Alexander Hebert
ECE 6390
Computer Project #3

Given the linear system,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.11323 & -0.98109 & -11.847 & -11.847 & -63.080 & -34.339 & -34.339 & -27.645 & 0 \\ 324.121 & -1.1755 & -29.101 & 0.12722 & 2.83448 & -967.73 & -678.14 & -678.14 & 0 & -129.29 \\ -127.30 & 0.46167 & 11.4294 & -1.0379 & 13.1237 & 380.079 & 266.341 & 266.341 & 0 & 1054.85 \\ -186.05 & 0.67475 & 16.7045 & 0.86092 & -17.068 & 555.502 & 389.268 & 389.268 & 0 & -874.92 \\ 341.917 & 1.09173 & 1052.75 & 756.465 & 756.465 & -29.774 & 0.16507 & 3.27626 & 0 & 0 \\ -30.748 & -0.09817 & -94.674 & -68.029 & -68.029 & 2.67753 & -2.6558 & 4.88497 & 0 & 0 \\ -302.36 & -0.96543 & -930.96 & -668.95 & -668.95 & 26.3292 & 2.42028 & -9.5603 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.6667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.000 \end{bmatrix} \tag{1a}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.66667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}^T \tag{1b}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.49134 & 0 & -0.63203 & 0 & 0 & -0.20743 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1c}$$

$$D = 0_{2\times 2} \tag{1d}$$

The original system has A with size 10 x 10, and it will be reduced using the matrix sign approach. The model reduction was coded in Python. LTI simulation, controller design, and Kalman filter design were coded in Matlab.

1)

A has the following eigenvalues:

```
-0.23448+0.00000j,
-0.34915+6.34401j,
-0.34915-6.34401j,
-1.04205+0.00000j,
-1.66670+0.00000j,
-10.00000+0.00000j,
-10.74614+0.00000j,
-17.66419+0.00000j,
-29.46253+313.93671j,
-29.46253-313.93671j
```

The last complex conjugate pair has very large imaginary components, which results in strong oscillations.

The mean of those eigenvalues is γ:

$$\gamma = \frac{\sum_n \sigma(A)}{n} = \quad 10.097693$$

The pole-zero map with the imaginary axis at γ is shown in Figure 1. There are six dominant poles/eigenvalues.

