Variational Principles & Cosmology On Manifolds With Boundary

J. LINLEY

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Abstract

New field equations for gravity are derived by implementing a certain method of variation of the Einstein-Hilbert action, proposed in particular so that boundary terms can be properly defined and explicitly evaluated. It is argued that this is the only well-posed method of varying *functions* on manifolds, as opposed to the variation of differential k-forms, for which the usual Euler-Lagrange equations are appropriate. The equations reduce to the Einstein Field Equations in regions of empty space, sufficiently distant from spacetime boundaries. When the equations are satisfied, the action vanishes, supporting the notion of the zero-energy universe.

Cosmological symmetry is applied to the field equations, and boundary terms are computed. The relatively simple cases of a real scalar field and sourced electromagnetic field are investigated as gravitational sources. It is found that a universe containing only uniform scalar fields has vanishing spatial curvature and exponential solutions. For the electromagnetic case, strict symmetry precludes interesting dynamics, nonetheless, it is a useful example of an action constructed from both functions and k-forms.

I. Introduction And The Einstein-Hilbert Action

Formulating theories of gravity and other physical systems is most often achieved in a modern and rigourous setting by applying the stationary action principle to an appropriate action. The Einstein Field Equations (EFEs) can be obtained from the Einstein-Hilbert (EH) action, well-known to be the simplest and most natural quantity containing all of the necessary information about the geometry of spacetime, together with a suitable matter fields action[1]. Proposals of other actions for gravity abound[2], but the extraordinary success of General Relativity (GR) and the failure of theories derived from alternative actions in superseding GR ensures that the EH action remains to be regarded as the safest candidate. However, a number of assumptions seem to be (inadvertently) ubiquitously applied when performing variations of actions that are not entirely satisfactory in a general context, which are to be examined in detail in this work.

Besides considerations of such techniques employed in variational calculus, there are features of GR that make the theory self-inconsistent and observations of large-scale cosmological dynamics that are incompatible with its theoretical predictions, which both urge further study. In particular, the Hawking-Penrose[3] singularity theorems guarantee problematic big-bang and black-hole singularities, and phenomena presently attributed to dark matter and dark energy cannot be explained without modifying our standard models of physics[4,5,6,7].

As we will see, the first problem that occurs when varying the EH action with respect to metric tensor components is that a boundary term is produced, which is not well-defined using the standard variational approach[8,9,10]. The EFEs are recovered only in the case that the spacetime manifold does not have a boundary. If the universe does have a boundary (in at least one of the space or time parts), then we retrieve *effective* EFEs from the EH action, by making the assumption that contributions of any boundary influences to the field equations on our local tests of gravity are negligible, since the scale of our tests compared to the cosmic scale is very small, and our location in spacetime appears to be very far from any such boundary. One might however expect that in studies of cosmology on appropriately large scales, potential boundary effects could produce significant contributions to the system dynamics, and must not be neglected. We should try to retain contributions to the field equations from boundaries in this case, to study cosmology on spacetimes with boundaries.

We start by stepping through the standard variation of the EH action, $\mathcal{S}_{EH}$, that leads to the EFEs. The EH action is defined as[1]

$$\mathcal{S}_{EH} \equiv \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, R, \tag{1}$$

where κ is Einstein's constant, $\mathcal{M}$ is the spacetime manifold with the metric $g_{\mu\nu}$ with determinant g, and R is the Ricci scalar, the twice contracted Riemann tensor $R^{\rho}{}_{\mu\sigma\nu}$

$$R \equiv g^{\mu\nu} R_{\mu\nu},$$

where $R_{\mu\nu} \equiv R^{\gamma}{}_{\mu\gamma\nu} = g^{\gamma\delta} R_{\delta\mu\gamma\nu}$ is the Ricci tensor. We now vary the EH action with respect to the metric

$$\begin{aligned} \delta\mathcal{S}_{EH} &= \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4x\, \delta(\sqrt{-g}\, R), \\ &= \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4x\; \delta(\sqrt{-g}) R + \sqrt{-g}\, \delta R. \end{aligned} \tag{2}$$

From the first term in the integrand, we have

$$\delta\sqrt{-g} = \frac{-\delta g}{2\sqrt{-g}} = -\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \,,$$

where the last step comes from Jacobi's formula for the variation of the metric's determinant[11]: $\delta g = -g\, g_{\mu\nu}\, \delta g^{\mu\nu}$.

With the definition of the Ricci scalar, the second term is

$$\begin{aligned} \delta R &= \delta(g^{\mu\nu} R_{\mu\nu}) \\ &= \delta(g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \,. \end{aligned}$$

Putting these together in (2) yields a variation of the EH action (the two distinct sets of summation indices used below will be useful later)

$$\delta\mathcal{S}_{EH} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4x \,\sqrt{-g}\,\left[\delta g^{\mu\nu} (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) + g^{\rho\sigma} \delta R_{\rho\sigma}\right] \,. \tag{3}$$

Let us now add the action of the matter fields to the EH action for the total action. The matter fields action is written as[1]

$$\mathcal{S}_M = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\mathcal{L}_M. \tag{4}$$

The variation of the matter action is then

$$\begin{aligned} \delta\mathcal{S}_M &= \int_{\mathcal{M}} \mathrm{d}^4x\ \delta(\sqrt{-g}\mathcal{L}_M) \\ &= \int_{\mathcal{M}} \mathrm{d}^4x\ \sqrt{-g}\ \left(-\tfrac{1}{2}\delta g^{\mu\nu} g_{\mu\nu}\mathcal{L}_M + \delta\mathcal{L}_M\right)\ . \end{aligned} \tag{5}$$

The total action is the sum $\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_M$, so varying the total action gives

$$\begin{aligned} \delta\mathcal{S} &= \delta\mathcal{S}_{EH} + \delta\mathcal{S}_M \\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\ \sqrt{-g}\left[\delta g^{\mu\nu}\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - \kappa g_{\mu\nu}\mathcal{L}_M\right) + g^{\rho\sigma}\,\delta R_{\rho\sigma} + 2\kappa\,\delta\mathcal{L}_M\right]\ . \end{aligned} \tag{6}$$

The integrand in (6) involves the quantity $\delta R_{\alpha\beta}$. Since $R_{\alpha\beta}$ is composed from the metric components and their derivatives, the variation includes factors of $\delta g_{\mu\nu}$, $\delta(\partial_\rho g_{\mu\nu})$, $\delta(\partial_\rho\partial_\sigma g_{\mu\nu})$ and similar variations of derivatives of inverse metric components. The appearance of variations of the derivatives is problematic with the standard variational approach, although solutions do exist to deal with $\delta R_{\alpha\beta}$ whether the manifold has a boundary or not.

Ordinarily, the variation of the field is assumed to vanish on the boundary, in which case $\delta R_{\alpha\beta}$ may be reformulated as a total derivative and rewritten as a boundary term using Stokes' theorem[11]. Then, if the manifold has no boundary, the integration region is the empty set, so this term vanishes. For manifolds with a boundary however, the action is usually altered by introducing the *Gibbons-Hawking-York* term[9], which cancels the ill-defined boundary term contribution from the variation. This revision keeps the variation well defined, but modifies the desirable EH action, and removes potential contributions to the equations of motion from the boundary that we wish to determine.

In the canonical derivation of the EFEs, the condition that variations of the metric vanish on the boundary is thought to be physically reasonable by analogy to other fields and in classical mechanics. The boundary points of trajectories of isolated particles, for example, must remain fixed under path variations, since variations of the endpoints would amount to solving for a different trajectory. The case made for continuous fields is that, for deriving equations of motion applicable in some local laboratory setting, the spacetime boundaries are so far from the region of interest that they will not have any significant effects on our experiments. Thus we can safely make the variations vanish there, knowing that the equations of motion obtained from the variational principle will be correct to a good approximation.

For general spacetimes, however, such arguments encounter problems. Take, for instance, some non-trivial action with cosmological symmetry. If the spacetime manifold is initially bounded in the time part by a spacelike hypersurface, then it is evident from the construction that matter is present on the boundary. In this case, placing any sort of constraint on variations of parts of the action, where it is known that matter exists, is not very well substantiated.

These reflections lead to the view that constraining variations at boundaries is unsuitable for the general case. Let us then relax this condition, so in general $\delta g^{\mu\nu}|_{\partial\mathcal{M}}$ is free and need not vanish. If we postulate that the universe is with boundary, then it is not possible to obtain the EFEs from the unmodified EH action, since the $\delta R_{\alpha\beta}$ term (hereafter referred to as the boundary term) cannot be discarded. We now investigate a means by which to obtain the equations of motion from the EH (and matter fields) action on a manifold with boundary.

It will be important to be vigilant in recognising that certain methods used to generate variations will require distinct physical interpretations in order to be treated appropriately. As we will see, some variations can be subject to certain constraints, and others may not be allowed to be performed at all.

Progress can be made with the stationary action principle if there can be found a method of variation which leads to a well-defined *quotient of variations* (not the so-called functional derivative) so that the object

$$\frac{\delta(...)}{\delta g^{\mu\nu}}\,,$$

has a precise meaning and mathematical expression, depending on the nature of the variation operator. If a well-defined quotient of variations can be constructed where the equation

$$\frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} = 1\,, \tag{7}$$

holds (where for the moment we avoid attaching meaning to matching indices) then we are able to write equation (6) as

$$\delta\mathcal{S} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + g^{\rho\sigma}\,\frac{\delta R_{\rho\sigma}}{\delta g^{\mu\nu}} - \kappa g_{\mu\nu}\mathcal{L}_M + 2\kappa\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\right)\,. \tag{8}$$

Further to these conditions, if it can be shown that all quotients of variations appearing in the brackets in (8) are independent of the arbitrary generator of the variation, then the variational principle holds, which will establish that for this action to be at a stationary point for arbitrary $\delta g^{\mu\nu}$, we must have that

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + g^{\rho\sigma}\,\frac{\delta R_{\rho\sigma}}{\delta g^{\mu\nu}} - \kappa g_{\mu\nu}\mathcal{L}_M + 2\kappa\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} = 0\,. \tag{9}$$

We therefore seek variational methods consistent with the requirements laid out above in order to retrieve equations of motion for a universe with boundary. Let us now consider a few approaches to generating variations. The following examples are not intended to provide an exhaustive list, but important cases pertinent to the main result are covered.

II. Variational Methods

1. Variation by diffeomorphism
Any smooth map $\phi : \mathcal{M} \to \mathcal{M}'$, with $\mathcal{M}'$ diffeomorphic to $\mathcal{M}$, can be generated by moving points on the manifold $p \in \mathcal{M}$ along a corresponding vector field V. We may then generate arbitrary variations from infinitesimal diffeomorphisms by moving points along the (smooth) vector field ϵV, for arbitrary generator V and infinitesimal ϵ. Thus points $p \to p' = \sigma_V(\epsilon, p)$, where $\sigma_V(\epsilon, p)$ denotes the flow along the integral curve, $\mathcal{C}(\lambda)$, of the vector field V through p, from p by a distance ϵ along λ. We find that variations of metric components are given by[12] their Lie derivatives with respect to the vector field ϵV, $\pounds_{\epsilon V}$, so that

$$\delta g_{\mu\nu} = -\epsilon \pounds_V g_{\mu\nu} + \mathcal{O}(\epsilon^2)\,. \tag{10}$$

Diffeomorphisms are equivalent to coordinate system transformations when the coordinates are moved in the opposite direction along the vector field V. The same variation in (10) can therefore be obtained if the coordinates transform as $x^\mu \to x'^\mu = x^\mu - \epsilon V^\mu(x)$.

Field equations cannot be obtained by diffeomorphism variations, since the EH action is diffeomorphism invariant. However, infinitesimal diffeomorphisms are important variations as they are used to show that the *stress-energy* tensor is conserved, although in the final section the standard proof of this is reviewed. There is no general intrinsic constraint forcing $\delta g^{\mu\nu}|_{\partial\mathcal{M}} = 0$ for diffeomorphisms, however diffeomorphism invariance of the action requires that the vector field generating the variation is tangent to boundaries on the boundaries.

2. Variation of geometry
Variations may be generated from physical perturbations of the geometry. As a result of a general perturbation, any combination of metric components may be affected. However, for perturbations affecting multiple components, it is not clear how to interpret the quotient of variations in (7). In order to satisfy our requirement that equation (7) holds, let us consider variations of the geometry that affect only one of the metric components. The μ, ν index sum in (6) then leaves one surviving term and the quotient in (7) will be well-defined when the indices belong to the varied component. Using this variation we can write $\delta\mathcal{S}$ as in (8), so long as we now take the μ, ν indices to represent a specific component and the ρ, σ indices to represent a summation.

Note that restricting the variation to a single metric component may impose constraints on the variation itself, depending on the geometry and the coordinate system. For example, the variation of Minkowski space in spherical coordinates $g_{\theta\theta}(r) = r^2 \to g_{\theta\theta}(r) + \delta g_{\theta\theta}(x) = r^2 + \delta g_{\theta\theta}(x)$, has the line element

$$ds'^{\,2} = -dt^2 + dr^2 + (r^2 + \delta g_{\theta\theta})\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,.$$

Demanding that the varied metric yields a physically admissible geometry, we must fix $\delta g_{\theta\theta}|_{r=0} = 0$, otherwise an observer making rotations at the origin would be attributed a non-zero spacetime interval. Strictly, we should not allow this variation, since it cannot be arbitrary everywhere.

However, suppose that our manifold admits a coordinate system that does not impose constraints on the variations. We still must deal with the existence of variations of the derivatives. As discussed, and since the variation operators commute with the partial derivative operators, such terms may, when integrated, be written as boundary terms of the integral. For manifolds without boundary, the boundary term vanishes. For manifolds with boundary however, these terms remain and must be evaluated. In the canonical methods of variational calculus, the action is formulated in such a way that when Lagrangians contain n^{th} order derivatives, the first $n-1$ derivatives of the variation on the boundary must be fixed[13]. This is generally deemed acceptable for the zeroth order fixing, but it is progressively more difficult to justify higher order fixings.

Without justification to fix the required parts of the variation, the stationary action principle fails, since the presence of the remaining terms renders impossible the manipulation of the integrand into a form that allows the fundamental lemma of variational calculus to be applied. However, the next type of variation we consider does not produce derivatives of the arbitrary generator, allowing for the arbitrary parts to cancel each other in quotients of variations. Finally, note that there is no intrinsic requirement that variations of the geometry vanish on the boundaries.

3. Variation by pull-back of metric components

We seek an approach to producing variations whereby quotients of variations of metric components and their derivatives, as they appear in the boundary term in (8), are independent of the arbitrary generator of the variation.

