Linear models

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1 Univariate models

1.1 White noise

- Uniform
  - $e_1 = uniform()$
  - $e_2 = e_1 * e1$
- Gaussian
  - $u = normal()$

1.2 MA(1)

- Invertible
  - $x_t = u_t + \theta u_{t-1}$, $u_t \sim N(0, 1), \theta = 0.8$
  - $y_t = u_t - \theta u_{t-1}$
- Not invertible
  - $x_t = u_t + (1/\theta)u_{t-1}$

1.3 AR(1)

- Stationary
  - $x_t = \alpha x_{t-1} + u_t$, $u_t \sim N(0, 1), \alpha = 0.8, x_1 = 100$
- Not stationary
  - $x_t = \alpha x_{t-1} + u_t$, $u_t \sim N(0, 1), \alpha = 1.0001$
- Random walk
  - $x_t = \alpha x_{t-1} + u_t$, $u_t \sim N(0, 1), \alpha = 1$
  - Wiener process

1.4 Markov chains

- Binary variable
  - $b_t = 1(u_t < 0.2)$, $u_t \sim U(0,1)$
- Binary Markov chain
  - $x_0 = 0$
  - $P(x_t = 0|x_{t-1} = 0) = p = .7$
  - $P(x_t = 1|x_{t-1} = 1) = q = .8$
  - $x_t = (1 - x_{t-1})1(u_t > p) + x_{t-1}1(u_t < q)$
- Unconditional probabilities under stationarity:
  - $P(x_t = 0) = P(x_t = 0|x_{t-1} = 0)P(x_{t-1} = 0) + P(x_t = 0|x_{t-1} = 1)P(x_{t-1} = 1)$
  - $\pi = p\pi + (1-q)(1-\pi) \Longrightarrow \pi = (1-q)/(2-p-q)$
- AR(1) specification:
  - $x_t = 1 - p + (p + q - 1)x_{t-1} + w_t$

1.5 ARMA(1,1)

- $x_t = \alpha x_{t-1} + u_t + \theta u_{t-1}$, $u_t \sim N(0,1), \alpha = 0.9, \theta = -0.8$

2 Multivariate models

2.1 White noise

- Final goal:
$$\mathbf{v}_t = \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} \sim N\left[\mathbf{0}, \boldsymbol{\Sigma}\right],$$
where
$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$
- Vector of independent variables:
$$\mathbf{w}_t = \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix} \sim N\left[\mathbf{0}, \mathbf{I}_2\right]$$
- How to obtain $\mathbf{v}_t$ from $\mathbf{w}_t$:

- $\mathbf{v}_t = \mathbf{H}\mathbf{w}_t$ such that

$$V(\mathbf{v}_t) = \mathbf{H}\mathbf{H}' = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

- Alternatives:
  - Cholesky factorisation: $\mathbf{H}_1$ lower triangular

$$\mathbf{H}_1 = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_{12}/\sigma_1 & \sqrt{\sigma_2^2 - (\sigma_{12}^2/\sigma_1^2)} \end{pmatrix}$$

  - Spectral decomposition:

$$\mathbf{\Sigma} = \mathbf{M}\mathbf{\Delta}\mathbf{M}',$$

    where

$$\mathbf{\Delta} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

    and the columns of $\mathbf{M} = [\mathbf{m}_1, \mathbf{m}_2]$ are the eigenvectors of λ_1 and λ_2, respectively. In this case, we multiply $\mathbf{w}_t$ by

$$\mathbf{H}_2 = \mathbf{M}\mathbf{\Delta}^{1/2}$$

    where

$$\mathbf{\Delta}^{1/2} = \begin{pmatrix} \lambda_1^{1/2} & 0 \\ 0 & \lambda_2^{1/2} \end{pmatrix}$$

  - Any orthogonal rotations over the above matrices:

$$\mathbf{R}(\theta) = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}$$

$$\bar{\mathbf{H}}_1(\theta) = \mathbf{H}_1\mathbf{R}(\theta)$$
$$\bar{\mathbf{H}}_1(\theta)\bar{\mathbf{H}}_1'(\theta) = \mathbf{H}_1\underbrace{\mathbf{R}(\theta)\mathbf{R}'(\theta)}_{\mathbf{I}_N}\mathbf{H}_1' = \mathbf{\Sigma}$$

- Example:

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & .6 \\ .6 & 1 \end{pmatrix}$$

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 0 \\ .6 & .8 \end{pmatrix}$$