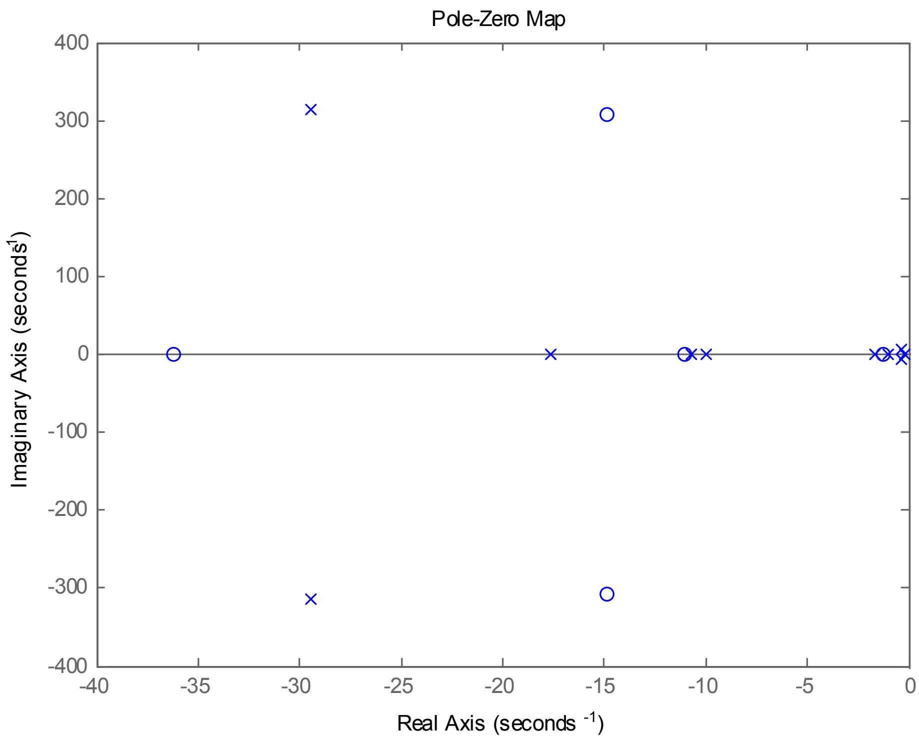

Figure 1. Pole-zero map with the imaginary axis at $\gamma = -10.097693$.

Since there are six dominant poles/eigenvalues and no other restrictions (e.g. realization/implementation), the reduced order model will be sixth order. After completing the matrix sign algorithm, the following results for A, B, C, and D are obtained.

Λ_1 =

```
[[  -1.69221    1.00404   -0.79535   -0.65912   -0.00000   -6.52054]
 [  15.27267   -0.21879  -26.78675  -28.84512  -27.64500  106.81115]
 [ 305.43843   -0.91883   54.21425   58.25123    0.00000 -345.25166]
 [-282.81502    0.85008  -50.51501  -54.27808    0.00000  442.60752]
 [   0.00000    0.00000    0.00000    0.00000   -1.66670    0.00000]
 [   0.00000    0.00000    0.00000    0.00000    0.00000  -10.00000]]
```

B_{d1} =

```
[[  -0.00000    0.00000]
 [   0.00000   -0.00000]
 [   0.00000   -0.00000]
 [  -0.00000    0.00000]
 [   1.66667    0.00000]
 [   0.00000   10.00000]]
```

C_{d1} =

```
[[   1.01935   -0.00168    0.01139    0.00707   -0.00486    1.42703]
 [  -0.11996   -0.00172    1.22169    1.31947   -0.00852   -9.89178]]
```

D^* =

```
[[   0.00073353   -0.981931]
 [   0.00155483    5.11005]]
```

The impulse response of the original system and 6th order reduced model is shown in Figure 2. The 6th order reduced model is very close compared to the original system.

