Testing for Low-Dimensional Chaos: Implications for Weak-Form Efficiency

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Abstract

Traditionally, financial markets are often assumed to be efficient, where asset prices fully reflect all available information, leaving no room for predictability. According to the weak form of the Efficient Market Hypothesis (EMH), changes in asset prices should follow a random walk, implying that no market participant can consistently outperform or out-predict the market. However, are financial markets unpredictable because of what is traditionally assumed, or because of something hidden and endogenous within the market that amplifies any small perturbation and external shocks causing asset prices to be unpredictable?

This thesis investigates whether the hourly EUR/USD exchange rate displays evidence of chaotic behavior, and what implications this may have for weak-form market efficiency. The analysis begins by removing any known structures, such as linear dependencies, unit roots, and conditional heteroskedasticity, from the dataset. Then a series of nonlinear tests—the BDS test, correlation dimension analysis, and largest Lyapunov exponent — are applied to the cleaned residual series to detect any further hidden structure in the EUR/USD exchange rates.

The results reveal strong evidence of chaotic behavior in the EUR/USD market, suggesting that the system is capable of generating large price fluctuations endogenously. These findings challenge the traditional view of EMH and instead support an interpretation that financial markets are locally predictable while remaining globally unpredictable.

Keywords – Chaos, Market Efficiency, Nonlinear Dynamics

1. Introduction

One of the most fundamental assumptions in finance is that financial markets are assumed to be efficient. Asset prices in an efficient market already reflects and incorporates all available information (Fama, 1970). In an efficient market, not only is the set of available information fully reflected in the current price of an asset, but the set of information is available and efficiently processed by all market agents.

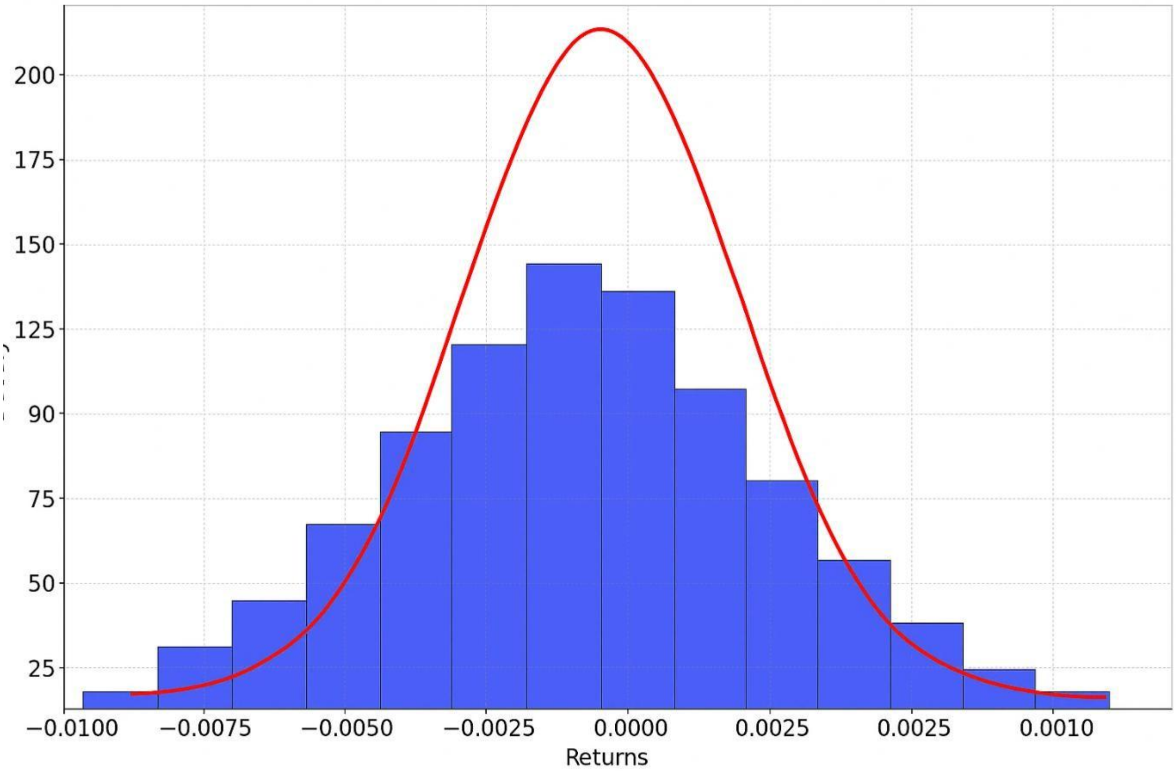
Therefore, there are two major implications in an efficient market – a. no one can consistently outperform the market, b. and any change in the price of an asset is purely random. The first implication of market efficiency states that while market agents can predict the future evolution of asset prices, and that from time to time, their predictions may be correct, however, no market agent can correctly predict the market consistently (Urrutia et al., 2002).

If at all a market agent can correctly forecast the market, it is purely because of luck and not due to skill. This is because the price of an asset evolves from P_t to P_{t+1} due to the arrival of some unexpected exogenous external shock, and by definition, no one can forecast or predict surprises.

The second implication of market efficiency is that asset prices follow a random walk, which means that any change in the price of an asset is purely random. $P_t = P_{t-1} + \varepsilon_t$, which implies $\Delta P_t = \varepsilon_t$ where ε_t is an IID random variable with a mean equal to zero, and because ε_t cannot be consistently forecasted or predicted, and neither can ΔP_t.

Therefore, in an efficient market, changes in asset prices are random and independent, driven by the arrival of new unexpected external shocks. In a simple random walk model with normally distributed shocks, most price changes would be expected to be small with large swings in prices to be rare. However, in reality, it is observed that large movements in asset prices occur much more than what the normal distribution predicts – markets have fat tails (Hseih, 1991). In other words, we observe much more wild swings in asset prices than what is predicted by the normal distribution.

EUR/USD Hourly Price Returns Distribution



This raises an important question – Are these large movements in asset prices caused simply due to unexpected exogenous and external shocks? Or are these frequent large movements in asset prices reflecting and revealing something else? Are these wild swings in asset prices pointing out that it isn't something exogenous but something endogenous that is generating these wild movements in prices? It is widely agreed that financial markets are unpredictable but are financial markets unpredictable because of what is traditionally assumed (that changes in asset prices are purely random) or because of an underlying instability within the market that amplifies any small perturbation and shocks?

If changes in asset prices aren't purely random but are internally generated – then there is some internal hidden complex structure that is generating these wild movements in asset prices – known as chaos (Jirasakuldech et al., 2012). The notion of chaos should not be confused with randomness, chaotic variables appear random, and they paint a picture of randomness, but they aren't random because chaotic variables are fundamentally driven by a deterministic structure. Secondly, chaotic variables are deterministic and not probabilistic, future evolutions of any chaotic variable are generated by a nonlinear function. Thirdly, the reason why chaotic variables appear random is because they are hypersensitive to their initial conditions.

