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k-equitable mean labeling

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ABSTRACT

Let $G = (V, E)$ be a graph with p vertices and q edges. Let $f : V \to \{0, 1, 2, \cdots, k\}$ $(1 \le k \le q)$ be a vertex labeling of G that induces an edge labeling $f^* : E \to \{0, 1, 2, \cdots, k\}$ be given by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. A labeling f is called $(k + 1)$-equitable mean labeling ($(k + 1)$-eml) if $|v_f(i) - v_f(j)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1$ for $i, j = 0, 1, 2, \cdots, k$ where $v_f(x)$ and $e_{f^*}(x)$, $x = 0, 1, 2, \cdots, k$ are the number of vertices and edges of G respectively with label x. A new concept namely k-equitable mean labeling of a graph is introduced in this paper.

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1 Introduction

Cahit [1] proposed the idea of distributing the vertex and edge labels among $\{0, 1, 2, \cdots, k-1\}$ as evenly as possible to obtain a generalization of graceful labeling as follows: For any graph $G(V, E)$ and for any positive integer k, assign vertex labels from $\{0, 1, 2, \cdots, k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with

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i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. A graph with such an assignment of labels is called k-equitable [2].
In [5] the notion of product cordial labeling was introduced. A product cordial labeling of a graph G with the vertex set V is a function f from V to {0, 1 } such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called product cordial graph.
Somasundaram and Ponraj [4] introduced the notion of mean labeling of graphs. A graph G with p vertices and q edges is called a mean graph if there is an injective function f from the vertices of G to $\{0, 1, 2, \cdots, q\}$ such that when each edge uv is labeled with $\lceil \frac{(f(u)+f(v)}{2} \rceil$, then the resulting edge labels are distinct.
A new concept namely k-equitable mean labeling of a graph is introduced in this paper. The graphs considered in this paper are finite simple graphs. Let $G = (V(G), E(G))$ be a graph of order p and size q. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The disjoint union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The disjoint union of m copies of the graph G is denoted by mG.The graph $G@P_n$ is obtained by identifying an end vertex of a path P_n with any vertex of G. Terms and notations not defined here are used in the sense of Harary[3].
For any integer n, $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n and $\lceil n \rceil$ denotes the least integer greater than or equal to n.

2 k-equitable mean labeling

Definition 2.1. *Let $G = (V, E)$ be a graph with p vertices and q edges. Let $f : V \to \{0, 1, 2, \cdots, k\} (1 \leq k \leq q)$ be a vertex labeling of G that induces an edge labeling $f^* : E \to \{0, 1, 2, \cdots, k\}$ be given by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. A labeling f is called $(k+1)$-equitable mean labeling ($(k+1)$-eml) if $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2, \cdots, k$ where $v_f(x)$ and $e_{f^*}(x)$, $x = 0, 1, 2, \cdots, k$ are the number of vertices and edges of G respectively with label x.*

A graph G that admits $(k + 1)$-equitable mean labeling is called a $(k + 1)$-equitable mean graph ($(k + 1) - emg$).

Theorem 2.2. *Let G be a (p, q)-connected graph. Then G is a $(q + 1)$-emg iff G is a mean graph.*

Proof: Suppose G is a $(q+1)$-emg. Then there is a vertex labeling $f : V \to \{0, 1, 2, \cdots, q\}$ that induces an edge labeling $f^* : E \to \{0, 1, 2, \cdots, q\}$ given by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2, \cdots, q$.
Since G has q edges and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$, the edge labels are distinct. Otherwise if $e_{f^*}(i) \geq 2$ for some i, then $e_{f^*}(j) = 0$ for at least one label $j \neq i$.
Since G is a connected graph, $q \geq p - 1$ and hence $q + 1 \geq p$. If $p = q + 1$ then all the vertex labels must be distinct and $v_f(i) = 1$ for all $i = 0, 1, 2, \cdots, q$.