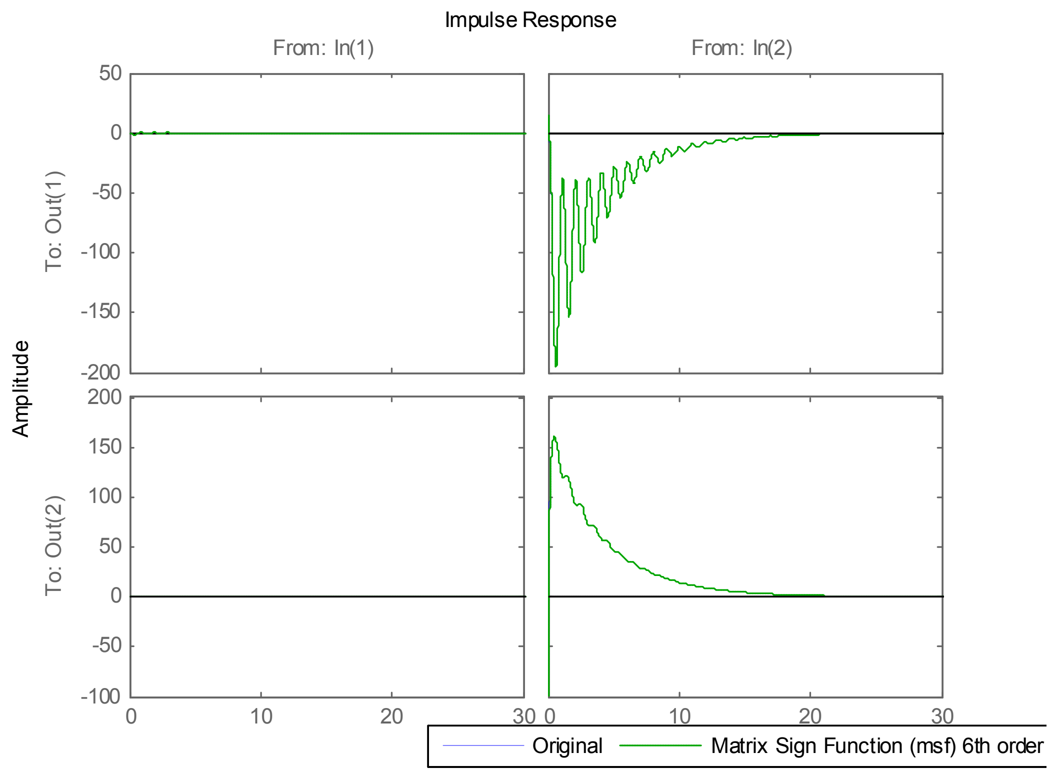

Figure 2. Impulse response of the original system and 6th order reduced model.

A state-feedback controller is designed using Matlab's *place.m* command. The set of desired poles/eigenvalues was determined by trial and error (looking at simulation response) with one constraint. Complex conjugate pairs must have real and imaginary parts with equal magnitude. That requirement corresponds to 45° angle with the real axis and the complex poles on the s-plane (and other characteristics).

The desired eigenvalues were selected as

```
  -0.5,
   -1,
   -2,
-40+40j,
-40-40j,
   -45
```

```
K = place(L1,Bd1,A_eigvals_desired)
```

yields

K =

```
[ -384.1933  -35.2865   -6.7088  -12.6971   38.6699  -39.7562
     6.3281    0.8159    0.3510    0.5159   -0.3026    5.0408]
```

If the states are assumed measurable and the controlled system ($A_{controlled} = A - BK$) is simulated, the impulse response is shown in Figure 3. The controlled response appears to be greatly improved.

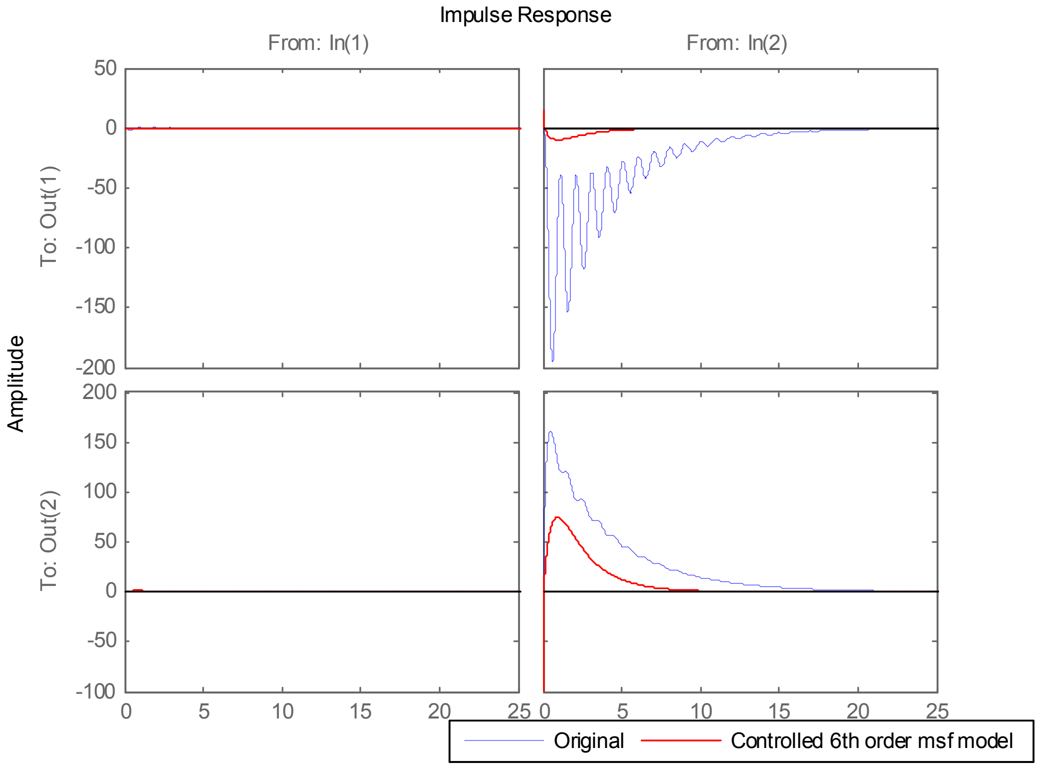

Figure 3. Impulse response of the original system and 6th order reduced model with control applied.

