3. Multivariate dependence

Javier Mencía
javier.mencia@bde.es

Banco de España, Financial Stability Department

Outline of the presentation

1. Introduction
2. Brief recap on covariance matrices
3. Principal component analysis
4. GARCH extensions and factor models
5. Asymmetric distributional dependence. Copulas

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Multivariate models

- As we have already seen, many economic and financial variables are interdependent, so that joint modelling may be necessary for certain applications.
- A multivariate time series $\mathbf{y}_t$ is a vector $(y_{1t}, y_{2t}, \cdots, y_{nt})'$ of n different time series processes that are measured concurrently.
- We have already seen linear models to characterise the dynamics of random vectors such as $\mathbf{y}_t$.
- In this session, we will first study ways to determine the main common factors driving changes in $\mathbf{y}_t$ by statistical methods (Principal components, factor models).
- We will then consider multivariate GARCH extensions.
- Finally, we will study asymmetric distributional dependence with Copulas, which goes beyond the pure variance-related dependence.

Outline

Brief recap on cross-covariances

- We can define the covariance matrix of $\mathbf{y}_t$ as

$$\mathbf{\Gamma}_0 = E[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_t - \boldsymbol{\mu})'] \text{ where } \boldsymbol{\mu} = E[\mathbf{y}_t].$$

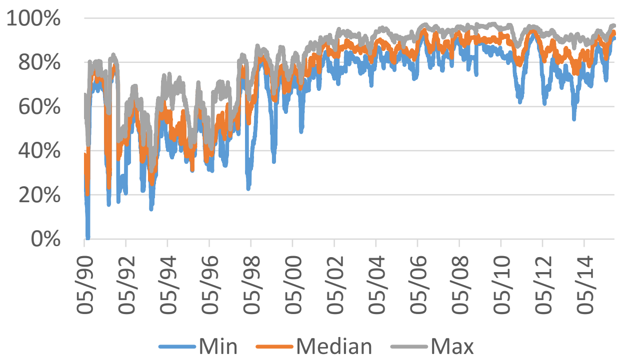
- Analogously, we can define the cross-covariance matrix of order l as

$$\mathbf{\Gamma}_l = E[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_{t-l} - \boldsymbol{\mu})'] \approx \frac{1}{T}\sum_{t=l+1}^{T}(\mathbf{y}_t - \bar{\mathbf{y}})(\mathbf{y}_{t-l} - \bar{\mathbf{y}})',$$

  where $\bar{\mathbf{y}}$ is the sample mean vector.
- In the previous sessions, we have assumed that covariances and correlations are constant over time.
- However, in practice they also show time varying features.

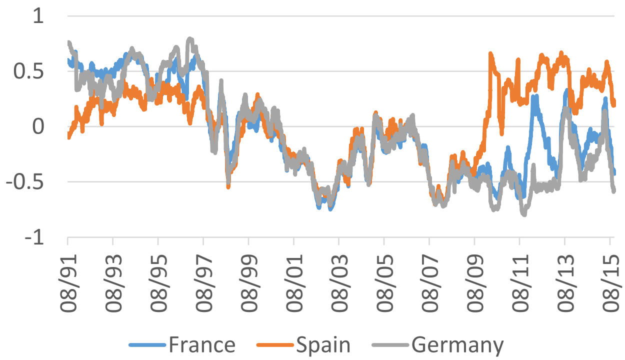
Correlations between the returns of some euro area reference stock indices

100-day moving window



Correlations between the returns of some euro area reference stock indices and their corresponding sovereign bond

100-day moving window



Outline

High dimensional problems

- Sometimes the dimension of $\mathbf{y}_t$ is very large, so that modelling all the possible interdependencies between its elements becomes too burdensome.
- Principal component analysis (PCA) is the most common technique to reduce the multivariate dimension.
- With PCA, we can obtain the most relevant factors driving the changes in $\mathbf{y}_t$.
- These factors are orthogonal (i.e. uncorrelated) and unobservable, so they have no economic or financial interpretation a priori.

