Pairs-Trading a Sparse Synthetic Control

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Abstract

This paper defines a novel approach to pairs trading by borrowing the concept of synthetic control from the treatment literature. In this paper we select a stock (termed "*target*") and construct its replica as a sparse linear combination of other assets (termed "*synthetic*"). Then, we perform pairs trading on the target vs. synthetic assets; for this purpose, we (nonlinearly) model their joint dependence via a Student-t copula and then construct mispricing indices from the implied conditional densities. Finally, we feed the dynamics of our miscpricing indices to a reinforcement learning agent. Our findings show that our RL agent succesfully implement statistical arbitrage based on our mispricing signals with a high net profitability out of sample.

JEL Codes: C14, C32, C58, C61, G12, G14

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1. Introduction

1.1 Relative pricing

Asset pricing can be viewed in absolute and relative terms.

- Absolute pricing
  - Absolute pricing values securities from fundamentals such as discounted future cash flow. This is a notoriously difficult process with a wide margin for error.
  - Articles by Bakshi and Chen (1997) and Lee et al. (1997), for example, are heroic attempts to build quantitative value-investing models.
- Relative pricing
  - Relative pricing is only slightly easier.
  - Relative pricing means that two securities that are close substitutes for each other should sell for the same price-it does not say what that price will be.
  - Thus, relative pricing allows for bubbles in the economy, but not necessarily arbitrage or profitable speculation.
  - The Law of One Price [LOP] and a "near-LOP" are applicable to relative pricing-even if that price is wrong. Ingersoll (1987) defines the LOP as the "proposition ... that two investments with the same payoff in every state of nature must have the same current value." In other words, two securities with the same prices in all states of the world should sell for the same amount.
  - Chen and Knez (1995) extend this to argue that "closely integrated markets should assign to similar payoffs prices that are close." They argue that two securities with similar, but not necessarily, matching payoffs across states should have similar prices. This is of course a weaker condition and subject to bounds on prices for unusual states; however, it allows the examination of "near-efficient" economies, or in Chen and Knez' case, near integrated markets. Notice that this theory corresponds to the desire to find two stocks whose prices move together as long as we can define states of nature as the time series of observed historical trading days.

We use an algorithm to choose pairs based on the criterion that they have had the same or nearly the same state prices historically. We then trade pairs whose prices closely match in historical state-space, because the LOP suggests that in an efficient market their prices should be nearly identical. In this framework, the current study can be viewed as a test of the LOP and near-LOP in the U.S. equity markets, under certain stationarity conditions. We are effectively testing the integration of very local markets-the markets for specific individual securities. This is similar in spirit to Bossaerts' (1988) test of co-integration of security prices at the portfolio level. We further conjecture that the marginal profits to be had from risk arbitrage of these temporary deviations is crucial to the maintenance of first-order efficiency. We could not have the first effect without the second.

2. Methodology

- We implement pairs trading in two stages. We form "*pairs*" over a 12-month formation period (train) and trade them in the next 6-month period (test). [Gatev et al (2006)]
- **Formation period**
  - In each pairs formation period, we screen out stocks from CRSP daily files that have one or more days with no trade. This serves to identify relatively liquid stocks as well as to facilitate pairs formation [Gatev et al (2006)]
  - Next we construct a cumulative total returns index for each stock over the formation period [Gatev et al (2006)]
  - We then choose a matching partner for the target stock by finding the linear combination of securities that minimize the sum of squared deviations between the two normalized price series. [Unrestricted]
    - We use this approach because it is consistent with the notion of cointegration. In fact, what we are doing is tracking the target asset with a synthetic asset. By construction, the tracking error is centered at 0 and mean reverting. In other words, there is a cointegration relationship between the target and synthetic assets.
  - In addition to "*unrestricted*" pairs, we will also present results by sector, where we restrict the synthetic asset to contain stocks belonging to the same industry as the target stock. The Industry categories employed are broad, as defined by Standard and Poors: Utilities, Transportation, Financial and Industrials. Each stock is assigned to one of these four groups based on the stock's SIC code. The minimum distance criterion is then used to match stocks within each of the groups. [Gatev et al (2006)]

