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Vertex equitable labeling of signed graphs

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Abstract

A *signed graph* (or, in short, *sigraph*) $S = (S^u, \sigma)$ consists of an underlying graph $S^u := G = (V, E)$ and a function $\sigma : E(S^u) \longrightarrow \{+, -\}$, called the signature of S. Let S be a signed graph with p vertices and q edges and let $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(S) \longrightarrow \mathcal{A}$ which is onto, is said to be a vertex equitable labeling of S if it induces a bijective edge labeling $f^* : E(S) \longrightarrow \{1, 2, ..., \mathfrak{m}, -1, -2, ..., -\mathfrak{n}\}$ defined by $f^*(uv) = \sigma(uv)(f(u) + f(v))$ such that $|v_f(a) - v_f(b)| \leq 1$, $\forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices with $f(v) = a$ and $\mathfrak{m}, \mathfrak{n}$ are number of positive and negative edges respectively in S. A signed graph S is said to be vertex equitable if it admits a vertex equitable labeling. In this paper, we initiate a vertex equitable labeling of signed graphs and study vertex equitable behavior of signed paths, signed stars and signed complete bipartite graphs $K_{2,n}$.

Keywords: vertex equitable labeling, vertex equitable signed graph.

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1 Introduction

For graph theoretical terminology, we refer to [2]. Graph labelings, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems but also of interest in their own right. The labels of the vertices induce labels of edges under certain conditions and there is an enormous amount of literature build up on several kind of numerical labeling of graphs and an interested reader is referred to [1].

In literature many types of graph labeling exist, i.e., graceful labeling, multiplicative labeling, vertex equitable labeling, harmonious labeling, cordial labeling, set labeling and so on. Here we introduce only a vertex equitable labeling in the realm of signed graphs.

A *signed graph* is an ordered pair $S = (S^u, \sigma)$, where $S^u := G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E(S^u) \longrightarrow \{+, -\}$ is a function, called the *signature* of S. In other terms, we say that the edges are *signed* by σ. $E^+(S) = \{e \in E(S^u) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(S^u) : \sigma(e) = -\}$. The elements of $E^+(S)$ $(E^-(S))$ are called *positive* (*negative*) *edges* of S and the set $E(S) = E^+(S) \cup E^-(S)$ is called the edge set of S. In this paper, we are taking $|E^+(S)| = \mathfrak{m}$ and $|E^-(S)| = \mathfrak{n}$. By a (p, q)- signed graph, we mean a signed graph S with $|V(S)| = p$ and $|E(S)| = q$. In a pictorial representation of a signed graph S, its positive edges are shown as solid line segments ('Jorden curves' drawn on the plane) and negative edges as dashed line segments as shown in **Figure 1**.

A signed graph in which all the edges are positive, is called an *all-positive signed graph* (*all-negative signed graph* is defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. By $d(v)$, we denote the degree of $v \in V(S)$, $d(v) = d^+(v) + d^-(v)$, here $d^+(v)$ $(d^-(v))$ denotes the positive (negative) degree of v.

The *negation* of a signed graph S, denoted by $\eta(S)$, is obtained by negating the sign of every edge of S, i.e., by changing the sign of every edge to its opposite [3].

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2 Vertex equitable labeling

In [4], Seenivasan and Lourdusamy introduced the idea of vertex equitable labeling of graphs as follows:

Definition 2.1 [4] Suppose G is a (p,q)-graph and $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(G) \longrightarrow \mathcal{A}$ which is onto, is said to be a *vertex equitable labeling* of G if it induces a bijective edge labeling $f^* : E(G) \longrightarrow \{1, 2, ..., q\}$ given by $f^*(uv) = f(u) + f(v)$ such that $|v_f(a) - v_f(b)| \leq 1 \; \forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices with $f(v) = a$. Here $\lceil n \rceil$ denotes the smallest integer greater than or equal to n. A graph G is said to be *vertex equitable* if it admits a vertex equitable labeling.

We extend the definition of a vertex equitable graph to the realm of a vertex equitable signed graph as follows:

Definition 2.2 Let S be a (p,q)-signed graph with $q = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}(\mathfrak{n})$ is the number of positive(negative) edges in S and $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(S) \longrightarrow \mathcal{A}$ which is onto, is said to be a *vertex equitable labeling* of S if it induces a bijective edge labeling $f^* : E(S) \longrightarrow \{1, 2, ..., \mathfrak{m}, -1, -2, ..., -\mathfrak{n}\}$ defined by $f^*(uv) = \sigma(uv)(f(u) + f(v))$ such that $|v_f(a) - v_f(b)| \leq 1$, $\forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices with $f(v) = a$. A signed graph S is said to be *vertex equitable* if it admits a vertex equitable labeling. In **Figure 1**, we show a vertex equitable signed graph.

