Non-stationary and non-linear models

Javier Mencía

CEMFI 2013/2014

1   Cointegrated VAR

- Example 1:

$$\left(\begin{array}{c} x_{1t} \\ x_{2t} \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right) \left(\begin{array}{c} x_{1t-1} \\ x_{2t-1} \end{array} \right) + \left(\begin{array}{c} u_{1t} \\ u_{2t} \end{array} \right)$$

Error correction model:

$$\left(\begin{array}{c} \Delta x_{1t} \\ \Delta x_{2t} \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \left(\begin{array}{cc} 1 & -1 \end{array} \right) \left(\begin{array}{c} x_{1t-1} \\ x_{2t-1} \end{array} \right) + \left(\begin{array}{c} u_{1t} \\ u_{2t} \end{array} \right)$$

where $\Delta x_{it} = x_{it} - x_{it-1}$. The cointegration vector is: $\left(\begin{array}{cc} 1 & -1 \end{array} \right)$

- Example 2:

$$\left(\begin{array}{c} x_{1t} \\ x_{2t} \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0.001 & 0.999 \end{array} \right) \left(\begin{array}{c} x_{1t-1} \\ x_{2t-1} \end{array} \right) + \left(\begin{array}{c} u_{1t} \\ u_{2t} \end{array} \right)$$

Error correction model:

$$\left(\begin{array}{c} \Delta x_{1t} \\ \Delta x_{2t} \end{array} \right) = \left(\begin{array}{c} 0 \\ 0.001 \end{array} \right) \left(\begin{array}{cc} 1 & -1 \end{array} \right) \left(\begin{array}{c} x_{1t-1} \\ x_{2t-1} \end{array} \right) + \left(\begin{array}{c} u_{1t} \\ u_{2t} \end{array} \right)$$

2   ARCH, GARCH, leptokurtic distributions

2.1   General framework

$$x_t = \mu_t + \sigma_t u_t$$

where u_t is *iid* with $E(u_t) = 0$ and $V(u_t) = 1$. Hence:

$$\begin{array}{rcl} E_{t-1}(x_t) & = & \mu_t \\ V_{t-1}(x_t) & = & \sigma_t^2 \end{array}$$

2.2 Distribution of the standardised residuals

2.2.1 Normal

$$u_{1,t} \sim N(0,1)$$

- Moments: $E(u_{1,t}) = 0$, $V(u_{1,t}) = 1$, $E(u_{1,t}^3) = 0$, $E(u_{1,t}^4) = 3$

2.2.2 Student t with ν degrees of freedom

- Usual parametrisation:

$$f(v_t) \propto \left(1 + \frac{v_t^2}{\nu}\right)^{\frac{-(\nu+1)}{2}}$$

  Then, for $\nu > 2$: $E(v_t) = 0$, $V(v_t) = \nu/(\nu - 2)$. Hence, $u_{2,t} = \sqrt{(\nu - 2)/\nu} v_t$ will have zero mean and unit variance.

- Simulation

$$u_{2,t} = \sqrt{\frac{\nu - 2}{z_t}} u_{1,t}$$

  where $u_{1,t} \sim N(0,1)$ and z_t is a Gamma variate with mean ν and variance 2ν. When ν is an integer, $z_t \sim \chi_\nu^2$

- For $\nu = 10$:

  ```
  genr chi2=randgen(X,10)
  series r=8/chi2
  series u2=r^.5 *u1
  ```

- Moments ($\nu > 4$): $E(u_{2,t}) = 0$, $V(u_{2,t}) = 1$, $E(u_{2,t}^3) = 0$, and

$$E(u_{2,t}^4) = 3\frac{\nu - 2}{\nu - 4} > 3$$

2.2.3 Asimmetric t with ν degrees of freedom

- Simulation

$$u_{3,t} = c(\beta, \nu)\beta \left(\frac{\nu - 2}{z_t} - 1\right) + \sqrt{\frac{\nu - 2}{z_t}} c(\beta, \nu) u_{1,t}$$

  where

$$c(\beta, \nu) = \frac{-1 + \sqrt{1 + \frac{8\beta^2}{\nu - 4}}}{\frac{4\beta^2}{\nu - 4}}$$

- Moments ($\nu > 8$): $E(u_{3,t}) = 0$, $V(u_{3,t}) = 1$,

$$E(u_{3,t}^3) = \begin{cases} > 0 & \text{si} \quad \beta > 0 \\ < 0 & \text{si} \quad \beta < 0 \end{cases}$$

and $E(u_{2,t}^4) > 3$

- For $\nu = 10$ and $\beta = -10$:

> series u3=-1.588534*(r-1)+ (r*.158853)^.5 *u1

2.2.4 Normal Inverse Gaussian

- Simulation

$$u_{4,t} = c(\beta, \gamma)\beta\left(\frac{\gamma}{w_t} - 1\right) + \sqrt{\frac{\gamma}{w_t}c(\beta, \nu)}u_{1,t}$$

where $\gamma \geq 0$,

$$c(\beta, \gamma) = \frac{-1 + \sqrt{1 + \frac{4\beta^2}{\gamma}}}{\frac{2\beta^2}{\gamma}}$$

and w_t is Inverse Gaussian or $IG(1, \gamma)$ for short.