2)

Assuming the state variables are not measurable, an observer based on a steady-state Kalman filter is designed for a generic noise profile. Therefore, $F = V = W = I_6$.

For steady-state, the algebraic Riccati equation (ARE) is

$$\dot{\hat{P}} = A\hat{P} + \hat{P}A^T \quad \hat{P}C^T W^{-1} C\hat{P} + FVF^T = 0$$

After adjusting the form to be compatible, the ARE is solved using Matlab's *care.m* function.

```
[P_hat,L,G,report] = care(Lambda1',Cd1',Q)  % where Q = FVF^T = I
```

yields $\hat{P}$ =

```
  3.2248e+000  1.2934e+001  3.2844e+000 -5.3859e+000 -8.8212e-002 -5.8182e-002
  1.2934e+001  4.2406e+002  2.0314e+003 -1.9062e+003 -2.8545e+000 -5.0729e-001
  3.2844e+000  2.0314e+003  1.4254e+004 -1.3212e+004 -1.1621e+001 -6.2528e-001
 -5.3859e+000 -1.9062e+003 -1.3212e+004  1.2258e+004  1.0740e+001  1.0387e+000
 -8.8212e-002 -2.8545e+000 -1.1621e+001  1.0740e+001  2.9383e-001  3.1520e-004
 -5.8182e-002 -5.0729e-001 -6.2528e-001  1.0387e+000  3.1520e-004  4.9172e-002
```

The Kalman gain is then

$$\widehat{K} = \left(\hat{P}C^T\right)W^{-1}$$

$\widehat{K} =$

3.1822	-2.9268
21.4238	-30.6732
68.0451	-17.4229
-64.6893	26.3607
-0.1425	-0.0154
0.0119	0.1281

The overall system is built from blocks as follows

$$\begin{bmatrix} \hat{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & & B_{d1}K & \\ \widehat{K}C_{d1} & \Lambda_1 & \widehat{K}C_{d1} & B_{d1}K \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_{d1} \\ B_{d1} \end{bmatrix} r$$

$$y = C_{d1}x + D$$

From Matlab,

a =

	x1	x2	x3	x4	x5
x1	-1.692	1.004	-0.7953	-0.6591	-2.352e-014
x2	15.27	-0.2188	-26.79	-28.85	-27.64
x3	305.4	-0.9188	54.21	58.25	2.43e-012
x4	-282.8	0.8501	-50.52	-54.28	8.44e-013
x5	0	0	0	0	-1.667
x6	1.918e-018	5.777e-020	1.74e-018	1.883e-018	1.127e-018
x7	3.595	-0.0003231	-3.539	-3.839	0.009463
x8	25.52	0.01666	-37.23	-40.32	0.1571
x9	71.45	-0.08444	-20.51	-22.51	-0.1823
x10	-69.1	0.06346	31.47	34.32	0.08989
x11	-0.1434	0.000266	-0.02047	-0.02136	0.0008242
x12	-0.003206	-0.00024	0.1567	0.1691	-0.001149