If $p < q + 1$ then there is at least one label say i with $v_f(i) = 0$. If any label $j \neq i$ occurs more than once, we get a contradiction to $|v_f(i) - v_f(j)| \leq 1$. Hence, the vertex labels are distinct. Thus f is a mean labeling. The converse part follows from the definition of the mean labeling of a graph. □

Theorem 2.3. *G is a 2-emg iff G is a product cordial graph.*

Proof: Let f be a 2-eml of G. Then $f : V(G) \to \{0, 1\}$ is a vertex labeling of G that induces an edge labeling f^* given by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and it satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1$.
Define $g : V(G) \to \{0, 1\}$ by $g(v) = 1 - f(v)$ and $g^*(uv) = 1 - f^*(uv)$. Then, g is a vertex labeling of G with $v_g(0) = v_f(1)$ and $v_g(1) = v_f(0)$. Hence, $|v_g(0) - v_g(1)| \leq 1$. Now $g(u)g(v) = (1 - f(u))(1 - f(v)) = 1 - (f(u) + f(v)) + f(u)f(v)$.
If $f(u) = 0 = f(v)$ then $f^*(uv) = 0$ and $f(u) + f(v) - f(u)f(v) = 0$. If both $f(u) = 1 = f(v)$ then $f^*(uv) = 1$ and $f(u) + f(v) - f(u)f(v) = 2 - 1 = 1$. If one of $f(u)$ and $f(v)$ is zero, say $f(u) = 0$ and $f(v) = 1$ then $f^*(uv) = 1$ and $f(u) + f(v) - f(u)f(v) = 1 + 0 = 1$. Hence, $1 - (f(u) + f(v)) + f(u)f(v) = 1 - f^*(uv)$. Thus, we have $g^*(uv) = 1 - f^*(uv) = g(u)g(v)$. Therefore, $e_{f^*}(0) = e_{g^*}(1)$ and $e_{f^*}(1) = e_{g^*}(0)$ which implies that $|e_{g^*}(0) - e_{g^*}(1)| \leq 1$. Hence, g is a product cordial labeling of G.
The proof of the converse is similar to the previous argument. □

3 3-equitable mean labeling of some standard graphs

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In this section, 3-equitable mean labeling of some families of graphs are exhibited.

Lemma 3.1. *If a (p, q)-graph G admits a 3-eml f then $v_f(i) \geq \left\lfloor \frac{p}{3} \right\rfloor$ and $e_{f^*}(i) \geq \left\lfloor \frac{q}{3} \right\rfloor$, $i = 0, 1, 2$.*

Theorem 3.2. *(i) For any (p, q)-graph G, the graph $3mG$ is a 3-equitable mean graph.*

(ii) For any (p, q)-3-equitable mean graph G, the graph $(3m + 1)G$ is a 3- equitable mean graph.

Proof: (i) Assign 0 to all the vertices of the first m copies of G, assign 1 to all the vertices of next m copies of G and assign 2 to all vertices of the remaining m copies of G. Thus, we have $v(0) = v(1) = v(2) = mp$ and $e(0) = e(1) = e(2) = mq$.
(ii) Assign 0 to all the vertices of first m copies of G, assign 1 to all the vertices of next m copies of G and assign 2 to all the vertices of last m copies of G. The remaining one copy of G has the given 3-eml. Hence, $(3m + 1)G$ is a 3-emg. □

Theorem 3.3. *Let H be a (p, q) graph and consider $3m$ copies of H as H_i, $1 \leq i \leq 3m$. Let G be a graph obtained by identifying a vertex of H_i with a vertex of H_{i+1} for $1 \leq i \leq 3m - 1$. Then G is a 3-emg.*

Proof: For the given graph G, we have $|V(G)| = 3mp-(3m-1) = 3m(p-1)+1$ and $|E(G)| = 3mq$. Let u_i be a vertex of H_i and u_{i+1} be a vertex of H_{i+1} such that u_i is identified with u_{i+1} for $1 \leq i \leq 3m-1$. Now, assign 0 to all the vertices of Hi for $1 \leq i \leq m$, assign 1 to all the vertices of H_i for $m+1 \leq i \leq 2m$ except for u_{m+1} and assign 2 to all the vertices of H_i for $2m+1 \leq i \leq 3m$ except for the vertex u_{2m+1}. Then we have $v(0) = m(p-1)+1$, $v(1) = m(p-1)$, $v(2) = m(p-1)$ and $e(0) = mq$, $e(1) = mq$, $e(2) = mq$. Hence, G is a 3-emg. □

Theorem 3.4. *If $G(p, q)$ is a 3-emg then $\Delta(G) \leq 2r + t$ where $q = 3r + t$, $t \in \{0, 1, 2\}$.*