Take two neighbouring points on a manifold, p_1 and p_2. Given a coordinate system, let us consider generating variations of metric components, their inverses and partial derivatives at p_1, by instead taking values of the respective functions at p_2. As we shall see, all variations, including variations of derivatives, depend only on the method by which p_2 is chosen, given p_1.

When a coordinate system is supplied, the metric components can be viewed simply as functions of those coordinates. Functions $\gamma_{\mu\nu}$ on $\mathcal{M}$ corresponding to the $g_{\mu\nu}$ may be defined using maps $\psi : \mathcal{M} \to \mathbb{R}^4$ via

$$g_{\mu\nu} \equiv \gamma_{\mu\nu} \cdot \psi^{-1} : \mathbb{R}^4 \to \mathbb{R} \ .$$

Now if we have a map $\phi : \mathcal{M} \to \mathcal{M}'$, where ϕ is an infinitesimal diffeomorphism, we can generate the pull-back of a *function*, which we will use as the *generator* of a metric component's variation, denoted with a prime

$$(g_{\mu\nu})' \equiv \phi^* g_{\mu\nu} \ .$$

Let us now take basis vectors $\hat{e}_{(\beta)}$ and coordinates x^β on $\mathcal{M}$, to basis vectors $\hat{e}_{(\beta')}$ and coordinates $x'^{\beta'}$ on $\mathcal{M}'$, using the diffeomorphism ϕ. By doing this, all of the primed metric components and their derivatives on $\mathcal{M}'$ will be the same functions of coordinates $x'^{\beta'}$ as the unprimed components and their derivatives on $\mathcal{M}$ will be of coordinates x^β.

In other words, we have generated a copy of the geometry and the coordinate system, thus giving rise to primed metric components (and their derivatives) on $\mathcal{M}'$ which are *form invariant* in relation to the unprimed quantities on $\mathcal{M}$. For instance, $g_{\mu\nu}(x) = g'_{\mu'\nu'}(x')$, where $x_p = x'_{p'}$. Similar relations apply to all metric components, their inverses, and all partial derivatives between the unprimed and primed copies of the objects. This setup allows pull-backs of the functions onto the primed spacetime to just be written down with primes, for example

$$\phi^*(g_{\mu\nu,\alpha})(x) = g'_{\mu'\nu',\alpha'}(x)\,.$$

If ϕ is generated by the infinitesimal vector field ϵV, then to first order in ϵ

$$x'^{\beta} = (\phi^* x)^{\beta} = x^{\beta} + \epsilon V^{\beta}\,,$$

so, finally varying, for example, a partial derivative of a metric component gives

$$\begin{aligned} g_{\mu\nu,\alpha}(x) \to (g_{\mu\nu,\alpha})'(x) &= \phi^*(g_{\mu\nu,\alpha})(x) = g'_{\mu'\nu',\alpha'}(x) \\ &= g'_{\mu'\nu',\alpha'}(x' - \epsilon V) \\ &= g'_{\mu'\nu',\alpha'}(x') - \epsilon V^{\sigma'}(x') g'_{\mu'\nu',\alpha'\sigma'}(x') \\ &= g_{\mu\nu,\alpha}(x) - \epsilon V^{\sigma}(x) g_{\mu\nu,\alpha\sigma}(x) \end{aligned}$$

to first order in ϵ, since $\epsilon V^{\sigma'}(x') = \epsilon V^{\sigma}(x) + \mathcal{O}(\epsilon^2)$ and using the invariance of the form of the metric components and their partial derivatives between unprimed and primed manifolds. Therefore the variation is

$$\delta(g_{\mu\nu,\alpha}) = (g_{\mu\nu,\alpha})' - g_{\mu\nu,\alpha} = -\epsilon V^{\sigma} g_{\mu\nu,\alpha\sigma} + \mathcal{O}(\epsilon^2)\,.$$

The variation of the first derivative of a metric component has been shown here to demonstrate the method, but the transformation applies in the same way to all dependants of the pulled back function, including the inverse metric component and any other derivatives. We therefore define the *variation operator* for pull-backs on functions and their dependants by

$$\delta \equiv -\epsilon V^{\sigma} \frac{\partial}{\partial x^{\sigma}}\,. \tag{11}$$

To be clear, what is meant by a 'dependant' of a metric component is any object in which a variation will be induced when that component is varied. Of course the inverse metric component will also vary, and so must any of their derivatives; to see this, we can take the familiar definition of the derivative of a function f at a with respect to parameter y. This is

$$f_{,y}(y)|_{y=a} = \lim_{h\to 0} \frac{f(y)|_{y=a+h} - f(y)|_{y=a}}{h}\,,$$

so if f and y are pulled back at a, the corresponding variation induced in $f_{,y}$ is

$$f_{,y}(y)|_{y=a} \to (f_{,y}(y)|_{y=a})' = \lim_{h\to 0} \frac{f'(y')|_{y'=a+h} - f'(y')|_{y'=a}}{h} \equiv f'_{,y'}(y')|_{y'=a}\,. \tag{12}$$

We can now choose the vector field V to contain a single non-vanishing (but arbitrary) element, so that the sum over indices in (11) picks out a single partial derivative. Then, for quotients of variations appearing in the boundary term in (8) where, say $V^\sigma = (f_0(x), 0, 0, 0)$, we may have the term, for example

$$\frac{\delta(g_{\mu\nu,\alpha})}{\delta g^{\mu\nu}} = \frac{-\epsilon f_0 g_{\mu\nu,\alpha x^0}}{-\epsilon f_0 {g^{\mu\nu}}_{,x^0}} = \frac{g_{\mu\nu,\alpha x^0}}{{g^{\mu\nu}}_{,x^0}} \,. \tag{13}$$

For (13) to hold, we require $f_0(x) \neq 0$. The quotient of variations generated by non-zero but otherwise arbitrary f_0, is now independent of f_0. It will be useful to make a further refinement of the definition of the variation operator in (11), by adding a subscript to denote which element of the vector field contains the non-zero function, thus

$$\delta_a \equiv -\epsilon V^a \partial_a \,,$$

where the a index is a coordinate label (not a sum). We can use this to formulate a more useful operator, reminding us which component has been pulled back

$$\frac{\delta_a}{\delta_a g^{\mu\nu}} \equiv \frac{\partial_a}{\partial_a g^{\mu\nu}} \,. \tag{14}$$

It is worth restating that this definition is only valid when it is acting directly on the component that has been pulled back and its dependants, and not in general when acting on objects constructed from those components. A generalised version that can be applied directly to any action may be defined by

$$\delta_a^{g^{\mu\nu}} \equiv -\epsilon V^a \left(\partial_a g^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}} + \partial_a \partial_\gamma g^{\mu\nu} \frac{\partial}{\partial(\partial_\gamma g^{\mu\nu})} + \partial_a \partial_\beta \partial_\gamma g^{\mu\nu} \frac{\partial}{\partial(\partial_\beta \partial_\gamma g^{\mu\nu})} + \ \dots \ \right) ,$$

where '...' stands for similar terms that correspond to any other metric dependants present in the operand. The $^{g^{\mu\nu}}$ superscript describes which fundamental object has been varied, but can be dropped when the context makes clear what that object is, in particular, when its variation appears as the denominator of a quotient, for example

$$\frac{\delta_a^{g^{\mu\nu}}}{\delta_a^{g^{\mu\nu}} g^{\mu\nu}} \equiv \frac{\delta_a}{\delta_a g^{\mu\nu}} \equiv \frac{\partial}{\partial g^{\mu\nu}} + \frac{\partial_a \partial_\gamma g^{\mu\nu}}{\partial_a g^{\mu\nu}} \frac{\partial}{\partial(\partial_\gamma g^{\mu\nu})} + \frac{\partial_a \partial_\beta \partial_\gamma g^{\mu\nu}}{\partial_a g^{\mu\nu}} \frac{\partial}{\partial(\partial_\beta \partial_\gamma g^{\mu\nu})} + \ \dots \ .$$

Let us imagine now that we have varied a single (inverse) metric component with indices g^{ab}. Then performing the summation over the μ and ν indices in (6) gives us

$$\delta_c \mathcal{S} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left[\delta_c g^{ab} \left(R_{ab} - \tfrac{1}{2} g_{ab} R - \kappa g_{ab} \mathcal{L}_M \right) + g^{\rho\sigma} \, \delta_c R_{\rho\sigma} + 2\kappa \, \delta_c \mathcal{L}_M \right] , \tag{15}$$

where we are still summing over ρ & σ. It is clear that

$$\frac{\delta_c g^{ab}}{\delta_c g^{ab}} = 1 \,,$$

holds and is well-defined (if $\partial_c g^{ab}$ exists), and with this we can write (15) as

$$\delta_c \mathcal{S} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g}\, \delta_c g^{ab} \left(R_{ab} - \tfrac{1}{2} g_{ab} R - \kappa g_{ab} \mathcal{L}_M + g^{\rho\sigma} \frac{\delta_c R_{\rho\sigma}}{\delta_c g^{ab}} + 2\kappa \frac{\delta_c \mathcal{L}_M}{\delta_c g^{ab}} \right) . \tag{16}$$

Now, $R_{\rho\sigma}$ and $\mathcal{L}_M$ are functions of the metric components and their derivatives (as well as matter fields for the matter Lagrangian). As demonstrated with (13), any quotient of variations from pull-backs of a metric component is independent of the vector field that generated it, so we can apply the fundamental lemma of variational calculus to (16) and obtain one of the equations of motion:

$$R_{ab} - \tfrac{1}{2} g_{ab} R - \kappa g_{ab} \mathcal{L}_M + g^{\rho\sigma} \frac{\delta_c R_{\rho\sigma}}{\delta_c g^{ab}} + 2\kappa \frac{\delta_c \mathcal{L}_M}{\delta_c g^{ab}} = 0\,. \tag{17}$$

To obtain other components of the field equations, we require repeat applications of the stationary action principle, by pulling back other metric components and by choosing different vector elements as generators of the pull-backs.

That the vector field generating the pull-back contains only one non-vanishing element means that constraints on the variations can arise from the coordinate system in use. Take a spherical coordinate system in 3+1 dimensions, (t, r, θ, ϕ). If we wish to pull back in the r direction, using vector field $V^\sigma = (0, f_1(x), 0, 0)$, then we must have $f_1(x)|_{r=0} = 0$, otherwise the pull-back has an indeterminate radial direction in which to move the origin O. This is a consequence of degeneracy in the coordinates at O, and we must conclude that the proposed vector field is not a suitable generator of arbitrary pull-backs in these coordinates.

Furthermore, degenerate coordinates give rise to metric components that are not well-defined under any variation. In spherical coordinates again, consider a spherically symmetric metric. The g_{rr} component gives the infinitesimal proper distance in the r-r direction, given by the angles θ and ϕ. At the origin O, the angles θ and ϕ are indeterminate, but the symmetry removes the degeneracy in $g_{rr}|_O$. However, if we make an arbitrary variation so that the spacetime is no longer spherically symmetric, then at O, the varied component $(g_{rr})'$ cannot tell us about the infinitesimal proper distance for *all* angles θ and ϕ, since it is a scalar valued function defined uniquely at each point on the manifold. Again we must conclude that the machinery developed here is not capable of allowing variations of metric components of degenerate coordinate charts to be well-defined.

Finally, consider another implication of the stationary total action using the variational method introduced here. Writing the stationary (varied) total action using (1) and (6) as

$$0 = \delta_\gamma \mathcal{S} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x\, \delta_\gamma g^{\mu\nu} \frac{\delta_\gamma}{\delta_\gamma g^{\mu\nu}} \left(\sqrt{-g}\, (R + 2\kappa \mathcal{L}_M) \right) \qquad \text{(no sum over } \mu,\ \nu,\ \gamma\text{)}$$

leads to equations of motion

$$\left(\frac{\delta_\gamma}{\delta_\gamma g^{\mu\nu}} - \tfrac{1}{2} g_{\mu\nu} \right) (R + 2\kappa \mathcal{L}_M) = 0\ . \tag{18}$$

But for matter Lagrangians depending at most on first derivatives of the metric (perhaps all matter Lagrangians) it will always be possible to find a coordinate chart such that at any given point p we can write

$$\frac{\delta_\gamma}{\delta_\gamma g^{\mu\nu}}(R + 2\kappa\mathcal{L}_M) = 0\ . \tag{19}$$

Since p is arbitrary, then using (19) in (18) it must hold everywhere that

$$R = -2\kappa\mathcal{L}_M\ . \tag{20}$$

Using this in equations of motion (18) indeed reduces them to (19), thus we may take the compactly written equations in (19) as the gravitational field equations.

Note that (20) means the total Lagrangian vanishes everywhere, implying a *zero-energy universe*; since $\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_M$, we have from (1) and (4) that the total action is stationary when it is identically zero:

$$(R + 2\kappa\mathcal{L}_M)|_{\delta\mathcal{S}=0} = 0 \qquad \Rightarrow \qquad \mathcal{S}|_{\delta\mathcal{S}=0} = 0\,.$$

III. Construction Of Actions And Applications Of The Euler-Lagrange Equation

As will be shown here, translating the variational method from the last part into the standard formalism for deriving the Euler-Lagrange equations introduces a means by which to produce conflicting equations of motion. This unacceptable situation demands a resolution, which will be attempted in this section.

The variations of metric components generated by pull-backs in the last part, the so-called 'active' coordinate transformations, are equivalent to applying the regular, 'passive' coordinate transformation[1] (in the opposite direction along V) to a single metric component. This passive coordinate transformation will now be used in the language of variational calculus as it is often applied in mechanics. Let us start with the simple action with one-dimensional Lagrangian

$$I[t,q] = \int_a^b d\tau\, L\Big(t(\tau), q(t(\tau)), q_{,t}(t(\tau))\Big)\,, \tag{21}$$

where the subscript $_{,t}$ denotes differentiation with respect to t.

