_i and v_j are joined by a path with exactly one negative line, then $v_i v_j$ is negative.
(3) Otherwise, $v_i v_j$ may be either positive or negative.

This corollary follows immediately from *Theorem 1*. The requirement that T have no cycle with exactly one negative line establishes statements (1) and (2). The same requirement for S guarantees that the conditions given in (1) and (2) cannot both hold for the same pair of points. If neither of these holds, all paths joining v_i and v_j in S have at least two negative lines and $v_i v_j$ may be either positive or negative, establishing statement (3).

Consider a sentimental structure S having two people, v_i and v_j, who at a given time neither like nor dislike each other. We may think of them as being in a potential relationship (v_i, v_j). Now if in the future v_i and v_j convert this potential relationship into an actual one, we may be able to predict its sign from the nature of S under the assumption of clustering. If the predicted sign is positive or negative, let us call the relationship (v_i, v_j) *latently positive* or *latently negative*, respectively. Otherwise, (v_i, v_j) is said to be *structurally free*.

The notion may be further expanded by assuming a graph which is neither complete nor perfectly clusterable. Its pair relationships may be classified as follows.

Pairs connected by a positive or negative line are: (1) *strained* if the line lies on a cycle with exactly one negative line; (2) *reinforced* if a reversal in sign would create a cycle with exactly one negative line; or (3) *free* if neither strained nor reinforced. Pairs which are not joined by a line are (4) *latently positive or negative* if their sign can be predicted from the corollary; (5) *latently strained* if either a positive or negative line would create a cycle with a single negative line, or (6) *free* if neither latently positive nor negative nor strained.

This leads to the omnibus hypothesis that compared with free lines of the same original value (positive, negative, or null): (1) reinforced lines will tend to remain the same; (2) latently positive or negative lines tend to shift to the appropriate sign; (3) strained lines will tend to shift in sign or become null; and (4) latently strained lines will tend to remain null.[3]

A second difference between the two theorems must be noted. Statements (3)

3. In sociological terms, (4) is the cross-pressure hypothesis (cf. Davis, 1963, pp. 450–1).

and (4) in the corollary, which are equivalent for complete graphs, are not equivalent for incomplete graphs.

In complete graphs clusterability may be determined by inspection of the 3-cycles, a notion akin to Flament's proof that a complete graph is *balanced* if and only if all its triangles are balanced (Flament, 1963, p. 94) and analogous to the theorem that a tournament is transitive if and only if each triple is transitive (Harary, Norman, & Cartwright, 1965, p. 298). In a 3-cycle each pair is connected by a line and also by a path of length 2. In 3-cycles with a single negative line there is a contradiction between the direct and indirect links for each pair. The pair connected by the negative line are also connected by an all-positive path, a contradiction; while the two pairs connected by direct lines are also connected by paths with a single negative line, also a contradiction. In sociological terms such situations may be interpreted as ones where a pair's direct relationship is inconsistent with that implied by their relationships with a third party.

In incomplete graphs, the absence of triangles with a single negative line is a necessary, but not sufficient condition for clusterability, because even if all triangles meet the criterion there may be longer cycles with a single negative line. Graphs in which all 3-cycles meet the criterion, while some longer cycles do not, may be interpreted as cases of *limited clusterability*, analogous to limited balance (Harary, Norman, & Cartwright, 1965, p. 352). In *Figure 1*, S_2 is an example of limited clusterability.

DISCUSSION

Clustering and balance. A clusterable graph and a balanced graph are related as follows: All balanced graphs are clusterable (a graph with even numbers of negative lines in all its cycles cannot have any cycles with a single negative line); while clusterable graphs may or may not be balanced, depending on the number of disjoint subsets of points.

This asymmetry has a number of substantive implications. First, it enables one to connect the social psychological development of balance theory with sociological hypotheses about subgroup formation. For example, in *The human group*, Homans (1950, p. 113) writes, 'the liking of friends within a group carries with it some dislike of outsiders. The greater the inward solidarity, the greater the outward hostility.' In the present context Homans is postulating tendencies toward clustering, but not necessarily toward balance.