Proof: Let f be a 3-eml of G and let S_0, S_1 and S_2 be the subgraphs of G induced by the edges of G that have labels 0,1 and 2 respectively. Then $|E(S_0)| + E(S_1)| + |E(S_2)| = q$. Let $v \in V(G)$. If $f(v) = 0$ then $v \in V(S_0) \cup V(S_1)$, if $f(v) = 1$ then $v \in V(S_1) \cup V(S_2)$ and if $f(v) = 2$ then $v \in V(S_1) \cup V(S_2)$. Hence, $deg(v) \leq |E(S_0)| + |E(S_1)|$ or $|E(S_1)| + |E(S_2)|$.
If $q = 3r$ then $|E(S_0)| = |E(S_1)| = |E(S_2)| = r$ and hence $deg(u) \leq 2r$. If $q = 3r + 1$ then $\{|E(S_0)|, |E(S_1)|, |E(S_2)|\} = \{r, r, r+1\}$ and hence $deg(u) \leq 2r+1$. If $q = 3r+2$ then $\{|E(S_0)|, |E(S_1)|, |E(S_2)|\} = \{r, r+1, r+1\}$ and hence $deg(u) \leq 2r+2$. Thus $\Delta(G) \leq 2r + t$. □

Theorem 3.5. *The cycle C_n is a 3-emg iff $n \not\equiv 0(mod 3)$.*

Proof: Suppose C_n is a 3-emg and $n \equiv 0(mod 3)$. Then $n = 3r$ and hence $e(0) = e(1) = e(2) = r$ and $v(0) = v(1) = v(2) = r$. If $v(0) = r$ then $e(0) \leq r - 1$ which is a contradiction. Therefore $n \not\equiv 0(mod 3)$. Conversely assume $n \not\equiv 0(mod 3)$. Let C_n be the cycle $v_1v_2v_3 \cdots v_nv_1$. We consider the following two cases.
Case (i): $n \equiv 1(mod 3)$. Hence, $n = 3r + 1$. Define a vertex labeling $f : V \to \{0, 1, 2\}$ by

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq r+1 \\ 1 \text{ if } r+2 \leq i \leq 2r+1 \\ 2 \text{ if } 2r+2 \leq i \leq 3r+1 \end{cases}$$

Now, $v_f(0) = r+1$, $v_f(1) = v_f(2) = r$, $e_{f^*}(0) = e_{f^*}(2) = r$ and $e_{f^*}(1) = r+1$. Thus $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2$. Therefore f is a 3-eml.
Case (ii): $n \equiv 2(mod 3)$. Hence, $n = 3r + 2$. Define f by

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq r+1 \\ 1 \text{ if } r+2 \leq i \leq 2r+1 \\ 2 \text{ if } 2r+2 \leq i \leq 3r+2 \end{cases}$$

Then, $v_f(0) = r+1 = v_f(2)$, $v_f(1) = r$, and $e_{f^*}(0) = r$, $e_{f^*}(1) = e_{f^*}(2) = r+1$. Hence, C_n is 3-emg. □

Theorem 3.6. *The path P_n is a 3-emg for all $n \geq 2$.*

Proof: Let a path P_n be $v_1v_2v_3 \cdots v_n$. Define a vertex labeling $f : V \to \{0, 1, 2\}$ as follows.
Case (i): $n \equiv 0(mod 3)$. Take $n = 3r$.

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq r \\ 1 \text{ if } r+1 \leq i \leq 2r \\ 2 \text{ if } 2r+1 \leq i \leq 3r \end{cases}$$

Now, $v_f(0) = v_f(1) = v_f(2) = r$, $e_{f^*}(0) = r - 1$ and $e_{f^*}(1) = r = e_{f^*}(2)$.
Case (ii): $n \equiv 1(mod3)$. Take $n = 3r + 1$.

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq r + 1 \\ 1 \text{ if } r + 2 \leq i \leq 2r + 1 \\ 2 \text{ if } 2r + 2 \leq i \leq 3r + 1 \end{cases}$$

Thus, $v_f(0) = r + 1, = v_f(1) = v_f(2) = r$ and $e_{f^*}(0) = e_{f^*}(1) = e_{f^*}(2) = r$.
Case (iii): $n \equiv 2(mod3)$. Take $n = 3r + 2$.