The thesis aims to solve a commonly researched question with a novel approach. In reality, we observe large movements in asset prices, much more than what the normal distribution predicts. Do these large movements in asset prices occur simply because of unexpected external shocks or are a part of these wild fluctuations generated internally and endogenously due to the presence of chaotic structure? Therefore, if price changes aren't entirely random, then what kind of structure is behind them? Finally, if there is a complex internal hidden structure in asset prices, then early versions of the Efficient Market Hypothesis (EMH) which imply that asset prices follow a random is called into question.

The research proceeds by first removing any known structure in the data – linear dependencies, structural breaks, unit roots (stochastic trends), deterministic trends, and conditional heteroskedasticity, after removing all known sources of structure, we want to find if there is any hidden structure left or if the set of cleaned residuals are IID random variables.

The remainder of the paper is structured as follows: Section 2 reviews the existing literature on chaos theory and market efficiency. Section 3 explains key concepts such as nonlinearity and chaotic variables. Section 4 discusses the data and EUR/USD exchange rate selection. Section 5 describes the methodology used to detect chaos. Section 6 presents the results, and Section 7 interprets the findings and discusses the key insights. Finally, Section 8 concludes the paper and discusses the implications of the results on market efficiency.

2. Literature Review

The Efficient Market Hypothesis (EMH), proposed by Fama (1970), is one of the cornerstone pillars of financial economics. It states that financial markets are "informationally efficient", which means that today's price of any asset fully reflects and incorporates all the available information. The weak form of market efficiency states that the current price of any asset fully reflects and incorporates all the available information from its past historical prices – implying that technical analysis cannot be used to consistently out-predict the market. Secondly, the weak form of market efficiency implies that asset prices follow a random walk, meaning that any change in the price of an asset is purely random and attributed to an unexpected external and exogenous shock. Therefore, if financial markets are weakly efficient, price changes should be purely stochastic and unpredictable.

However, a growing body of literature has challenged the notion that financial markets behave purely random. It is widely agreed that financial markets are unpredictable, but this growing body of literature argues that financial markets are unpredictable, not because price changes are random, but that price changes are driven by a complex and hidden mechanism within the market. This is the notion of chaos theory which asserts that large movements in asset prices could be generated endogenously due to an underlying nonlinear process that is sensitive to its initial conditions causing this unpredictable behavior in asset prices. These systems are nonlinear, meaning that a small change in their inputs, can lead to disproportionately large changes in their outputs. Therefore, chaos theory asserts an alternate interpretation of why financial markets are unpredictable. Chaos theory suggests that changes in asset prices aren't driven by exogenous random shocks, but due to a hidden and complex mechanism within the system (Brock, 1986; Hseih, 1991). Chaotic variables appear random and paint a picture of randomness, but they are deterministic and not probabilistic because chaotic variables are governed by a set of underlying rules. The data from chaotic variables are generated from a nonlinear process that is sensitive to its initial conditions where even small changes in its initial conditions can lead to completely different outcomes – the "butterfly effect". Therefore, if financial markets are chaotic, then asset prices should have some degree of predictability – at least in the short term.

Numerous studies have attempted to test for chaos in various financial markets and instruments. Hseih (1991) was among the first to study U.S. exchange rates for evidence of chaos. The existing literature states that it is arduous to detect chaos with aggregated data. In order to detect chaos, one needs to look at disaggregated data with a large sample size. Hseih (1991) concluded that while there is strong evidence of nonlinear structure in asset prices, conclusive proof of deterministic chaos remains elusive. Brock and Sayers (1988) suggested that traditional linear models fail to capture the full dynamics of financial markets, asserting the use of nonlinear and chaotic models. More recent work by Chatrath et al. (2001) tested for chaotic behavior in gold and silver futures (financial derivatives) and found evidence consistent for low-degree chaos, further supporting the hypothesis that financial markets may not be purely random and stochastic.

Along the same lines in the literature of chaos is the broader discussion of nonlinearity in exchange rate behavior. Exchange rate prices have been traditionally modelled using linear models, such as ARMA or GARCH models. However, several studies have found that these models often fail to capture the full dynamics in currency data. Studies by De Grauwe and Dewachter (1993) and Sarantis (2001) provide evidence that nonlinear models outperform linear alternatives in explaining and forecasting exchange rate movements, particularly during periods of financial stress and structural breaks. In summary, the literature has grown and evolved from believing that asset prices follow a random walk to a growing recognition that financial markets may exhibit nonlinear and potentially chaotic behavior.

3. Theoretical Framework

Before examining the existence of chaos in financial markets and how it relates to market efficiency, it is important to understand the difference between chaos and randomness. Although closely related, they differ in their implications. A random process exhibits total unpredictability and lacks any systematic structure. In these systems, each result is uninfluenced by past values, and no amount of previously recorded data can assist in forecasting future values. This is consistent with the notion of a purely stochastic process, such as white noise or a fair coin toss.

On the other hand, chaotic systems are deterministic, not probabilistic, meaning that the data generated by a chaotic variable is driven by a set of underlying deterministic rules and formulas. Chaotic variables appear random, and they paint a picture of randomness, but they aren’t random because all chaotic processes are fundamentally governed by a set of deterministic structures. The reason why chaotic variables are highly unpredictable and appear random is because chaotic processes are extremely sensitive to their starting conditions – the “butterfly effect.” This sensitivity means that small fluctuations in the initial values will result in different final values over an extended period.

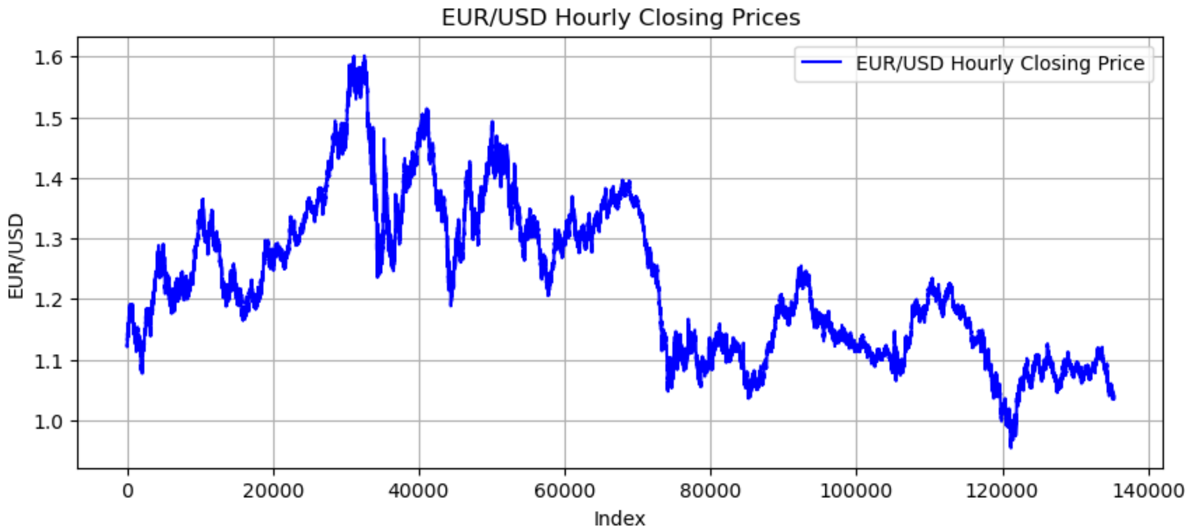
A system is typically classified as chaotic if it exhibits three key properties: determinism, nonlinearity, and sensitivity to initial conditions (Brock and Sayers, 1988). Determinism ensures that the system is not governed by chance but by fixed rules. Nonlinearity means that small changes in input can lead to disproportionately large changes in output. Sensitivity to initial conditions creates unpredictability over time, even in the absence of external shocks. These characteristics combine to make chaotic systems unpredictable in practice, even though they are not random in theory.