Second, the difference between the two structural principles raises the following question for the theoretician: Because 'consistency' (tendencies toward positive relationships within groups and negative ones between groups) is a necessary but not sufficient condition for polarization, what are the principles which explain why some groups polarize and others divide into multiple cliques? We pose the question and do not pretend to answer it, but the following observations may be relevant.

In the case of symmetrical sentiments (friends and enemies) the Cartwright and Harary balance theorem, as Rapoport (1963, p. 541) and others have noted, generates a number of aphorisms regarding friends and enemies.

(1) A friend of a friend will be a friend, not an enemy.
(2) An enemy of a friend will be an enemy, not a friend.

(3) A friend of an enemy will be an enemy, not a friend.
(4) An enemy of an enemy will be a friend.

It can be shown that the first three propositions will continue to hold under the clustering theorems. The fourth, however, depends on the number of subsets. It holds in the case of two subsets, but not where there are three or more subsets. Putting it another way, group polarization (in a complete graph) is formally identical with clustering plus a tendency for those with a common enemy to become friends, i.e. *coalition formation.*

Third, the clustering theorem gives some perspective on the problem of 'all-negative triangles'. Under the balance theorem, triangles with three negative sides are impossible. Heider himself, however, writes (1958, p. 206), 'If two negative relations are given, balance can be obtained *either* when the third relationship is positive or when it is negative, though there appears to be a preference for the positive alternative.' This is equivalent to two hypotheses: a strong tendency toward clustering and among the clustered states a less strong tendency toward balance.

Clustering and connectedness. A rather different interpretation of clustering is suggested by Bott's well-known study, *Family and social network* (1957, pp. 59–60). She writes:

> 'Although all the research families belonged to networks . . . there was considerable variation in the "*connectedness*" of their networks. By connectedness I mean the extent to which the people known by a family know and meet one another independently of the family. . . . When many of the people a person knows interact with one another . . . the person's network is close-knit.'

Consider a signed graph S constructed in the following way: We let the points of S represent the families of a community, and we join two points by a positive line if the corresponding families know each other and by a negative line otherwise. Clearly, S is complete. Let us assume that this acquaintance network satisfies the extreme degree of 'connectedness' suggested by Bott—namely, every two families known by a third family know each other. In other words, S contains no 3-cycle with exactly one negative line. By *Theorem 2*, S is clusterable, and the families of the community can be partitioned uniquely into subgroups all of whose members know each other and none of whom know anyone outside the subgroup. Moreover, there will be more than one such subgroup except when everyone in the community knows everyone else. Thus unless one is willing to assume extremely large acquaintance volumes, a social system which is highly 'connected' must be one consisting of a number of relatively small, completely connected subgroups that are isolated from one another; a formulation which appears to fit Bott's description of some of the working class neighbourhoods in her study.

Such a social system is somewhat paradoxical. (1) At the level of the individual, the system is highly connected, for he lies at the center of a dense network of direct and indirect social relationships. (2) At the level of the total system it is highly disconnected, for there are many pairs who have neither direct nor indirect relationships. (3) It is one in which individuals tend to have many ties to a few people and few or no ties to many others.

SUMMARY

Building from the Cartwright and Harary balance theorem, which treats structures that are polarized into two groups, we introduced a new concept, clusterability or division into two or more groups, and a theorem stating the necessary and sufficient condition for it: a signed graph is clusterable if and only if it contains no cycle with exactly one negative line.

When comparing clustering with balance we noted that the idea of clustering helps to build a bridge between social psychological developments in balance theory and sociological theories of subgroup formation. At the same time, it raises a new question: when will a group polarize? A review of aphorisms associated with balance theory suggested that theories of coalition formation may be useful in answer to the question. We also noted that the new theorem may be of use in interpreting Heider's comments on all-negative triangles.

A second theorem was presented showing that a complete graph is clusterable if none of its 3-cycles (triangles) has a single negative line. A comparison of the two theorems led to the notion of *latent, strained, reinforced,* and *free* social relationships. This in turn led to a set of hypotheses about the dynamics of group structures.

Finally, in a different substantive context, we pointed out some formal analogies between clustering and connectedness in a graph. It was shown that a number of the concepts in Bott (1957) can be reinterpreted in terms of clusterability.

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