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq r + 1 \\ 1 \text{ if } r + 2 \leq i \leq 2r + 2 \\ 2 \text{ if } 2r + 3 \leq i \leq 3r + 2 \end{cases}$$

Now, $v_f(0) = v_f(1) = r + 1$, $v_f(2) = r$ and $e_{f^*}(0) = e_{f^*}(2) = r$, $e_{f^*}(1) = r + 1$.
In the above three cases f satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2$. Hence, P_n is a 3-emg. □

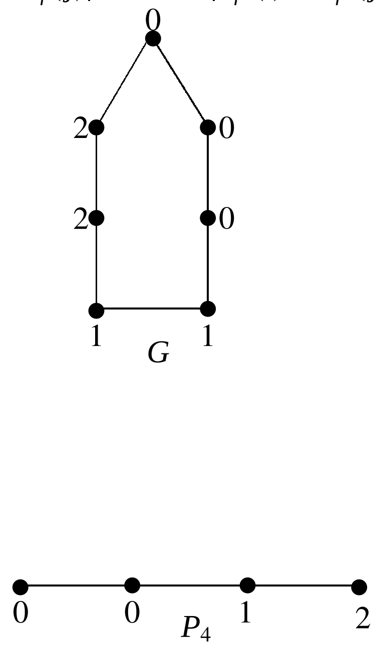

Figure 1

Theorem 3.7. *If G is a 3-emg then $G@P_n$, where $n \equiv 1(mod3)$ is a 3 - emg.*

Proof: Let P_n be a path $u_1u_2u_3 \cdots u_n$ and f be a 3-eml of P_n as Theorem 3.6. By the Case (ii) of Theorem 3.6 $v_f(0) = r + 1, = v_f(1) = v_f(2) = r$ and $e_{f^*}(0) = e_{f^*}(1) = e_{f^*}(2) = r$. Let g be a 3-eml of G and $u \in V(G)$ with $g(u) = 0$. Now, identify the vertex u with an end vertex of $P_n$whose label is 0. Define a labeling $h : V(G@Pn) \to \{0, 1, 2\}$ by

$$h(v) = \begin{cases} g(v) \text{ if } v \in V(G) \\ f(v) \text{ if } v \in P_n \end{cases}$$

Now $v_h(0) = v_g(0)+v_f(0)-1 = v_g(0)+r$, $v_h(1) = v_g(1)+v_f(1) = v_g(0)+r$, $v_h(2) = v_g(2)+v_f(2) = v_g(2)+r$, $e_{h^*}(0) = e_{g^*}(0) + r$, $e_{h^*}(1) = e_{g^*}(1) + r$ and $e_{h^*}(2) = e_{g^*}(2) + r$. Thus $|v_h(i) - v_h(j)| = |v_g(i) - v_g(j)| \le 1$ and $|e_{h^*}(i) - e_{h^*}(j)| = |e_{g^*}(i) - e_{g^*}(j)| \le 1$. Hence, h is a 3-eml of $G@P_n$. □

An example for a 3 - emg with $G = C_7$ and $n = 4$ is given in Figure 1 and Figure 2.

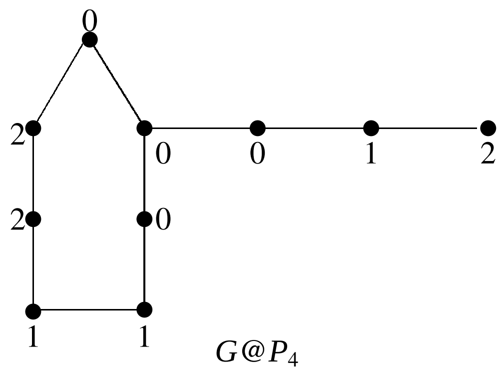

Figure 2

Theorem 3.8. *The bistar $B(m, n)$ with $m \ge n$ is 3-emg iff $n \ge \left\lfloor \frac{q}{3} \right\rfloor$.*

Proof: Let $V(B(m, n)) = \{u, u_i : 1 \le i \le n\} \cup \{v, v_i : 1 \le i \le m\}$ and $E(B(m, n)) = \{uv\} \cup \{uu_i : 1 \le i \le n\} \cup \{vv_i : 1 \le i \le m\}$. Thus $p = m + n + 2$ and $q = m + n + 1$.
Define a vertex labeling f as follows:
Case (i): Suppose $q = 3r$.

