Mathematical Statistics I

Chapter 4: Expected Values

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Outline

1. Discrete random variables

2. Continuous random variables

Discrete random variables

Introduction

- This material comes primarily from Rice (2007, Chapter 4), though I'm going to deviate slightly.
- We will cover the ideas of expected value, variance, as well has higher-order moments.
- This includes topics such as conditional expectation, which is one of the fundamental ideas behind many branches of statistics and machine learning.
- For instance, most regression / prediction algorithms are built with the idea of minimizing some conditional expectation.

Expectation: Discrete random variables

- We will begin by defining the expectation for discrete random variables.
- I'm going to deviate from our textbook in this definition.

Expectation: Discrete random variables II

Definition: Expectation of discrete random variables

Let X be a discrete random variable with pmf $p(x)$, which takes values in the space $\mathcal{X}$. The expected value of $g(X)$ is

$$E\big(g(X)\big) = \sum_{x \in \mathcal{X}} g(x)\, p(x).$$

In particular, for $g(x) = x$, we have

$$E(X) = \sum_{x \in \mathcal{X}} x\, p(x).$$

Expectation: Discrete random variables III

- This is not the most mathematically precise definition of expectation, but a more complete treatment of the topic is outside the scope of this course (See Resnick, 2019).
- The definition is only applicable if the sum is finite.
- The concept of the expected value parallels the notion of a *weighted average*.
- That is, we weight each possibility $x \in \mathcal{X}$ by their corresponding probability: $\sum_x x\, p(x)$.
- $E(X)$ is also referred to as the mean of X, and is typically denoted μ or μ_X.

Expectation: Discrete random variables IV

- If the function p is thought of as a weight, then $E(X)$ is the center; that is, if we place the mass $p(x_i)$ at the points x_i, then the balancing point is $E(X)$.
- Like with the pmf and cdf, we often use subscripts to denote which probability law we are using for the expectation, it if is not clear: $E_X(X)$.

Expectation: Discrete random variables V

Roulette

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet \$1 that an odd number comes up, you win or lose \$1 according to whether that event occurs. If X denotes your net gain, $X = 1$ with probability $18/38$ and $X = -1$ with probability $20/28$. The expected value of X is

$$E(X) = 1 \times \frac{18}{38} + (-1) \times \frac{20}{38} = -\frac{1}{19}.$$

- As you might imagine, the expected value coincides in the limit with the actual average loss per game, if you play many games (Chapter 5).

Expectation: Discrete random variables VI

- Most casino games have a negative expected value by design; you may win some money, but if a large number of games are played, the house will come out on top.

Expectation: Discrete random variables VII

Geometric Random Variable

Suppose that items are produced in a plant are independently defective with probability p. If items are inspected one by one until a defective item is found, then how many items must be inspected on average?

Solution:

Expectation: Discrete random variables VIII

Poisson Distribution

The Poisson(λ) distribution has pmf $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, for all $k \geq 0$. Thus, if $X \sim \text{Pois}(\lambda)$, then what is $E[X]$?

Solution:

Continuous random variables

Expectation: Continuous random variables

References and Acknowledgements

Resnick S (2019). *A probability path*. Springer.

Rice JA (2007). *Mathematical statistics and data analysis*, volume 371. 3 edition. Thomson/Brooks/Cole Belmont, CA.

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