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¡presentation¿ Outline
Convergence Concepts
allowframebreaks]Introduction
This material comes primarily from [][Chapter 5]rice07, but will be supplemented with material from [][Chapter 5]casell
In this chapter, we are interested in the convergence of sequences of random variables.
In particular, we are interested in the convergence of the sample mean, $\bar{X}_n = (X_1 + X_2 + \ldots + X_n)/n$, as the number o
Because $\bar{X}_n$ is itself a random variable, we have to carefully define what it means for the convergence of a random varia
In this class, we are mainly concerned with three types of convergence.
Because convergence of random variables is a tricky topic, we will treat them in varying amounts of detail.
allowframebreaks]Convergence in Probability
The first type of convergence is one of the weaker types, and is usually easy(ish) to verify.
Definition: Convergence in Probability A sequence of random variables $X_1, X_2, \ldots$ converges in probability to a random

¡article¿ The idea for this proof is tho show that, for some neighborhood around $t = 0$, the mgf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ conve
Let $Z_i = (X_i - \mu)/\sigma$. Using the properties of mgf, the mgf of Z_i, denoted Z_i exists for $|t| < \sigma t$. The exact form of $M_Z($

Thus, using the properties of the mgf, we have $\mathrm{M}_{\sqrt{n}(\bar{X}_n-\mu)/\sigma}(t) = M_{\sum_{i=1}^n Z_i/\sqrt{n}}(t)$
$= M_{\sum_{i=1}^n Z_i}\big(t/\sqrt{n}\big)$
$= \Big(M_Z\big(t/\sqrt{n}\big)\Big)^n$. We now do a second order Taylor-Expansion of $M_Z\big(t/\sqrt{n}\big)$ about 0: $\mathrm{M}_Z\big(t/\sqrt{n}\big) = M_Z(0) + M_Z^{(1)}(0)$
$= 1 + \frac{t^2}{2n} + R_2\big(t/\sqrt{n}\big)$ Now by Taylor's theorem, there exists some function $h_2(x)$ such that

where

or equivalently,

Since t is fixed, (and because the below equality holds when $t = 0$), this implies:

Finally, returning to the mgf of $\sqrt{n}(\bar{X}_n - \mu)$, we take the limit as $n \to \infty$: $\lim_{n\to\infty} M_{\sqrt{n}(\bar{X}_n-\mu)/\sigma}(t) = \lim_{n\to\infty} \Big(M_Z\big(t$
$= \lim_{n\to\infty} \Big(1 + \frac{t^2}{2n} + R_2\big(t/\sqrt{n}\big)\Big)^n$
$= \lim_{n\to\infty} \Big(1 + \frac{1}{n}\Big[\frac{t^2}{2} + nR_2\big(t/\sqrt{n}\big)\Big]\Big)^n$
$= \lim_{n\to\infty} \Big(1 + \frac{a_n}{n}\Big)^n$, where $a_n = \frac{t^2}{2} + nR_2\big(t/\sqrt{n}\big)$. Here, we note that $a_n \to \frac{t^2}{2}$ due to the convergence of $nR_2\big(t/\sqrt{n}\big)$

Therefore, we get

Thus, by the continuity theorem,

One practical implication of the CLT is that, for large n, we can approximate

if X_i are independent and identically distributed with finite mean and variance.
In practice, $n \approx 30$ has been found to lead to good approximations, but it depends heavily on the distribution of X_i.
A further investigation of the CLT proof shows that the convergence towards the normal distribution happens at a rate
If we used a Taylor-series approximation with one additional order, we could derive a more accurate approximation und
Example: Binomial-Normal Approximation Let $X \sim Binomial(n, p)$. For any $k \in \{0, 1, \ldots\}$, approximate $P(X \leq k)$.
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Note that if X is binomial distributed, it has the same distribution as the sum of Bernoulli random variables.
Let $X_1, X_2, \ldots, X_n$ be Bernoulli(p) random variables, and then $X d{=} \sum_i X_i$.
Recall $E[X_i] = p$, $\mathrm{Var}(X_i) = p(1 - p)$.
By considering $\bar{X}_n = \frac{1}{n}\sum_i X_i$, we can use the CLT to approximate:

Therefore,

and

Using this approximation, we have

One thing to note is we can make a fairly simple improvement to this approximation by doing a continuity correction.
Specifically, X only takes on real values, so if we approximate it with a continuous distribution $X d{\approx} Y$, we generally get
allowframebreaks]Slutsky's Theorem
The following theorem is useful for our notes and supporting other ideas we will cover. However, we won't discuss the p
Theorem: Slutsky's Theorem If $X_n d{\to} X$ and $Y_n p{\to} a$ for some random variable X and constant a, then
$Y_n X_n d{\to} aX$.
$X_n + Y_n d{\to} X + a$.

Acknowledgments
Compiled on using Rversion 4.5.1.

We acknowledge https://jeswheel.github.io/4450$_f$25/*acknowledge.htmlstudentsandinstructorsforpreviousversionsoft*