Mathematical Statistics I

Chapter 6: Distributions Derived from the Normal Distribution

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1. χ^2 distributions

χ^2 **distributions**

Introduction

- This material comes primarily from Rice (2007, Chapter 6).

- Here, we introduce several important distributions that arise from transformations applied to normal distributions.

- Many of these distributions form the basis of traditional statistical inference procedures that are taught in introductory statistics courses.

- They are very useful in practice due to the central limit theorem: with enough observations, the limiting behavior of nearly all distributions is normal, so distributions that come from the normal distribution arise in practice as well.

χ^2_ν Distribution

- The first distribution we will consider is the χ^2_1 (Chi-square with 1 degree of freedom).

Definition: χ^2_1 distribution

If Z is a standard normal random variable, then $X = Z^2$ is called the chi-square distribution with 1 degree of freedom.

- We typically use the notation $X \sim \chi^2_1$ (in LaTeX: \chi).

χ^2_ν Distribution II

The pdf of χ^2_1

Let X follow a χ^2_1 distribution. Then, the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}.$$

χ^2_ν Distribution III

- In Chapter 2, we previously noted that that $f_X(x)$ is an example of a Gamma distribution.

- Specifically, the *kernel* of the Gamma density is x raised to some power, and e raised to some multiple of x:

$$f_{\text{Gamma}}(x) \propto x^{\alpha-1}e^{-\lambda x}.$$

- Thus, ignoring the constant for a moment, if $\alpha = 1/2$, $\lambda = 1/2$, then the pdf of $X \sim \chi^2_1$ is just this Gamma density:

$$f_X(x) \propto x^{-1/2}e^{-x/2} = x^{\alpha-1}e^{-\lambda x}.$$

- Since both functions are proper probability density functions, they have to integrate to one, so the normalizing constant *must* be the same.

χ^2_ν Distribution IV

- This is also easily verified. The normalizing constant of the Gamma distribution is $\lambda^\alpha/\Gamma(\alpha)$.

- With our specific values of $\lambda = \alpha = 1/2$, and recalling that $\Gamma(1/2) = \sqrt{\pi}$,

$$\frac{1}{\sqrt{2\pi}} = \frac{(1/2)^{(1/2)}}{\Gamma(1/2)} = \frac{\lambda^\alpha}{\Gamma(\alpha)}$$

Rice JA (2007). *Mathematical statistics and data analysis*, volume 371. 3 edition. Thomson/Brooks/Cole Belmont, CA.

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