Chapter 1 :

- Random Experiment. E.g. coin flips

- Sample space Ω : the set of all possible outcomes.

$$\Omega = \{H, T\}$$

$$\Omega = (0, \infty) \leftarrow \text{time in min waiting for a bus.}$$

- Events are subsets of Ω,

$$A = \{HH, TT\} \subseteq \{TT, TH, HT, HH\}$$

- Ω - uncountable
  - Vitali sets $\leftarrow$ unmeasureable.

- ~~Alien~~ . Event is a measureable subset. $A \subseteq \Omega$.

- Prob Space $(\Omega, \underset{\substack{\uparrow \\ \text{measurable sets}}}{\mathcal{F}}, P)$

$\mathbb{P}$:

(1) $P(\Omega) = 1$

(2) A is Measureable, $P(A) \ge 0$

(3) $A_1, A_2, \ldots$ disjoint

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

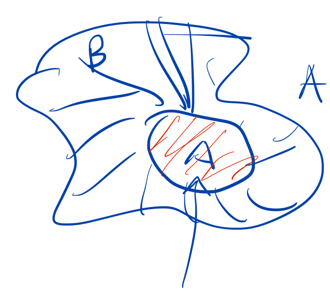
$A \subseteq \Omega$, $\quad P(A^c) = 1 - P(A)$.

(1) $$P(\Omega) = 1$$
$$\| $$
$$P(A \cup A^c) = 1$$

(3) $$P(A \cup A^c) = P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A). \qquad \square$$

$A \subseteq B$, $\quad P(A) \le P(B)$.



$A \subset B$

$$B = (B \cap A^c) \cup \overset{=A}{(B \cap A)}$$

(3) $$P(B) = P(B \cap A^c) + P(B \cap A)$$
$$= \underbrace{P(B \cap A^c)}_{\ge 0} + P(A) \ge P(A) \qquad \square$$

(2)

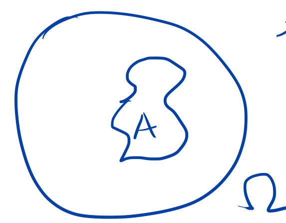
$$P(B) \ge P(A)$$

Uniform prob.

- equal weight to all outcomes $\omega \in \Omega$.

$$\begin{cases} P(A) = \dfrac{\#\ A \text{ can happen}}{\#\text{ of events}} & \text{Discrete} \\[2ex] P(A) = \dfrac{|A|}{|\Omega|} \end{cases}$$

$\rightarrow$



$\nearrow \dfrac{\text{area}(A)}{\text{area}(\Omega)}$

- $A = \{a_1, a_2, \ldots, a_n\}$
- pick r things, <u>order them</u>

$$\begin{cases} (n)(n)\ldots(n) = n^r & \leftarrow \text{ w/ replacement} \\ (n)(n-1)\ldots(n-r+1) = \dfrac{n!}{(n-r)!} = nP_r(n,r) & \text{w/o replacement.} \end{cases}$$

- $\binom{n}{r} \rightarrow$ # of ways to pick r things from a group of n, order doesn't matter. $\}$

- how many ways to order r things?

$r! = r(r-1)\ldots(2)(1).$

$$\binom{n}{r} = nPr(n,r) / r! = \boxed{\frac{n!}{(n-r)!\,r!}}$$

"n choose r".