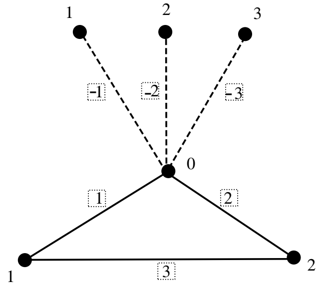

Fig. 1. A vertex equitable signed graph

In order to investigate vertex equitable behavior of signed paths, signed stars and signed complete bipartite graph $K_{2,n}$ we need the following results:

Theorem 2.3 ([4]) *The path P_n is vertex equitable.*

Theorem 2.4 ([4]) *Complete bipartite graph $K_{2,n}$ is vertex equitable.*

3 Main Results

Theorem 3.1 *If a signed graph S is vertex equitable then $\eta(S)$ is also vertex equitable.*

Proof. Let a signed graph S be vertex equitable. There exists a vertex labeling $f : V(S) \longrightarrow \mathcal{A}$, where $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$ such that f induces an edge labeling f^* given by $f^*(uv) = \sigma(uv)(f(u) + f(v))$ and $|v_f(a) - v_f(b)| \leq 1$ $\forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices with $f(v) = a$. $f^*(E) = \{1, 2, ..., \mathfrak{m}, -1, -2, ..., -\mathfrak{n}\}$. Since in $\eta(S)$, only signs of edges are opposite, i.e., in $\eta(S)$, $f^*(E) = \{-1, -2, ..., -\mathfrak{m}, 1, 2, ..., \mathfrak{n}\}$, this vertex labeling f is also a vertex equitable labeling for $\eta(S)$. Hence the result follows. □

Theorem 3.2 *Homogenous path P_n is vertex equitable.*

Proof. The result follows from Theorem 2.3 and Theorem 3.1. □

Now, a natural question is to determine heterogeneous vertex equitable paths. The following is a partial answer to this question:

Theorem 3.3 *A signed path P_n having a negative pendant edge and all other edges positive, is vertex equitable.*

Proof. Let $P_n = v_1 v_2 v_3 ... v_n$ be a signed path with n vertices and a negative pendant edge $v_1 v_2$. We define $f : V(P_n) \longrightarrow \{0, 1, 2, ..., \lceil \frac{n-1}{2} \rceil\}$ as

$$f(v_1) = 0, f(v_2) = 1, f(v_3) = 0 \text{ and}$$

$$f(v_i) = f(v_{i-2}) + 1, \text{ for } 4 \leq i \leq n.$$

It is easy to see that f is a vertex equitable labeling. Thus the result follows. □

A vertex equitable labeling for signed path P_9 having a negative pendant edge and other edges positive is shown in **Figure 2**.

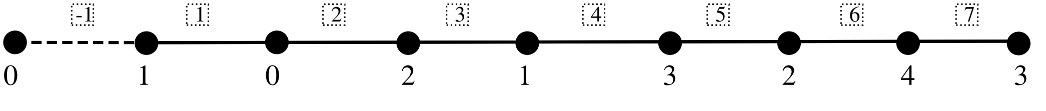

Fig. 2. A vertex equitable signed path

Corollary 3.4 *A signed path P_n having a positive pendant edge and all other edges negative is vertex equitable.*

Now, in **Figure 3**, we give some vertex equitable heterogeneous signed paths.

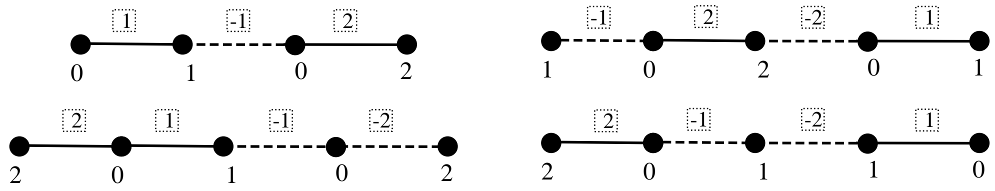

Fig. 3. Some vertex equitable signed paths

In **Figure 3**, we have shown a signed path P_5 having one negative and one positive section each of length two which is vertex equitable. Note that the signed paths shown in **Figure 4** are not vertex equitable.