Notice that this Lagrangian does not depend explicitly on τ, which is just an integration variable. Rather it depends on the function $t(\tau)$, acting as a general parameterisation of the integration range to a chosen reference coordinate. The action is now a functional of q and t. Suppose the action in (21), given coordinates $t(\tau)$ and function $q(t)$, is at a stationary point. Then for any function f and infinitesimal ϵ, we assert that[13]

$$I[t,q] = I[t + \epsilon f, q]\,.$$

Varying the coordinate function $t \to t + \epsilon f$ in (21) gives

$$
\begin{aligned}
I[t+\epsilon f, q] &= \int_a^b d\tau \, L\Big(t+\epsilon f, q(t+\epsilon f), q_{,t}(t+\epsilon f)\Big) \\
&= \int_a^b d\tau \, L\Big(t, q(t), q_{,t}(t)\Big) + \epsilon f \left(\frac{\partial L}{\partial t} + \frac{dq}{dt}\frac{\partial L}{\partial q} + \frac{d^2 q}{dt^2}\frac{\partial L}{\partial(\frac{dq}{dt})} \right) + \mathcal{O}(\epsilon^2) \, ,
\end{aligned}
$$

and so given our assertion that $I[t+\epsilon f, q] = I[t, q]$, we find at $\mathcal{O}(\epsilon)$ that

$$
\frac{\partial L}{\partial t} + \frac{dq}{dt}\frac{\partial L}{\partial q} + \frac{d^2 q}{dt^2}\frac{\partial L}{\partial(\frac{dq}{dt})} = 0 \, . \tag{22}
$$

Equation (22) is analogous to the Euler-Lagrange (E-L) equation. Indeed, it *is* one of the E-L equations, since we may write the Lagrangian as a function of t only, $L(t, q(t), q_{,t}(t)) \equiv L_q(t)$. Then the E-L equation is simply

$$
\frac{\partial L_q(t)}{\partial t} = 0 \, ,
$$

and we may expand the dependency on t, writing

$$
\frac{\partial L_q(t)}{\partial t} = \frac{dL_q(t)}{dt} = \frac{dL(t, q(t), q_{,t}(t))}{dt} \, ,
$$

where the term on the right is equivalent to the left hand side of equation (22).

We have not required any boundary fixing of the variation. This is important for making sense of variations of our cosmological solution which must contain matter on the boundary. The form of the Lagrangian appearing in the action in (21) is appealing in this respect, but moreso because of the fact that Newtonian mechanics is recovered when the Lagrangian is equal to the total energy of the system, $L = E_{tot} = T + U$, where T is the kinetic energy and U is the potential energy, rather than the difference of these quantities as is ordinarily required[14]. For example, for the motion of a particle in arbitrary potential $U(q)$, where the Lagrangian is given by $L = \frac{1}{2}m(q_{,t})^2 + U$, equation (22) gives

$$
0 = q_{,t}\frac{dU}{dq} + m q_{,t} q_{,tt} \qquad \Rightarrow \qquad m q_{,tt} = -\frac{dU}{dq} \, ,
$$

where we identify the right hand side with the force on the particle. If $L = E_{tot}$, we also see that equation (22) just expresses conservation of energy.

So far, this seems like a satisfactory way to build and vary actions. However, it appears that we now have two distinct approaches to doing this: that which we have defined above in (21), or the canoncial construction. These approaches are incompatible, and an attempt at an explanation, on mathematical or physical grounds, of the 'correct' manner of construction of an action is in order.

Two types of stationary point problems will be addressed here: finding those *functions* at stationary points of an action, and finding the *curves* that minimise time or distance (geodesics). The following arguments illustrate that functionals and curve lengths as actions require different approaches in their construction.

i. Stationary point of the action (integration of functions)
A resolution of the apparent contradictions perhaps lies in a subtle detail of the construction of the action. It will suffice to present the main argument in terms of ordinary calculus.

In the canonical approach of defining an action[9,13,15], we take a given path q_0 and its derivative $\dot{q}_0$, with respect to a parameter t, with which we define the Lagrangian $L_{q_0}(x) = L(x, q_0(x), \dot{q}_0(x))$. This is a function

$$L_{q_0} \,:\, [a,b] \to X_{q_0} \subset \mathbb{R}\,.$$

The associated action $I[q_0]$ is the integral of L_{q_0} over $[a,b] \subset \mathcal{M} \sim \mathbb{R}$, but here we make sure to remain explicit by introducing a new symbol, for example k, to denote the dummy integration variable. The action for an arbitrary q is then

$$I[q] = \int_a^b dk\, L_q(k) = \int_a^b dk\, L(k, q(k), \dot{q}(k))\,.$$

Now, varying $q \to q + \epsilon\eta$, for some arbitrary function η, induces the variation $\dot{q} \to \dot{q} + \epsilon\dot{\eta}$, where the dot still denotes the derivative with respect to parameter t. Then $I[q] \to I[q+\epsilon\eta]$, where

$$\begin{aligned} I[q+\epsilon\eta] &= \int_a^b dk\, L(k, q(k) + \epsilon\eta(k), \dot{q}(k) + \epsilon\dot{\eta}(k)) \\ &= \int_a^b dk\, L(k, q(k), \dot{q}(k)) + \epsilon\left(\eta(k)\frac{\partial L}{\partial q} + \dot{\eta}(k)\frac{\partial L}{\partial \dot{q}}\right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

so that the variation, $\delta I[q] = I[q+\epsilon\eta] - I[q]$, to $\mathcal{O}(\epsilon)$ is

$$\begin{aligned} \delta I[q] &= \epsilon \int_a^b dk\; \eta(k)\frac{\partial L}{\partial q} + \dot{\eta}(k)\frac{\partial L}{\partial \dot{q}} \\ &= \epsilon \int_a^b dk\; \eta(k)\frac{\partial L}{\partial q} + \frac{d\eta}{dk}(k)\frac{dk}{dt}\frac{\partial L}{\partial \dot{q}}\,. \end{aligned}$$

Imposing boundary conditions $\eta(a) = \eta(b) = 0$, then integrating by parts gives:

$$\delta I[q] = \epsilon \int_a^b dk\; \eta(k)\left(\frac{\partial L}{\partial q} - \frac{d}{dk}\left(\frac{dk}{dt}\frac{\partial L}{\partial \dot{q}}\right)\right)\,,$$

If we were to perform this integral, we would be free to choose $k = t$ to simplify the calculation. However, we do not integrate. Instead, the fundamental lemma of variational calculus is applied, meaning we must be mindful of the integration variable k. Our equations of motion then turn out to be

$$\begin{aligned} 0 &= \frac{\partial L}{\partial q} - \frac{d}{dk}\left(\frac{dk}{dt}\frac{\partial L}{\partial \dot{q}}\right) \\ &= \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\ddot{k}}{\dot{k}}\frac{\partial L}{\partial \dot{q}}\,. \end{aligned} \tag{23}$$

We find that, even after fixing the boundary of the variation, the equations of motion depend on the relationship between two coordinate parameterisations, t and k, of the range $[a,b]$. We may write $k = k(t)$ and demand C^2 differentiability of k, but since k originated as a dummy integration variable, the relationship is otherwise arbitrary. For (23) to be true for all $k(t)$ then, we must have that

$$\frac{\partial L}{\partial \dot{q}} = 0 \qquad \Rightarrow \qquad \frac{\partial L}{\partial q} = 0\,.$$

This means that the Lagrangian which extremises the action is independent of q and $\dot{q}$. But perhaps we should not be surprised by this; the action principle gives us the form of the Lagrangian that minimises the variation. For arbitrary path q, this is the Lagrangian for which the dependence on q and $\dot{q}$ vanishes, since any given variation will incur the smallest change to the Lagrangian that depends on q and $\dot{q}$ the least. Setting $\delta I = 0$ finds the particular solution for the form of the Lagrangian that we might call the 'trivial', or 'empty' Lagrangian, rather than a general relationship between q and $\dot{q}$ that we seek. Defining actions in this way cannot constrain dynamics of the system, only the system itself, so we conclude that the proposed form of the action is not of an appropriate form for our requirements.

Let us instead now look to the Lagrangian as it is defined in (21). Here, given a coordinate t_1, a function q_1, and its derivative with respect to the coordinate $\partial_{t_1} q_1$, we define the Lagrangian $L_{t_1,q_1}(x) = L(t_1(x), q_1(x), \partial_{t_1} q_1(x))$, which again is just a function

$$L_{t_1,q_1} \,:\, [a,b] \to X_{t_1,q_1} \subset \mathbb{R}\,.$$

In this case, the associated action $I[t_1, q_1]$ is the integral of L_{t_1,q_1} over our range $[a,b] \subset \mathcal{M} \sim \mathbb{R}$. The action for arbitrary t and q is then given by (21).

We have already seen that, for any function q, the choice of coordinates that minimises the action is that parameter t which satisfies (22). But we may also vary $I[t,q]$ by varying q. However, following the usual procedure of deriving the equation of motion then appears to give a result similar to (23), and yields the trivial solution. This would contradict (22) and undermine our proposed form of construction of the Lagrangian, so we must try to find fault in the derivation, to ensure our Lagrangian is not forced to be empty.

The strongest argument presently known to me is concerned with how the boundary points of the function are set. We suppose that, for some function t, the action is defined with (21). Then varying $q \to q + \epsilon f$, the variation is

$$\begin{aligned}\delta I[t,q] &= \epsilon \int_{t(a)}^{t(b)} d\tau\; f\,\frac{\partial L}{\partial q} + \frac{df}{d\tau}\frac{d\tau}{dt}\frac{\partial L}{\partial (q_{,t})} \\ &= \epsilon \int_{t(a)}^{t(b)} d\tau\; f\left(\frac{\partial L}{\partial q} - \frac{d}{d\tau}\left(\frac{d\tau}{dt}\frac{\partial L}{\partial (q_{,t})}\right)\right) + \epsilon\left[f\,\frac{d\tau}{dt}\frac{\partial L}{\partial (q_{,t})}\right]_{t(a)}^{t(b)}\,.\end{aligned}$$

Now if q describes the path of a particle, we might state that the position at time $t(a)$ is given by $q(t(a))$, and at time $t(b)$ is given by $q(t(b))$. The boundary of the variation would then be fixed: $f(t(a))=f(t(b))=0$. The problem, however, is that $t(\tau)$ is a placeholder for any arbitrary function of τ, so for boundary constraints $f(t(a))=f(t(b))=0$ to be consistent for all functions t, the variation f must vanish everywhere. We thus conclude that fixing the boundary points for this sort of Lagrangian is not a feasible condition of the variation. It is not possible to state that the boundary term above vanishes, so we cannot apply the fundamental lemma of variational calculus to determine equations of motion from actions like (21) by direct variations of q.

ii. Minimisation of time/distance (integration of curves)
One of the first applications of variational calculus was to the brachistochrone problem[16], which is to find the shape of the curve that results in the shortest time taken for a bead to slide along a wire, without friction, from a point p_1, to a (not higher) point p_2, starting from rest, and under a constant acceleration due to gravity. If ds is the differential arc length of the curve, and v is the speed of the bead, then ds/v is the differential arc time, so the time taken is given by

$$t_{12} = \int_{p_1}^{p_2} \frac{ds}{v} \, .$$

A similar setup is used to find the path minimising the distance between points p_1' and p_2': the geodesic curve. If $d\lambda$ is the differential arc length of the curve, then its total length is given by

$$l_{12} = \int_{p_1'}^{p_2'} d\lambda \, .$$

The differentials here are purely geometric and independent of a coordinate chart; they are examples of *one-forms* and may be expressed in any coordinates. For example, for the brachistochrone curve, the solution is most straightforward if a y-coordinate is chosen such that the gravitational potential energy is given by mgy, where m is the mass of the bead and g is the gravitational acceleration. We may choose any arbitrary parameterisation of the orthogonal direction as our x-coordinate. The differential $ds=\sqrt{dx^2+dy^2}=\sqrt{1+y_x^2}\,dx$, and conservation of energy gives us $v=\sqrt{2gy}$, so that

$$t_{12} = \int_{p_1}^{p_2} \sqrt{\frac{1+y_x^2}{2gy}} \, dx \, .$$

We can see that, contrary to what was found for the integral of a function, a dummy integration variable is not introduced for integrals of curves; changing the integration variable $x \to x'$ amounts to a coordinate transformation, which resulting E-L equations will reflect. The E-L equation describing the minimising path is found in the usual way, and it appears that the usual derivation remains well posed: the geodesic equations need not be reviewed.

IV. The Field Equations And Cosmology

Using the stationary total action as given in (16), we calculate here equations of motion that result from a homogeneous and isotropic energy distribution. It is well-known that such symmetry gives rise to the Robertson-Walker (RW) metric, the line element of which, in reduced-circumference spherical coordinates (t', r, θ, ϕ), is given by[17]

$$ds^2 = -u(t')\,dt'^2 + a'(t')^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{24}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$ and k is called the curvature parameter; $k>0$, $k=0$, and $k<0$ correspond to positive, flat, and negatively curved spaces respectively. The term in the square brackets is inherited from the metric of the maximally symmetric 3D spatial subspace, to be denoted h_{ij}. The metric may be written

$$g_{\mu\nu} = \begin{pmatrix} -u(t') & 0 \\ 0 & a'(t')^2 h_{ij} \end{pmatrix},$$

where

$$h_{ij} = \begin{pmatrix} (1-kr^2)^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{25}$$

We now start to retrieve available equations from (16) that this coordinate system will allow. Starting with the $\delta_{t'} = -\epsilon V^{t'}\partial_{t'}$ variation operator, pulling back the $g^{t't'}$ component along the vector field $V^a = (f_{t'}(x), 0, 0, 0)$ for arbitrary $f_{t'}(x)$, we obtain

$$R_{t't'} - \tfrac{1}{2}g_{t't'}R + g^{\rho\sigma}\frac{\delta_{t'}R_{\rho\sigma}}{\delta_{t'}g^{t't'}} - \kappa g_{t't'}\mathcal{L}_M + 2\kappa\frac{\delta_{t'}\mathcal{L}_M}{\delta_{t'}g^{t't'}} = 0\,. \tag{26}$$

Coordinates were chosen so that our metric in (24) takes a form where the component $g_{t't'}$, and so its inverse $g^{t't'}$, are time-dependent functions, enabling us to apply the stationary action principle by making arbitrary variations of the function with the operator $\delta_{t'}$. If the standard coordinates were to be used, where $g_{t't'} = -1$, independent of t', then we would not be able to produce a variation using our $\delta_{t'}$ operator, hence we would not be able to obtain equation (26). Our initial coordinate choice granted the acquisition of (26), which subsequently can be expressed in new coordinates (t, r, θ, ϕ) whereby the line-element takes the form

$$ds^2 = -\,dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]\,. \tag{27}$$

In fact it is not quite so straightforward, since if $g_{tt} = -1$, then $\delta_t g^{tt} = 0$. The expansion of the $\delta_t R_{\rho\sigma}/\delta_t g^{tt}$ term gives rise to terms like $\delta_t g^{tt}/\delta_t g^{tt}$, so with $g_{tt} = -1$, this term would yield an undefined division by zero. We can easily get around this by putting the line-element in the effectually equivalent form

$$ds^2 = -\lim_{\varepsilon\to 0} e^{\varepsilon t}\,dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]\,, \tag{28}$$

where it is clear that we can evaluate this kind of term, for example

$$\frac{\delta_t g^{tt}}{\delta_t g^{tt}} = \frac{{g^{tt}}_{,t}}{{g^{tt}}_{,t}} = \lim_{\varepsilon\to 0} \frac{\partial_t\, e^{-\varepsilon t}}{\partial_t\, e^{-\varepsilon t}} = 1\,.$$

It will also be useful to note that terms like $\delta_t {g^{tt}}_{,t}/\delta_t g^{tt}$ and $\delta_t {g^{tt}}_{,tt}/\delta_t g^{tt}$ vanish:

$$\frac{\delta_t {g^{tt}}_{,t}}{\delta_t g^{tt}} = \frac{{g^{tt}}_{,tt}}{{g^{tt}}_{,t}} = \lim_{\varepsilon\to 0} \frac{\partial_t^2\, e^{-\varepsilon t}}{\partial_t\, e^{-\varepsilon t}} = \lim_{\varepsilon\to 0} \mathcal{O}(\varepsilon) = 0\,.$$

For the coordinate transformation, we introduce the function $v(t')$ defined by

$$u(t') \equiv \lim_{\varepsilon\to 0}\ v_{,t'}(t')^2\, e^{\varepsilon v(t')}\,, \tag{29}$$

then the new coordinate t is defined by

$$\int u(t')^{1/2}\, dt' = \lim_{\varepsilon\to 0} \int v_{,t'}(t')\, e^{\varepsilon v(t')/2}\, dt' \equiv \lim_{\varepsilon\to 0} \int e^{\varepsilon t/2}\, dt \tag{30}$$

so that $t = v(t')$, and which gives $u(t')\, dt'^2 = \lim_{\varepsilon\to 0} e^{\varepsilon t} dt^2$. Next we define the new function a by setting

$$a(t) \equiv a'(t')\,,$$

then the line-element becomes exactly that written in (28) as intended.