- For $\gamma = 1$ and $\beta = -10$:

```
genr v=normal()
genr s=randgen(X,1)
series q=1-.5*((s^2 +4*s)^.5-s)
series w=(v<=(1/(1+q)))*q+(v>(1/(1+q)))*(1/q)
series u4=-0.951249*(w-1)+ (.0951249*w)^.5 *u1
```

2.3   Models for the mean μ_t

- See session 1. We will assume that $\mu_t = 0, \forall t$

2.4   Models for σ_t^2

2.4.1   ARCH(1)

- Model: $x_t = \sigma_t u_t$

$$\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2$$

- Kurtosis

$$E(x_t^4) = \frac{\kappa\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - \kappa\alpha_1^2)}$$

where $\kappa = E(u_t^4)$.

- Let $\alpha_0 = .6, \alpha_1 = .4$

- Simulation:

```
smpl 1 100000
series sig2arch1=1
smpl 2 100000
sig2arch1=.6+.4*sig2arch1(-1)*u1(-1)^2
smpl 1 100000
series x1=sig2arch1^0.5*u1
```

2.4.2 GARCH(1,1)

- Model: $x_t = \sigma_t u_t$

$$\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$$

- Kurtosis

$$E(x_t^4) = \frac{\kappa \alpha_0^2 (1 + \alpha_1 + \alpha_2)}{(1 - \alpha_1 - \alpha_2)(1 - \kappa \alpha_1^2 - \alpha_2^2 - 2\alpha_1 \alpha_2)}$$

where $\kappa = E(u_t^4)$.

- Let $\alpha_0 = .05, \alpha_1 = .1, \alpha_2 = .85$

- Simulation:

```
smpl 1 100000
series sig2garch1=1
smpl 2 100000
sig2garch1=.05+.1*sig2garch1(-1)*u1(-1)^2+.85*sig2garch1(-1)
smpl 1 100000
series x2=sig2garch1^0.5*u1
```

3   ARG

- y_t follows an autoregressive Gamma process of order 1, or *ARG(1)* for short, if the distribution of $x_t = 2y_t/c$ conditional on information known at $t-1$ is a non-central chi-square with non-centrality parameter $2\beta y_{t-1}$ and degrees of freedom 2δ.

- The ARG(1) is an autoregressive heteroskedastic process for positive distributions:

$$
\begin{aligned}
E_{t-1}(y_t) &= c\delta + c\beta y_{t-1}, \\
V_{t-1}(y_t) &= c^2\delta + 2c^2\beta y_{t-1}, \\
E(y_t) &= \frac{c\delta}{1 - c\beta}, \\
V(y_t) &= c^2\delta \frac{1 + c\beta}{(1 - c\beta)(1 - c^2\beta^2)}
\end{aligned}
$$

- In terms of autocorreltions, it behaves as an AR(1):

$$
corr(y_t, y_{t-k}) = (c\beta)^k.
$$

- Simulation. Let $\delta = 1/2$, then we can simulate y_t as

$$
y_t = \frac{c}{2} z_t^2,
$$

where $z_t \sim N(\sqrt{2\beta y_{t-1}}, 1)$.

4 AR with Markov chain

$$
y_t = \begin{cases} \phi_{00} + \phi_{10} y_{t-1} + \sigma_0 u_t^0 & \text{si} \quad s_t = 0 \\ \phi_{01} + \phi_{11} y_{t-1} + \sigma_1 u_t^1 & \text{si} \quad s_t = 1 \end{cases}
$$

where s_t is a Markov chain with $s_1 = 0$ and

$$
\begin{aligned}
\Pr(s_t = 1 | s_{t-1} = 0) &= 1 - p \\
\Pr(s_t = 1 | s_{t-1} = 1) &= q
\end{aligned}
$$

- Let

$$
\begin{array}{ll}
\phi_{00} = -.00001 & \phi_{01} = .06 \\
\phi_{10} = .9999 & \phi_{11} = .4 \\
\sigma_0^2 = 1.28 & \sigma_1^2 = .16 \\
p = .95 & q = .95
\end{array}
$$

