

0.1 Copula calibration algorithms

Algorithm 1. Calibrating Archimedean Copulas

Require: Pseudo-observations from training data: $\mathbf{u}, \mathbf{v} \in [0, 1]^{T^{tr}}$

- 1: Compute $N_C \leftarrow \sum_{i=1}^{T^{tr}} \sum_{j=i+1}^{T^{tr}} \mathbb{1}((u_i - u_j)(v_i - v_j) > 0)$
- 2: Compute $N_D \leftarrow \sum_{i=1}^{T^{tr}} \sum_{j=i+1}^{T^{tr}} \mathbb{1}((u_i - u_j)(v_i - v_j) < 0)$
- 3: Compute Kendall's $\hat{\tau} \leftarrow \frac{2(N_C - N_D)}{n(n-1)}$
- 4: Retrieve $\hat{\theta} \leftarrow f^{-1}(\hat{\tau})$

Ensure: $\hat{\theta}$

Where we leverage the fact that, for Archimedean copulas, Kendall's tau verifies (?; p.280):

$$\tau = f(\theta) := \left(1 + 4 \int_0^1 \frac{\varphi(t; \theta)}{\varphi'(t; \theta)} dt \right)$$

And θ is retrieved from the inversion of this relationship:

<i>Clayton</i>	$\theta = 2\tau(1 - \tau)^{-1}$	Analytical inverse
<i>Gumbel</i>	$\theta = (1 - \tau)^{-1}$	Analytical inverse
<i>Joe</i>	$\theta = f^{-1}(\tau)$	Numerical inverse
<i>N14</i>	$\theta = (1 + \tau)/(2 - 2\tau)$	Analytical inverse

Algorithm 2. Calibrating Gaussian Copula

Require: Pseudo-observations from training data: $\mathbf{u}, \mathbf{v} \in [0, 1]^{T^{tr}}$

- 1: Transform \mathbf{u} and \mathbf{v} into standard normal variates $\mathbf{x} = \Phi^{-1}(\mathbf{u})$ and $\mathbf{y} = \Phi^{-1}(\mathbf{v})$
- 2: Obtain the empirical covariance matrix

$$\hat{\Sigma} := \begin{bmatrix} \hat{\sigma}_x^2 & \hat{\sigma}_{yx} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{bmatrix} = \frac{1}{T_{tr} - 1} \begin{bmatrix} \mathbf{x}^\top \mathbf{x} & \mathbf{x}^\top \mathbf{y} \\ \mathbf{y}^\top \mathbf{x} & \mathbf{y}^\top \mathbf{y} \end{bmatrix}$$

- 3: Set $\hat{\rho} \leftarrow \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x \hat{\sigma}_y}$

Ensure: $\hat{\rho}$

Algorithm 3. Calibrating the Student- t Copula**Require:** Pseudo-observations from training data: $\mathbf{u}, \mathbf{v} \in [0, 1]^{T_{tr}}$

- 1: **for** each $\nu \in \mathcal{V} := [1, 15]$ **do**
- 2: Transform \mathbf{u} and \mathbf{v} into Student- t variates: $\mathbf{x}_\nu = t_\nu^{-1}(\mathbf{u})$; $\mathbf{y}_\nu = t_\nu^{-1}(\mathbf{v})$
- 3: Obtain the empirical covariance matrix

$$\hat{\Sigma}(\nu) := \begin{bmatrix} \hat{\sigma}_x^2(\nu) & \hat{\sigma}_{yx}(\nu) \\ \hat{\sigma}_{xy}(\nu) & \hat{\sigma}_y^2(\nu) \end{bmatrix} = \frac{1}{T_{tr} - 1} \begin{bmatrix} \mathbf{x}_\nu^\top \mathbf{x}_\nu & \mathbf{x}_\nu^\top \mathbf{y}_\nu \\ \mathbf{y}_\nu^\top \mathbf{x}_\nu & \mathbf{y}_\nu^\top \mathbf{y}_\nu \end{bmatrix}$$

- 4: Evalutate the log-likelihood of the t -copula

$$\ell(\nu; \mathbf{u}, \mathbf{v}) := \sum_{t \in T_{tr}} \log c(u_t, v_t; \nu, \hat{\rho}(\nu)) \quad \text{where} \quad \hat{\rho}(\nu) = \frac{\hat{\sigma}_{xy}(\nu)}{\hat{\sigma}_x(\nu) \hat{\sigma}_y(\nu)}$$

- 5: **end for**
- 6: Set $\nu^* \leftarrow \arg \max_{\nu \in \mathcal{V}} \ell(\nu; \mathbf{u}, \mathbf{v})$, and $\hat{\rho}^* \leftarrow \hat{\rho}(\nu^*)$

Ensure: $(\nu^*, \hat{\rho}^*)$

0.2 Copula sampling algorithms

0.2.1 Conditional sampling

Algorithm 4. Conditional sampling**Require:** Conditional copula distribution $C_{V|U}(v|u)$ with calibrated $\hat{\theta}$

- 1: Generate two independent variates $u, w \sim \mathcal{U}(0, 1)$
- 2: Keep the first variate u as is
- 3: Transform the second variate w into v using the inverse of the conditional distribution:

$$v = C_{V|U}^{-1}(w|u)$$

Ensure: The resulting pair (u, v) will follow the desired copula distribution

We applied this method to sample from the Clayton copula, whose conditional distribution is given by:

$$C_{V|U}(v|u) = \frac{\partial}{\partial u} (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} = u^{-(1+\theta)} (u^{-\theta} + v^{-\theta} - 1)^{-(1+\theta)/\theta}, \quad (1)$$

and setting (1) equal to the fixed probability q delivers the Clayton copula q -quantile curve. Solving for v delivers

$$v = C_{V|U}^{-1}(q|u) = \left[1 + u^{-\theta}(q^{-\theta/(1+\theta)} - 1)\right]^{-1/\theta}.$$

For more details, check p.275 in ?.

