Mathematical Statistics I

Chapter 6: Distributions Derived from the Normal Distribution

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Introduction

- This material comes primarily from Rice (2007, Chapter 6).
- Here, we introduce several important distributions that arise from transformations applied to normal distributions.
- Many of these distributions form the basis of traditional statistical inference procedures that are taught in introductory statistics courses.
- They are very useful in practice due to the central limit theorem: with enough observations, the limiting behavior of nearly all distributions is normal, so distributions that come from the normal distribution arise in practice as well.

χ^2_ν Distribution

- The first distribution we will consider is the χ^2_1 (Chi-square with 1 degree of freedom).

Definition: χ^2_1 distribution
If Z is a standard normal random variable, then $X = Z^2$ is called the chi-square distribution with 1 degree of freedom.

- We typically use the notation $X \sim \chi^2_1$ (in LaTeX: `\chi`).

The pdf of χ^2_1
Let X follow a χ^2_1 distribution. Then, the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}.$$

Proof. There are a few ways to show this is the case, and was one of the early examples we saw in Chapter 2. For practice, we repeat this example here.

- By definition, X has the same distribution of Z^2, where Z is a standard normal.

- Recall the standard normal density is:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- Using the CDF method, we write

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(Z^2 \leq x) \\
&= P(-\sqrt{x} \leq Z \leq \sqrt{x}) \\
&= P(Z \leq \sqrt{x}) - P(Z \leq -\sqrt{x}) \\
&= \Phi(\sqrt{x}) - \Phi(-\sqrt{x}),
\end{aligned}$$

- Where $\Phi(z)$ is the cdf of Z.
- Taking the derivative of both sides of the equation, the chain rule gives us

$$\begin{aligned}
f_X(x) &= \frac{1}{2} x^{-1/2} \phi(\sqrt{x}) + \frac{1}{2} x^{-1/2} \phi(-\sqrt{x}) \\
&= x^{-1/2} \phi(\sqrt{x}),
\end{aligned}$$

- where the last step is a result of the symmetry of $\phi(x)$, noting $\phi(-x) = \phi(x)$ for all $x \in \mathbb{R}$.
- Thus, replacing $\phi(\sqrt{x})$ with the definition,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}$$

$\square$

- In Chapter 2, we previously noted that that $f_X(x)$ is an example of a Gamma distribution.
- Specifically, the *kernel* of the Gamma density is x raised to some power, and e raised to some multiple of x:

$$f_{\text{Gamma}}(x) \propto x^{\alpha-1} e^{-\lambda x}.$$

- Thus, ignoring the constant for a moment, if $\alpha = 1/2$, $\lambda = 1/2$, then the pdf of $X \sim \chi^2_1$ is just this Gamma density:

$$f_X(x) \propto x^{-1/2} e^{-x/2} = x^{\alpha-1} e^{-\lambda x}.$$

- Since both functions are proper probability density functions, they have to integrate to one, so the normalizing constant *must* be the same.
- This is also easily verified. The normalizing constant of the Gamma distribution is $\lambda^\alpha / \Gamma(\alpha)$.
- With our specific values of $\lambda = \alpha = 1/2$, and recalling that $\Gamma(1/2) = \sqrt{\pi}$,

$$\frac{1}{\sqrt{2\pi}} = \frac{(1/2)^{(1/2)}}{\Gamma(1/2)} = \frac{\lambda^\alpha}{\Gamma(\alpha)}$$

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References

Rice JA (2007). *Mathematical statistics and data analysis*, volume 371. 3 edition. Thomson/Brooks/Cole Belmont, CA. 1