Each of the terms in our equations of motion now need to be expressed in terms of the transformed geometric objects in the new coordinate system. Using the tensor transformation law[11] on a tensor component $T_{t't'}$, we have

$$T_{t't'} = \frac{\partial x^\alpha}{\partial t'} \frac{\partial x^\beta}{\partial t'} T_{\alpha\beta} = \left(\frac{\partial t}{\partial t'}\right)^2 T_{tt}\,, \tag{31}$$

since our transformation does not mix coordinates, and simply reparameterises the time coordinate. The transformations of quotients of variations must also be evaluated. For a pulled back metric component $g^{\mu\nu}$,

$$\frac{\delta_{t'}}{\delta_{t'} g^{\mu\nu}} = \frac{\partial_{t'}}{\partial_{t'}\left(\frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} g^{\alpha\beta}\right)}\,,$$

so the pull-back of $g^{t't'}$ with respect to t', expressed in terms of the pull-back of g^{tt} with respect to t, is

$$\frac{\delta_{t'}}{\delta_{t'} g^{t't'}} = \frac{\partial x^\alpha}{\partial t'} \frac{\partial x^\beta}{\partial t'} \frac{\frac{\partial t}{\partial t'} \partial_t}{\frac{\partial t}{\partial t'} \partial_t g^{\alpha\beta}} = \left(\frac{\partial t}{\partial t'}\right)^2 \frac{\partial_t}{\partial_t g^{tt}} = \left(\frac{\partial t}{\partial t'}\right)^2 \frac{\delta_t}{\delta_t g^{tt}}\,. \tag{32}$$

Now the t'-t' components of the tensors and quotients of variations in (26) can be replaced using (31) and (32), so that (26) becomes:

$$\left(\frac{\partial t}{\partial t'}\right)^2 \left(R_{tt} - \tfrac{1}{2} g_{tt} R + g^{\rho\sigma} \frac{\delta_t R_{\rho\sigma}}{\delta_t g^{tt}} - \kappa g_{tt} \mathcal{L}_M + 2\kappa \frac{\delta_t \mathcal{L}_M}{\delta_t g^{tt}}\right) = 0\,,$$

from which we find:

$$R_{tt} - \tfrac{1}{2}g_{tt}R + g^{\rho\sigma}\,\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{tt}} - \kappa g_{tt}\mathcal{L}_M + 2\kappa\frac{\delta_t\mathcal{L}_M}{\delta_t g^{tt}} = 0\ . \tag{33}$$

We have just obtained an equation which appears to have come from the pull-back of the essentially constant g^{tt} function. One will readily acknowledge that a variation generated by moving a constant around on the manifold vanishes, but as we have seen, it can be useful to find coordinate systems such that one can generate non-zero variations to obtain an equation of motion, which then easily transform into more manageable coordinates.

Similar equations can be obtained by pull-backs of spatial components with the δ_t operator. We can start with the coordinates (t, r, θ, ϕ), since the spatial components are already time dependent. However, recall from the discussion following (18) that the pull-back of g^{rr} is not allowed, due to the coordinate degeneracy at $r=0$. A similar argument reveals that the $g^{\theta\theta}$ component is not able to be pulled back, since arbitrary variations are not well-defined where $\theta=0$ and $\theta=\pi$ because the angle ϕ is indeterminate there. The ϕ coordinate has no degenerate points however, so variations of $g^{\phi\phi}$ are legitimate, so long as they are produced with an admissible operator. We obtain our second equation of motion by varying $g^{\phi\phi}$ with the δ_t operator:

$$R_{\phi\phi} - \tfrac{1}{2}g_{\phi\phi}R + g^{\rho\sigma}\,\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{\phi\phi}} - \kappa g_{\phi\phi}\mathcal{L}_M + 2\kappa\frac{\delta_t\mathcal{L}_M}{\delta_t g^{\phi\phi}} = 0\ . \tag{34}$$

A further equation can be obtained using the variation of $g^{\phi\phi}$, by repeating a similar procedure to that used to obtain (32). That is, perform the coordinate transformation $\phi \to \phi'$ such that $h_{\phi\phi} \to h_{\phi'\phi'} = h_{\phi\phi}\xi(\phi')$ for some non-constant function $\xi(\phi')$. Then we can vary the action by pulling back $g^{\phi'\phi'}$ using the $\delta_{\phi'}$ operator, to obtain the ϕ'-ϕ' equation. Using the tensor tranformation relations as we did earlier, rewrite the ϕ'-ϕ' equation as $(\partial\phi/\partial\phi')^2$ times the ϕ-ϕ equation. Dividing through by $(\partial\phi/\partial\phi')^2$ then gives us our second ϕ-ϕ equation:

$$R_{\phi\phi} - \tfrac{1}{2}g_{\phi\phi}R + g^{\rho\sigma}\,\frac{\delta_\phi R_{\rho\sigma}}{\delta_\phi g^{\phi\phi}} - \kappa g_{\phi\phi}\mathcal{L}_M + 2\kappa\frac{\delta_\phi\mathcal{L}_M}{\delta_\phi g^{\phi\phi}} = 0\ . \tag{35}$$

We have implicitly assumed that the inverse transformation is not exact and in fact takes $h_{\phi'\phi'}$ back to the equivalent form $h_{\phi\phi} = \lim_{\varepsilon\to 0} e^{\varepsilon\phi} r^2 \sin^2\theta$, ensuring that divisions by zero do not occur in our quotients of variations.

Now we evaluate the boundary terms in equations (33)-(35). We require the coordinate basis expression of the Riemann tensor, which is given by

$$R^{\lambda}{}_{\rho\gamma\sigma} \equiv \partial_\gamma\Gamma^{\lambda}{}_{\sigma\rho} - \partial_\sigma\Gamma^{\lambda}{}_{\gamma\rho} + \Gamma^{\lambda}{}_{\gamma\delta}\Gamma^{\delta}{}_{\sigma\rho} - \Gamma^{\lambda}{}_{\sigma\delta}\Gamma^{\delta}{}_{\gamma\rho}\,,$$

so that the Ricci tensor, $R_{\rho\sigma} \equiv R^{\lambda}{}_{\rho\lambda\sigma}$, is

$$R_{\rho\sigma} = \partial_\lambda\Gamma^{\lambda}{}_{\sigma\rho} - \partial_\sigma\Gamma^{\lambda}{}_{\lambda\rho} + \Gamma^{\lambda}{}_{\lambda\delta}\Gamma^{\delta}{}_{\sigma\rho} - \Gamma^{\lambda}{}_{\sigma\delta}\Gamma^{\delta}{}_{\lambda\rho}\,,$$

where ${\Gamma^\delta}_{\alpha\beta}$ are the Christoffel symbols given by

$$
{\Gamma^\delta}_{\alpha\beta} = \tfrac{1}{2} g^{\delta\gamma}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) \,.
$$

The non-vanishing Christoffel symbols for (28) in the limit as $\varepsilon \to 0$ are:

$$
{\Gamma^t}_{ij} = -\tfrac{1}{2} g^{tt} g_{ij,t} \;, \qquad {\Gamma^i}_{tj} = \tfrac{1}{2} g^{ki} g_{kj,t} \;, \qquad {\Gamma^i}_{jk} = {\tilde{\Gamma}^i}{}_{jk} \tag{36}
$$

where ${\tilde{\Gamma}^i}{}_{jk}$ is the Christoffel symbol of the 3D spatial geometry. The components of the Ricci tensor are

$$
\begin{aligned}
R_{tt} &= -\partial_t {\Gamma^i}_{it} - {\Gamma^i}_{tj} {\Gamma^j}_{it} \\
&= -\tfrac{1}{2} g^{ij} g_{ij,tt} - \tfrac{1}{2} {g^{ij}}_{,t} g_{ij,t} - \tfrac{1}{4} g^{ki} g^{lj} g_{kj,t} g_{li,t}
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
R_{ij} &= \tilde{R}_{ij} + \partial_t {\Gamma^t}_{ij} + {\Gamma^k}_{kt} {\Gamma^t}_{ij} - {\Gamma^t}_{jk} {\Gamma^k}_{ti} - {\Gamma^k}_{jt} {\Gamma^t}_{ki} \\
&= \tilde{R}_{ij} - \tfrac{1}{2} g^{tt} g_{ij,tt} - \tfrac{1}{4} g^{tt} g^{lk} g_{lk,t} g_{ij,t} + \tfrac{1}{2} g^{tt} g^{lk} g_{kj,t} g_{li,t}
\end{aligned} \tag{38}
$$

where $\tilde{R}_{ij}$ is the Ricci tensor of the 3D spatial subspace h_{ij}. The g^{tt} components are kept explicitly in (38) so terms are not missed when we vary with respect to the g^{tt} component. Similarly, although we know that $\tilde{R}_{ij}$ vanishes everywhere for a flat space, it is nevertheless composed of metric components, so a pull-back in the spatial part could vary this term. Of course the remaining R_{ti} are comprised of metric components, so they too do not necessarily vanish under variations, but they will not be computed because their variations only appear in the sum $g^{\rho\sigma}\delta R_{\rho\sigma}$, to which, since $g^{ti}=0$, they do not contribute.

We are now ready calculate the boundary terms in our set of equations. Beginning with the t-t component as it appears in (33), we operate with $\delta_t/\delta_t g^{tt}$,

$$
\begin{aligned}
g^{\rho\sigma} \frac{\delta_t R_{\rho\sigma}}{\delta_t g^{tt}} &= g^{tt} \cancelto{0}{\frac{\delta_t R_{tt}}{\delta_t g^{tt}}} + g^{ij} \frac{\delta_t R_{ij}}{\delta_t g^{tt}} \\
&= g^{ij} \left(-\tfrac{1}{2} g_{ij,tt} - \tfrac{1}{4} g^{\delta k} g_{\delta k,t} g_{ij,t} + \tfrac{1}{2} g^{\delta k} g_{kj,t} g_{\delta i,t} \right) \\
&= -6 \frac{\dot{a}^2}{a^2} - 3 \frac{\ddot{a}}{a} \,.
\end{aligned} \tag{39}
$$

The spatial boundary term component in (34) is more cumbersome to compute. Working on the $g^{tt} \frac{\delta_t}{\delta_t g^{\phi\phi}} R_{tt}$ and $g^{mn} \frac{\delta_t}{\delta_t g^{\phi\phi}} R_{mn}$ parts separately:

$$
\begin{aligned}
g^{tt} \frac{\delta_t R_{tt}}{\delta_t g^{\phi\phi}} &= g^{tt} \frac{\delta_t}{\delta_t g^{\phi\phi}} \left(-\tfrac{1}{2} g^{mn} g_{mn,tt} - \tfrac{1}{2} {g^{mn}}_{,t} g_{mn,t} - \tfrac{1}{4} g^{km} g^{ln} g_{kn,t} g_{lm,t} \right) \\
&= \tfrac{1}{2} \Bigg(g_{\phi\phi,tt} + g^{mn} \frac{\delta_t g_{mn,tt}}{\delta_t g^{\phi\phi}} + \frac{\delta_t {g^{mn}}_{,t}}{\delta_t g^{\phi\phi}} g_{mn,t} + {g^{mn}}_{,t} \frac{\delta_t g_{mn,t}}{\delta_t g^{\phi\phi}} \\
&\qquad + g^{ln} g_{in,t} g_{lj,t} + g^{km} g^{ln} \frac{\delta_t g_{kn,t}}{\delta_t g^{\phi\phi}} g_{lm,t} \Bigg) \,.
\end{aligned} \tag{40}
$$

The second piece is:

$$
\begin{aligned}
g^{mn}\frac{\delta_t R_{mn}}{\delta_t g^{\phi\phi}} &= g^{mn}\frac{\delta_t}{\delta_t g^{\phi\phi}}\left(\tilde{R}_{mn} - \tfrac{1}{2}g^{tt}g_{mn,tt} - \tfrac{1}{4}g^{tt}g^{\delta k}g_{\delta k,t}g_{mn,t} + \tfrac{1}{2}g^{tt}g^{\delta k}g_{kn,t}g_{\delta m,t}\right) \\
&= g^{mn}\frac{\delta_t \tilde{R}_{mn}}{\delta_t g^{\phi\phi}} + \tfrac{1}{2}g^{mn}\frac{\delta_t g_{mn,tt}}{\delta_t g^{\phi\phi}} + \tfrac{1}{4}g^{mn}g_{mn,t}g_{\phi\phi,t} + \tfrac{1}{2}g^{mn}g_{mn,t}g^{\delta k}\frac{\delta_t g_{\delta k,t}}{\delta_t g^{\phi\phi}} \\
&\quad - \tfrac{1}{2}g^{mn}g_{jn,t}g_{im,t} - g^{mn}g^{\delta k}\frac{\delta_t g_{kn,t}}{\delta_t g^{\phi\phi}}g_{\delta m,t}\,. \qquad (41)
\end{aligned}
$$

