Mathematical Statistics I

Chapter 5: Limit theorems

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1 Convergence Concepts

Introduction

- This material comes primarily from Rice (2007, Chapter 5), but will be supplemented with material from Casella and Berger (2024, Chapter 5).
- In this chapter, we are interested in the convergence of sequences of random variables.
- In particular, we are interested in the convergence of the sample mean, $\bar{X}_n = (X_1+X_2+\ldots+X_n)/n$, as the number of samples n grows.
- Because $\bar{X}_n$ is itself a random variable, we have to carefully define what it means for the convergence of a random variable.
- In this class, we are mainly concerned with three types of convergence.
- Because convergence of random variables is a tricky topic, we will treat them in varying amounts of detail.

Convergence in Probability

- The first type of convergence is one of the weaker types, and is usually easy(ish) to verify.

  Definition: Convergence in Probability

  A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\epsilon > 0$,
  $$\lim_{n\to\infty} P\big(|X_n - X| \geq \epsilon\big) = 0$$
  or, equivalently,
  $$\lim_{n\to\infty} P\big(|X_n - X| < \epsilon\big) = 1.$$

- We often use the shorthand $X_n \overset{P}{\to} X$ to denote "X_n converges in probability to X as n goes to infinity".

- Note that the X_i in the definition above do *not* need to be independent and identically distributed.
- The distribution of X_n changes as the subscript changes, and each of the convergence concepts we will discuss will describe different ways in which the distribution of X_n converges to some limiting distribution as the subscript becomes large.
- A special case is when the limiting random variable X is a constant.

Example: The (Weak) Law of Large Numbers
Let $X_1, X_2, \ldots$ be iid random variables with $E[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Define $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$. Then $\bar{X}_n \xrightarrow{P} \mu$.

Proof. The proof is a straightforward application of Chebychev's Inequality.

- We want to show that
$$\lim_{n\to\infty} P\big(|\bar{X}_n - \mu| \geq \epsilon\big) = 0.$$
- For every $\epsilon > 0$, Chebychev's inequality gives us:
$$\begin{aligned} P\big(|\bar{X}_n - \mu| \geq \epsilon\big) &= P\big((\bar{X}_n - \mu)^2 \geq \epsilon^2\big) \\ &\leq \frac{E\big[(\bar{X}_n - \mu)^2\big]}{\epsilon^2} \\ &= \frac{\mathrm{Var}(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}. \end{aligned}$$
- Thus, taking the limit, we have $\lim_{n\to\infty} P\big(|\bar{X}_n - \mu| \geq \epsilon\big) = 0$

$\square$

2 Test Section

Test Frame

Acknowledgments

- Compiled on October 20, 2025 using R version 4.5.1.

- We acknowledge students and instructors for previous versions of this course / slides.

References

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