- **Trading period**
  - Once we have paired up all liquid stocks in the formation period, we study the top 5 and 20 pairs with the smallest historical distance measure (in our case, with the *smallest historical tracking error*), in addition to the 20 pairs after the top 100 (pairs 101-120).
    - This last set is valuable because most ot the top pairs share certain characteristics
  - On the day following the last day of the pairs formation period, we begin to trade according to a prespecified rule.
  - We base our rules for opening and closing positions on a standard deviation metric. We open a position in a pair when prices diverge by more than 2 historical standard deviations, as esitmated during the pairs formation period. We unwind the position at the next crossing of the prices. If prices do not cross before the end of the trading interval, gais or losses are calculated at the end of the last trading day of the trading interval. If a stock in a pair is delisted from CRSP, we close the position in that pair, using the delising return, or the last available price.
    - A potential problem arises if inaccurate and stale prices exaggerate the excess returns and bias the estimated return of a long position in a plummeting stock. To address this potential concern, we reestimate our results under the extreme assumption that only a long stock experiences a -100% return when it is delisted. This zero-price extreme includes, among other things, the possibility that of nontrading due to the lack of liquidity. Because selective loss on the long position always harms the pair profit, this extreme assumption biases the results against profitability.
  - We report the payoff by going one dollar shot in the higher-priced stock and one dollar long in the lower-priced stock
- **Excess return computation**
  - Pairs that open and converge during the trading interval will have positive cash flows. Because pairs can reopen after initial convergence, they can have multiple positive cash flows during the trading interval.
  - Pairs that open but do not converge will only have cash flows on the last day of the trading interval when all positions are closed out.
  - Therefore, the payoffs to a pairs trading strategies are a set of positive cash flows that are randomly distributed throughout the trading period, and a set of cash flows at the end of the trading interval that can be either positive or negative.

– For each pair we can have multiple cash flows during the trading interval, or we may have none in the case when prices never diverge by more than 2 historical standard deviations during the trading interval

– Because the trading gains and losses are computed over long-short positions of one dollar, the payoffs have the interpretation of excess returns.

– The excess return on a pair during a trading interval is computed as the reinvested payoffs during the trading interval.

  * This is a conservative approach to computing the excess return, because it implicitly assumes that all cash earns zero interest rate when not invested in an open pair. Because any cash flow during the trading interval is positive by construction, it ignores the fact that these cash flows are received early and understates the computed excess returns.

– In particular, the long and short portfolio positions are marked-to-market daily.

– The daily returns on the long and short positions are calculated as value-weighted returns

$$r_{P,t} = \frac{\sum_{i \in P} w_{i,t} r_{i,t}}{\sum_{i \in P} w_{i,t}} \qquad w_{i,t} = w_{i,t-1}(1 + r_{i,t-1}) = (1 + r_{i,1}) \cdots (1 + r_{i,t-1})$$

where r refines returns and w defines weights, and the daily returns are compounded to obtain monthly returns. This has the simple interpretation of a buy-and-hold strategy.

A. Appendix

A.1 Cointegration Meets Synthetic Controls: A Formal Equivalence

In this appendix section, we develop a formal argument showing how, under some stringent assumptions, our notion of *synthetic control* can be viewed as a special case of *cointegration*. This connection underlies the intuition that, when one normalizes the first variable of a cointegrated system to 1, the remaining cointegration relationships effectively produce the *synthetic* version of the first variable when the cointegration vector satisfies a specific restriction.

Let $\{y_{i,t}\}_{t=1}^{T}$ denote the time series sequence of log-prices for each asset $i \in \{1, \dots, N\}$. Throughout, we assume each $y_{i,t}$ is an $I(1)$ process (integrated of order 1). Formally, an $I(1)$ process is one that becomes *stationary* (and typically ergodic) upon differencing once: $\Delta y_{i,t} := y_{i,t} - y_{i,t-1} \sim I(0)$. The notion of cointegration, due to Engle and Granger, is central in analyzing potentially long-run equilibria among these variables.