	x6	x7	x8	x9	x10
x1	-6.521	-3.447e-012	-4.396e-013	-1.864e-013	-2.748e-013
x2	106.8	5.614e-010	7.224e-011	3.1e-011	4.559e-011
x3	-345.3	3.365e-009	4.332e-010	1.86e-010	2.734e-010
x4	442.6	-3.142e-009	-4.044e-010	-1.736e-010	-2.553e-010
x5	0	640.3	58.81	11.18	21.16
x6	-10	-63.28	-8.159	-3.51	-5.159
x7	33.49	-5.287	1.004	2.744	3.18
x8	334	-10.25	-0.2354	10.44	11.48
x9	269.4	234	-0.8344	74.72	80.76
x10	-353.1	-213.7	0.7866	-81.98	-88.6
x11	-0.05081	640.5	58.81	11.2	21.18
x12	-1.25	-63.28	-8.158	-3.667	-5.328

```
                 x11          x12
   x1    1.715e-013  -2.657e-012
   x2   -2.704e-011   4.447e-010
   x3   -1.619e-010   2.668e-009
   x4    1.512e-010  -2.491e-009
   x5        -64.45        66.26
   x6         3.026       -50.41
   x7     -0.009463       -40.01
   x8         -27.8       -227.2
   x9        0.1823       -614.7
   x10     -0.08989        795.7
   x11       -66.12        66.31
   x12        3.028       -59.16

b =
                  u1           u2
   x1   -3.305e-016   5.246e-013
   x2    9.446e-015  -8.814e-011
   x3     4.93e-014  -5.288e-010
   x4   -4.528e-014   4.938e-010
   x5         1.667            0
   x6    3.224e-019           10
   x7   -3.305e-016   5.246e-013
   x8    9.446e-015  -8.814e-011
   x9     4.93e-014  -5.288e-010
   x10  -4.528e-014   4.938e-010
   x11        1.667            0
   x12   3.224e-019           10

c =
               x1          x2          x3          x4          x5          x6
   y1       1.019   -0.001681     0.01139    0.007068   -0.004861       1.427
   y2       -0.12   -0.001717       1.222       1.319   -0.008518      -9.892

               x7          x8          x9         x10         x11         x12
   y1       1.019   -0.001681     0.01139    0.007068   -0.004861       1.427
   y2       -0.12   -0.001717       1.222       1.319   -0.008518      -9.892

d =
               u1          u2
   y1   0.0007335     -0.9819
   y2    0.001555        5.11
```

The impulse response of the original system and overall system (6^{th} order reduced model with steady-state Kalman filter and control applied) is shown in Figure 4. The overall system response appears to be acceptable (without additional information).