The implications of chaos for financial theory are significant, particularly about the Efficient Market Hypothesis (EMH). The weak form of EMH asserts that all past price information is already reflected in current prices, implying that asset prices follow a random walk and cannot be predicted using historical data. This assumption rules out the possibility of deterministic structures like chaos in price movements. However, if asset prices exhibit chaotic behavior, then they are not entirely random. Instead, they are influenced by complex, nonlinear behavior that evolves from within the system itself.

The presence of chaos would challenge the core premise of weak-form efficiency. It would suggest that past prices may hold some short-term predictive power—not due to linear trends, but because of nonlinear, deterministic dynamics that traditional models fail to capture. In this sense, chaos introduces a third category of behavior that lies between complete randomness and perfect predictability.

4. Data

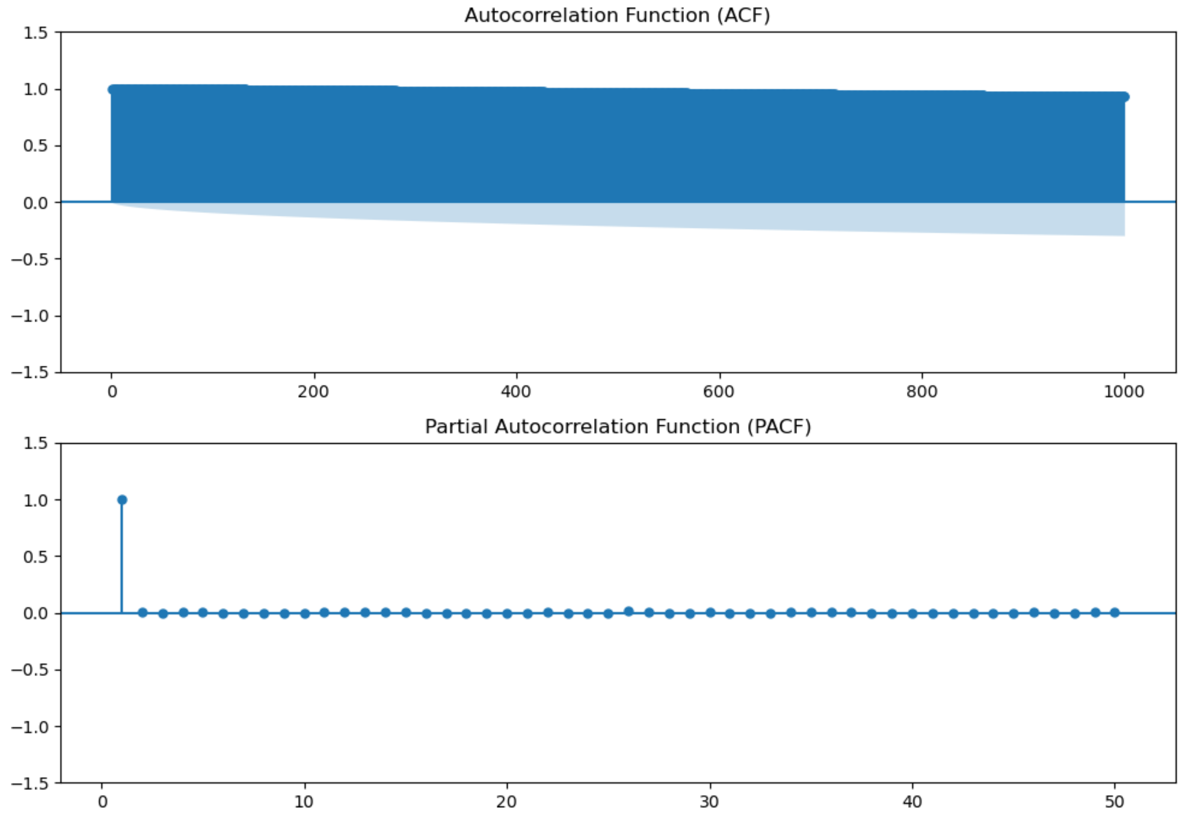
This research is conducted on the EUR/USD exchange rates at their hourly time frame. The sample size for this study spans from 2003 to 2025. The dataset consists of roughly 1,30,000 observations, allowing for ample degrees of freedom. The large sample size is necessary for detecting chaos because chaos can only be detected at higher embedding dimensions. A chaotic process will always appear random at lower embedding dimensions; therefore, the large sample size makes it suitable for detecting the hidden endogenous structure which could potentially explain why asset prices fluctuate so much. The EUR/USD currency pair has been selected for three main reasons: a. firstly, the EUR/USD currency pair is one of the most globally traded & liquid instruments, therefore, any artificial price jumps due to illiquidity will not be captured, b. secondly, there is minimal government intervention in this particular currency pair, therefore, this asset is minimally influenced by external distortions, c. and finally, the data that is drawn from this currency pair is rich, continuous, and clean.