PCA intuition (i)

- Let $\boldsymbol{\Sigma}$ define the covariance matrix of $\mathbf{y}_t$: $V(\mathbf{y}_t) = \boldsymbol{\Sigma}$.
- The idea behind PCA is to express the elements in $\mathbf{y}_t$ as a linear combination of a set of orthogonal factors $\mathbf{f}_t$:

$$y_{it} = \alpha_{i1} f_{1t} + \alpha_{i2} f_{2t} + \cdots + \alpha_{in} f_{nt}.$$

- These factors are ordered in decreasing order of importance, such that $V(f_{1t}) > V(f_{2t}) > \cdots > V(f_{nt})$.
- Thus, we should be able to approximate reasonably well the main common drivers of $\mathbf{y}_t$ by just using the most important factors. For instance:

$$\mathbf{y}_t \approx \boldsymbol{\alpha}_1 f_{1t} + \boldsymbol{\alpha}_2 f_{2t} + \boldsymbol{\alpha}_3 f_{3t}.$$

PCA intuition (ii)

- But, what are exactly these factors?
- In fact, they are just linear combinations of the variables:

$$f_{it} = \boldsymbol{\omega}_i' \mathbf{y}_t = \omega_{1i} y_{1t} + \cdots + \omega_{ni} y_{nt}.$$

- Using the properties of linear combinations, we have

$$V(f_{it}) = \boldsymbol{\omega}_i' \boldsymbol{\Sigma} \boldsymbol{\omega}_i, cov(f_{it}, f_{jt}) = \boldsymbol{\omega}_i' \boldsymbol{\Sigma} \boldsymbol{\omega}_j.$$

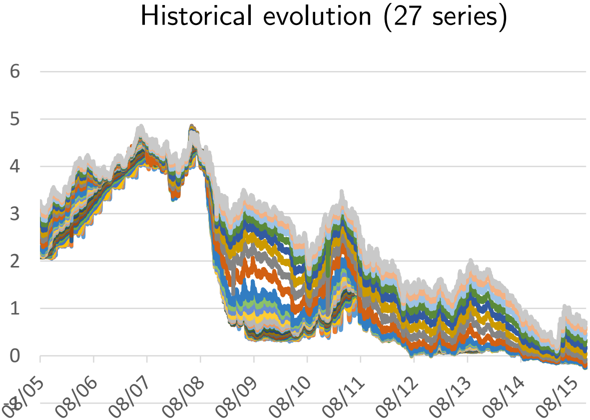
- The first PC is the linear combination that maximises $V(f_{1t})$, subject to the scale costraint $\boldsymbol{\omega}_1' \boldsymbol{\omega}_1 = 1$.
- The second PC maximises $V(f_{2t})$ subject to $\boldsymbol{\omega}_2' \boldsymbol{\omega}_2 = 1$ and $cov(f_{1t}, f_{2t}) = 0$.
- The ith PC maximises $V(f_{it})$ subject to $\boldsymbol{\omega}_i' \boldsymbol{\omega}_i = 1$ and $cov(f_{it}, f_{jt}) = 0$ for $j = 1, 2, \cdots, i - 1$.

PCA formal definition

- The principal components can be obtained from the spectral decomposition of $\boldsymbol{\Sigma}$, which can always be computed since covariance matrices are non-negative definite.
- In particular, it can be shown that $\boldsymbol{\Sigma} = \boldsymbol{\Omega}\boldsymbol{\Lambda}\boldsymbol{\Omega}'$.
- $\boldsymbol{\Omega}$ is the matrix of eigenvectors, where $\boldsymbol{\Omega}\boldsymbol{\Omega}' = \mathbf{I}_n$.
- $\boldsymbol{\Lambda}$ is a diagonal matrix, whose diagonal terms are known as eigenvalues: $\lambda_1, \lambda_2, \cdots, \lambda_n$.
- Assuming that the elements in the diagonal of $\boldsymbol{\Lambda}$ are in decreasing order, it can be shown that:
  1. The ith PC can be expressed as $f_{it} = \boldsymbol{\omega}_i'\mathbf{y}_t = \omega_{1i}y_{1t} + \cdots + \omega_{ni}y_{nt}$, where $\boldsymbol{\omega}_i$ is the ith column of $\boldsymbol{\Omega}$.
  2. $V(f_{it}) = \lambda_i$, where λ_i is the ith eigenvalue.

Example: Overnight index swap (OIS)

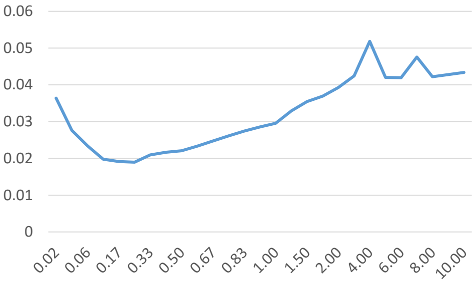
Current preferred proxy of risk-free rate



Example: Overnight index swap (OIS)

- PCA can by carried out based on either the covariance or the correlation matrix.
- The more constant is the term structure of standard deviation, the more equivalent the two approaches are.