Noting that $\tilde{R}_{ij}$ is independent[17] of t, we find that the $\delta_t\tilde{R}_{ii}$ terms for $i \in r, \theta, \phi$ vanish. Using (40) and (41), and since $g^{ij}\delta_t\tilde{R}_{ij}$ vanishes, we have

$$
\begin{aligned}
g^{\rho\sigma}\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{\phi\phi}} &= \tfrac{1}{2}g_{\phi\phi,tt} + g^{mn}\frac{\delta_t g_{mn,tt}}{\delta_t g^{\phi\phi}} + \tfrac{1}{2}\frac{\delta_t g^{mn}{}_{,t}}{\delta_t g^{\phi\phi}}g_{mn,t} + \tfrac{1}{2}g^{mn}{}_{,t}\frac{\delta_t g_{mn,t}}{\delta_t g^{\phi\phi}} \\
&\quad - \tfrac{1}{2}g^{km}g^{ln}\frac{\delta_t g_{kn,t}}{\delta_t g^{\phi\phi}}g_{lm,t} + \tfrac{1}{4}g^{mn}g_{mn,t}g_{\phi\phi,t} + \tfrac{1}{2}g^{mn}g_{mn,t}g^{lk}\frac{\delta_t g_{lk,t}}{\delta_t g^{\phi\phi}} \qquad (42)
\end{aligned}
$$

Applying the variation operator definition (14) to (42) yields

$$
\begin{aligned}
g^{\rho\sigma}\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{\phi\phi}} &= \tfrac{1}{2}g_{\phi\phi,tt} + g^{\phi\phi}\frac{g_{\phi\phi,ttt}}{g^{\phi\phi}{}_{,t}} + \tfrac{1}{2}\frac{g^{\phi\phi}{}_{,tt}}{g^{\phi\phi}{}_{,t}}g_{\phi\phi,t} + \tfrac{1}{2}g^{\phi\phi}{}_{,t}\frac{g_{\phi\phi,tt}}{g^{\phi\phi}{}_{,t}} \\
&\quad - \tfrac{1}{2}g^{\phi\phi}g^{\phi\phi}\frac{g_{\phi\phi,tt}}{g^{\phi\phi}{}_{,t}}g_{\phi\phi,t} + \tfrac{1}{4}g^{mn}g_{mn,t}g_{\phi\phi,t} + \tfrac{1}{2}g^{mn}g_{mn,t}g^{\phi\phi}\frac{g_{\phi\phi,tt}}{g^{\phi\phi}{}_{,t}} \\
&= g_{\phi\phi,tt} + g^{\phi\phi}\frac{g_{\phi\phi,ttt}}{g^{\phi\phi}{}_{,t}} + \tfrac{1}{2}\frac{g^{\phi\phi}{}_{,tt}}{g^{\phi\phi}{}_{,t}}g_{\phi\phi,t} + \tfrac{1}{4}g^{mn}g_{mn,t}g_{\phi\phi,t} \\
&\quad + \tfrac{1}{2}\frac{g_{\phi\phi,tt}}{g^{\phi\phi}{}_{,t}}\left(g^{mn}g_{mn,t}g^{\phi\phi} - g^{\phi\phi}g^{\phi\phi}g_{\phi\phi,t}\right)\,. \qquad (43)
\end{aligned}
$$

The various factors appearing in (43) evaluate to:

$$
g_{\phi\phi,t} = 2a\dot{a}h_{\phi\phi} \qquad g_{\phi\phi,tt} = 2(\dot{a}^2 + a\ddot{a})h_{\phi\phi} \qquad g_{\phi\phi,ttt} = (6\dot{a}\ddot{a} + 2a\dddot{a})h_{\phi\phi}
$$

$$
g^{\phi\phi}{}_{,t} = -2\dot{a}a^{-3}h^{\phi\phi} \qquad g^{\phi\phi}{}_{,tt} = (6\dot{a}^2a^{-4} - 2\ddot{a}a^{-3})h^{\phi\phi}
$$

so finally, the ϕ-ϕ boundary term component is

$$
g^{\rho\sigma}\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{\phi\phi}} = -\left(2a\ddot{a} + a^2\frac{\dddot{a}}{\dot{a}}\right)h_{\phi\phi}\,. \qquad (44)
$$

The final boundary term we require, appearing in (35), is given by

$$
\begin{aligned}
g^{\rho\sigma}\,\frac{\delta_\phi R_{\rho\sigma}}{\delta_\phi g^{\phi\phi}} &= g^{tt}\,\cancelto{0}{\frac{\delta_\phi R_{tt}}{\delta_\phi g^{\phi\phi}}} + g^{ij}\,\frac{\delta_\phi R_{ij}}{\delta_\phi g^{\phi\phi}} \\
&= g^{ij}\,\frac{\delta_\phi}{\delta_\phi g^{\phi\phi}}\left(\tilde{R}_{ij} - \tfrac{1}{2}g^{tt}g_{ij,tt} - \tfrac{1}{4}g^{tt}g^{lk}g_{lk,t}g_{ij,t} + \tfrac{1}{2}g^{tt}g^{lk}g_{kj,t}g_{li,t}\right)
\end{aligned}
$$

Since[17] $\tilde{R}_{ij}=2kh_{ij}$, we have $\delta_\phi \tilde{R}_{ij}=0$ for $i,j\neq\phi$, so that

$$\begin{aligned}
g^{\rho\sigma}\,\frac{\delta_\phi R_{\rho\sigma}}{\delta_\phi g^{\phi\phi}} &= g^{\phi\phi}\,\frac{\delta_\phi \tilde{R}_{\phi\phi}}{\delta_\phi g^{\phi\phi}} + \tfrac{1}{2}g^{\phi\phi}\,\frac{\delta_\phi g_{\phi\phi,tt}}{\delta_\phi g^{\phi\phi}} + \tfrac{1}{4}g^{\phi\phi}\,g^{lk}g_{lk,t}\frac{\delta_\phi g_{\phi\phi,t}}{\delta_\phi g^{\phi\phi}} - \tfrac{1}{2}(g^{\phi\phi})^2 g_{\phi\phi,t}\,\frac{\delta_\phi g_{\phi\phi,t}}{\delta_\phi g^{\phi\phi}} \\
&= 2kg^{\phi\phi}\,\frac{h_{\phi\phi,\phi}}{{g^{\phi\phi}}_{,\phi}} + \tfrac{1}{2}g^{\phi\phi}\,\frac{g_{\phi\phi,tt\phi}}{{g^{\phi\phi}}_{,\phi}} + \tfrac{1}{4}g^{\phi\phi}\,g^{lk}g_{lk,t}\frac{g_{\phi\phi,t\phi}}{{g^{\phi\phi}}_{,\phi}} - \tfrac{1}{2}(g^{\phi\phi})^2 g_{\phi\phi,t}\,\frac{g_{\phi\phi,t\phi}}{{g^{\phi\phi}}_{,\phi}} \\
&= -2kh_{\phi\phi} - \tfrac{1}{2}g_{\phi\phi,tt} - \tfrac{1}{4}g^{lk}g_{lk,t}g_{\phi\phi,t} + \tfrac{1}{2}g^{\phi\phi}g_{\phi\phi,t}g_{\phi\phi,t} \\
&= -\left(2\dot{a}^2 + a\ddot{a} + 2k\right)h_{\phi\phi}\,. \qquad (45)
\end{aligned}$$

We have been unable to obtain r-r and θ-θ equations by variations of g^{rr} and $g^{\theta\theta}$ components, due to the fact that pull-backs along single component vector fields are incompatible with degenerate coordinates. The spherical coordinate system has proved useful for appreciating some subtleties in applying variations, however, note that for a flat 3D spatial subspace with non-degenerate Cartesian coordinates (x,y,z), the coordinate transformation procedure can easily produce equations for variations of all diagonal metric components:

$$R_{ii} - \tfrac{1}{2}g_{ii}R + g^{\rho\sigma}\,\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{ii}} - \kappa g_{ii}\mathcal{L}_M + 2\kappa\frac{\delta_t \mathcal{L}_M}{\delta_t g^{ii}} = 0\,, \qquad (46)$$

$$R_{ii} - \tfrac{1}{2}g_{ii}R + g^{\rho\sigma}\,\frac{\delta_i R_{\rho\sigma}}{\delta_i g^{ii}} - \kappa g_{ii}\mathcal{L}_M + 2\kappa\frac{\delta_i \mathcal{L}_M}{\delta_i g^{ii}} = 0\;, \qquad (47)$$

where $i \in x,y,z$. There will be three terms like (44) given by the relations

$$g^{\rho\sigma}\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{ii}} = -\left(2a\ddot{a} + a^2\frac{\dddot{a}}{\dot{a}}\right)h_{ii}\;, \qquad (48)$$

and three terms like (45) where of course $k=0$, given by the relations

$$g^{\rho\sigma}\,\frac{\delta_i R_{\rho\sigma}}{\delta_i g^{ii}} = -\left(2\dot{a}^2 + a\ddot{a}\right)h_{ii}\,. \qquad (49)$$

In general (for any k) the solutions of equations (33)-(35) require the terms

$$R_{tt} = -3\frac{\ddot{a}}{a}\,, \qquad R_{ij} = \left(2\frac{k}{a^2} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)g_{ij}\;, \qquad R = 6\left(\frac{k}{a^2} + \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\;, \qquad (50)$$

computed from (37) and (38). Then from (39), (44), (45) and (50), we find that

$$R_{tt} - \tfrac{1}{2}g_{tt}R + g^{\rho\sigma}\,\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{tt}} = -3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - \frac{k}{a^2}\right)\;, \qquad (51)$$

$$R_{\phi\phi} - \tfrac{1}{2}g_{\phi\phi}R + g^{\rho\sigma}\,\frac{\delta_t R_{\rho\sigma}}{\delta_t g^{\phi\phi}} = -\left(\dot{a}^2 + 4a\ddot{a} + a^2\frac{\dddot{a}}{\dot{a}} + k\right)h_{\phi\phi}\,, \qquad (52)$$

$$R_{\phi\phi} - \tfrac{1}{2}g_{\phi\phi}R + g^{\rho\sigma}\,\frac{\delta_\phi R_{\rho\sigma}}{\delta_\phi g^{\phi\phi}} = -3\left(\dot{a}^2 + a\ddot{a} + k\right)h_{\phi\phi}\,. \qquad (53)$$

These are the analogues of the Einstein tensor parts (the 'left hand side') of the Friedmann equations[18].

V. The Cosmological Matter Lagrangian

The geometrical parts of our equations have been found in (51)-(53), we must now look at the matter parts. We have purposefully avoided writing the matter parts of the field equations in accord with usual practice in GR, which is by means of the introduction of the *stress-energy tensor*, $T_{\mu\nu}$, usually defined as[19]

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \,. \tag{54}$$

This is because, as we have seen throughout these investigations, different types of variations such as pull-backs along different vector field elements, will result in a quotient of variations which has a form determined by the variation operator that generated it. For example, (44) and (45) are both boundary terms for the ϕ-ϕ equations, but the different generators yield different terms. The variations in (54) have not been uniquely prescribed, so we cannot presume a definite form of the stress-energy tensor that is independent of the method of variation. Thus the definition appears to be incomplete, insofar as the variational techniques we have considered here are concerned, since the variations represent not just one, but a set of distinct operators.

Since the above definition of the stress-energy tensor is incompatible with our field equations we are led to consider the possibility that it is not a fundamental physical entity from the perspective of the theory put forth here. Of course, this is not to say a tensor field cannot be used to construct a matter Lagrangian, nor that a tensor cannot be defined, for practical purposes, that encodes information on energy flow in the way the stress-energy tensor does. However, defining an object to encode chosen information in some way, then demanding that Nature's operations accord with that choice is not a reliable approach to finding laws of physics. Since our variation operator allows us to compute direct variations of $\mathcal{L}_M$ with respect to metric components explicitly, this is what we shall do here instead.

Before presenting potential formulations of $\mathcal{L}_M$, let us note some important constraints on $\mathcal{L}_M$ that result directly from the cosmological symmetry. Notice that the right hand sides of (51) and (53) provide a relationship between two variations of $\mathcal{L}_M$. From (33) and (51) we have

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - \frac{k}{a^2}\right) = -\kappa g_{tt}\mathcal{L}_M + 2\kappa \frac{\delta_t \mathcal{L}_M}{\delta_t g^{tt}} \equiv \kappa\rho - 6\frac{k}{a^2} \,, \tag{55}$$

where we have defined the new function, ρ, in reference to the energy density[20] of the fluid as it is defined in FRW cosmologies. From (47) and (53), we have

$$\begin{aligned} &3\left(\dot{a}^2 + a\ddot{a} + k\right) h_{\phi\phi} = -\kappa g_{\phi\phi}\mathcal{L}_M + 2\kappa \frac{\delta_\phi \mathcal{L}_M}{\delta_\phi g^{\phi\phi}} \\ &\Rightarrow 3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - \frac{k}{a^2}\right) = -\kappa\mathcal{L}_M + 2\kappa g^{\phi\phi} \frac{\delta_\phi \mathcal{L}_M}{\delta_\phi g^{\phi\phi}} - 6\frac{k}{a^2} \end{aligned} \tag{56}$$

Combining these two equations provides a relationship between the Lagrangian and two of its variations:

$$
\begin{aligned}
-\kappa g_{tt}\mathcal{L}_M + 2\kappa\frac{\delta_t\mathcal{L}_M}{\delta_t g^{tt}} &= -\kappa\mathcal{L}_M + 2\kappa g^{\phi\phi}\frac{\delta_\phi\mathcal{L}_M}{\delta_\phi g^{\phi\phi}} - 6\frac{k}{a^2} \\
\Rightarrow \mathcal{L}_M &= g^{\phi\phi}\frac{\delta_\phi\mathcal{L}_M}{\delta_\phi g^{\phi\phi}} - \frac{\delta_t\mathcal{L}_M}{\delta_t g^{tt}} - \frac{3k}{\kappa a^2} \qquad (57)
\end{aligned}
$$

Whatever the final form of $\mathcal{L}_M$, it must be consistent with (57). But notice that, with (50), the left-hand-side of (55) is equal to $\frac{1}{2}R - 6k/a^2$. Then recalling that $R = -2\kappa\mathcal{L}_M$, (55) becomes

$$
\begin{aligned}
-\kappa\mathcal{L}_M - 6\frac{k}{a^2} &= -\kappa g_{tt}\mathcal{L}_M + 2\kappa\frac{\delta_t\mathcal{L}_M}{\delta_t g^{tt}} \\
\Rightarrow \frac{\delta_t\mathcal{L}_M}{\delta_t g^{tt}} &= -\mathcal{L}_M - \frac{3k}{\kappa a^2}\,, \qquad (58)
\end{aligned}
$$

which, in turn, combined with (57) tells us that

$$
\frac{\delta_\phi\mathcal{L}_M}{\delta_\phi g^{\phi\phi}} = 0\,. \qquad (59)
$$