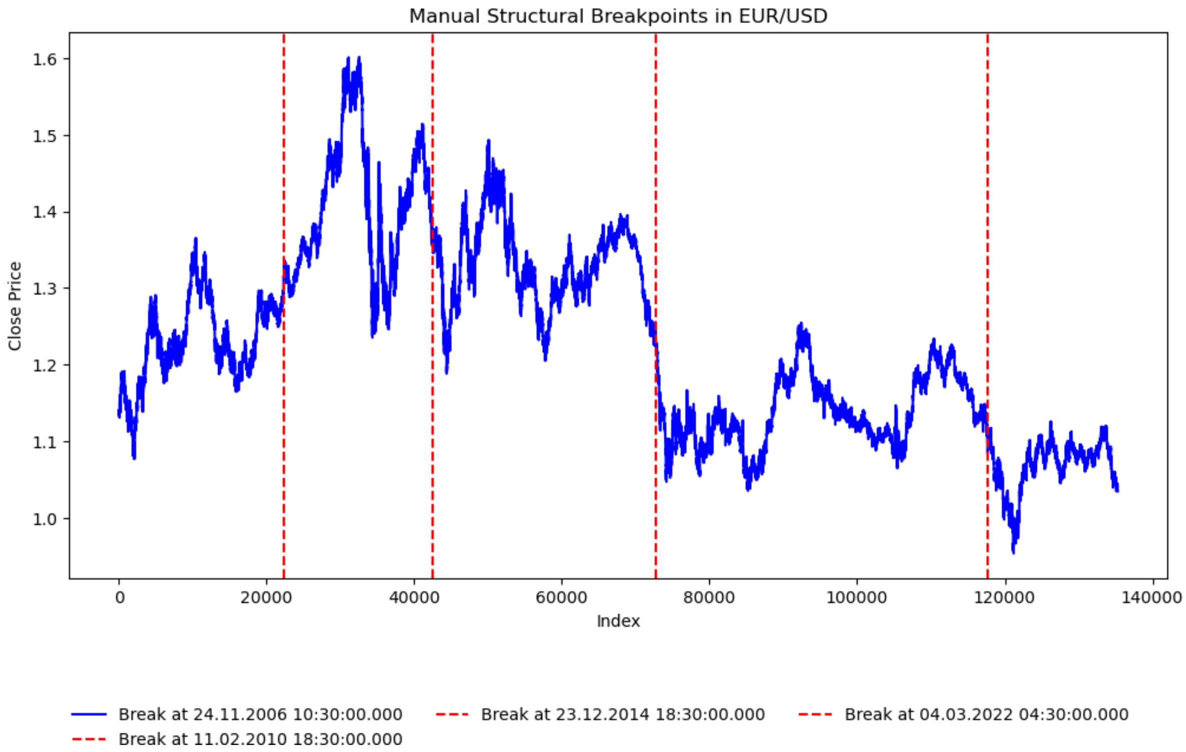
5. Methodology

A. Removing the Known Structure

Before addressing structural breaks, it is important to understand whether the EUR/USD currency pair is non-stationary in its raw form. The most fundamental method way to examine this is by inspecting the correlograms. For a stationary variable, the Auto-Correlation Function (ACF) decays rapidly, indicating that past shocks die down, and past values do not influence the current price of the asset. However, a non-stationary variable would show a slow, persistent, and gradual decay in the ACF, reflecting the presence of a unit root (stochastic trend) or deterministic trend. A slow and gradual decay in the ACF would mean that the system still remembers its past shocks, that the past shocks have not fully died down, and past values are still affecting the current price of the asset. The ACF of the EUR/USD exchange rate showed a slow, gradual decline — indicating non-stationarity, while the PACF showed a single significant spike.



Accounting for Structural Breaks



Identifying and modelling major structural breakpoints is the most fundamental step to correctly establish the source of non-stationarity. If structural breaks are not accounted for, the subsequent results will be misleading and biased. Structural breaks are those points where the behavior of the variable changes abruptly causing the mean or trend of the variable to change. These breakpoints can arise due to macroeconomic shocks, changes in policy regimes, financial crises, or technological changes in the forex market. If structural breaks are not addressed, then the source of non-stationarity will be established incorrectly, and the variable will be transformed incorrectly to remove the non-stationarity.

To identify and detect major structural changes in the EUR/USD dataset, the study uses the binary segmentation method from the ruptures Python package. Binary segmentation works by first looking at the entire dataset and finding the point where the biggest shift or "break" occurs by analyzing the largest squared difference between two consecutive points. Once that break is found, the method splits the entire dataset at that point and then repeats the same process on each of the new segments. It keeps going until no more significant breaks are found. In large datasets, several changes may have happened over time, and other methods to detect structural breaks like Bai-perron's test are computationally infeasible, however binary segmentation is a method that is both accurate and computationally effective in detecting change points in large datasets.

Once the structural breaks are detected, each structural break is modelled using two dummy variables – one dummy variable to capture any change in the intercept (or level), and another dummy to capture changes in the trend of the variable. As a result, any detected nonlinearity or chaos is more likely to be genuine, rather than a byproduct of uncorrected structural instability.

Structural Break	**Breakpoint**
1	24.11.2006
2	11.02.2010
3	23.12.2014
4	04.03.2022

Removing the Source of Non-Stationarity

After detecting and accounting for structural breaks, the next important step is to determine the source of non-stationarity. EUR/USD prices may be non-stationary due to the presence of a stochastic trend or a deterministic trend. A stationary series has statistical properties, such as mean and variance, which remain constant over time. In contrast, a non-stationary series contains unit roots and deterministic trends, causing the variable to wander further and further away, not allowing the variable to revert to its mean.