Standard deviation of yield daily changes



Example: Overnight index swap (OIS)

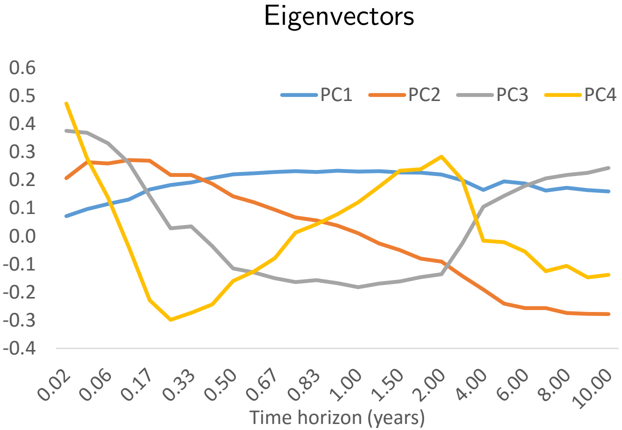
Eigenvalues (Correlation matrix based PCA)

```
Principal components/correlation                  Number of obs    =      2666
                                                  Number of comp.  =         4
                                                  Trace            =        27
    Rotation: (unrotated = principal)             Rho              =    0.8803
```

Component	Eigenvalue	Difference	Proportion	Cumulative
Comp1	15.883	11.0686	0.5883	0.5883
Comp2	4.81435	2.57814	0.1783	0.7666
Comp3	2.23621	1.40258	0.0828	0.8494
Comp4	.83363	.100896	0.0309	0.8803
Comp5	.732734	.291943	0.0271	0.9074
Comp6	.440791	.147113	0.0163	0.9237
Comp7	.293678	.0283208	0.0109	0.9346
Comp8	.265357	.0684286	0.0098	0.9444
Comp9	.196929	.0138881	0.0073	0.9517
Comp10	.183041	.00550182	0.0068	0.9585

Example: Overnight index swap (OIS)

Correlation matrix based PCA



Outline

GARCH extensions. Constant conditional correlation (CCC).

- A simple way of generalising the GARCH model to a multivariate setting is to assume that the correlation matrix is constan.
- Consider, for simplicity, a bivariate model: $\mathbf{y}_t = (y_{1t}, y_{2t})'$.
- Then, we assume that each of the series follows a GARCH(1,1) process

$$\begin{aligned} V(y_{1t}|I_{t-1}) = \sigma^2_{1t} &= \alpha_{10} + \alpha_{11}(y_{1,t-1} - \mu_{1,t-1})^2 + \beta_{11}\sigma^2_{1,t-1}, \\ V(y_{2t}|I_{t-1}) = \sigma^2_{2t} &= \alpha_{20} + \alpha_{21}(y_{2,t-1} - \mu_{2,t-1})^2 + \beta_{21}\sigma^2_{2,t-1}. \end{aligned}$$

- However, the correlation is assumed to be constant:

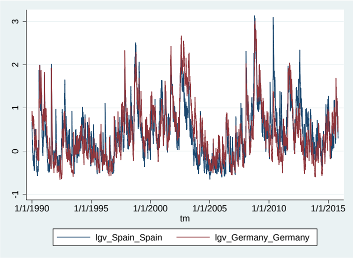
$$cor(y_{1t}, y_{2t}|I_{t-1}) = \rho$$

CCC Example. Estimates

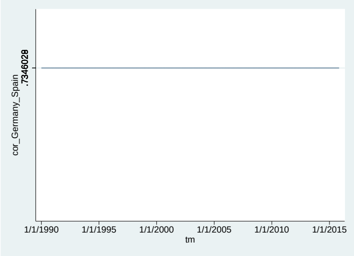
	Coef.	Std. Err.	z	P>|z|	[95% Conf. Interv
Spain					
_cons	.0669547	.0132693	5.05	0.000	.0409473 .092
ARCH_Spain					
arch					
L1.	.0822672	.0060716	13.55	0.000	.0703671 .094
garch					
L1.	.8901459	.007847	113.44	0.000	.8747661 .905
_cons	.0467493	.0053378	8.76	0.000	.0362874 .057
Germany					
_cons	.0797108	.0131111	6.08	0.000	.0540135 .105
ARCH_Germany					
arch					
L1.	.069142	.0053744	12.87	0.000	.0586084 .079
garch					
L1.	.9057437	.007043	128.60	0.000	.8919397 .919
_cons	.0398282	.0044429	8.96	0.000	.0311203 .04
corr(Spain, Germany)	.709379	.0061418	115.50	0.000	.6973413 .72

CCC Example. Volatilities and correlation

Volatility (in logs)



Correlation



GARCH extensions. Dynamic conditional correlation.

