Mathematical Statistics I

Chapter 6: Distributions Derived from the Normal Distribution

Jesse Wheeler

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1 χ^2 distributions

Introduction

- This material comes primarily from Rice (2007, Chapter 6).
- Here, we introduce several important distributions that arise from transformations applied to normal distributions.
- Many of these distributions form the basis of traditional statistical inference procedures that are taught in introductory statistics courses.
- They are very useful in practice due to the central limit theorem: with enough observations, the limiting behavior of nearly all distributions is normal, so distributions that come from the normal distribution arise in practice as well.

χ^2_ν Distribution

- The first distribution we will consider is the χ^2_1 (Chi-square with 1 degree of freedom).

Definition: χ^2_1 distribution
If Z is a standard normal random variable, then $X = Z^2$ is called the chi-square distribution with 1 degree of freedom.

- We typically use the notation $X \sim \chi^2_1$ (in LaTeX: `\chi`).

The pdf of χ^2_1
Let X follow a χ^2_1 distribution. Then, the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}.$$

Proof. There are a few ways to show this is the case, and was one of the early examples we saw in Chapter 2. For practice, we repeat this example here.

- By definition, X has the same distribution of Z^2, where Z is a standard normal.

- Recall the standard normal density is:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- Using the CDF method, we write

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(Z^2 \leq x) \\ &= P(-\sqrt{x} \leq Z \leq \sqrt{x}) \\ &= P(Z \leq \sqrt{x}) - P(Z \leq -\sqrt{x}) \\ &= \Phi(\sqrt{x}) - \Phi(-\sqrt{x}), \end{aligned}$$

- Where $\Phi(z)$ is the cdf of Z.
- Taking the derivative of both sides of the equation, the chain rule gives us

$$\begin{aligned} f_X(x) &= \frac{1}{2} x^{-1/2} \phi(\sqrt{x}) + \frac{1}{2} x^{-1/2} \phi(-\sqrt{x}) \\ &= x^{-1/2} \phi(\sqrt{x}), \end{aligned}$$

- where the last step is a result of the symmetry of $\phi(x)$, noting $\phi(-x) = \phi(x)$ for all $x \in \mathbb{R}$.
- Thus, replacing $\phi(\sqrt{x})$ with the definition,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}$$

□

- In Chapter 2, we previously noted that that $f_X(x)$ is an example of a Gamma distribution.
- Specifically, the *kernel* of the Gamma density is x raised to some power, and e raised to some multiple of x:

$$f_{\text{Gamma}}(x) \propto x^{\alpha-1} e^{-\lambda x}.$$

- Thus, ignoring the constant for a moment, if $\alpha = 1/2$, $\lambda = 1/2$, then the pdf of $X \sim \chi^2_1$ is just this Gamma density:

$$f_X(x) \propto x^{-1/2} e^{-x/2} = x^{\alpha-1} e^{-\lambda x}.$$

- Since both functions are proper probability density functions, they have to integrate to one, so the normalizing constant *must* be the same.
- This is also easily verified. The normalizing constant of the Gamma distribution is $\lambda^\alpha / \Gamma(\alpha)$.
- With our specific values of $\lambda = \alpha = 1/2$, and recalling that $\Gamma(1/2) = \sqrt{\pi}$,

$$\frac{1}{\sqrt{2\pi}} = \frac{(1/2)^{(1/2)}}{\Gamma(1/2)} = \frac{\lambda^\alpha}{\Gamma(\alpha)}$$

MGF of χ_1^2
We previously derived the MGF of a Gamma(α, λ) distribution: $M(t) = \big(\lambda/(\lambda - t)\big)^{\alpha}$. Thus, the MGF of a Chi-square(1) distribution is

$$M(t) = (1 - 2t)^{-1/2}, \quad t < 1/2.$$

Definition
If $U_1, U_2, \ldots, U_n$ are n independent χ_1^2 random variables, then

$$V = U_1 + U_2 + \ldots + U_n$$

then the distribution of V is called the Chi-square distribution with n degrees of freedom, denoted χ_n^2.