Definition 1 (Engle and Granger (1987)). *The components of $\mathbf{y}_t := [y_{1t}, ..., y_{Nt}]$ are said to be cointegrated of order d, b, denoted $\mathbf{y}_t \sim CI(d, b)$, if (a) all components of $\mathbf{y}_t$ are $I(d)$ and (b) a vector $\boldsymbol{\beta} \neq 0$ exists so that $\boldsymbol{\beta}'\mathbf{y}_t \sim I(d-b)$, $b > 0$. The vector $\boldsymbol{\beta}$ is called the cointegrating vector.*

Definition 2 (Synthetic Control). *Let $\{y_1, y_2, \dots, y_n\}$ be a collection of random variables, where y_1 is the "target" variable and $\mathbf{y}_{2:n} = (y_2, \dots, y_n)$ constitute the "donor pool". A* synthetic control *for y_1 is constructed by choosing weights $\mathbf{w}$ in the $(n-1)$-dimensional space $\mathcal{W} := \{\mathbf{w} \in \mathbb{R}_+^{n-1} : \sum_{j=2}^{n} w_j = 1\}$ that satisfy $\mathbf{w} = \arg\min_{w \in \mathcal{W}} \sum_{t=1}^{T}(y_{1,t} - \mathbf{w}'\mathbf{y}_{2:n,t})^2$.*

Given that cointegration relationships prevail up to scale and sign changes, then, under suitable conditions on the cointegration vector, there exists a nontrivial constant κ that allows us to reinterpret the cointegration relationship as one of a synthetic control. In particular,

Proposition 1. *For a cointegrated vector $\mathbf{y}$ with rank r, if (at least) one of the cointegrating vectors $\boldsymbol{\beta}$ satisfies the restriction $\mathcal{R} = \{\mathbf{1}'\boldsymbol{\beta} = 0\}$, then we can scale the cointegration vector by $\kappa = 1/\beta_i$ such that $\kappa\boldsymbol{\beta}'\mathbf{y}$ is stationary and describes a "synthetic control" relationship (as per Definition 2) between y_i and $\mathbf{y}_{-i}$.*

Proof. The proof is straightforward. For a cointegration vector $\boldsymbol{\beta}$ where $\mathcal{R}$ holds, we have that $\mathbf{1}'\boldsymbol{\beta} = \sum_{j=1}^{n} \beta_j = 0$, which trivially implies $\beta_i = -\sum_{j \neq i} \beta_j$. For the sake of the proof, set that β_i to the first component (β_1). Then $\beta_1 = -\sum_{j=2}^{n} \beta_j$ and $\kappa = (\beta_1)^{-1} = -(\sum_{j=2}^{n} \beta_j)^{-1}$

$$\kappa\boldsymbol{\beta}'\mathbf{y} = \frac{1}{\beta_1}[\beta_1 \quad \boldsymbol{\beta}_{2:n}]\mathbf{y}_t = \begin{bmatrix} 1 & \frac{-\boldsymbol{\beta}_{2:n}'}{\sum_{j=2}^{n} \beta_j} \end{bmatrix} \begin{bmatrix} y_1 \\ \mathbf{y}_{2:n} \end{bmatrix} = y_1 - \frac{\beta_2}{\sum_{j=2}^{n} \beta_j} y_2 - \dots - \frac{\beta_n}{\sum_{j=2}^{n} \beta_j} y_n \sim I(0)$$

describes a stationary cointegration relationship in $\mathbf{y}$, and since

$$\begin{aligned} y_1 &= \frac{\beta_2}{\sum_{j=2}^{n} \beta_j} y_2 + \cdots + \frac{\beta_n}{\sum_{j=2}^{n} \beta_j} y_n + \epsilon \\ &= \mathbf{w}' \mathbf{y}_{2:n} + \epsilon \end{aligned}$$

with $\epsilon \sim I(0)$ and $\mathbf{w} := \left(\frac{\beta_2}{\sum_{j=2}^{n} \beta_j}, ..., \frac{\beta_n}{\sum_{j=2}^{n} \beta_j} \right)' \in \mathcal{W}$, then this relationship is endowed with a synthetic control structure. A similar reasoning applies to any other β_i different from β_1. $\square$