Mathematical Statistics I

Chapter 6: Distributions Derived from the Normal Distribution

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Outline

1. χ^2 distributions

2. The t and F distributions

3. Sampling Distributions

χ^2 distributions

Introduction

- This material comes primarily from Rice (2007, Chapter 6).
- Here, we introduce several important distributions that arise from transformations applied to normal distributions.
- Many of these distributions form the basis of traditional statistical inference procedures that are taught in introductory statistics courses.
- They are very useful in practice due to the central limit theorem: with enough observations, the limiting behavior of nearly all distributions is normal, so distributions that come from the normal distribution arise in practice as well.

χ^2_ν Distribution

- The first distribution we will consider is the χ^2_1 (Chi-square with 1 degree of freedom).

Definition: χ^2_1 distribution

If Z is a standard normal random variable, then $X = Z^2$ is called the chi-square distribution with 1 degree of freedom.

- We typically use the notation $X \sim \chi^2_1$ (in LaTeX: `\chi`).

χ^2_ν Distribution II

The pdf of χ^2_1

Let X follow a χ^2_1 distribution. Then, the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}.$$

χ^2_ν Distribution III

- In Chapter 2, we previously noted that that $f_X(x)$ is an example of a Gamma distribution.
- Specifically, the *kernel* of the Gamma density is x raised to some power, and e raised to some multiple of x:

$$f_{\text{Gamma}}(x) \propto x^{\alpha-1} e^{-\lambda x}.$$

- Thus, ignoring the constant for a moment, if $\alpha = 1/2$, $\lambda = 1/2$, then the pdf of $X \sim \chi^2_1$ is just this Gamma density:

$$f_X(x) \propto x^{-1/2} e^{-x/2} = x^{\alpha-1} e^{-\lambda x}.$$

- Since both functions are proper probability density functions, they have to integrate to one, so the normalizing constant *must* be the same.

χ^2_ν Distribution IV

- This is also easily verified. The normalizing constant of the Gamma distribution is $\lambda^\alpha/\Gamma(\alpha)$.
- With our specific values of $\lambda = \alpha = 1/2$, and recalling that $\Gamma(1/2) = \sqrt{\pi}$,

$$\frac{1}{\sqrt{2\pi}} = \frac{(1/2)^{(1/2)}}{\Gamma(1/2)} = \frac{\lambda^\alpha}{\Gamma(\alpha)}$$

MGF of χ^2_1

We previously derived the MGF of a Gamma(α, λ) distribution: $M(t) = \big(\lambda/(\lambda - t)\big)^\alpha$. Thus, the MGF of a Chi-square(1) distribution is

$$M(t) = (1 - 2t)^{-1/2}, \quad t < 1/2.$$

χ^2_ν Distribution V

Definition

If $U_1, U_2, \ldots, U_n$ are n independent χ^2_1 random variables, then

$$V = U_1 + U_2 + \ldots + U_n$$

then the distribution of V is called the Chi-square distribution with n degrees of freedom, denoted χ^2_n.

- There are a few different ways of deriving the pdf of a χ^2_n random variable. Here, we will use the MGF uniqueness theorem.

χ^2_ν Distribution VI

- Let $M_i(t)$ denote the MGF of U_i, where $U_i \sim \chi^2_1$. Then, due to independence,

$$M_V(t) = M_{\sum_i U_i}(t) = \prod_{i=1}^{n} M_i(t) = \big(M_t(t)\big)^n = (1-2t)^{-n/2}$$

- Compare this to the Gamma MGF: $M(t) = \big(\lambda/(\lambda - t)\big)^\alpha$. Then, setting $\lambda = 1/2$, $\alpha = n/2$, we see that V has a Gamma$(n/2, 1/2)$ distribution.
- Thus, the pdf of V is given by:

$$f_V(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}.$$

χ^2_ν Distribution VII

- The expected value and variance of the χ^2_n distribution can easily be found then by using the fact that it is a special case of a Gamma distribution.

The t and F distributions

The Student's t distributions

The Student's t distribution

If $Z \sim N(0,1)$ and $U \sim \chi^2_n$, and Z and U are independent, then the distribution of T, where

$$T = \frac{Z}{\sqrt{U/n}},$$

is called the Student's t distribution (or simply the t distribution) with n degrees of freedom, which is often denoted t_n

- Students often forget the make sure that Z and U in the definition of the t distribution are independent.
- The t distribution is the distribution used to perform the famed "t-test".