- There are a few different ways of deriving the pdf of a χ_n^2 random variable. Here, we will use the MGF uniqueness theorem.
- Let $M_i(t)$ denote the MGF of U_i, where $U_i \sim \chi_1^2$. Then, due to independence,

$$M_V(t) = M_{\sum_i U_i}(t) = \prod_{i=1}^{n} M_i(t) = \big(M_t(t)\big)^n = (1 - 2t)^{-n/2}$$

- Compare this to the Gamma MGF: $M(t) = \big(\lambda/(\lambda - t)\big)^{\alpha}$. Then, setting $\lambda = 1/2$, $\alpha = n/2$, we see that V has a Gamma$(n/2, 1/2)$ distribution.
- Thus, the pdf of V is given by:

$$f_V(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}.$$

- The expected value and variance of the χ_n^2 distribution can easily be found then by using the fact that it is a special case of a Gamma distribution.

2 The t and F distributions

The Student's t distributions

The Student's t distribution
If $Z \sim N(0, 1)$ and $U \sim \chi_n^2$, and Z and U are independent, then the distribution of T, where

$$T = \frac{Z}{\sqrt{U/n}},$$

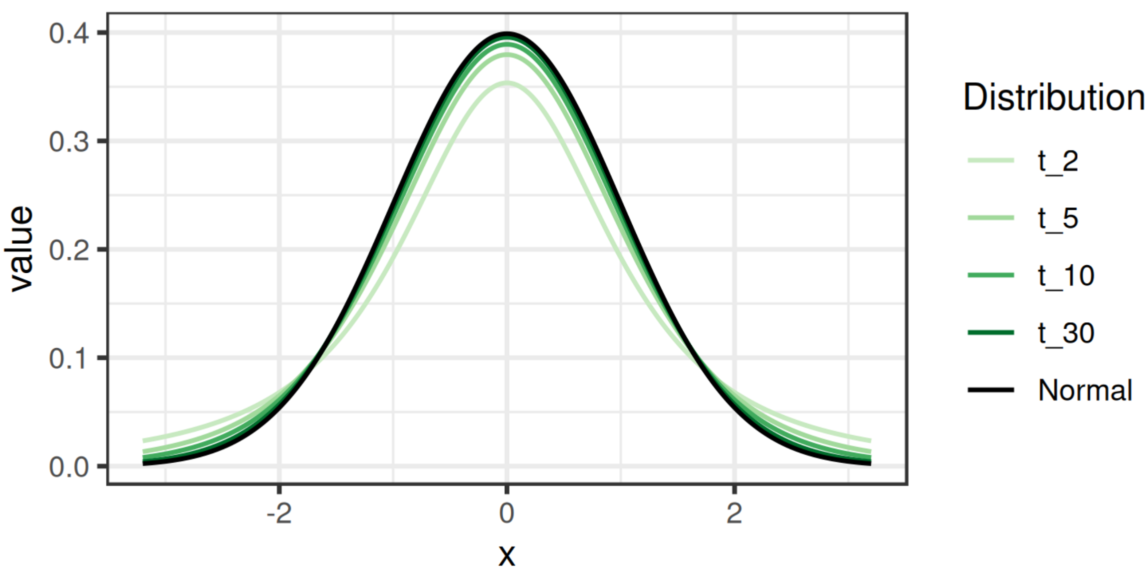
is called the Student's t distribution (or simply the t distribution) with n degrees of freedom, which is often denoted t_n

- Students often forget the make sure that Z and U in the definition of the t distribution are independent.
- The t distribution is the distribution used to perform the famed "t-test".

The density of the t_n ***distribution***
The pdf of the t distribution with n degrees of freedom is:

$$f(t) = \frac{\Gamma\big((n+1)/2\big)}{\sqrt{n\pi}\Gamma(n/2)} \Big(1 + \frac{t^2}{n}\Big)^{-(n+1)/2}$$

- The derivation of the pdf of a t distribution is a good practice exercise.
- Recall it is defined as the ratio of two independent random variables; in Chapter 3, we derived a formula for computing densities of random variables of this form.
- Note that $f(t) = f(-t)$, and so f is symmetric about zero.
- It also has a bell-curve shape similar to a normal distribution.
- You can see as $n \to \infty$, the t_n distribution converges to the standard normal (e.g., use Slutsky's theorem, good practice).



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References

Rice JA (2007). *Mathematical statistics and data analysis*, volume 371. 3 edition. Thomson/Brooks/Cole Belmont, CA. 1