Using (58) and (59), the equations (55) and (56) may both simply be written

$$
3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + \frac{k}{a^2}\right) = -\kappa\mathcal{L}_M = \kappa\rho\,. \qquad (60)
$$

Whilst (59) is not surprising given the symmetry, (58) is interesting, because it tells us that the curvature depends entirely on the form of $\mathcal{L}_M$. Equation (58) will prove useful in constraining $\mathcal{L}_M$. Substituting for k in (60) using (58) gives

$$
3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = \kappa\frac{\delta_t\mathcal{L}_M}{\delta_t g^{tt}}\,.
$$

We can also take the time derivative of (60)

$$
-\kappa\dot{\mathcal{L}_M} = 3\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2} + \frac{\dddot{a}}{\dot{a}} - 2\frac{k}{a^2}\right)\,, \qquad (61)
$$

and since from (34) and (52) we have

$$
\frac{\dddot{a}}{\dot{a}} = -\kappa\mathcal{L}_m + 2\kappa g^{\phi\phi}\frac{\delta_t\mathcal{L}_M}{\delta_t g^{\phi\phi}} - \frac{\dot{a}^2}{a^2} - 4\frac{\ddot{a}}{a} - \frac{k}{a^2}\,,
$$

we obtain the following useful relationship

$$
\begin{aligned}
-\kappa\dot{\mathcal{L}_M} &= 3\frac{\dot{a}}{a}\left(-3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + \frac{k}{a^2}\right) - \kappa\mathcal{L}_m + 2\kappa g^{\phi\phi}\frac{\delta_t\mathcal{L}_M}{\delta_t g^{\phi\phi}}\right) \\
\Rightarrow \dot{\mathcal{L}_M} &= -6\frac{\dot{a}}{a}\frac{\delta_t\mathcal{L}_M}{\delta_t g^{\phi\phi}}g^{\phi\phi}\,. \qquad (62)
\end{aligned}
$$

We will see the consequences of the constraints in (58), (59) and (62) on different matter Lagrangians, but note that (62) is a particularly stringent constraint: the Lagrangian is constant if not dependent on time derivatives of the metric.

The function called ρ was defined earlier in order to point out an interesting result, which we now take a moment for: using (34) and (52), we have that

$$\frac{\dot{a}^2}{a^2} + 4\frac{\ddot{a}}{a} + \frac{\dddot{a}}{\dot{a}} + \frac{k}{a^2} = -\kappa\mathcal{L}_M + 2\kappa g^{\phi\phi}\frac{\delta_t\mathcal{L}_M}{\delta_t g^{\phi\phi}} \equiv -\kappa P \,, \tag{63}$$

where we have defined another new function, P, in reference to the fluid pressure from FRW cosmologies.

These definitions are rather arbitrary, and there is no reason to suppose these are acceptable choices. We will discuss later what is perhaps a more fundamental way of accounting for the effects of pressure, by working with particle velocities directly. Nevertheless, it is interesting to note that these such straightforward definitions of the functions ρ and P lead to the same expression of conservation encountered in FRW universes.

Using $\mathcal{L}_M = -\rho$ in (61) and substituting for $\dddot{a}/\dot{a}$ from (63), we obtain

$$\begin{aligned}\kappa\dot{\rho} &= -3\frac{\dot{a}}{a}\left(3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + \frac{k}{a^2}\right) + \kappa P\right) \\ &= -3\kappa\frac{\dot{a}}{a}\left(\rho + P\right) ,\end{aligned}$$

which, defining w using the cosmological equation of state $P \equiv w\rho$, leads to the familiar form of the FRW conservation relation[17]

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho\left(1 + w\right) .$$

Continuing to borrow from FRW cosmology, using $w = 0$ for a universe filled with dust and $w = \frac{1}{3}$ for a universe filled with radiation[21], we find (after making simplifying choices of integration constants) that

$$\dot{a} \propto \begin{cases} a^{-1/2} & \text{for} \quad w = 0 \ , \\ a^{-1}\sqrt{\ln a} & \text{for} \quad w = \frac{1}{3} \ . \end{cases}$$

The 'radiation' case in particular is interesting, since as $a \to 1$, then $\dot{a} \to 0$, and an initial singularity does not occur. Presently these results are only curiosities however; we have no grounds on which to make the statement that the functions ρ and P represent the energy density and pressure, respectively, as they do in FRW cosmology, that the values of w are appropriate, or even that the equation of state is valid. This approach shall hence not be explored further here; we will proceed with exploring the form of $\mathcal{L}_M$.

On the macroscopic scale, the fluid may be attributed with a 4-velocity and pressure. This implies the fluid is considered as a singular entity, and is the view taken in determining the elements of the FRW stress-energy tensor. However, fluid pressure is an emergent property, appearing due to the microscopic motions of the particulate constituents of the fluid. At the microscopic level, the fluid particles move randomly in all directions, with varying speeds. Before assigning a fluid velocity or pressure, let us associate a *characteristic speed* to particles in each small region of space, then simplify the model of the fluid by assuming all particles in that region have precisely this speed.

Since we define the fluid to be homogeneous and isotropic, it follows that there will be equal numbers of particles moving in all directions, at all points on $\mathcal{M}$. So let us consider decomposing the field into a continuum of homogeneous fields, each 'pointing' toward a unique direction on the sky, where each field represents all the particles moving precisely parallel to that direction. The sum of all such 'fields' then cover the sky and represent all particles moving in all directions. Individual fields themselves are not isotropic, but the total matter field will be the sum of all such fields, whereby isotropy will be recovered. Setting up the field in this way is intended to capture the general nature of the microscopic motions of the particles that make up the fluid, without having to define a pressure.

At any point p, a comoving observer may choose spherical coordinates (θ, ϕ) to cover their sky. Then the proportion of the total matter that has a velocity parallel to the direction given by, say, (θ_0, ϕ_0), which is taken as a distinct field, can be approximated by the ratio of a small (unit) sky area element $\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$ at (θ_0, ϕ_0), to the area of the whole sky, 4π.

Denoting the field describing particles moving parallel to the (θ, ϕ) direction by $\mathfrak{T}^{\theta,\phi}$ (this could be a scalar, vector, tensor etc.) the proportion of the total matter assigned to this field element is approximated by $(4\pi)^{-1}\mathfrak{T}^{\theta,\phi}\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$. Integrating all (θ, ϕ) fields over the sky where the approximations become exact, gives the total matter field

$$\mathfrak{T} \equiv \frac{1}{4\pi}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\sin\theta\,\mathfrak{T}^{\theta,\phi}\,. \tag{64}$$

However, this is only practically applicable if an 'effective' field can somehow be derived that captures the properties of 'wave-packet ensembles', in a way that faithfully reproduces appropriate cosmological dynamics, whilst maintaining the homogeneity and isotropy required for derivation of the RW metric. For fields $\mathfrak{T}^{\theta,\phi}$ with perfect homogeneity, as we deal with here, there are no wave-packets, and one may perhaps foresee that the extremely high degree of symmetry could preclude any interesting dynamics; the superposition of homegeneous fields from the integral leads to a complete cancellation in the field values. Nevertheless we will use (64) to ensure isotropy in this simple presentation.

One of the first issues we must consider is what happens to the matter fields when metric components are varied. In fact, with the picture described earlier, in which a copy of the manifold is generated by a vector field ϵV and a metric component is varied by pulling back that component from the copy, one can see that there is no reason for the matter fields to vary if they are not defined in terms of the metric: recall that, with a given coordinate system, the action is a functional of the 10 'functions' that are the independent metric components, plus those objects that describe matter (scalars, differential form fields and so on), varying one of these at a time. Matter fields remain fixed as a background upon which metric components vary, whereby the equations of motion arise.

I shall now briefly investigate two Lagrangians: one constructed from a scalar field and one from a one-form field (Electromagnetism). However, note that the field superposition considered in (64) is not relevant for scalar fields, since the velocities of uniform scalar fields are meaningless.

Real Scalar Field

It follows immediately from (58) that a matter Lagrangian of the form $\mathcal{L}_M = \varrho$, where ϱ is a function on $\mathcal{M}$, is then only possible when $\varrho = -3k/\kappa a^2$. The next guess is that

$$\mathcal{L}_M = \tfrac{1}{2} g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - \tfrac{1}{2} m^2 \psi^2 \,,$$

for a scalar field ψ and a constant m, and where m^2 may be positive or negative. The cosmological symmetry implies ψ is a function of t only, and since covariant derivatives of scalars are simply partial derivatives, this reduces to

$$\mathcal{L}_M = \tfrac{1}{2} g^{tt} \dot{\psi}^2 - \tfrac{1}{2} m^2 \psi^2 \,, \tag{65}$$

This is compatible with (59), and the constraint (58) means we must have $m = 0$. Of course we know that m corresponds to the mass of the field, and so it might seem too restrictive to only be allowed massless scalar fields. However, we are dealing with a purely classical field theory with the cosmological symmetry. Although we say the field has a mass of m, it is the momentum eigenstates that result from quantisation of the field that we interpret as particles, which have a mass[22] of m. So it might be argued that, strictly, under cosmological symmetry, since there are no wave-packets (or 'distinguishable harmonic oscillators') to be interpreted as particles, the field mass is somewhat ambiguous. The Lagrangian should nevertheless be able to reproduce a Klein-Gordon like equation for scalar fields in a general, non cosmologically symmetric case.

The imposed masslessness of the field by (58) could potentially be rectified by modifying the Lagrangian, including a factor of $-v_\mu v_\nu g^{\mu\nu}$ with the mass term, where v_μ is the 4-velocity of the fluid (represented in the cotangent space), with the understanding that v_μ will not be varied as we vary the metric. However, we have discussed the meaninglessness of the velocities of uniform scalar fields. We might imagine tracking the evolution of scalar field wave-packets using a vector field that is taken as the fluid 4-velocity, defining an effective fluid as in (64), but this approach will not be pursued here.

It is perhaps more reasonable to consider that the potential term includes a factor of $-u_\mu u_\nu g^{\mu\nu}$, where $u^\mu = (\sqrt{-g^{tt}}, 0, 0, 0)$ is the 4-velocity of an observer who is supposed to be comoving with the fluid. Since the fluid comoves with the cosmological frame, this choice of observer will define the cosmological frame. The covector u_μ also does not vary with the metric by the preceeding arguments so we can write $u_\mu = (-1,0,0,0)$. From these reflections, we will write $\mathcal{L}_M$ as

$$\begin{aligned}\mathcal{L}_M &= \tfrac{1}{2} g^{tt}\dot{\psi}^2 + \tfrac{1}{2} m^2 \psi^2 u_\mu u_\nu g^{\mu\nu} \\ &= \tfrac{1}{2} g^{tt}(\dot{\psi}^2 + m^2\psi^2)\,. \end{aligned} \tag{66}$$

Note that we must use the covector u_μ rather than the 4-velocity vector u^μ. This is presumed to be because we are working in the dual space varying inverse metric components, rather than metric components.

For a scalar field Lagrangian, we have $\delta_t \mathcal{L}_M / \delta_t g^{ii} = 0$, so the last constraint we wrote in (62) means that the matter Lagrangian is a constant. Thus indeed a Klein-Gordon equation is retrieved by taking the time-derivative of (66)

$$(\partial_t^2 + m^2)\psi = 0\,. \tag{67}$$

This can also be obtained by varying the action from a pull-back of ψ. Using a vector field of the form $V^\mu = (\xi(x), 0, 0, 0)$, for arbitrary $\xi(x)$, for generation of the copy of $\mathcal{M}$ will cause ψ to vary. The EH action is invariant under variations of ψ, so the variation of the action by varying $\psi(t) \to \psi(t + \epsilon\xi(x))$ reduces to

$$\begin{aligned} 0 = \delta_t^\psi \mathcal{S}_{\mathrm{tot}} = \delta_t^\psi \mathcal{S}_M &= \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \delta_t^\psi \mathcal{L}_M \\ &= \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \epsilon\xi(x) g^{tt} \dot{\psi} \left(\ddot{\psi} + m^2 \psi \right)\,, \end{aligned}$$

where the superscript $^\psi$ reminds us that we have pulled-back the varied ψ field and not the metric. For an arbitrary $\xi(x)$, this again yields (67), from which the field has plane wave solutions for $m^2 > 0$. Then, since the matter Lagrangian is a constant, a solution to (60) is

$$a(t) \propto e^{\pm\sqrt{-\kappa \mathcal{L}_M / 6}\, t}\,. \tag{68}$$

A further possible solution to (60) for $\mathcal{L}_M = 0$ is

$$a(t) = c_1 (t + t_0)^{1/2} + c_2\,, \tag{69}$$

for constants t_0, c_1 & c_2. However, in general for a scalar field Lagrangian, we can always combine (60) and (63) to obtain

$$\begin{aligned} \frac{\dot{a}^2}{a^2} + 4\frac{\ddot{a}}{a} + \frac{\dddot{a}}{\dot{a}} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) \\ \Rightarrow \quad a\dot{a}\ddot{a} - 2\dot{a}^3 + a^2 \dddot{a} &= 0\,, \end{aligned}$$

which gives $c_2 = 0$ if $c_1 \neq 0$, or vice versa. Note that for this Lagrangian we have the flat cosmology, $k = 0$, using (58).

Other configurations of the field potential may provide alternative behaviour of the scale factor however, and for scalars in particular we are interested in the

Higgs field. Spontaneous symmetry breaking (SSB) potentials come with a large potential difference between the symmetric and broken phases[23,24]. Written in terms of even powers of the field, either a constant must be added to bring the vacuum state energy to an appropriate level, or the vacuum state contributes a large, negative (effective) constant to the potential. For example, consider the SSB potential (dropping the factor of u^2 momentarily)

$$V_{\mathrm{SSB}}(\psi) = -\tfrac{1}{2}\mu^2\psi^2 + \tfrac{1}{4}\lambda\psi^4 \,.$$

Expanding this potential around its minima yields a constant term and mass and interaction terms. However, it is well-known that this constant is unsuitable as a candidate for the cosmological constant (at least for the proper treatment of the Higgs field as a complex scalar; we take a real field here for simplicity). We now briefly discuss a means by which to remove the vacuum energy contribution from the field via defining an 'effective field' ψ, in terms of a true, fundamental field ψ'. Such an operation might be deemed trivial from the particle physics perspective, but for gravity, a constant in the Lagrangian is highly relevant.