- The CCC model is too restrictive if correlation changes over time (see above example).
- In the DCC model, we also assume that each of the series follows a GARCH(1,1) process

$$\begin{aligned} V(y_{1t}|I_{t-1}) = \sigma_{1t}^2 &= \alpha_{10} + \alpha_{11}(y_{1,t-1} - \mu_{1,t-1})^2 + \beta_{11}\sigma_{1,t-1}^2, \\ V(y_{2t}|I_{t-1}) = \sigma_{2t}^2 &= \alpha_{20} + \alpha_{21}(y_{2,t-1} - \mu_{2,t-1})^2 + \beta_{21}\sigma_{2,t-1}^2. \end{aligned}$$

- However, the correlation is assumed to be time varying:

$$cor(y_{1t}, y_{2t}|I_{t-1}) = \rho_t = exp(q_t)/(1 + exp(q_t)),$$

where

$$q_t = \omega_0 + \omega_1 \rho_{t-1} + \omega_2 \frac{(y_{1,t-1} - \mu_{1,t-1})(y_{2,t-1} - \mu_{2,t-1})}{\sigma_{1,t-1}\sigma_{2,t-1}}$$

DCC Example. Estimates

```
Dynamic conditional correlation MGARCH model

Sample: 1 - 6624                                Number of obs    =       (
Distribution: Gaussian                          Wald chi2(.)     =
Log likelihood = -18755.05                      Prob > chi2      =
```

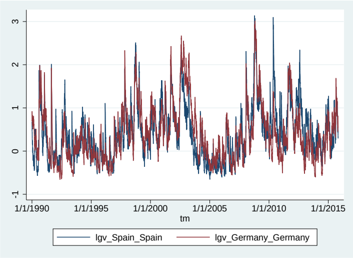
	Coef.	Std. Err.	z	P>|z|	[95% Conf. Interv
corr(Spain, Germany)	.794841	.0183568	43.30	0.000	.7588623	.830(
Adjustment						
lambda1	.0524026	.004478	11.70	0.000	.0436259	.061:
lambda2	.9326267	.0061032	152.81	0.000	.9206647	.944!

- Likelihood ratio test with respect to the CCC:

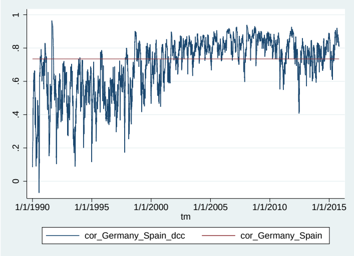
$$LR = 2(-18755.05 + 19173.97) = 837.84 \sim \chi_2$$

DCC Example. Volatilities and correlation

Volatility (in logs)



Correlation



Outline

Limitations of the Gaussian-Elliptical framework

- Correlation is the most commonly used measure of dependence, but it only describes well certain types of dependence (Gaussian, elliptical)
- Classical theories of portfolio and risk management and risk management assume multivariate normal returns, but financial markets do not behave according to these idealised conditions
- For any two random variables x_t and y_t, copulas are a way to isolate their dependence from the marginal distributions $f(x_t)$ and $f(y_t)$
- We can use copulas to specify a joint distribution in a two stage proces:
  1. Specify the marginal distributions
  2. Specify the copula distribution

- Let us study the dependence between several equity returns indices: ES, FR, DE, IT
- We are going to study conditional dependence, by eliminating the effect of time varying volatility:

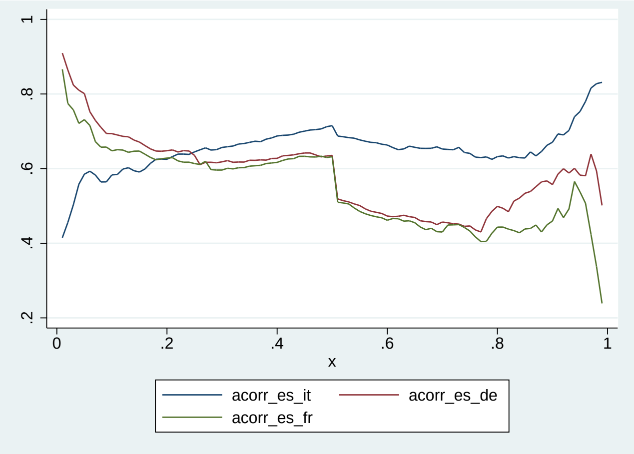
$$\varepsilon_{yt}^{*} = \frac{y_t - \mu}{\sigma_t}$$

