0.1 Copula calibration algorithms

Algorithm 1. Calibrating Archimedean Copulas

Require: Pseudo-observations from training data: $\mathbf{u}, \mathbf{v} \in [0,1]^{T^{tr}}$

1: Compute $N_C \leftarrow \sum_{i=1}^{T^{tr}} \sum_{j=i+1}^{T^{tr}} \mathbb{1}((u_i - u_j)(v_i - v_j) > 0)$

2: Compute $N_D \leftarrow \sum_{i=1}^{T^{tr}} \sum_{j=i+1}^{T^{tr}} \mathbb{1}((u_i - u_j)(v_i - v_j) < 0)$

3: Compute Kendall's $\hat{\tau} \leftarrow \dfrac{2(N_C - N_D)}{n(n-1)}$

4: Retrieve $\hat{\theta} \leftarrow f^{-1}(\hat{\tau})$

Ensure: $\hat{\theta}$

Where we leverage the fact that, for Archimedean copulas, Kendall's tau verifies (**?**; p.280):

$$\tau = f(\theta) := \left(1 + 4\int_0^1 \frac{\varphi(t;\theta)}{\varphi'(t;\theta)} dt\right)$$

And θ is retrieved from the inversion of this relationship:

Clayton	$\theta = 2\tau\,(1-\tau)^{-1}$	Analytical inverse
Gumbel	$\theta = (1-\tau)^{-1}$	Analytical inverse
Joe	$\theta = f^{-1}(\tau)$	Numerical inverse
N14	$\theta = (1+\tau)/(2-2\tau)$	Analytical inverse

Algorithm 2. Calibrating Gaussian Copula

Require: Pseudo-observations from training data: $\mathbf{u}, \mathbf{v} \in [0,1]^{T^{tr}}$

1: Transform $\mathbf{u}$ and $\mathbf{v}$ into standard normal variates $\mathbf{x} = \Phi^{-1}(\mathbf{u})$ and $\mathbf{y} = \Phi^{-1}(\mathbf{v})$

2: Obtain the empirical covariance matrix

$$\widehat{\mathbf{\Sigma}} := \begin{bmatrix} \hat{\sigma}_x^2 & \hat{\sigma}_{yx} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{bmatrix} = \frac{1}{T_{tr}-1} \begin{bmatrix} \mathbf{x}^\top \mathbf{x} & \mathbf{x}^\top \mathbf{y} \\ \mathbf{y}^\top \mathbf{x} & \mathbf{y}^\top \mathbf{y} \end{bmatrix}$$

3: Set $\hat{\rho} \leftarrow \dfrac{\hat{\sigma}_{xy}}{\hat{\sigma}_x \hat{\sigma}_y}$

Ensure: $\hat{\rho}$

Algorithm 3. Calibrating the Student-t Copula

Require: Pseudo-observations from training data: $\mathbf{u}, \mathbf{v} \in [0,1]^{T^{tr}}$

1: **for** each $\nu \in \mathcal{V} := [1, 15]$ **do**

2: Transform $\mathbf{u}$ and $\mathbf{v}$ into Student-t variates: $\mathbf{x}_\nu = t_\nu^{-1}(\mathbf{u})$; $\mathbf{y}_\nu = t_\nu^{-1}(\mathbf{v})$

3: Obtain the empirical covariance matrix

$$\widehat{\mathbf{\Sigma}}(\nu) := \begin{bmatrix} \hat{\sigma}_x^2(\nu) & \hat{\sigma}_{yx}(\nu) \\ \hat{\sigma}_{xy}(\nu) & \hat{\sigma}_y^2(\nu) \end{bmatrix} = \frac{1}{T_{tr} - 1} \begin{bmatrix} \mathbf{x}_\nu^\top \mathbf{x}_\nu & \mathbf{x}_\nu^\top \mathbf{y}_\nu \\ \mathbf{y}_\nu^\top \mathbf{x}_\nu & \mathbf{y}_\nu^\top \mathbf{y}_\nu \end{bmatrix}$$

4: Evalutate the log-likelihood of the t-copula

$$\ell(\nu; \mathbf{u}, \mathbf{v}) := \sum_{t \in \mathcal{T}_{tr}} \log c(u_t, v_t; \nu, \hat{\rho}(\nu)) \quad \text{where} \quad \hat{\rho}(\nu) = \frac{\hat{\sigma}_{xy}(\nu)}{\hat{\sigma}_x(\nu)\hat{\sigma}_y(\nu)}$$

5: **end for**

6: Set $\nu^\star \leftarrow \arg\max_{\nu \in \mathcal{V}} \ell(\nu; \mathbf{u}, \mathbf{v})$, and $\hat{\rho}^\star \leftarrow \hat{\rho}(\nu^\star)$