It perhaps seems more natural if the potential does not involve a constant that is not in some way coupled to the field. Suppose there exists a fundamental field ψ', then, with a potential of the form

$$V'(\psi') = \alpha_2\psi'^2 + \alpha_3\psi'^3 + \alpha_4\psi'^4 \,, \tag{70}$$

where the α_n might be temperature dependent self-coupling parameters and the ψ'^n are powers of the field. Now suppose we wish to derive a low-temperature, 'effective field' ψ, with an SSB potential, forming a Lagrangian that is invariant under change of sign $\psi \to -\psi$.

If the fundamental field is related to the effective field by

$$\psi = \psi(\psi') \equiv \psi' - \psi_0 \,, \tag{71}$$

for a constant offset of centre of symmetry ψ_0, the field's derivative terms will be invariant. Thus for the Lagrangian to be invariant, we must have $V'(\psi') = V(\psi)$. The global $\mathbb{Z}_2$ symmetry is maintained if the effective potential contains terms of even powers in the effective field only, so let us define

$$V(\psi) \equiv \Lambda' - \tfrac{1}{2}\mu^2\psi^2 + \tfrac{1}{4}\lambda\psi^4 \,, \tag{72}$$

where $\Lambda' \equiv \Lambda + \mu^4/4\lambda$ and Λ is a constant, defined so the vacuum energy is given by Λ. Writing $V(\psi)$ in terms of ψ' from (71)

$$\begin{aligned} V(\psi) = \Lambda' &- \tfrac{1}{2}\mu^2\psi_0^2 + \tfrac{1}{4}\lambda\psi_0^4 + (\mu^2\psi_0 - \lambda\psi_0^3)\psi' \\ &+ \tfrac{1}{2}(3\lambda\psi_0^2 - \mu^2)\psi'^2 - \lambda\psi_0\psi'^3 + \tfrac{1}{4}\lambda\psi'^4 \,, \end{aligned} \tag{73}$$

we find the 'constants' α_n by matching powers of ψ' between (70) and (73). We have $\mu^2 = \lambda\psi_0^2$ and the constant Λ' is found to be

$$\Lambda' = \mu^4/4\lambda \qquad \Rightarrow \qquad \Lambda = 0 \,, \tag{74}$$

so the vacuum energy can be made to vanish when non-symmetric fundamental fields transition into symmetric effective fields. For vanishing Λ in the vacuum state, $V(\psi)$ contains the constant $\mu^4/4\lambda$, but the fundamental field potential is still written in terms of the field only:

$$V'(\psi')|_{\Lambda=0} = \mu^2\psi'^2 \mp \sqrt{\lambda}\mu\psi'^3 + \tfrac{1}{4}\lambda\psi'^4 \,. \tag{75}$$

Thus, the vacuum expectation value (VEV) of the fundamental field is either at $\psi'_{vac}=0$ or $\psi'_{vac}=\pm 2\mu/\sqrt{\lambda}$, where the sign corresponds to that appearing in the above potential, whereas the VEV of the effective field is $\psi_{vac}=\pm\mu/\sqrt{\lambda}$.

The point is that there seems to be a natural way (via field couplings only) to zero the vacuum state energy, or to make it very small when the symmetry is approximate. Presumably this procedure of offsetting the centre of symmetry can be straightforwardly applied to other fields, of particular interest of course is the Higgs scalar field, in which case the '$\mathbb{Z}_2$ field offset' above would become a (global) '$U(1)$ field offset', leading to the Higgs vacuum manifold becoming a disk rather than a circle.

Additionally, note that temperature dependent self-coupling parameters could drastically affect generation of *topological defects* such as *cosmic strings*, since, if the field evolves slowly enough, it would tend to fall into the 'central well' of the potential everywhere, before the vacuum manifold has fully developed.

Suppose our field is in its vacuum state. Then $\mathcal{L}_M = u^2\Lambda$, and (60) becomes

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = \kappa\Lambda = 0 \,.$$

In general, as for the purely quadratic potential described earlier, the Lagrangian with SSB potential is constant and does not dilute with the scale factor. Vacuum fluctuations of the field will contribute a non-vanishing positive value which may contribute to exponential solutions as found in (68).

Maxwell Electromagnetism

The final action to be considered will provide a good opportunity to review the different kinds of action that we introduced in part 3, and relevant methods of variation. Recall that actions were described either as integrals of functions or integrals of curves, and each of these types requires a distinct sort of variation in order to provide any equations of motion. For the action of electromagnetism (EM) and gravity, these two types appear simultaneously.

First, let us clear up some potential confusion with the form of the EH action. Looking at the simple action defined in (23), we might say the Lagrangian has a sort of 'internalised' coordinate system (as opposed to integration 'coordinates'). But we have not yet explicitly rewritten the EH action this way: as an integral, delimited using a volume element defined in some coordinate system, of a Lagrangian that has its 'internal components' defined with respect to another coordinate system. Until now, it has not been necessary to do this as we have been able to make progress without having to be explicit about it. As we shall see, this is due to the difference between 'active' and 'passive' transformations of metric components. Indeed, the variations we applied to the EH action already implied the presence of these so-called internal coordinates, although we could say they were hidden in notation under the active transformation.

Written explicitly then, we can take the gravitational action to be

$$\mathcal{S}_{Grav} \equiv \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 y \sqrt{-g(y^\alpha)}\, R(x^\mu(y^\alpha))\,, \tag{76}$$

so that the Ricci scalar is defined as some function of metric components, which themselves are defined with respect to coordinates x^μ. The Ricci scalar is integrated over the manifold with respect to different coordinates y^α, using volume element $\mathrm{d}^4 y\sqrt{-g(y)}$, so that the integrand is diffeomorphism invariant. Here, either of the coordinate systems x^μ or y^α could be transformed; the integrand will remain invariant. Note that upon integration, we write the coordinates x^μ as functions of coordinates y^α.

The action in (76) seems to be a natural definition, but let us consider its variation. The object that we integrate over the manifold to obtain the action is the Ricci scalar, which is varied by changing some chosen metric component. The varied action is then given by the integral of the varied Ricci scalar, but we must also be aware that varying the metric $g_{\mu\nu}(x)$ will induce a variation in the volume element $\mathrm{d}^4 y\sqrt{-g(y)}$, even though the metrics and their determinants are defined with different coordinate systems.

Keeping track of all of the terms that arise in the variation from the metrics in different coordinate systems would be a very messy and time consuming task, but we can save a good deal of effort by choosing to study the special case where the coordinate systems coincide, $x^\mu = y^\mu$. In that case, the action can be written in a way that departs from the more natural definition given in (76), but which is qualitatively equivalent and much simpler in practical applications:

$$\mathcal{S}_{EH} \equiv \int_{\mathcal{M}} \mathrm{d}^4 y \sqrt{-g(x^\mu(y^\alpha))}\, R(x^\mu(y^\alpha)) \Big|_{x^\mu = y^\mu} \tag{77}$$

The condition $x^\mu = y^\mu$ ensures diffeomorphism invariance of the Lagrangian.

In deriving equations (16)-(20), it was automatically implied that variations were to be interpreted as pullbacks of metric components. In this case, it is not necessary to consider a separate coordinate system. However, if the variations are interpreted as coming from changes to a metric component's parameters, then equation (77) provides the clearest rendition of the action, with a separate coordinate system shown explicitly. By defining the metric determinant in the volume element as being dependent on the x^μ coordinates means we can apply parameter variation operators in a straightforward way to this action.

For what follows, it is important to stress the equivalence between these two approaches. We can then talk about the variations only in terms of this latter, parameter variation interpretation. In terms of clarity of both the notation and visualisation, this is the most convenient view.

To see this, let us consider both the active and passive methods of performing the variation. The passive method is very simple, since it consists of varying a coordinate dependency of a chosen metric component (and all its dependants), for example: $g_{12}(x^0, x^1, x^2, x^3) \to g_{12}(x^0, x^1, x^2 + \epsilon f(x), x^3)$. On the other hand, the active kind requires a little more care. Instead of varying coordinate *values* of a component, we must simultaneously vary the coordinate *system* along with the metric component itself. This approach was demonstrated in equation (12). However, we must only vary the coordinate system for that single component (and its derivatives). These two approaches are equivalent, so we should rather use the simpler variation of parameter.

The reason for trying to be clear about these details is because we will now be faced with a variety of possibilities for the form of the EH+EM action and the methods by which to vary them, and we must be able to select the appropriate objects and operators. For example, one may suppose that variations of the EM action might also be performed in the same way in which we have varied the EH action, that is, by pulling back the fundamental field that the EM action is constructed from and replacing a component. Doing this will not be effective in producing equations of motion however; we will see that the parameter variation view can clearly show why. Let us now step through the possible options.

We have looked at actions as integrals of functions or curves. In part 3, the curve length was the integral of a differential form of rank 1, or *one-form*. This exemplified a *top-form*[25] in that 1-dimensional case. In the case of gravity and matter in 3+1 dimensions, the analogous top-forms are *four-forms*.

The EH action can be written in the language of differential forms as

$$\mathcal{S}_{EH} = \frac{1}{2\kappa} \int_{\mathcal{M}} \star R\,, \tag{78}$$

where $\star$ is the Hodge-star operator. The integrand can be seen as the top-form $\star 1$, scaled by the Ricci scalar R. This is not a useful representation for variation with respect to a metric component however, as we need to know the variations of all affected components, for which (77) is suitable.

From the discussion on forming actions by integrating Lagrangians defined in terms of functions and their derivatives (which are the basic objects we want to vary here) it is only possible to do to using scalars, such as the Ricci scalar, that can be ascribed with some 'internal' coordinates (so that the functions are in $\mathbb{R}^m$ and not directly on $\mathcal{M}$). The discussion around equation (23) contains relevant arguments behind this statement. It was also reasoned there that, in that case, only variations by pull-back of those functions could lead to equations of motion from the stationary action principle.

There are more subtleties with the EM action however. Let us consider the sourceless EM action for now, and start with the 'coordinate description'. This is given in terms of local coordinates by

$$\mathcal{S}_{EM} \equiv -\tfrac{1}{4} \int_{\mathcal{M}} \mathrm{d}^4x \, \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu}\,, \tag{79}$$

where $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field strength tensor, and A is the one-form known as the vector potential. There is ambiguity in this notation however. It is not manifest in the above statement that the μ, ν indices of the field strength tensor correspond to the same coordinate system that $\mathrm{d}^4x\,\sqrt{-g}$ is defined with respect to. Indeed, $F_{\mu\nu}F^{\mu\nu}$ is a Lorentz scalar, so we should surely be able to choose any two different sets of coordinates for each of the Lorentz invariants: $\mathrm{d}^4x\,\sqrt{-g}$ and $F_{\mu\nu}F^{\mu\nu}$.

Therefore, the way this action is written suggests there exists a scalar field, $F_{\mu\nu}F^{\mu\nu}$, that is integrated to give the EM action in the same way as the Ricci scalar is to give the EH action. If this were the case, then we should be able to vary components A_μ, just as we varied components $g_{\mu\nu}$ that define the Ricci scalar. This can be attempted, where, say, the A_b component and its dependants transform as $A_b(x) \to A_b(x+\epsilon\xi(x)) = A_b(x) + \epsilon\xi^\sigma\partial_\sigma A_b(x) + \mathcal{O}(\epsilon^2)$, for some vector field ξ. Redundancies in ξ allow for setting collections of terms in the varied Lagrangian to zero, whilst leaving a remaining arbitrary component of ξ that allows the fundamental lemma of variational calculus to be applied to some left-over, 'isolated' terms. The resulting equations are highly restrictive, and it should be concluded that the treatment of the action with $F_{\mu\nu}F^{\mu\nu}$ as a scalar field on the manifold is not the correct one.

Of course we should hope to obtain Maxwell's equations from the EM action. Writing the EM action with sources in differential form notation, we have[26]

$$\mathcal{S}_{EM} \equiv \int_{\mathcal{M}} -\tfrac{1}{2}F\wedge\star F - A\wedge\star J\,, \tag{80}$$

where $F = \mathrm{d}A$, J is the source term and $\wedge$ is the exterior product. It is well-known that the canonical E-L equations of this action give rise to the Maxwell equations. Recall that variations that lead to the E-L equations are interpreted as a physical perturbation of the field, in this case, the one-form A. Since this way of writing the action is coordinate independent, the interpretation of the type of physical object that exists on the manifold is unambiguous.

One might still imagine that different components of A can be varied through our method of replacing components of A by relevant pull-backs. In that case, derivatives would vary in the same way as the components do (as is the case for the metric components), for example, $\delta\partial_a A_b = \epsilon\xi^\sigma\partial_\sigma\partial_a A_b$, rather than varying as per the derivative of the variation itself, $\delta\partial_a A_b = \partial_a\delta A_b$. To explain why pull-back variations do not work on actions constructed from differential forms, we can use equivalence between the pull-back and parameter variations.

First, write the one-form in some local coordinate frame, so that

$$A = A_\mu \mathrm{d}x^\mu\,.$$

Now we can treat the A_μ coefficients as functions. If A is varied by variation of these functions, $A' \equiv A'_\mu \mathrm{d}x^\mu$, the result is a physical perturbation of the field so the ordinary E-L equations must be employed, since a coordinate chart and the integration differentials $\mathrm{d}x^\mu$ are already defined.

On the other hand, if we tried to generate a variation by copying the manifold with all of its contents and dragging it along a vector field by a small amount, as we did for variation of the metric components, then by equivalence of pull-back and parameter transformation interpretations, the varied field will be given by

$$A' \equiv A_\mu(x')\mathrm{d}x'^\mu = A_\mu(x)\mathrm{d}x^\mu = A\,,$$

so this variation vanishes identically for actions defined from differential forms. The term ‘stationary point’ of such an action under this type of variation loses its meaning, so the action principle cannot be applied. The variations must be generated in the usual way that leads to the canonical E-L equations.

It has been argued that metric components, as functions on $\mathbb{R}^4$, can only be varied by variations of their parameter dependencies, and that the EM vector potential, as a differential form, can only be varied from physical perturbations to the field. Thus, the EH+EM action has various ways of being written that are suitable for the different types of variation. To obtain Maxwell’s equations, it is useful to write

$$\mathcal{S}_{EH+EM} \equiv \int_{\mathcal{M}} \tfrac{1}{2\kappa} \star R - \tfrac{1}{2} F \wedge \star F - A \wedge \star J\,. \tag{81}$$

This can be varied in the usual way for Maxwell’s equations: $A \to A' = A + \delta A$.

To vary with respect to metric components however, we must revert to the coordinate description of the vector potential. If we are to write down an EM action like that in (79), it should be made explicit that both Lorentz invariants, $\mathrm{d}^4x\,\sqrt{-g}$ and $F_{\mu\nu}F^{\mu\nu}$, are defined with respect to the same coordinates, so that it precisely and unambiguously coincides with the action given by the differential form notation. This is done simply by writing dependencies explicitly:

$$\mathcal{S}_{EM} \equiv \int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g(x)}\,\left(-\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + J^\mu(x)A_\mu(x)\right)\,, \tag{82}$$

where we have now included EM sources.