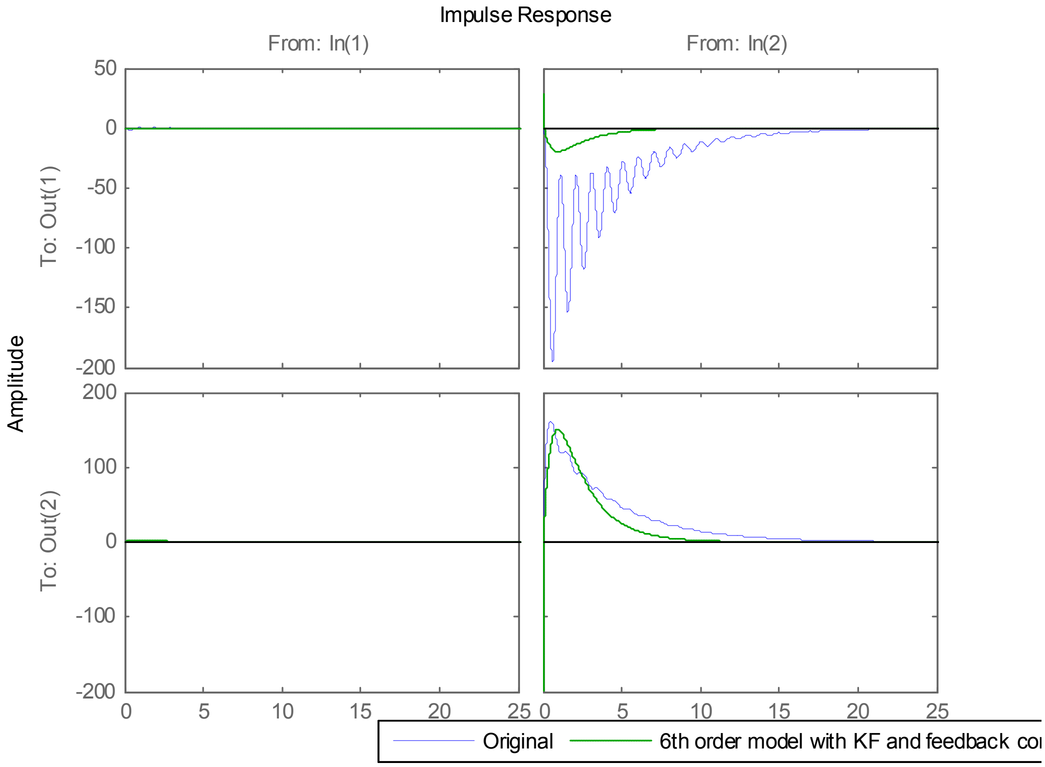

Figure 4. The impulse response of the original system and overall system (6[th] order reduced model with steady-state Kalman filter and control applied).

Appendix

Matlab code

```
% Alexander Hebert
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% Computer Project #3 continued from Python

clear; clc;

A = load('A.txt');
B = load('B.txt');
C = load('C.txt');

disp('Original system:')
sys_original = ss(A,B,C,0)

figure(1)
pzmap(sys_original)


L1 = dlmread('Lambda1.txt',' '); % L = capital lambda
Bd1 = dlmread('Bd1.txt',' ');
Cd1 = dlmread('Cd1.txt',' ');
Dstar = dlmread('Dstar.txt',' ');

disp(' ')
disp('Matrix Sign Function (msf) 4th order reduced model:')
sys_msf_2nd = ss(L1,Bd1,Cd1,Dstar)

figure(2)
impulse(sys_original,'b',sys_msf_2nd,'g--')
legend('Original','Matrix Sign Function (msf) 6th
order','Location','SouthOutside','Orientation','horizontal')


% L1_eigvals = [...
% -3.491460840764012419e-01+6.344006476918209181e+00j,...
% -3.491460840764012419e-01-6.344006476918209181e+00j,...
% -1.042053905840172989e+00,...
% -2.344831273439368091e-01,...
% -1.000000000000000355e+01,...
% -1.666700000000000070e+00]

% Decide on desired pole/eigenvalue locations (determined by trial and error).
% Complex conjugate pair must have equal real and imaginary parts.
A_eigvals_desired = [-0.5, -1, -2, -40+40j, -40-40j, -45]

% Design controller for reduced order model.
% Compute state-feedback matrix K using Matlab's place command
K = place(L1,Bd1,A_eigvals_desired)

A_control = L1 - Bd1*K;
sys_msf_2nd_control = ss(A_control,Bd1,Cd1,Dstar)

figure(3)
```

```
impulse(sys_original,'b',sys_msf_2nd_control,'r--')
legend('Original','Controlled 6th order msf
model','Location','SouthOutside','Orientation','horizontal')

% Design steady-state Kalman filter with generic noise profile.
% F = V = W = I

% Steady-state Kalman filter
% Solve ARE for error covariance matrix P
% For Matlab's care() function, R^-1 = W^-1 = I

% For compatibility with class notes:
% A_care = A'
% B_care = C'
% X_care = P
% Q = F*V*F' = I
Q = eye(6);

[P_hat,L,G,report] = care(L1',Cd1',Q);
report

% Steady-state Kalman gain
K_hat = P_hat*Cd1' % W^-1 = I so it is left out of the product

% Overall system with Kalman filter and state-feedback control:
A_co = [L1, -Bd1*K; K_hat*Cd1, (L1 - K_hat*Cd1 - Bd1*K)];
B_co = [Bd1; Bd1];
C_co = [Cd1, Cd1];

sys_co = ss(A_co,B_co,C_co,Dstar)

figure(4)
impulse(sys_original,'b',sys_co,'g--')
legend('Original','6th order model with KF and feedback
control','Location','SouthOutside','Orientation','horizontal')
```