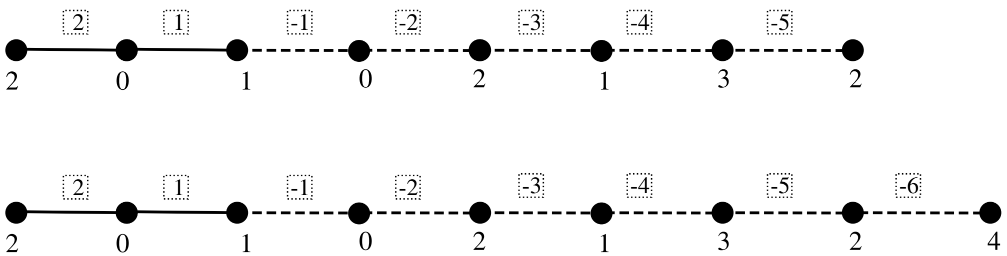

Fig. 4. Signed paths which are not vertex equitable

Now, we pose the following conjecture:

Conjecture 3.5 *A signed path P_n having one positive section of length two having a pendant edge and one negative section of length greater than 2; and the negation of this path are not vertex equitable.*

Now naturally, we have the following problem:

Problem 3.6 *Find all non-isomorphic heterogeneous signed paths P_n which are vertex equitable.*

Theorem 3.7 *Star $K_{1,n}$ is vertex equitable if and only if $n \leq 3$.*

Proof. Let (V_1, V_2) be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$. Since $K_{1,n}$ has n edges, $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{n}{2} \rceil\}$. To get edge label 1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1.

Assume that $K_{1,n}$ is vertex equitable.

Let $f(u) = 0$ then $f(v_i) = i$, $1 \leq i \leq n$.

If $n \geq 2$ then we can not assign this labelling as $f(v_n) = n$ and $n \notin \mathcal{A}$. Hence

$f(u) \neq 0$.
For $f(u) = 1$, we have $f(v_i) = i - 1$ for $1 \leq i \leq n$.

Since $K_{1,n}$ is vertex equitable, we must have

$$n - 1 \leq \lceil \tfrac{n}{2} \rceil.$$

Clearly, this holds for $n = 1, 2, 3$. Further we can easily see that when $n = 1$ or 2 or 3, $K_{1,n}$ admits a vertex equitable labeling. Thus, the star $K_{1,n}$ is vertex equitable if and only if $n \leq 3$. This completes the proof. □

We have signed stars $K_{1,n}$, for $n \geq 4$, which are vertex equitable as shown below.

Theorem 3.8 *A signed star $K_{1,n}$ where $n = \mathfrak{m} + \mathfrak{n}$, is vertex equitable if and only if*

(i) $|\mathfrak{m} - \mathfrak{n}| \leq 1$ *or*

(ii) *If $\mathfrak{m} = 1$ then $\mathfrak{n} \leq 4$ and if $\mathfrak{n} = 1$ then $\mathfrak{m} \leq 4$.*

Proof. Let (V_1, V_2) be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{u_1, u_2, u_3, ..., u_{\mathfrak{m}}, v_1, v_2, v_3, ..., v_{\mathfrak{n}}\}$, where u_i and v_j are the vertices adjacent to u through positive and negative edges respectively. As $K_{1,n}$ has $\mathfrak{m} + \mathfrak{n}$ edges, $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil\}$.
Necessity:
To get the edge labels 1 and -1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1. Now, we have the following two cases:

Case I: f(u)= 0 then to find edge labels of pendant edges, we must assign

$$f(u_i) = i \text{ for } 1 \leq i \leq \mathfrak{m} \text{ and}$$
$$f(v_j) = j \text{ for } 1 \leq j \leq \mathfrak{n},$$

as shown in **Figure 5**.
As $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil\}$, we must have
$\mathfrak{m} \leq \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil$ and $\mathfrak{n} \leq \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil$

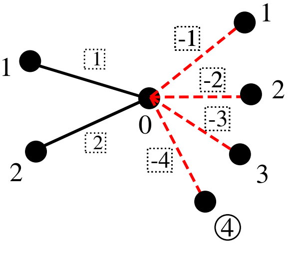

Fig. 5: Signed star $K_{1,6}$

- If $\mathfrak{m}$ and $\mathfrak{n}$ both are even or odd then we must have
  $\mathfrak{m} \leq \frac{\mathfrak{m}+\mathfrak{n}}{2}$ and $\mathfrak{n} \leq \frac{\mathfrak{m}+\mathfrak{n}}{2}$
  Clearly, this holds for $\mathfrak{n} = \mathfrak{m}$
- If $\mathfrak{m}$ is even and $\mathfrak{n}$ is odd or $\mathfrak{n}$ is even and $\mathfrak{m}$ is odd then
  $\mathfrak{m} \leq \frac{\mathfrak{m}+\mathfrak{n}+1}{2}$ and $\mathfrak{n} \leq \frac{\mathfrak{m}+\mathfrak{n}+1}{2}$. That is
  $2\mathfrak{m} \leq \mathfrak{m} + \mathfrak{n} + 1$ and $2\mathfrak{n} \leq \mathfrak{m} + \mathfrak{n} + 1$

$\mathfrak{m} \leq \mathfrak{n} + 1$ and $\mathfrak{n} \leq \mathfrak{m} + 1$
$\mathfrak{m} - \mathfrak{n} \leq 1$ and $\mathfrak{n} - \mathfrak{m} \leq 1$
This holds only when $|\mathfrak{m} - \mathfrak{n}| \leq 1$. Thus, **(i)** follows.