The Student's t distributions II

The density of the t_n distribution

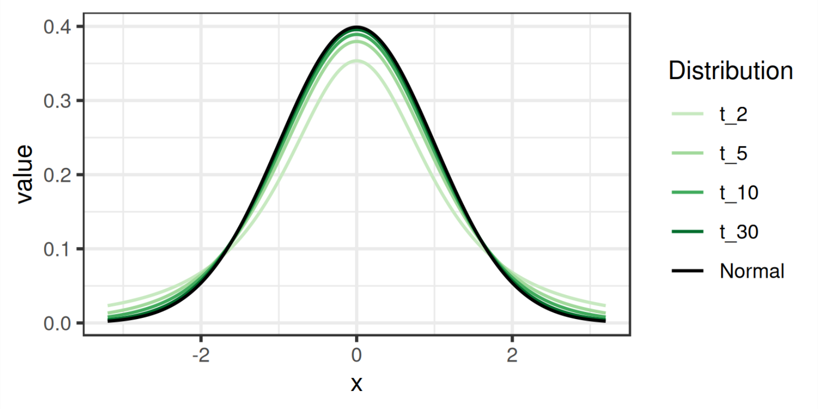
The pdf of the t distribution with n degrees of freedom is:

$$f(t) = \frac{\Gamma\big((n+1)/2\big)}{\sqrt{n\pi}\Gamma(n/2)}\Big(1+\frac{t^2}{n}\Big)^{-(n+1)/2}$$

- The derivation of the pdf of a t distribution is a good practice exercise.
- Recall it is defined as the ratio of two independent random variables; in Chapter 3, we derived a formula for computing densities of random variables of this form.
- Note that $f(t) = f(-t)$, and so f is symmetric about zero.
- It also has a bell-curve shape similar to a normal distribution.

The Student's t distributions III

- You can see as $n \to \infty$, the t_n distribution converges to the standard normal (e.g., use Slutsky's theorem, good practice).



The F distributions

Sampling Distributions

The sample mean

- In what follows, we'll assume that we are taking samples $X_1, X_2, \ldots, X_n$ from a larger population.
- These samples can be repeated experiments, or repeated observations. However, we will assume in general that the samples are independent and identically distributed, unless stated otherwise.
- For the remainder of the chapter, we will also assume $X_i \sim N(\mu, \sigma^2)$ for all i.

The sample mean II

- As a reminder from earlier chapters, linear combinations of independent normal random variables are also normally distributed. Thus, if $X_1, X_2, \ldots, X_n$ are iid normal, then $\bar{X}_n$ is also normally distributed.

Sampling distribution of the mean

If X_i are iid $N(\mu, \sigma^2)$, then $\bar{X}_n$ is normal, with

$$E\Big[1/n \sum_i X_i\Big] = (1/n) \sum_i \mu = \mu,$$

$$\mathrm{Var}\big(1/n \sum_i X_i\big) = 1/n^2 \sum_i \sigma^2 = \sigma^2/n.$$

Thus, $\bar{X}_n \sim N(\mu, \sigma^2/n)$.

The sample mean III

Lemma 6.1: Independent Normal RVs

Let X and Y be normally distributed random variables. Then X and Y are independent, if and only if

$$\mathrm{Cov}(X, Y) = 0.$$

- The above statement can be proved using the factorization theorem, and considering the MGF or pdf of a bivariate normal distribution.
- Recall that for most distributions, independence implies $\mathrm{Cov}(X, Y) = 0$, but not the other way around.
- It turns out that the normal distribution is the only distribution that has this property.

The sample mean IV

Theorem 6.1: Independence of Deviations

Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables. Then, $\bar{X}_n$ is independent of the vector of random variables called the *deviations*, $\big(X_i - \bar{X}_n\big)_{i=1}^{n}$.

Proof.

The sample mean V

Corollary 6.1

If the X_i are iid $N(\mu, \sigma^2)$, then $\bar{X}_n$ is independent of the sample variance S^2, defined by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

The sample mean VI

Theorem 6.2

If the X_i are iid normal, then $(n-1)S^2/\sigma^2$ has a chi-square distribution with $n-1$ degrees of freedom.

The sample variance

References and Acknowledgements

Rice JA (2007). *Mathematical statistics and data analysis*, volume 371. 3 edition. Thomson/Brooks/Cole Belmont, CA.

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