0.2.2 Theorem 4.3.7 in ?

From Theorem 4.3.7 in ? (p.129), it can be shown that Algorithm 5. generates random variates (u, v) whose joint distribution function is an Archimedean copula C with generator φ :

Algorithm 5. Sampling from Archimedean Copulas using Theorem 4.3.7 in ?

Require: Generator function φ of the desired Archimedean C -copula and its calibrated $\hat{\theta}$

- 1: Generate two independent variates $s, t \sim \mathcal{U}(0, 1)$
- 2: Set $w \leftarrow K_C^{(-1)}(t; \hat{\theta})$, where $K_C(w; \hat{\theta}) = t - \frac{\varphi(w; \hat{\theta})}{\varphi'(w^+; \hat{\theta})}$
- 3: Set $u \leftarrow \varphi^{[-1]}(s\varphi(w; \hat{\theta}))$ and $v \leftarrow \varphi^{[-1]}((1-s)\varphi(w; \hat{\theta}))$

Ensure: The pair (u, v) will follow the desired copula distribution

Note that K_C is given by Theorem 4.3.4. (p.127) and it denotes the C -measure of the set $\{(u, v) \in [0, 1]^2 \mid C(u, v) \leq w\}$

$$K_C(w; \theta) := t - \frac{\varphi(w; \theta)}{\varphi'(w^+; \theta)},$$

and $\varphi'(w^+)$ denotes the right-sided derivative of the generator.

In our case, we use this algorithm to sample from Gumbel, Joe and N14. Their C -measures are given below. Note that there is no analytical solution for their inverse, so we have to resort to numerical inversion.

$$\begin{array}{ll} \text{Gumbel} & K_C(w; \theta) = w \cdot \left[1 - \frac{\log(w)}{\theta}\right] \\ \text{Joe} & K_C(w; \theta) = w - \frac{1}{\theta} \cdot \frac{\log[1 - (1-w)^\theta] \cdot [1 - (1-w)^\theta]}{(1-w)^{\theta-1}} \\ \text{N14} & K_C(w; \theta) = -w \cdot (-2 + w^{1/\theta}) \end{array}$$

Algorithm 6. Sampling from Gaussian Copula**Require:** Calibrated $\hat{\Sigma}$

- 1: Generate correlated Gaussian pairs $(x, y) \sim \mathcal{N}(\mathbf{0}, \hat{\Sigma})$
- 2: Transform the Gaussian pairs into uniform variates $u = \Phi(x)$ and $v = \Phi(y)$.

Ensure: The pair (u, v) follows the Gaussian copula distribution.**Algorithm 7.** Sampling from Student- t Copula**Require:** Calibrated ν^* and $\hat{\Sigma}(\nu^*)$

- 1: Sample from a bivariate normal: $(x_1, x_2) \sim \mathcal{N}(\mathbf{0}, \hat{\Sigma}(\nu^*))$
- 2: Sample from a chi-square distribution with ν^* degrees of freedom: $\chi \sim \chi_{\nu^*}^2$
- 3: Compute the Student- t variates: $y_1 = x_1 / \sqrt{\chi / \nu^*}$ and $y_2 = x_2 / \sqrt{\chi / \nu^*}$
- 4: Transform the Student- t variates into uniform variates $u = t_{\nu}(y_1)$ and $v = t_{\nu}(y_2)$

Ensure: The pair (u, v) follows the Student- t copula distribution

TABLE 1: Performance Metrics by Copula

Copula	Total Return (%)	Ann. Return (%)	Ann. Vol. (%)	Sharpe Ratio	Sortino Ratio	Calmar Ratio	Max DD (%)	Win Rate (%)	Profit Factor	VaR-95 (%)
Gumbel	72.39	16.07	4.56	3.53	4.32	7.41	2.17	79.56	2.26	-0.35
Frank	63.02	13.99	3.91	3.58	4.45	10.28	1.36	82.64	2.42	-0.30
Clayton	70.37	15.62	4.11	3.80	4.96	10.27	1.52	82.11	2.52	-0.31
Joe	77.57	17.22	4.54	3.79	5.47	12.01	1.43	77.00	2.28	-0.35
N14	77.82	17.28	4.35	3.97	5.76	11.26	1.53	79.56	2.50	-0.32
Gaussian	62.70	13.92	4.43	3.14	4.10	8.13	1.71	79.30	2.07	-0.35
Student-t	66.49	14.76	4.49	3.29	4.05	4.88	3.02	78.77	2.14	-0.34