Case II: Suppose **f(u)= 1** then we must assign $f(u_1) = f(v_1) = 0$. We can not assign $f(u_2) = f(v_2) = 1$ as f will not be a vertex equitable labeling. Hence $\mathfrak{m} = 1$ or $\mathfrak{n} = 1$.

- If $\mathfrak{m} = 1$ then vertex labeling is
$$f(u_1) = 0,\ f(v_i) = i - 1,\ 1 \leq i \leq \mathfrak{n}.$$
Thus, $\mathfrak{n} - 1 \leq \lceil \frac{\mathfrak{n}+1}{2} \rceil$. This holds only for $\mathfrak{n} \leq 4$.

On the other hand

- If $\mathfrak{n} = 1$ then similarly we have $\mathfrak{m} - 1 \leq \lceil \frac{\mathfrak{m}+1}{2} \rceil$ and this too holds only for $\mathfrak{m} \leq 4$. Thus, **(ii)** follows.

Hence, the necessity follows.

Sufficiency:
Suppose conditions hold. We define the vertex labeling $f : V(K_{1,n}) \longrightarrow \mathcal{A}$ as

- If $|\mathfrak{m} - \mathfrak{n}| \leq 1$ then
$$f(u) = 0,\ f(u_i) = i,\ 1 \leq i \leq \mathfrak{m} \text{ and}$$
$$f(v_j) = j,\ 1 \leq j \leq \mathfrak{n}.$$
- If $\mathfrak{m} = 1$ and $\mathfrak{n} = 3$ or 4 then
$$f(u) = 1,\ f(u_1) = 0 \text{ and } f(v_j) = j - 1,\ 1 \leq j \leq \mathfrak{n}$$
- If $\mathfrak{n} = 1$ and $\mathfrak{m} = 3$ or 4 then
$$f(u) = 1,\ f(v_1) = 0 \text{ and } f(u_i) = i - 1,\ 1 \leq i \leq \mathfrak{m}.$$

It is easy to see that f is a vertex equitable labeling. This completes the proof. □

Theorem 3.9 *Signed complete bipartite graphs $K_{2,n}$ are vertex equitable if and only if it is any one of the following:*

1. *homogeneous or*

2. *heterogeneous, in which for each vertex v_i of degree 2, $d^+(v_i) = d^-(v_i) = 1$.*

Proof. Let (V_1, V_2) be the bipartition of signed $K_{2,n}$ with $V_1 = \{u, v\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$.

Necessity:
We prove the necessity by contrapositive, i.e., we prove that heterogeneous signed complete bipartite graphs $K_{2,n}$, in which for at least one vertex v_i, $d^+(v_i) \neq d^-(v_i)$, is not vertex equitable.

To get the edge labels 1 and -1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1 and $f(v) = 0$ or 1. Now, in all the cases some edge labels are repeated. Hence, signed $K_{2,n}$ are not vertex equitable. Thus, the necessity follows.

Sufficiency:

Suppose conditions hold. Now, we have the following two cases:

Case I: If $K_{2,n}$ are homogeneous then by Theorem 2.4 and Theorem 3.1, these are vertex equitable signed graphs.

Case II: If $K_{2,n}$ are heterogeneous, in which for each vertex v_i of degree 2, $d^+(v_i) = d^-(v_i) = 1$, we define $f : V(K_{2,n}) \longrightarrow \{0, 1, 2, ..., n\}$ as

$$f(u) = f(v) = 0 \text{ and } f(v_i) = i \text{ for } 1 \leq i \leq n.$$

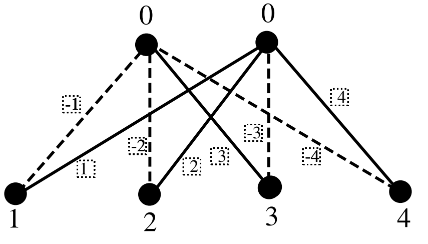

Fig. 6. Vertex equitable signed $K_{2,4}$

It can be easily seen that f is a vertex equitable labeling, as shown in **Figure 6**. $\square$

Further work on vertex equitable labeling of signed graphs will be reported in our subsequent papers.

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