$$f(u) = 0;\ f(u_i) = \begin{cases} 0 \text{ if } 1 \le i \le r \\ 1 \text{ if } r + 1 \le i \le n \end{cases};\ f(v) = 1;\ f(v_i) = \begin{cases} 1 \text{ if } 1 \le i \le 2r - n - 1 \\ 2 \text{ if } 2r - n \le i \le m \end{cases}$$

Hence, $v_f(0) = r + 1$, $v_f(1) = v_f(2) = r$ and $e_{f^*}(0) = e_{f^*}(1) = e_{f^*}(2) = r$.
Case (ii): Suppose $q = 3r + 1$.

$$f(u) = 0;\ f(u_i) = \begin{cases} 0 \text{ if } 1 \le i \le r \\ 1 \text{ if } r + 1 \le i \le n \end{cases};\ f(v) = 1;\ f(v_i) = \begin{cases} 1 \text{ if } 1 \le i \le 2r - n \\ 2 \text{ if } 2r - n + 1 \le i \le m \end{cases}$$

Hence, $v_f(0) = r + 1$, $v_f(1) = r + 1$, $v_f(2) = r$ and $e_{f^*}(0) = r$, $e_{f^*}(1) = r + 1$ and $e_{f^*}(2) = r$.
Case (iii): Suppose $q = 3r + 2$.

$$f(u) = 0;\ f(u_i) = \begin{cases} 0 \text{ if } 1 \le i \le r \\ 1 \text{ if } r + 1 \le i \le n \end{cases};\ f(v) = 1;\ f(v_i) = \begin{cases} 1 \text{ if } 1 \le i \le 2r - n \\ 2 \text{ if } 2r - n + 1 \le i \le m \end{cases}$$

Hence, $v_f(0) = r + 1$, $v_f(1) = r + 1$, $v_f(2) = r + 1$ and $e_{f^*}(0) = r$, $e_{f^*}(1) = e_{f^*}(2) = r + 1$.
In all the above three cases f satisfies $|v_f(i) - v_f(j)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1$ for $i, j = 0, 1, 2$. Thus f is a 3-eml of $B(m, n)$.
Conversely, suppose that $m \ge n$ and $n < \left\lfloor \frac{q}{3} \right\rfloor$. We have $q = 3\left\lfloor \frac{q}{3} \right\rfloor + t$ where $t \in \{0, 1, 2\}$. Then $2\left\lfloor \frac{q}{3} \right\rfloor + t = q - \left\lfloor \frac{q}{3} \right\rfloor < q - n = m + n + 1 - n = m + 1 = \Delta(G)$. Hence, $\Delta(G) > 2\left\lfloor \frac{q}{3} \right\rfloor + t$. By Theorem 3.4, $B(m, n)$ is not 3-emg. □

Theorem 3.9. *$K_{1,n}$ is 3-emg iff $n \le 2$.*

Proof: Suppose that $n \le 2$. When $n = 1$, $K_{1,n} \cong P_2$ and when $n = 2$, $K_{1,n} \cong P_3$. Hence, by Theorem 3.6, $K_{1,n}$ is 3-emg.
Suppose $K_{1,n}$ is 3-emg. Then $\Delta(K_{1,n}) \le 2\left\lfloor \frac{q}{3} \right\rfloor + t$ where $t \in \{0,1,2\}$. Here $\Delta(K_{1,n}) = n = q$. By Theorem 3.4, $n \le 2\left\lfloor \frac{q}{3} \right\rfloor + t \Rightarrow 3\left\lfloor \frac{q}{3} \right\rfloor + t \le 2\left\lfloor \frac{q}{3} \right\rfloor + t \Rightarrow \left\lfloor \frac{q}{3} \right\rfloor \le 0 \Rightarrow \left\lfloor \frac{q}{3} \right\rfloor = 0$. Hence, $n = t \le 2$. □

4 3-equitable mean labeling of $T_n^{(k)} (n > 1)$

We define the graph $T_n^{(k)}$ to be the graph with the vertex set $V(T_n^{(k)}) = \{v_1, v_2, v_3, \cdots, v_k; v_{k+1} v_{k+2}, \cdots, v_{2k-1}; v_{2k}, v_{2k+1} \cdots, v_{3k-2}; \cdots; v_{(k-1)(n-1)+2}; \cdots v_{(k-1)n+1}\}$ and with the edge set $E(T_n^{(k)}) = \{v_i v_{i+1} : 1 \le i \le (k-1)n\} \cup \{v_1 v_k, v_k v_{2k-1}, v_{2k-1} v_{3k-2}, \cdots, v_{(k-1)(n-1)+1} v_{(k-1)n+1}\}$. Hence, we have $p = (k-1)n + 1$ and $q = kn$.