However, standard unit root tests like the Augmented Dickey-Fuller (ADF) test may fail to correctly determine the source of non-stationarity when structural breaks are present. This is because these breaks can mask the true nature of the process, making a stationary series appear non-stationary. To address this, the study uses Perron’s (1989) unit root test, which is specifically designed to test for the presence of a unit root in the midst of structural breaks. The test includes dummy variables (level and trend dummies) to account for changes in intercept and trend at known breakpoints.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                delta_y   R-squared:                       0.002
Model:                            OLS   Adj. R-squared:                  0.001
Method:                 Least Squares   F-statistic:                     3.382
Date:                Fri, 02 May 2025   Prob (F-statistic):           6.34e-19
Time:                        03:59:40   Log-Likelihood:             6.9044e+05
No. Observations:              135238   AIC:                        -1.381e+06
Df Residuals:                  135171   BIC:                        -1.380e+06
Df Model:                          66
Covariance Type:            nonrobust
=================================================================================
                    coef    std err          t      P>|t|      [0.025      0.975]
---------------------------------------------------------------------------------
const             0.0005   9.13e-05      5.041      0.000       0.000       0.001
Close_Lag1       -0.0004    7.5e-05     -5.046      0.000      -0.001      -0.000
Time_Trend     1.118e-09   1.56e-09      0.717      0.473   -1.94e-09    4.17e-09
D_L1           6.334e-05   2.97e-05      2.135      0.033     5.2e-06       0.000
D_L2          -1.623e-05   2.77e-05     -0.587      0.557   -7.04e-05     3.8e-05
D_L3          -7.393e-05    2.7e-05     -2.742      0.006      -0.000   -2.11e-05
D_L4          -5.155e-05   2.77e-05     -1.861      0.063      -0.000    2.75e-06
D_T1          -2.211e-09   2.35e-09     -0.941      0.347   -6.82e-09    2.39e-09
D_T2           3.484e-10   2.04e-09      0.171      0.865   -3.66e-09    4.35e-09
D_T3           1.408e-09   1.11e-09      1.267      0.205   -7.71e-10    3.59e-09
D_T4           3.545e-10   2.23e-09      0.159      0.874   -4.03e-09    4.73e-09
delta_y_Lag1     -0.0081      0.003     -2.989      0.003      -0.013      -0.003
delta_y_Lag2      0.0018      0.003      0.673      0.501      -0.004       0.007
delta_y_Lag3     -0.0054      0.003     -1.980      0.048      -0.011   -5.32e-05
delta_y_Lag4     -0.0018      0.003     -0.660      0.509      -0.007       0.004
delta_y_Lag5      0.0006      0.003      0.214      0.830      -0.005       0.006
delta_y_Lag6      0.0077      0.003      2.816      0.005       0.002       0.013
delta_y_Lag7      0.0013      0.003      0.463      0.644      -0.004       0.007
delta_y_Lag8      0.0031      0.003      1.134      0.257      -0.002       0.008
delta_y_Lag9      0.0022      0.003      0.827      0.408      -0.003       0.008
delta_y_Lag10    -0.0023      0.003     -0.857      0.391      -0.008       0.003
delta_y_Lag11    -0.0039      0.003     -1.434      0.152      -0.009       0.001
delta_y_Lag12    -0.0115      0.003     -4.243      0.000      -0.017      -0.006
delta_y_Lag13    -0.0033      0.003     -1.202      0.229      -0.009       0.002
delta_y_Lag14    -0.0015      0.003     -0.535      0.593      -0.007       0.004
delta_y_Lag15     0.0006      0.003      0.238      0.812      -0.005       0.006
delta_y_Lag16     0.0047      0.003      1.715      0.086      -0.001       0.010
delta_y_Lag17     0.0007      0.003      0.256      0.798      -0.005       0.006
delta_y_Lag18     0.0079      0.003      2.913      0.004       0.003       0.013
delta_y_Lag19     0.0025      0.003      0.927      0.354      -0.003       0.008
delta_y_Lag20     0.0030      0.003      1.110      0.267      -0.002       0.008
delta_y_Lag21    -0.0020      0.003     -0.727      0.468      -0.007       0.003
delta_y_Lag22     0.0080      0.003      2.936      0.003       0.003       0.013
delta_y_Lag23     0.0002      0.003      0.057      0.954      -0.005       0.005
delta_y_Lag24     0.0056      0.003      2.061      0.039       0.000       0.011
delta_y_Lag25    -0.0174      0.003     -6.389      0.000      -0.023      -0.012
delta_y_Lag26    -0.0044      0.003     -1.617      0.106      -0.010       0.001
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delta_y_Lag27      0.0022      0.003      0.812      0.417      -0.003      0.008
delta_y_Lag28      0.0007      0.003      0.254      0.800      -0.005      0.006
delta_y_Lag29     -0.0071      0.003     -2.619      0.009      -0.012     -0.002
delta_y_Lag30      0.0099      0.003      3.637      0.000       0.005      0.015
delta_y_Lag31      0.0041      0.003      1.494      0.135      -0.001      0.009
delta_y_Lag32     -0.0009      0.003     -0.348      0.728      -0.006      0.004
delta_y_Lag33     -0.0006      0.003     -0.210      0.834      -0.006      0.005
delta_y_Lag34     -0.0041      0.003     -1.491      0.136      -0.009      0.001
delta_y_Lag35     -0.0041      0.003     -1.510      0.131      -0.009      0.001
delta_y_Lag36     -0.0017      0.003     -0.615      0.538      -0.007      0.004
delta_y_Lag37     -0.0019      0.003     -0.691      0.490      -0.007      0.003
delta_y_Lag38      0.0014      0.003      0.499      0.618      -0.004      0.007
delta_y_Lag39   7.974e-05      0.003      0.029      0.977      -0.005      0.005
delta_y_Lag40      0.0015      0.003      0.544      0.587      -0.004      0.007
delta_y_Lag41      0.0050      0.003      1.822      0.068      -0.000      0.010
delta_y_Lag42      0.0011      0.003      0.397      0.692      -0.004      0.006
delta_y_Lag43      0.0028      0.003      1.017      0.309      -0.003      0.008
delta_y_Lag44      0.0030      0.003      1.104      0.269      -0.002      0.008
delta_y_Lag45      0.0002      0.003      0.068      0.946      -0.005      0.006
delta_y_Lag46      0.0027      0.003      0.977      0.328      -0.003      0.008
delta_y_Lag47      0.0053      0.003      1.951      0.051   -2.56e-05      0.011
delta_y_Lag48     -0.0048      0.003     -1.783      0.075      -0.010      0.000
delta_y_Lag49     -0.0069      0.003     -2.530      0.011      -0.012     -0.002
delta_y_Lag50     -0.0022      0.003     -0.824      0.410      -0.008      0.003
delta_y_Lag51     -0.0097      0.003     -3.559      0.000      -0.015     -0.004
delta_y_Lag52      0.0071      0.003      2.593      0.010       0.002      0.012
delta_y_Lag53      0.0010      0.003      0.357      0.721      -0.004      0.006
delta_y_Lag54     -0.0014      0.003     -0.519      0.604      -0.007      0.004
delta_y_Lag55     -0.0058      0.003     -2.144      0.032      -0.011     -0.001
delta_y_Lag56      0.0055      0.003      2.012      0.044       0.000      0.011
==============================================================================
Omnibus:                    28633.954   Durbin-Watson:                   2.000
Prob(Omnibus):                  0.000   Jarque-Bera (JB):          1215581.654
Skew:                          -0.000   Prob(JB):                         0.00
Kurtosis:                      17.688   Cond. No.                     7.87e+07
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 7.87e+07. This might indicate that there are
strong multicollinearity or other numerical problems.
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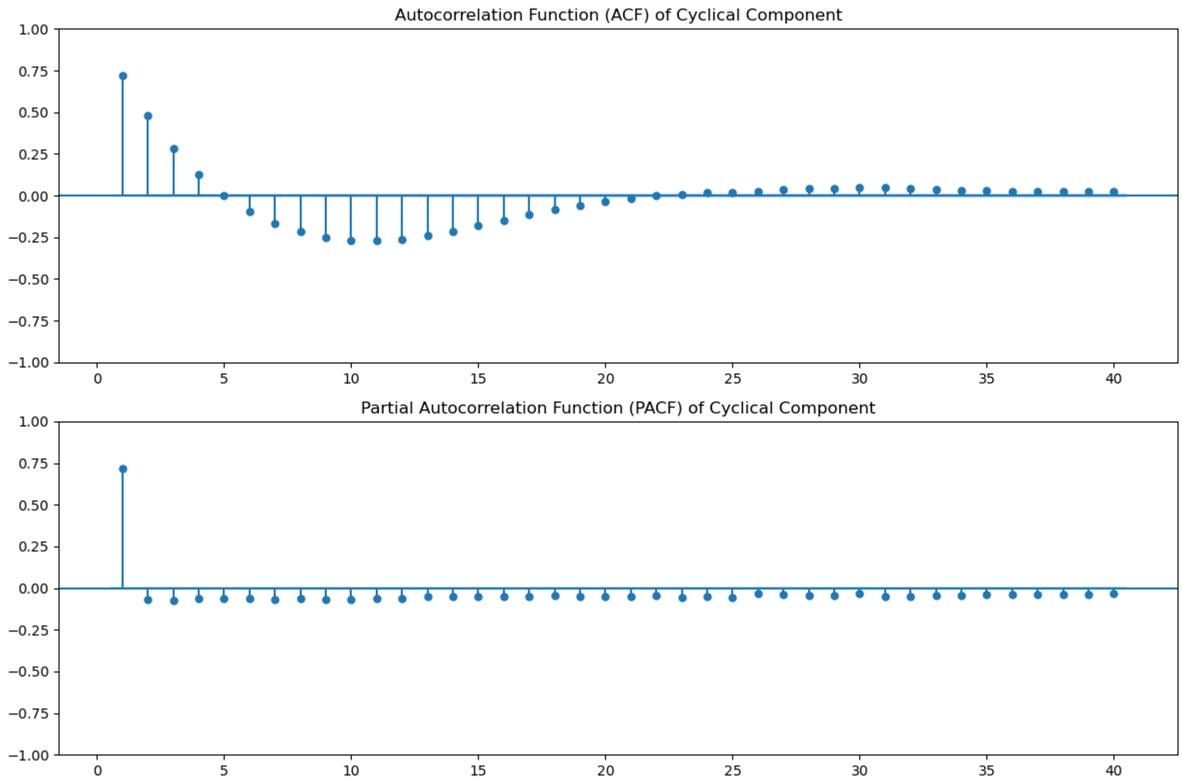
Perron's version of the Augmented Dickey-Fuller test was conducted on the tau-tau form, which includes an intercept, a deterministic trend term that travels with time, fifty-six lagged terms selected (based on AIC) to whiten the white noise process, and the trend dummies & intercept dummies. It is ideal to begin with the least restrictive model to accommodate for everything. The coefficient of y_lagged indicates that the null hypothesis of a unit root could be rejected even at the 1% level. This means that after accounting for structural breaks, a deterministic trend is responsible for causing non-stationarity in the EUR/USD exchange rate series. The EUR/USD exchange rate series evolves around a deterministic trend, in other words, the EUR/USD exchange rate is stationary around its deterministic trend. The deterministic trend was detrended using a Hodrick-Prescott filter – which fits a quadratic trend to the data and filters out the cyclical residuals.