- Hence:

$$
\begin{array}{ll}
E(y_t | s_t = 0) = -.1 & V(y_t | s_t = 0) = 2 \\
E(y_t | s_t = 1) = .1 & V(y_t | s_t = 1) = 1
\end{array}
$$

- Simulation

```
smpl 1 100000
series s=0
series y=0
series b=uniform()
smpl 2 100000
s=(1-s(-1))*(b<.05)+s(-1)*(b<.95)
y=(1-s)*(-.00001+.9999*y(-1)+1.28^.5 *u1)+s*(.06+.4*y(-1)+.16^.5 *u1)
```

- Filtering: probability of the regimes conditional on observable information.

$$
\begin{aligned}
\Pr(s_t = 1 | I_{t-1}) &= \Pr(s_t = 1 | s_{t-1} = 1, I_{t-1}) \Pr(s_{t-1} = 1 | I_{t-1}) \\
&\quad + \Pr(s_t = 1 | s_{t-1} = 0, I_{t-1})[1 - \Pr(s_{t-1} = 1 | I_{t-1})], \\
&= q \Pr(s_{t-1} = 1 | I_{t-1}) + (1 - p)[1 - \Pr(s_{t-1} = 1 | I_{t-1})],
\end{aligned}
$$

$$
\Pr(s_t = 1 | I_t) = \frac{f(y_t | s_t = 1, I_{t-1}) \Pr(s_t = 1 | I_{t-1})}{f(y_t | s_t = 0, I_{t-1})[1 - \Pr(s_t = 1 | I_{t-1})] + f(y_t | s_t = 1, I_{t-1}) \Pr(s_t = 1 | I_{t-1})},
$$

where $I_t = \{y_t, y_{t-1}, y_{t-2}, \cdots\}$. We initialise the filter with the unconditional probability

$$
\Pr(s_0 = 1 | I_0) = \frac{1 - p}{2 - p - q}.
$$

A Gamma distribution

- Notation: $z \sim \Gamma(a, b)$

- Density function:
$$\frac{a^b}{\Gamma(b)} z^{b-1} \exp(-az)$$
where $\Gamma()$ is the Gamma function.

- Moments:
$$E(z^r) = a^{-r} \frac{\Gamma(b + r)}{\Gamma(b)}$$

- For the Student t and the Asymmetric t, $a = \nu/2$ y $b = 1/2$. If ν is an integer, $z \sim \chi_\nu^2$.

B Inverse Gaussian distribution

- Notation: $w \sim IG(1, \gamma)$

- Density:
$$\frac{\exp(\gamma)}{\sqrt{2\pi}} w^{-3/2} \exp\left[-\frac{1}{2}(w^{-1} + \gamma^2 w)\right]$$
where $\gamma \geq 0$ y $w > 0$.

- Moments:

$$
\begin{aligned}
E(w) &= \gamma^{-1} \\
E(w^2) &= \gamma^{-2}(1 + \gamma^{-1}) \\
E(w^3) &= \gamma^{-3}(1 + 3\gamma^{-1} + 3\gamma^{-2})
\end{aligned}
$$

- Simulation:

  1. Generate $u \sim Unif(0, 1)$ and $s \sim \chi_1^2$ independent of u.
  2. Let
  $$q = 1 - (1/2)\gamma^{-1}\left(\sqrt{s^2 + 4\gamma s} - s\right)$$
  3. Finally:
  $$w = \begin{cases} \frac{q}{\gamma} & \text{si} \quad u \leq (1 + q)^{-1} \\ \frac{1}{\gamma q} & \text{si} \quad u > (1 + q)^{-1} \end{cases}$$

C   Non-central chi-square distribution

- Two parameters:

    - Degrees of freedom: ν,
    - Non-centrality: λ.

- Probability density function:

$$f_{NC2}(x;\nu,\lambda) = \sum_{j=0}^{\infty} \frac{(\lambda/2)^j \exp(-\lambda/2)}{j!} f_{\chi^2}(x;\nu+2j)$$

for $x \geq 0$, where $f_{\chi^2}(\cdot;\nu)$ is the pdf of a chi-square distribution with ν degrees of freedom.

- First non-centred moments:

$$\begin{aligned} \mu_1' = E(x) &= \nu + \lambda, \\ \mu_2' = E(x^2) &= 2(2\lambda+\nu) + (\lambda+\nu)^2. \end{aligned}$$

- Property: If $\nu = k$ is an integer, then it is possible to generate this distribution as

$$\sum_{i=1}^{k} x_i^2$$

where $x_i \sim N(\mu_i, 1)$, so that $\lambda = \sum_{i=1}^{k} \mu_i^2$.