  - $v_{1t} = w_{1t}$
  - $v_{2t} = 0.6w_{1t} + 0.8w_{2t}$

$$\mathbf{\Delta} = \begin{pmatrix} 1.6 & 0 \\ 0 & 0.4 \end{pmatrix}; \mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}} \Rightarrow \mathbf{H}_2 = \begin{pmatrix} \sqrt{.8} & \sqrt{.2} \\ \sqrt{.8} & -\sqrt{.2} \end{pmatrix}$$

2.2 VMA(1)

$$\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} + \mathbf{B} \begin{pmatrix} v_{1t-1} \\ v_{2t-1} \end{pmatrix}$$

where

$$\mathbf{B} = \begin{pmatrix} 0.2 & 0.3 \\ 0.6 & 0.5 \end{pmatrix}$$

2.3 VAR(1)

$$\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} \tag{1}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- $\mathbf{A}$ is lower or upper triangular:

$$\mathbf{A} = \begin{pmatrix} 0.9 & 0 \\ 1 & 0 \end{pmatrix}$$

```
smpl 1 100000
series x1=0
series x2=0
smpl 2 100000
x1=0.9*x1(-1)+v1
x2=x1(-1)+v2
```
(2)

- When $\mathbf{A}$ is not triangular, there are (at least) two ways to generate (1) in Gretl:

$$\mathbf{A} = \begin{pmatrix} .4 & .16 \\ .36 & .4 \end{pmatrix}$$

1.) Transform the model into a triangular one:

$$\begin{aligned} \mathbf{x}_t &= \mathbf{A}\mathbf{x}_{t-1} + \mathbf{v}_t \\ \underbrace{\mathbf{P}\mathbf{x}_t}_{\mathbf{x}_t^*} &= \underbrace{\mathbf{P}\mathbf{A}\mathbf{P}^{-1}}_{\mathbf{A}^*}\underbrace{\mathbf{P}\mathbf{x}_{t-1}}_{\mathbf{x}_{t-1}^*} + \underbrace{\mathbf{P}\mathbf{v}_t}_{\mathbf{v}_t^*} \end{aligned}$$

where $\mathbf{P}$ is such that $\mathbf{A}^*$ is lower triangular.

- Simulate $\mathbf{x}_t^*$ using (2), and then generate $\mathbf{x}_t = \mathbf{P}^{-1}\mathbf{x}_t^*$

- In practice, there are infinite possible choices for $\mathbf{P}$. Example:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} p_{11} & p_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -p_{12} \\ 0 & p_{11} \end{pmatrix} \frac{1}{p_{11}} &= \begin{pmatrix} a_{11}^* & 0 \\ a_{21}^* & a_{22}^* \end{pmatrix} \\ (a_{22} - a_{11}) p_{11} p_{12} + a_{12} p_{11}^2 - a_{21} p_{12}^2 &= 0 \end{aligned}$$

$$\Downarrow$$

$$\mathbf{P} = \begin{pmatrix} .6 & .4 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{PAP^{-1}} = \begin{pmatrix} .64 & 0 \\ .6 & .16 \end{pmatrix}$$

```
smpl 1 100000
matrix A2={.4,.16;.36,.4}
matrix P={.6,.4;0,1}
matrix A2s=P*A2*inv(P)
series x1=0
series x2=0
series x1s=0
series x2s=0
smpl 2 100000
x1s=A2s[1,1]*x1s(-1)+P[1,1]*v1+P[1,2]*v2
x2s=A2s[2,1]*x1s(-1)+A2s[2,2]*x2s(-1)+P[2,1]*v1+P[2,2]*v2
matrix Pinv=inv(P)
x1=Pinv[1,1]*x1s+Pinv[1,2]*x2s
x2=Pinv[2,1]*x1s+Pinv[2,2]*x2s
```

2.) Programming with loops

3 Kalman filter

$$\begin{aligned} y_t &= \mu_t + u_t, \\ \mu_t &= \mu_{t-1} + v_t, \end{aligned}$$

where

$$\begin{aligned} u_t &\sim iid\ N(0, \sigma_u^2), \\ v_t &\sim iid\ N(0, \sigma_v^2). \end{aligned}$$

- Kalman filter predictions of y_t:

$$y_t^* = y_{t-1}^* + (1-\delta)(y_{t-1} - y_{t-1}^*)$$

where

$$\begin{aligned}\delta &= \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2 + p}, \\ p &= \frac{-\sigma_v^2 + \sqrt{\sigma_v^4 + 4\sigma_u^2\sigma_v^2}}{2}.\end{aligned}$$