Ensure: $(\nu^\star, \hat{\rho}^\star)$

0.2 Copula sampling algorithms

0.2.1 Conditional sampling

Algorithm 4. Conditional sampling

Require: Conditional copula distribution $C_{V|U}(v|u)$ with calibrated $\hat{\theta}$

1: Generate two independent variates $u, w \sim \mathcal{U}(0,1)$

2: Keep the first variate u as is

3: Transform the second variate w into v using the inverse of the conditional distribution:

$$v = C_{V|U}^{-1}(w|u)$$

Ensure: The resulting pair (u, v) will follow the desired copula distribution

We applied this method to sample from the Clayton copula, whose conditional distribution is given by:

$$C_{V|U}(v|u) = \frac{\partial}{\partial u}(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} = u^{-(1+\theta)}(u^{-\theta} + v^{-\theta} - 1)^{-(1+\theta)/\theta}, \tag{1}$$

and setting (1) equal to the fixed probability q delivers the Clayton copula q-quantile curve. Solving for v delivers

$$v = C^{-1}_{V|U}(q|u) = \left[1 + u^{-\theta}(q^{-\theta/(1+\theta)} - 1)\right]^{-1/\theta}.$$

For more details, check p.275 in **?**.

0.2.2 Theorem 4.3.7 in ?

From Theorem 4.3.7 in **?** (p.129), it can be shown that Algorithm 5. generates random variates (u, v) whose joint distribution function is an Archimedean copula C with generator φ:

Algorithm 5. Sampling from Archimedean Copulas using Theorem 4.3.7 in **?**

Require: Generator function φ of the desired Archimedean C-copula and its calibrated $\hat{\theta}$

1: Generate two independent variates $s, t \sim \mathcal{U}(0, 1)$

2: Set $w \leftarrow K_C^{(-1)}(t; \hat{\theta})$, where $K_C(w; \hat{\theta}) = t - \dfrac{\varphi(w; \hat{\theta})}{\varphi'(w^+; \hat{\theta})}$

3: Set $u \leftarrow \varphi^{[-1]}(s\varphi(w; \hat{\theta}))$ and $v \leftarrow \varphi^{[-1]}((1 - s)\varphi(w; \hat{\theta}))$

Ensure: The pair (u, v) will follow the desired copula distribution

Note that K_C is given by Theorem 4.3.4. (p.127) and it denotes the C-measure of the set $\{(u, v) \in [0, 1]^2 \mid C(u, v) \leq w\}$

$$K_C(w; \theta) := t - \frac{\varphi(w; \theta)}{\varphi'(w^+; \theta)},$$

and $\varphi'(w^+)$ denotes the right-sided derivative of the generator.

In our case, we use this algorithm to sample from Gumbel, Joe and N14. Their C-measures are given below. Note that there is no analytical solution for their inverse, so we have to resort to numerical inversion.

$$\begin{aligned}
&\textit{Gumbel} \qquad K_C(w; \theta) = w \cdot \left[1 - \frac{\log(w)}{\theta}\right] \\
&\textit{Joe} \qquad K_C(w; \theta) = w - \frac{1}{\theta} \cdot \frac{\log[1 - (1 - w)^{\theta}] \cdot [1 - (1 - w)^{\theta}]}{(1 - w)^{\theta - 1}} \\
&\textit{N14} \qquad K_C(w; \theta) = -w \cdot (-2 + w^{1/\theta})
\end{aligned}$$

Algorithm 6. Sampling from Gaussian Copula

Require: Calibrated $\hat{\mathbf{\Sigma}}$

1: Generate correlated Gaussian pairs $(x, y) \sim \mathcal{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$

2: Transform the Gaussian pairs into uniform variates $u = \Phi(x)$ and $v = \Phi(y)$.

Ensure: The pair (u, v) follows the Gaussian copula distribution.

Algorithm 7. Sampling from Student-t Copula

Require: Calibrated $\nu^\star$ and $\hat{\mathbf{\Sigma}}(\nu^\star)$

1: Sample from a bivariate normal: $(x_1, x_2) \sim \mathcal{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}}(\nu^\star))$

2: Sample from a chi-square distribution with $\nu^\star$ degrees of freedom: $\chi \sim \chi^2_{\nu^\star}$

3: Compute the Student-t variates: $y_1 = x_1/\sqrt{\chi/\nu^\star}$ and $y_2 = x_2/\sqrt{\chi/\nu^\star}$

4: Transform the Student-t variates into uniform variates $u = t_\nu(y_1)$ and $v = t_\nu(y_2)$

Ensure: The pair (u, v) follows the Student-t copula distribution