- More specifically, we are going to exceedance correlations to assess the presence of asymmetric and tail dependence:

$$EC(\tau) = \begin{cases} corr(\varepsilon_{xt}, \varepsilon_{yt} | \varepsilon_{xt} < \tau, \varepsilon_{yt} < \tau) & if \quad \tau < 0 \\ corr(\varepsilon_{xt}, \varepsilon_{yt} | \varepsilon_{xt} > \tau, \varepsilon_{yt} > \tau) & if \quad \tau > 0 \end{cases}$$

Why correlation may describe dependence inaccurately?

Asymmetric and tail dependence



- Consider a variable x, whose conditional density function is $F(x)$
- It can be shown that the cdf transform of x is always a uniform distribution between 0 and 1

$$u = F(x) \sim Uniform(0, 1)$$

- The u quantile of X is the value x_u such that $u = Pr(X < x_u) = F(x_u)$
- Intuition:

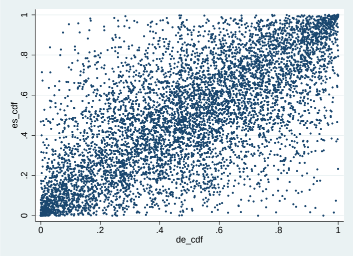
$$Pr(U < u_0) = Pr(X < x_0) = F(x_0) = u_0$$

Copulas

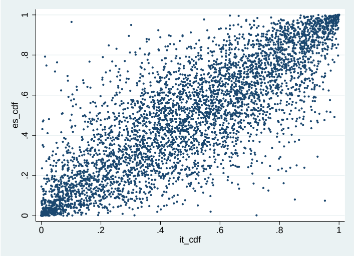
- The copula is the joint distribution of the cdf transforms of two variables x and y
- If we denote the cdf transforms as $u = F_1(x)$ and $v = F_2(y)$, then the copula would be the joint distribution of u and v
- The effect of the marginal distributions has been completely separated, since both u and v are uniform variables, regardless of the distributions of x and y

Scatter plots of cdf transforms

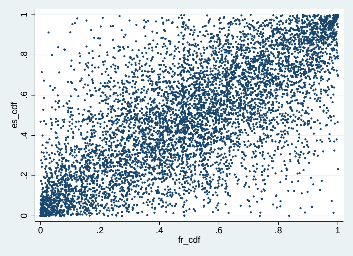
ES-DE



ES-IT



ES-FR



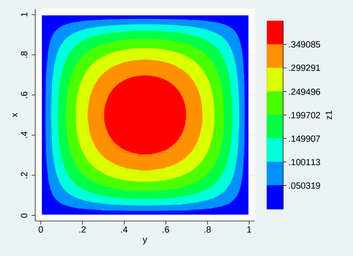
Independent, symmetric and asymmetric copulas

- Gaussian independent copula: $x \sim N(\mathbf{0}, \mathbf{I})$
- Gaussian correlated copula: $x \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = (1, \rho; \rho, 1)$
- Symmetric mixture: $p \cdot N(\mathbf{0}, \boldsymbol{\Sigma}_1) + (1-p) \cdot N(\mathbf{0}, \boldsymbol{\Sigma}_2)$, with ρ_1, ρ_2 and p as free parameters
- Asymmetric mixture: $p \cdot N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) + (1-p) \cdot N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$, such that $p\boldsymbol{\mu}_1, +(1-p)\boldsymbol{\mu}_2, = 0$, with the elements of $\boldsymbol{\mu}_1$ as additional free parameters

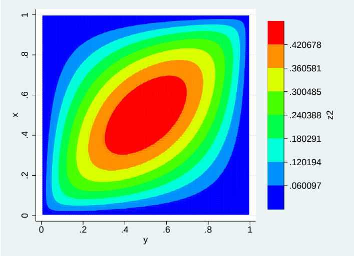
Contour plots of the copula densities

Spanish vs. German equity index returns

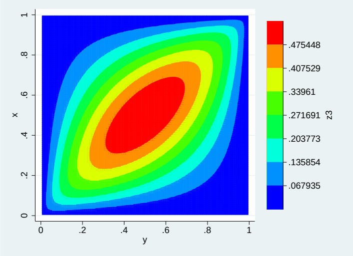
Indepdendent normals



Correlated normals



Symmetric mixture



Asymmetric mixture