Mathematical Statistics I

Chapter 5: Limit theorems

Jesse Wheeler

Outline

Convergence Concepts

Introduction

- This material comes primarily from Rice (2007, Chapter 5), but will be supplemented with material from Casella and Berger (2024, Chapter 5).
- In this chapter, we are interested in the convergence of sequences of random variables.
- In particular, we are interested in the convergence of the sample mean, $\bar{X}_n = (X_1 + X_2 + \ldots + X_n)/n$, as the number of samples n grows.
- Because $\bar{X}_n$ is itself a random variable, we have to carefully define what it means for the convergence of a random variable.
- In this class, we are mainly concerned with three types of convergence.

Introduction II

- Because convergence of random variables is a tricky topic, we will treat them in varying amounts of detail.

Convergence in Probability

- The first type of convergence is one of the weaker types, and is usually easy(ish) to verify.

Definition: Convergence in Probability

A sequence of random variables $X_1, X_2, \ldots$ converges in probability to a random variable X if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} P\big(|X_n - X| \geq \epsilon\big) = 0$$

or, equivalently,

$$\lim_{n\to\infty} P\big(|X_n - X| < \epsilon\big) = 1.$$

Convergence in Probability II

- We often use the shorthand $X_n \xrightarrow{P} X$ to denote "X_n converges in probability to X as n goes to infinity".
- Note that the X_i in the definition above do *not* need to be independent and identically distributed.
- The distribution of X_n changes as the subscript changes, and each of the convergence concepts we will discuss will describe different ways in which the distribution of X_n converges to some limiting distribution as the subscript becomes large.
- A special case is when the limiting random variable X is a constant.

Convergence in Probability III

Example: The (Weak) Law of Large Numbers

Let $X_1, X_2, \ldots$ be iid random variables with $E[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Define $\bar{X}_n = (1/n) \sum_{i=1}^{n} X_i$. Then $\bar{X}_n \xrightarrow{P} \mu$.

Test Section

Test Frame

References and Acknowledgements

Casella G, Berger R (2024). *Statistical inference*. Chapman and Hall/CRC.

Rice JA (2007). *Mathematical statistics and data analysis*, volume 371. 3 edition. Thomson/Brooks/Cole Belmont, CA.

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References and Acknowledgements II

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