Fitting a Box-Jenkins to Remove Linear Dependence



Once the exchange rates have been detrended using a Hodrick-Prescott filter and the cyclical residual component has been extracted, the next step is to inspect if the variable displays any linear dependence on its past values and error shocks. This is because today's price of an asset can be linearly dependent on several of its past values (autoregressive terms) and past error shocks (moving average terms). Therefore, a Box-Jenkins model needs to be fitted that fits an appropriate AMRA (p,q) model to account for the linear dependence in the exchange rates. The correlograms – ACF & PACF, indicated that today's price of an asset is linearly dependent on one of its past values, and four of its past shocks, therefore an ARMA (1,4) model was fitted to account for this linear structure. If these linear dependencies are not accounted for, they may be mistaken for chaos of nonlinearity in the subsequent steps. The AIC revealed that an ARMA (1,4) is the best-fitted model, and the residuals from this model should be used to test for volatility clustering.

$$P_t = a_0 + a_1 P_{t-1} + b_1 \varepsilon_{t-1} + b_2 \varepsilon_{t-2} + b_3 \varepsilon_{t-3} + b_4 \varepsilon_{t-4} + \varepsilon_t$$

Removing Conditional Heteroskedasticity

```
                     Constant Mean - GARCH Model Results
==============================================================================
Dep. Variable:                   None   R-squared:                       0.000
Mean Model:             Constant Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:                723731.
Distribution:                  Normal   AIC:                      -1.44745e+06
Method:            Maximum Likelihood   BIC:                      -1.44741e+06
                                        No. Observations:               135296
Date:                Tue, Apr 29 2025   Df Residuals:                   135295
Time:                        11:56:36   Df Model:                            1
                                 Mean Model
=============================================================================
                 coef    std err          t      P>|t|       95.0% Conf. Int.
-----------------------------------------------------------------------------
mu         2.6718e-08  1.611e-06  1.658e-02      0.987 [-3.131e-06,3.184e-06]
                              Volatility Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
omega      3.5624e-08  8.980e-11    396.727      0.000 [3.545e-08,3.580e-08]
alpha[1]       0.1000  3.978e-03     25.138 1.919e-139   [9.220e-02,  0.108]
beta[1]        0.8800  3.396e-03    259.141      0.000     [  0.873,  0.887]
============================================================================
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After fitting an ARMA (1,4) model to remove the linear dependence of the variable with its past values and shocks, the correlograms of the squared residuals indicated that the volatility of the variable isn't constant. Therefore, the issue of conditional heteroskedasticity needs to be addressed. Volatility clustering is a common notion in financial distributions, especially in exchange rates. Volatility in financial asset prices tends to cluster, for example, periods of high volatility are often followed by periods of high volatility, and periods of low volatility are often followed by periods of low volatility. The correlograms of the squared residuals revealed heteroskedastic variance, and a parsimonious GARCH (1,1) model was fitted to fit a model to the variance. The final remaining cleaned residuals serve as the cleanest possible representation of the EUR/USD exchange rates as it is stripped of all its known structure – structural breaks, deterministic trends, linear dependencies, and GARCH errors. This remaining set of cleaned residuals should be inspected for any further hidden structure.

B. Detecting for Any Further Hidden Structure – Chaos

Correlation Dimension

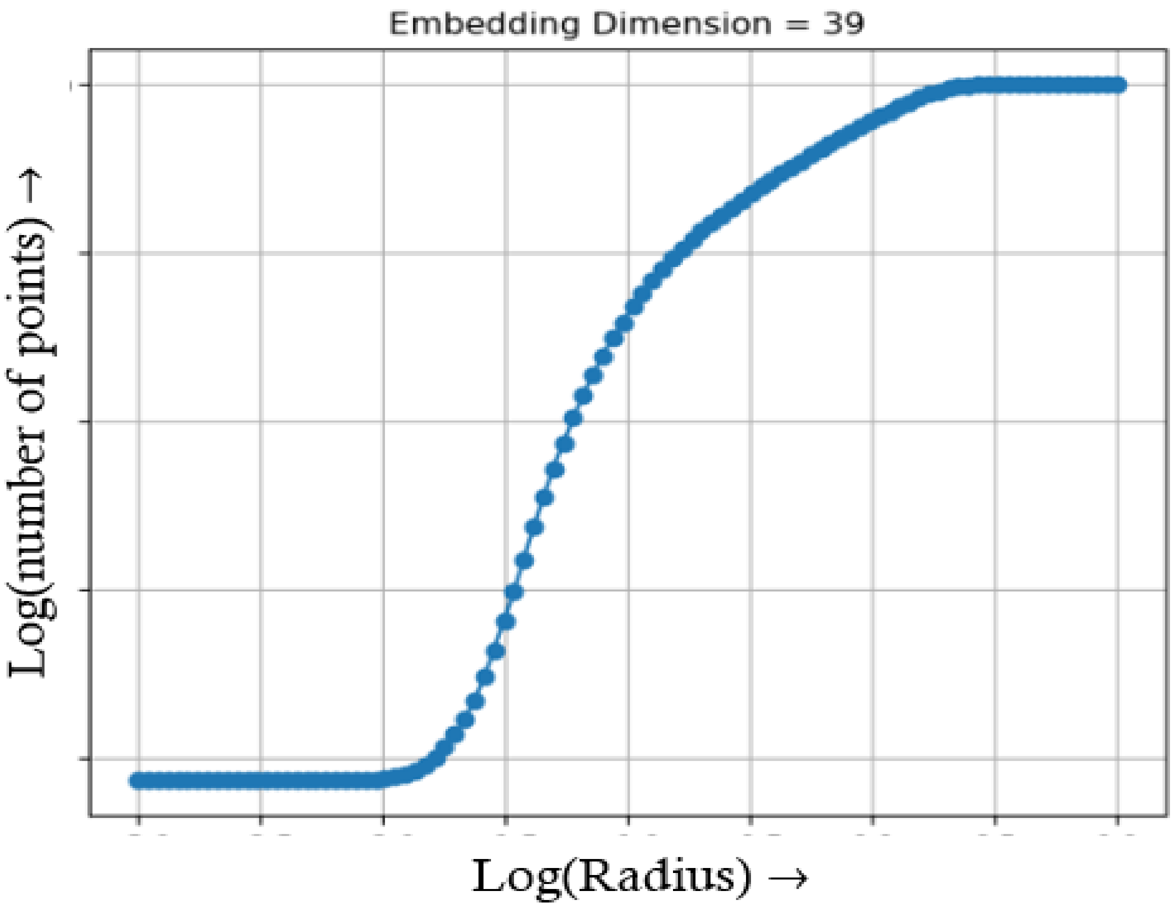
After removing the deterministic trend, linear dependencies, and non-constant volatility effects from the data, the next step was to check if there was any further hidden structure left — something that traditional models could not explain. For this, the study used a method called the correlation dimension. This tool helps to understand the complexity of a system by seeing how many variables or "dimensions" are needed to describe its behavior. If a time series is completely random, the correlation dimension keeps increasing as more dimensions are added. But if the system is driven by some hidden rules — like in chaotic systems — then after a certain point, adding more dimensions does not change the result much.

Chaotic processes are extremely complex; hence, a chaotic process can fill up the first n-dimensions uniformly but have large holes in the $(n+1)^{st}$ dimension. This is precisely done by the correlation dimension test (Hsieh, 1991). The test can be applied to each dimension to measure how clustered the data points are in a given space.

$$C_n(\varepsilon) = \lim_{T\to\infty} \# \frac{\{(t,s), 0 < t, s < T: \left\| x_t^n - x_s^n \right\| < \varepsilon\}}{T^2}$$

$C_n(\varepsilon)$ is called the correlation integral. The correlation integral measures how clustered (within a small distance – ε) a pair of points are in a given space (n). As $T \rightarrow \infty$, the correlation integral counts the number of pairs of points that are within a small distance ε. For a random process, this integral would keep increasing with n because there is no limit to how much space the process can occupy. However, for a chaotic process, this integral would only increase till a particular level, after which, even if the distance ε increases, the number of points clustering around that distance would remain the same. The correlation dimension test provides the intuition and basis for detecting chaos; however, this is not a statistical test as the accuracy of the correlation test cannot be quantified from a statistical viewpoint.

In the analysis, it was found that the correlation integral increased at first but then started to flatten out, forming a kind of plateau and stagnation. This Correlation Dimension (CD) plot suggests that the system is not purely random, but might be following a complex yet deterministic process, which is a key feature of chaos. This finding gave early evidence that the cleaned EUR/USD residual series might have underlying chaotic behavior.



BDS Test