Lemma 4.1. *If $k \equiv 1(mod3)$ then $T_n^{(k)}$ is a 3-emg.*

Proof: Since $k \equiv 1(mod3)$, $p \equiv 1(mod3)$. Define a labeling $f : V(T_n^{(k)}) \to \{0,1,2\}$ as follows:

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \le i \le \left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 1 \text{ if } \left\lfloor \frac{p}{3} \right\rfloor + 2 \le i \le 2\left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 2 \text{ if } 2\left\lfloor \frac{p}{3} \right\rfloor + 2 \le i \le p \end{cases}$$

Thus $v_f(0) = \left\lfloor \frac{p}{3} \right\rfloor + 1$, $v_f(1) = \left\lfloor \frac{p}{3} \right\rfloor$, $v_f(2) = \left\lfloor \frac{p}{3} \right\rfloor$. To find the values of $e_{f^*}(0)$, $e_{f^*}(1)$ and $e_{f^*}(2)$ we consider the following three cases.
Case (i): Suppose $n \equiv 0(mod3)$. Take $n = 3r$. Then $p = 3r(k-1) + 1$ and $\left\lfloor \frac{p}{3} \right\rfloor = r(k-1)$. Hence, $e_{f^*}(0) = rk$ and $v_f(1) = v_f(2) = r(k-1)$ which implies that $e_{f^*}(1) = e_{f^*}(2) = rk$.
Case (ii): Suppose $n \equiv 1(mod3)$. Take $n = 3r + 1$. Then $\left\lfloor \frac{p}{3} \right\rfloor = \left\lfloor \frac{(3r+1)(k-1)+1}{3} \right\rfloor = \left\lfloor \frac{3r(k-1)+k-1+1}{3} \right\rfloor = r(k-1) + \left\lfloor \frac{k}{3} \right\rfloor = r(k-1) + \frac{k-1}{3}$. Hence, $v_f(0) = r(k-1) + \frac{k-1}{3} + 1$ which implies $e_{f^*}(0) = rk + \frac{k-1}{3}$. Again $v_f(1) = v_f(2) = r(k-1) + \frac{k-1}{3}$ implies that $e_{f^*}(1) = (3r+1)k - rk - \frac{k-1}{3} - rk - \frac{k-2}{3} = rk + \frac{k-1}{3}$ and $e_{f^*}(2) = rk - 1 + \frac{k-1}{3} + 2 = rk + \frac{k+2}{3}$.
Case (iii): Suppose $n \equiv 2(mod3)$. Take $n = 3r + 2$. Then $\left\lfloor \frac{p}{3} \right\rfloor = \left\lfloor \frac{(3r+2)(k-1)+1}{3} \right\rfloor = r(k-1) + \left\lfloor \frac{2(k-1)}{3} \right\rfloor$. Hence, $v_f(0) = r(k-1) + \frac{2(k-1)}{3} + 1$ implies that $e_{f^*}(0) = rk + \frac{2(k-1)}{3}$, $v_f(1) = v_f(2) = r(k-1) + \frac{2(k-1)}{3}$ implies that $e_{f^*}(2) = rk - 1 + \frac{2(k-1)}{3} + 2 = rk + 1 + \frac{2(k-1)}{3}$ and $e_{f^*}(1) = 3rk + 2k - 2rk - 1 - \frac{4(k-1)}{3} = rk + 1 + \frac{2(k-1)}{3}$.
In the above three cases f satisfies $|v_f(i) - v_f(j)| \le 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \le 1$ for $i, j = 0,1,2$. Hence, f is a 3-eml of $T_n^{(k)}$. □

Lemma 4.2. *If $k \equiv 0(mod3)$ and $n \equiv 0(mod3)$ then $T_n^{(k)}$ is a 3-emg.*