Binomial Theorem

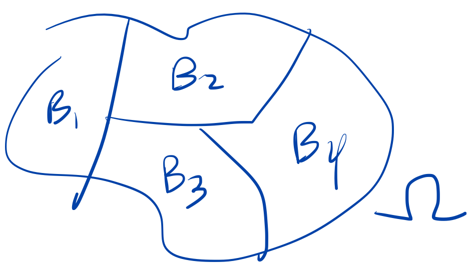
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

conditional Prob:

$$P(A|B) := \begin{cases} \dfrac{P(A\cap B)}{P(B)} & \text{if } P(B) \neq 0, \\ 0 & \text{o.w.} \end{cases}$$

$$P(A|B)\,P(B) = P(A\cap B), \quad P(B) \neq 0.$$

law of total prob:



$B_1, B_2, \ldots, B_n$ be a partition of Ω.

$$\bigcup B_i = \Omega, \quad B_i \cap B_j = \emptyset.$$

$$\begin{aligned} P(A) &= P(A\cap\Omega) \\ &= P\left(A \cap \left(\textstyle\bigcup_i B_i\right)\right) && \text{(partition)} \\ &= P\left(\textstyle\bigcup_i (A\cap B_i)\right) && \text{de Morgans} \\ &= \sum P(A\cap B_i) && \text{(axiom 3 + partition)} \end{aligned}$$

$$p(A) = \sum p(A|B_i)\,p(B_i)$$

Baye's rule:

$$p(B_j|A) = \frac{p(B_j \cap A)}{p(A)}$$

$$p(B_j|A) = \frac{p(A|B_j)\,p(B_j)}{\sum p(A|B_i)\,p(B_i)}$$

Independence: $A \perp B$ if

$$p(A \cap B) = p(A)\,p(B).$$

Pairwise: $A_1, A_2, \ldots, A_n$

$$p(A_i \cap A_j) = p(A_i)\,p(A_j) \quad i \neq j$$

mutual: $i_1, i_2, \ldots, i_m$ be any subset of $\{1, 2, \ldots, n\}$

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_m})$$

$$= P(A_{i_1}) P(A_{i_2}) \dots P(A_{i_m})$$

Chapter 2

* **R.V.** $\quad (\Omega, \mathcal{F}, \mathbb{P})$

$$X : \mathcal{F} \to \mathbb{R},$$

Discrete R.V. $\quad \Omega$

pmf: $\quad P(X = x) = p(x).$

$$= P(\{\omega : X(\omega) = x\})$$

<u>cdf:</u> $$F(x) = P(X \le x) = \sum_{y=-\infty}^{x} P(X = y)$$

- Bernoulli → indicator
- $X \to 0$ (prob $1-p$)

  $X \to 1$ (prob p)

- <u>Binomial</u> (n,p) X_i are <u>ind.</u>

$$\sum_{i=1}^{n} X_i \sim Bin(n,p)$$

- Pois, Negative Bin, Geometric, Hypergeometric

Cont. Ω uncountable,

$\underbrace{F(x) = P(X \le x)}_{F}$, exist for all $x \in \mathbb{R}$,

(1) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

(2) $F(x)$ is non-decreasing

$A \subset B$, $P(A) \le P(B)$

(3) $F(x)$ is right-cont.

$\lim_{x \downarrow x_0} F(x) = F(x_0)$ everywhere, $\forall x_0$

$X \stackrel{d}{=} Y$ $\quad X \sim Y$

iff $\quad$ Theorem.

$F_X(x) = F_Y(y)$

→ continuous,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$\frac{d}{dx} F_X(x) = f_X(x)$$

(FTC).

$f(x)$:

(1) $f(x) \geq 0$

(2) : $\begin{cases} \int_{-\infty}^{\infty} f(x)\,dx = 1 \\ \sum_{x} f(x) = 1 \quad \text{(discrete)} \end{cases}$

$Y = g(X)$ What is Distn of Y?

• cdf method:

$Y = X^2$

$$\boxed{F_Y(y)} = P(Y \le y)$$

$$= P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$

$$\boxed{= F_X(\sqrt{y}) - F_X(-\sqrt{y})}$$

take derivatives

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$\rightarrow \; = \overbrace{f_X(\sqrt{y}) \cdot \frac{d}{dy}\sqrt{y}}\; + \overbrace{f_X(-\sqrt{y})\left|\frac{d}{dy}\sqrt{y}\right|}$$

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$