A critical drawback of the Correlation Dimension (CD) plot is that the CD plot isn't a statistical test. Therefore, the presence of a chaotic structure cannot be concluded with any statistical confidence. To resolve this problem and test for any remaining hidden structure in the series, the Brock-Dechert-Scheinkman (BDS) test was applied. This test checks whether a variable is a random process or if the variable is dependent on its past values with the moments of the variable changing over time. If the BDS test rejects the null hypothesis, it suggests that the series has a hidden, nonlinear structure that is not captured by traditional models.

The BDS statistic tests if the residuals (data set) are independent and identically distributed. The BDS test does not tell you if the data is stochastic or chaotic, the BDS test only tells you whether the residuals are IID random variables or not (Hsieh, 1991). If $\{x_t\colon\ t = 1, 2, 3, 4, \dots, T\}$ is a random sample of IID observations, then:

$$C_n(\varepsilon) = C_1(\varepsilon)^n$$

$C_n(\varepsilon)$ → measures how dense a pair of points are close to each other within a distance of ε. It counts how many pairs of points in the time series are within a small distance of ε. It measures how close the points are within a distance of ε. This is called the correlation integral which measures how "clustered" the data points are in a given space.

$$BDS\ Statistic\colon \quad W_{n,s} = \frac{C_n(\varepsilon) - C_1(\varepsilon)^n}{\sigma_{n,s}} \sim N(0,1)$$

$$H_0\colon\ The\ dataset\ is\ independently\ and\ identically\ distributed$$

$$H_A\colon\ H_0\ is\ false$$

The BDS statistic asymptotically follows a standard normal distribution (Hseih, 1991). If the null hypothesis fails to be rejected, then this means that the data is IID and there is no dependence or non-linearity in the residuals.

In this study, the BDS test was conducted across multiple embedding dimensions to check how the results evolved. Interestingly, up to the 33rd dimension, the test did not reject the null hypothesis, meaning that the data appeared random (and without structure) at lower embedding dimensions. However, at dimension 34 and beyond, the null was rejected, revealing the presence of non-random structures at higher dimensions.

This result is consistent with the behavior of chaotic systems, which often appear random in lower dimensions, even though they are governed by deterministic rules. One of the key challenges in detecting chaos is that it may remain hidden until the system is examined in a high enough dimension. A chaotic variable may look completely random in dimensions 1 through n, but in dimension $(n + 1)$, the hidden structure begins to emerge — sometimes described as the system "showing holes" in its structure. The BDS test's ability to detect this shift at higher dimensions strengthens the evidence that the EUR/USD series may be driven by deterministic chaos, not pure randomness.

Embedding Dimension	P-values
2	0.5885
10	0.9099
20	0.5559
30	0.1556
33	0.0633
34	0.0414
35	0.0260

Lyapunov Exponent

The BDS test identifies a hidden nonlinear structure in the set of cleaned residuals, however, the BDS test fails to specify whether the nonlinear structure is chaotic or not. Not all nonlinear processes are chaotic, chaotic processes are just a small subset of nonlinear processes. Therefore, to determine whether the hidden structure is chaotic or not, the second property of chaotic variables needs to be tested. The sensitivity of the process to its initial conditions needs to be tested. To accomplish this, the Lyapunov exponent was calculated. The Lyapunov exponent measures how quickly two points that start at similar states diverge from each other, in other words, the Lyapunov exponent measures the sensitivity of the process to its starting conditions. The Lyapunov exponent measures how quickly two nearly identical points eventually evolve into vastly different outcomes. If small differences in starting points grow exponentially over time, the system is said to be chaotic.

$$d(t) = d(0).\, e^{\lambda t}$$

$$\Longrightarrow \log[d(t)] = \log[d(0)] + \lambda t$$

$d(t)$: Distance between two initially close points at time t.

$d(0)$: Initial distance between the two points.

λ: Lyapunov exponent — measures the average rate of divergence between the two close points.

t: Time

log $[d(t)]$: Log of the distance at time t, showing exponential growth

log $[d(0)]$: Log of the initial distance

If the value of lambda, λ is positive, this is considered strong evidence for chaos because it suggests that the system is highly sensitive to its initial conditions causing the variable to be unpredictable in the long run. A positive Lyapunov exponent implies that two nearly identical points diverge very quickly from each other with time – a hallmark of chaos. A negative exponent implies that two points converge towards each other with time, and a zero exponent suggests a periodic or neutral system.

After removing structural breaks, non-stationarity, volatility clustering, and linear structures, the set of remaining residuals was used to calculate the Lyapunov exponent. The Lyapunov exponent was found to be positive, and the p-value suggests that the chaotic structure is not due to chance or random noise but that the variable is driven by a nonlinear process that is sensitive to its initial conditions.

Particulars	Values
Lyapunov Exponent	0.1041
R-squared	0.75
P-value	0.0012

6. Results

The primary aim of the research is to understand why asset prices fluctuate so much. Therefore, the analysis began by looking at the time plot of the exchange rates to visually understand any trends and patterns. The correlograms were inspected to understand if the system remembers its past shocks. The ACF of the EUR/USD exchange rates displayed a slow and gradual decay which means that the past shocks in the system have not fully died down and that past values are still influencing today's price of the asset. Therefore, there is a strong presence of non-stationarity in the variable which needs to be either detrended or differenced. However, before establishing the source of non-stationarity, structural breaks need to be accounted for since the span of the data set is quite long and ranges from 2003 to 2025. Structural breakpoints were identified using the Ruptures library in Python, and the breakpoints were detected based on binary segmentation. Four major structural breakpoints were identified, and each breakpoint was modelled using two dummy variables – one dummy to capture any change in the intercept, and another dummy to capture any change in the slope (or trend) of the variable. Perron's unit root test, tested for the presence of a unit root (stochastic trend) in the midst of all dummy variables and, the null hypothesis of a unit root could be confidently rejected, implying that the series was driven by a trend stationary process. The series was detrended and the cyclical component was extracted using a Hodrick-Prescott filter and an ARMA (1,4) model was fitted to remove the linear dependencies of the variable with its past prices