Proof: If $k \equiv 0(mod3)$ then $q \equiv 0(mod3)$. Since $n \equiv 0(mod3)$, take $n = 3r$. Define a labeling f as follows:

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq \left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 1 \text{ if } \left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq 2\left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 2 \text{ if } 2\left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq p \end{cases}$$

Thus $v_f(0) = \left\lfloor \frac{p}{3} \right\rfloor + 1$, $v_f(1) = v_f(2) = \left\lfloor \frac{p}{3} \right\rfloor$. Here $\left\lfloor \frac{p}{3} \right\rfloor = \left\lfloor \frac{(k-1)3r+1}{3} \right\rfloor = r(k-1)$. Hence, $v_f(0) = r(k-1)+1$, $v_f(1) = v_f(2) = r(k-1)q$. Thus $e_{f^*}(1) = 3rk - 2rk = rk$, $e_{f^*}(2) = rk$ and $e_{f^*}(3) = rk$. Hence, f satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2$. Therefore, f is a 3-eml of $T_n^{(k)}$. □

Lemma 4.3. *If $k \equiv 0(mod3)$ and $n \equiv 0(mod3)$ then $T_n^{(k)}$ is a 3-emg.*

Proof: If $k \equiv 0(mod3)$ then $q \equiv 0(mod3)$. Since $n \equiv 0(mod3)$, take $n = 3r$. Define a labeling f as follows:

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq \left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 1 \text{ if } \left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq 2\left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 2 \text{ if } 2\left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq p \end{cases}$$

Thus $v_f(0) = \left\lfloor \frac{p}{3} \right\rfloor + 1$, $v_f(1) = v_f(2) = \left\lfloor \frac{p}{3} \right\rfloor$. Here $\left\lfloor \frac{p}{3} \right\rfloor = \left\lfloor \frac{(k-1)3r+1}{3} \right\rfloor = r(k-1)$. Hence, $v_f(0) = r(k-1)+1$, $v_f(1) = v_f(2) = r(k-1)q$. Thus $e_{f^*}(1) = 3rk - 2rk = rk$, $e_{f^*}(2) = rk$ and $e_{f^*}(3) = rk$. Hence, f satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2$. Therefore, f is a 3-eml of $T_n^{(k)}$. □

Lemma 4.4. *If $k \equiv 0(mod3)$ and $n \equiv 1(mod3)$ then $T_n^{(k)}$ is not a 3-emg.*

Proof: Let f be a 3-eml of $T_n^{(k)}$. Then $v_f(0)$ is either $\left\lfloor \frac{p}{3} \right\rfloor$ or $\left\lfloor \frac{p}{3} + 1 \right\rfloor$. If we take $n = 3r + 1$ then $v_f(0)$ is either $r(k-1) + \frac{k}{3}$ or $r(k-1) + \frac{k}{3} + 1$. Hence, $e_{f^*}(0)$ is either $rk + \frac{k}{3} - 1$ or $rk + \frac{k}{3}$. Since $k \equiv 0(mod3)$, we must have $e_{f^*}(0) = \frac{kn}{3}$. □

Lemma 4.5. *If $k \equiv 0(mod3)$ and $n \equiv 2(mod3)$ then $T_n^{(k)}$ is not a 3-emg.*

Proof: Let f be a 3-eml of $T_n^{(k)}$. Then $v_f(0)$ is either $\left\lfloor \frac{p}{3} \right\rfloor$ or $\left\lfloor \frac{p}{3} \right\rfloor + 1$. If we take $n = 3r + 2$ then $v_f(0)$ is either $r(k-1) + \frac{2k-3}{3}$ or $r(k-1) + \frac{2k-3}{3} + 1$. Hence, $e_{f^*}(0)$ is either $rk - 2 + \frac{2k}{3}$ or $rk - 1 + \frac{2k}{3}$. Since $k \equiv 0(mod3)$, we must have $e_{f^*}(0) = \frac{k(3r+2)}{3}$. which gives a contradiction. Thus $T_n^{(k)}$ is not a 3-emg. □

Lemma 4.6. *If $k \equiv 2(mod3)$ and $n \equiv 0(mod3)$ then $T_n^{(k)}$ is a 3-emg.*

Proof: Define a labeling f as follows:

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq \left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 1 \text{ if } \left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq 2\left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 2 \text{ if } 2\left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq p \end{cases}$$

Thus $v_f(0) = \left\lfloor \frac{p}{3} \right\rfloor + 1$, $v_f(1) = v_f(2) = \left\lfloor \frac{p}{3} \right\rfloor$ and $e_{f^*}(0) = e_{f^*}(1) = e_{f^*}(2) = \left\lfloor \frac{q}{3} \right\rfloor = \frac{kn}{3}$ Hence, f satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2$. Therefore, f is a 3-eml of $T_n^{(k)}$. □

Lemma 4.7. *If $k \equiv 2(mod3)$ and $n \equiv 1(mod3)$ then $T_n^{(k)}$ is a 3-emg.*

Proof: Define a labeling f as follows:

$$f(v_i) = \begin{cases} 0 \text{ if } 1 \leq i \leq \left\lfloor \frac{p}{3} \right\rfloor + 1 \\ 1 \text{ if } \left\lfloor \frac{p}{3} \right\rfloor + 2 \leq i \leq 2\left\lfloor \frac{p}{3} \right\rfloor + 2 \\ 2 \text{ if } 2\left\lfloor \frac{p}{3} \right\rfloor + 3 \leq i \leq p \end{cases}$$

Then $v_f(0) = v_f(1) = \left\lfloor \frac{p}{3} \right\rfloor + 1$, $v_f(2) = \left\lfloor \frac{p}{3} \right\rfloor$. If we take $n = 3r + 1$ then $e_{f^*}(0) = rk + \frac{k-2}{3}$ and $e_{f^*}(1) = e_{f^*}(2) = rk + \frac{k-2}{3} + 1$. Hence, f satisfies $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2$. Therefore, f is a 3-eml of $T_n^{(k)}$. □

Lemma 4.8. *If $k \equiv 2(mod3)$ and $n \equiv 2(mod3)$ then $T_n^{(k)}$ is not a 3-emg.*

Proof: Since $k \equiv 2(mod3)$ and $n \equiv 2(mod3)$, $k - 1 \equiv 1(mod3)$ and $p \equiv n(k - 1) + 1 \equiv n + 1 \equiv 3 \equiv 0(mod3)$. Hence, $v_f(0) = v_f(1) = v_f(2) = \left\lfloor \frac{p}{3} \right\rfloor = \left\lfloor \frac{(k-1)(3r+2)+1}{3} \right\rfloor = r(k-1) + \left\lfloor \frac{2k-1}{3} \right\rfloor = r(k-1) + \frac{2(k-2)}{3} + 1$. So, $e_{f^*}(0) = rk + \frac{2(k-2)}{3}$ and $e_{f^*}(2) = rk + 2 + \frac{2(k-2)}{3}$ which implies that $|e_{f^*}(0) - e_{f^*}(2)| = 2$. Thus, $T_n^{(k)}$ is not a 3-emg. □

From the above lemmas we have the following theorem.

Theorem 4.9. *$T_n^{(k)}$ is a 3-emg if*

(i) $k \equiv 1(mod3)$

(ii) $k \equiv 0(mod3)$ *and* $n \equiv 0(mod3)$

(iii) $k \equiv 2(mod3)$ *and* $n \equiv 1(mod3)$

(iv) $k \equiv 2(mod3)$ *and* $n \equiv 0(mod3)$.

Theorem 4.10. *$K_{1,2n} \cup K_{1,n}$ is a 3-emg.*

Proof: Let u and v be the central vertices of the star graphs $K_{1,2n}$ and $K_{1,n}$ respectively, $u_1, u_2, \cdots, u_{2n}$ be the vertices incident with u and $v_1, v_2, \cdots, v_n$ be the vertices incident with v. Hence, $p = 3n + 2$ and $q = 3n$. Now assign 0 to all the vertices of $K_{1,n}$, 1 to the vertices $u, u_1, u_2, \cdots, u_n$ and 2 to the vertices $u_{n+1}, u_{n+2}, \cdots, u_{2n}$. Hence, $v_f(0) = n + 1$, $v_f(1) = v_f(2) = n$, $e_{f^*}(0) = e_{f^*}(1) = e_{f^*}(2) = n$. Thus, $K_{1,2n} \cup K_{1,n}$ is a 3-emg. □

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