and shocks. Finally, since the volatility of the variable changes over time, a GARCH (1,1) model was fitted to remove the conditional heteroskedasticity in the variable. After all known structures were removed – structural breaks, deterministic trend, linear dependencies, conditional heteroskedasticity, the remaining set of residuals could be tested if any further hidden structure remained. As all linear structures have been removed, the remaining residuals need to be tested if they are IID random variables or if some hidden nonlinear structure remains. The Correlation Dimension (CD) plot showed an upward trending line, meaning that as the radius was increased – the number of points clustering around that radius would also increase, but after a certain level, the upward trending line plateaued, indicating the presence of low-degree chaos in the system. The BDS test could not reject the null hypothesis of IID variables at lower dimensions, but at higher dimensions, the null hypothesis could be rejected indicating the presence of a nonlinear non-random structure. The BDS test reveals the presence of a hidden nonlinear structure, but the test fails to reveal if the nonlinear structure is chaotic or not. To understand whether this nonlinear structure is chaotic, we need to test if it is sensitive to its initial conditions. The Lyapunov exponent was significantly positive, implying that the nature of the nonlinear structure is chaotic, and the p-value indicates that this isn't just due to noise.

S. No.	Stage	Method	Result
1	Initial Diagnosis	ACF & PACF	Slow decay – evidence of non-stationarity.
2	Structural Breaks	Ruptures (Binary Segmentation)	Four breakpoints were detected and modelled with level & trend dummies.
3	Non-stationarity	Perron's Unit Root Test	Trend stationary with breaks
4	Linear Dependence	Fitting a Box-Jenkins	ARMA (1,4)
5	Conditional Heteroskedasticity	ARCH/GARCH	GARCH (1,1)
6	Chaos Detection – Step 1	Correlation Dimension	Plateauing curve
7	Chaos Detection – Step 2	BDS Test	Rejects IID at higher dimensions – nonlinear hidden structure detected.
8	Chaos Detection – Step 3	Lyapunov Exponent	Positive value – confirms sensitivity to initial conditions.

7. Findings and Insights

The question that the study aimed to address is why asset prices fluctuate so much. The findings of this research provide a clear understanding as to why we observe such large movements in asset prices. Asset prices not only fluctuate due to random external shocks but due to chaotic structure. The reason why financial distributions have such heavy fat tails is due to an endogenous nonlinear structure with sensitive initial conditions which is driving these large movements in asset prices. Even after removing structural breaks, non-stationarity, linear dependencies, and conditional heteroskedasticity, the remaining set of cleaned residuals still concealed a hidden nonlinear chaotic structure.

These findings challenge the Efficient Market Hypothesis (EMH) in its weak form which implies that asset prices follow a random walk and future price changes cannot be consistently forecasted using past historical prices. However, the findings of this research indicate that changes in asset prices are not purely random and stochastic as a nonlinear chaotic structure also drives movements in asset prices. Therefore, it would be inaccurate to model asset prices using traditional random walk models, and secondly, nonlinear chaotic models would yield better short-term forecasting results. Since changes in asset prices are not purely stochastic, the presence of chaos questions early forms of the Efficient Market Hypothesis (EMH) and provides an alternate and accurate explanation that changes in asset prices are also endogenously generated. Finally, the presence of chaos would mean that asset prices could be locally predictable while remaining globally unpredictable. This supports the view that market efficiency may vary over time – that the notion of market efficiency is not a constant due to the bifurcation property of chaotic systems.

8. Conclusion and Future Research

The results of the study present a strong case that fluctuations in asset prices are endogenously generated due to a complex nonlinear process with highly sensitive initial conditions. There is strong evidence for chaotic structure in the EUR/USD currency pair even after accounting for structural breaks, non-stationarity (deterministic trend), linear dependencies, and conditional heteroskedasticity. The cleaned residuals displayed evidence of structure at higher embedding dimensions, and the Lyapunov exponent further confirmed that the remaining hidden structure is nonlinear and chaotic. Therefore, asset prices that appear random on the surface conceal a hidden and complex structure underneath which is driving these large movements in asset prices. These findings question the assumptions of the Efficient Market Hypothesis (EMH), especially in its weak form, and challenge the notion that asset prices follow a random walk because of the nonlinear structure captured in asset prices. Therefore, it is inaccurate to model asset prices through traditional linear models because wild fluctuations in asset prices are also partly endogenously generated which can only be captured through sophisticated nonlinear models.

The implications of our findings are critical – if financial markets are chaotic, then nonlinear models should have better out-of-sample predictive power in the short run, however, forecasting power in the long run will remain low due to the hypersensitive initial conditions of the chaotic process. This opens the door to a richer understanding of the financial market – where the degree of market efficiency changes over time as the bifurcations of chaotic processes will not allow the notion of market efficiency to be constant across all market phases.

This research provides the base for future research in quite a few areas. Firstly, the methodology of these results should be applied to various other currency pairs and financial assets across developing and developed countries to generalize our findings at a broader scale. Secondly, the nature of the structure driving movements in asset prices is established – that a hidden chaotic process is causing large movements in asset prices, however, the exact process of the chaotic structure needs to be established. Once the form of the chaotic process is established, it can be used for out-of-sample forecasting to check if it yields superior results. Finally, the degree of market efficiency cannot be the same across all market phases simply because chaotic processes bifurcate with time. Therefore, the degree of market efficiency needs to be objectively and statistically tested using Permutation Entropy (PE).

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