Further steps are required, however, in order to be able to perform variations of (82) with respect to metric components in a straightforward manner. Again, rather than pulling back relevant metric components and the coordinate charts they are defined by, we can use the equivalent approach of varying parameters. To do this, as we did in (77), we can replace all occurrences of metric components and their dependants defined using coordinate chart y^α, by the corresponding objects defined by coordinate chart $x^\mu(y^\alpha)$ when $x^\mu = y^\mu$, thus

$$\mathcal{S}_{EM} \equiv \int_{\mathcal{M}} \mathrm{d}^4y\,\sqrt{-g(x^\mu(y^\alpha))}\,\left(-\tfrac{1}{4}F_{\mu\nu}(y)F^{\mu\nu}(y) + J^\mu(y)A_\mu(y)\right)\Big|_{x^\mu=y^\mu} \tag{83}$$

We should not forget about metric components that appear in contractions of indices in $F_{\mu\nu}F^{\mu\nu}$ and in $J^\mu A_\mu$, and of course, from Maxwell’s equations that give $J^\nu = \nabla_\mu F^{\mu\nu}$, the metric dependants that appear in the Christoffel symbols. However, the notation is becoming cumbersome, and since it would be difficult to write everything out explicitly, keeping track of the different coordinate systems in order to apply parameter variations, we will now omit writing details of the internal coordinates and conditions on them, and simply recall how variations operate on the metric components as they arise.

With an understanding of the machinery of variations, and taking account of the equivalence between parameter variations and pull-backs of components, we finally write the EH+EM action as

$$\mathcal{S}_{EH+EM} \equiv \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\Big(\tfrac{1}{2\kappa}R - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\nu \nabla_\mu F^{\mu\nu}\Big)\,, \tag{84}$$

where we have used Maxwell's equations to replace J^ν in the last term.

We have taken a long but necessary detour to apply the reasoning developed in section three to higher dimensional Lagrangians, and now return to the task of finding cosmological solutions with the EM action. As mentioned previously, the cosmological symmetry leading to equation (62) means that any Lagrangian independent of time derivatives of the metric (which seems to naturally appear only via covariant derivatives) must always equal a constant. Thus, if we were to consider the sourceless case of (84), that is, a uniform distribution of 'radiation' only, then we can immediately say the scale factor is either like (68) for $\mathcal{L}_M \neq 0$, and like (69) for $\mathcal{L}_M = 0$.

Let us compute explicitly the Maxwell terms in (84) to see the implications of the cosmological constraints found earlier. For the pure gauge part, we have

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &= g^{tt}g^{tt}F_{tt}F_{tt} + g^{tt}g^{ij}F_{ti}F_{tj} + g^{ij}g^{tt}F_{it}F_{jt} + g^{ij}g^{kl}F_{ik}F_{jl} \\ &= 2g^{tt}g^{ij}F_{ti}F_{tj} + g^{ij}g^{kl}F_{ik}F_{jl}\,. \end{aligned}$$

For the source term, we have

$$\begin{aligned} A_\nu \nabla_\mu F^{\mu\nu} &= g^{\mu\rho}g^{\nu\sigma}A_\nu \nabla_\mu F_{\rho\sigma} \\ &= g^{\mu\rho}g^{\nu\sigma}A_\nu(\partial_\mu F_{\rho\sigma} - \Gamma^\alpha{}_{\mu\rho}F_{\alpha\sigma} - \Gamma^\alpha{}_{\mu\sigma}F_{\rho\alpha}) \\ &= g^{tt}g^{tt}A_t\nabla_t F_{tt} + g^{ij}g^{tt}A_i(\partial_t F_{tj} - \Gamma^\alpha{}_{tt}F_{\alpha j} - \Gamma^\alpha{}_{tj}F_{t\alpha}) \\ &\quad\quad + g^{tt}g^{ij}A_t(\partial_i F_{jt} - \Gamma^\alpha{}_{ij}F_{\alpha t} - \Gamma^\alpha{}_{it}F_{j\alpha}) \\ &\quad\quad + g^{kl}g^{ij}A_i(\partial_k F_{lj} - \Gamma^\alpha{}_{kl}F_{\alpha j} - \Gamma^\alpha{}_{kj}F_{l\alpha}) \\ &= g^{ij}g^{tt}A_i(\partial_t F_{tj} - \Gamma^k{}_{tj}F_{tk}) \\ &\quad\quad + g^{tt}g^{ij}A_t(\partial_i F_{jt} + \Gamma^k{}_{ij}F_{tk}) \\ &\quad\quad + g^{kl}g^{ij}A_i(\partial_k F_{lj} - \Gamma^t{}_{kl}F_{tj} + \Gamma^t{}_{kj}F_{tl}) \\ &= g^{tt}g^{ij}A_i(\partial_t F_{tj} - g^{kl}g_{lj,t}F_{tk} + \tfrac{1}{2}g^{kl}g_{kl,t}F_{tj}) \\ &\quad\quad + g^{ij}g^{kl}A_i\partial_k F_{lj} + g^{tt}g^{ij}A_t(\partial_i F_{jt} + \Gamma^k{}_{ij}F_{tk})\,, \end{aligned} \tag{85}$$

and thus

$$\begin{aligned} \tilde{\mathcal{L}}_{EM} &= -\tfrac{1}{2}g^{tt}g^{ij}F_{ti}F_{tj} - \tfrac{1}{4}g^{ij}g^{kl}F_{ik}F_{jl} \\ &\quad\quad + g^{tt}g^{ij}A_i(\partial_t F_{tj} - g^{kl}g_{lj,t}F_{tk} + \tfrac{1}{2}g^{kl}g_{kl,t}F_{tj}) \\ &\quad\quad + g^{ij}g^{kl}A_i\partial_k F_{lj} + g^{tt}g^{ij}A_t(\partial_i F_{jt} + \Gamma^k{}_{ij}F_{tk})\,. \end{aligned} \tag{86}$$

This expression for $\tilde{\mathcal{L}}_{EM}$ is useful as it is valid with any coordinate choice for the 3D spatial part. Using this in (58), we obtain the following relation between the EM field strength tensor and the scaled curvature parameter

$$\frac{3k}{\kappa a^2} = \tfrac{1}{4}F_{ij}F^{ij} - g^{ij}g^{kl}A_i\partial_k F_{lj} \tag{87}$$

Suppose we take usual coordinates expressed in (27), then we can also apply the EM Lagrangian to (59) to obtain

$$\begin{aligned} 0 = -\tfrac{1}{2}g^{tt}F_{t\phi}F_{t\phi} - \tfrac{1}{2}g^{ij}F_{i\phi}F_{j\phi} + g^{tt}A_\phi\left(\partial_t F_{t\phi} + \frac{\dot{a}}{a}F_{t\phi}\right) \\ + g^{kl}A_\phi\partial_k F_{l\phi} + g^{tt}A_t(\partial_\phi F_{\phi t} + \Gamma^k{}_{\phi\phi}F_{tk}) + g^{ij}A_i\partial_\phi F_{\phi j}\,, \end{aligned} \tag{88}$$

which is valid for any EM matter Lagrangian that is cosmologically symmetric in the weakest sense, that is, taking only $\tilde{\mathcal{L}}_{EM} = \tilde{\mathcal{L}}_{EM}(t)$.

Now, it is certainly the case, given that $R \propto \tilde{\mathcal{L}}_{EM}$ from (20), and $R = R(t)$, that $\tilde{\mathcal{L}}_{EM} = \tilde{\mathcal{L}}_{EM}(t)$. We have not yet imposed constraints on the fields A & F; for simplicity of modelling, let us now demand $F_{\mu\nu} = F_{\mu\nu}(t)$. However, for these both to stand, then (if $J^\mu \neq 0$) it must also be that $A_\mu = A_\mu(t)$. Consequentially, the Lagrangian reduces to

$$\begin{aligned} \tilde{\mathcal{L}}_{EM}|_{A_\mu = A_\mu(t)} = -\tfrac{1}{2}g^{tt}g^{ij}F_{ti}F_{tj} + g^{tt}g^{ij}A_t\Gamma^k{}_{ij}F_{tk} \\ + g^{tt}g^{ij}A_i(\partial_t F_{tj} - g^{kl}g_{lj,t}F_{tk} + \tfrac{1}{2}g^{kl}g_{kl,t}F_{tj})\,, \end{aligned} \tag{89}$$

and from (87) we have that $k|_{A_\mu = A_\mu(t)} = 0$. Let us use Cartesian coordinates for the flat spacelike part so that purely spatial Christoffel symbols vanish. The Lagrangian becomes

$$\tilde{\mathcal{L}}_{EM}|_{A_\mu(t)} = -\tfrac{1}{2}g^{tt}g^{ij}F_{ti}F_{tj} + g^{tt}g^{ij}A_i\left(\partial_t F_{tj} - g^{kl}g_{lj,t}F_{tk} + \tfrac{1}{2}g^{kl}g_{kl,t}F_{tj}\right)\,. \tag{90}$$

In order for A_μ to be isotropic, we must have that $A_i = 0$, leading to $\tilde{\mathcal{L}}_{EM} = 0$, thus metric solutions are of the form (69). We can try to constrain this solution further however by considering a gauge transformation[27]

$$A_\mu \to A_\mu - \partial_\mu\lambda \tag{91}$$

for arbitrary λ. The EM source term is not gauge invariant, since

$$\begin{aligned} A_\nu\nabla_\mu F^{\mu\nu} &\to (A_\nu - \partial_\nu\lambda)\nabla_\mu(\partial^\mu(A^\nu - \partial^\nu\lambda) - \partial^\nu(A^\mu - \partial^\mu\lambda)) \\ &= A_\nu\nabla_\mu F^{\mu\nu} - \partial_\nu\lambda\nabla_\mu F^{\mu\nu}\,, \end{aligned}$$

but the A_μ equations of motion do not vary under gauge transformations as we will see. The extra term arising from the transformation can be written as

$$\partial_\nu\lambda\nabla_\mu F^{\mu\nu} = \nabla_\nu\lambda\nabla_\mu F^{\mu\nu} = \nabla_\nu(\lambda\nabla_\mu F^{\mu\nu}) - \lambda\nabla_\nu\nabla_\mu F^{\mu\nu}$$

The covariant derivative commutation relation for (2,0) tensors can be written[11]

$$(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu)T^{\alpha\beta} = R^\alpha{}_{\gamma\nu\mu}T^{\gamma\beta} + R^\beta{}_{\gamma\nu\mu}T^{\alpha\gamma}\,.$$

We can then write

$$\begin{aligned}\nabla_\nu\nabla_\mu F^{\mu\nu} &= \tfrac{1}{2}(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu)F^{\mu\nu}\\ &= \tfrac{1}{2}(R^\mu{}_{\gamma\mu\nu}F^{\gamma\nu} + R^\nu{}_{\gamma\mu\nu}F^{\mu\gamma})\\ &= \tfrac{1}{2}(R^\mu{}_{\gamma\mu\nu}F^{\gamma\nu} - R^\nu{}_{\gamma\mu\nu}F^{\gamma\mu})\\ &= \tfrac{1}{2}(R^\mu{}_{\gamma\mu\nu} - R^\mu{}_{\gamma\nu\mu})F^{\gamma\nu}\\ &= R_{\gamma\nu}F^{\gamma\nu}\,,\end{aligned}$$

which vanishes since $R_{\mu\nu}$ is symmetric and $F_{\mu\nu}$ is antisymmetric. Thus

$$\begin{aligned}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,A_\nu\nabla_\mu F^{\mu\nu} &\to \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,(A_\nu\nabla_\mu F^{\mu\nu} - \nabla_\nu(\lambda\nabla_\mu F^{\mu\nu}))\\ &= \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,A_\nu\nabla_\mu F^{\mu\nu} - \int_{\mathcal{M}} \mathrm{d}^4x\,\partial_\nu(\sqrt{-g}\,\lambda\nabla_\mu F^{\mu\nu})\,.\end{aligned}$$

The last term vanishes upon variation of A_μ, since it contributes only a boundary term and recall we argued that, for differential form fields, variations are fixed necessarily on boundaries. The Maxwell equations are therefore invariant.

However, for a general A_μ, the last term does not necessarily vanish when varying $g_{\mu\nu}$. That is, of course, unless the gauge freedom is spoiled by placing highly restrictive and specific constraints on λ. The field equations for gravity are therefore not invariant under gauge transformations, so the EM source part of this Lagrangian is not suitable: ultimately we require the action to be gauge invariant under variation of any field. Note that boundary terms from non gauge invariant theories invalidate the zero-action universe condition: $\mathcal{S}|_{\delta\mathcal{S}=0}=0$.

We might still be able to determine how this particular Lagrangian constrains metric solutions since we found that $A_i=0$, and with $A_t=A_t(t)$, we have $F_{\mu\nu}=0$, so the boundary term vanishes. Not surprisingly, this empty solution seems to be gauge invariant (even though the theory is not) so let us try

$$\lambda = -(x+y+z)\mathcal{A}(t)$$

for an arbitrary function of time $\mathcal{A}(t)$. Then, in components, the vector potential A_μ will transform to

$$(A_t,0,0,0) \to A_\mu - \partial_\mu\lambda = (A_t - \partial_t\lambda, \mathcal{A}, \mathcal{A}, \mathcal{A})\,.$$

Putting this into the general Lagrangian form (86), we have

$$\mathcal{L}_{EM} = \delta_{ij}g^{tt}\left[-\tfrac{1}{2}g^{ij}\dot{\mathcal{A}}^2 + g^{ij}\ddot{\mathcal{A}}\mathcal{A} + (\tfrac{1}{2}g^{ij}g^{kl}g_{kl,t} - g^{ik}g^{jl}g_{lk,t})\dot{\mathcal{A}}\mathcal{A}\right]\,. \tag{92}$$

Since the spatial part uses Cartesian coordinates, (59) can be generalised as the set of three equations

$$\frac{\delta_i\mathcal{L}_M}{\delta_i g^{ii}} = 0\,, \qquad \text{for } i\in(x,y,z)\,. \tag{93}$$

Inserting (92) into (93) then gives three copies of:

$$\tfrac{1}{2}\dot{\mathcal{A}}^2 = \mathcal{A}\left(\ddot{\mathcal{A}} + \frac{\dot{a}}{a}\dot{\mathcal{A}}\right) . \tag{94}$$

Substituting this back into (92) then gives

$$\mathcal{L}_{EM} = 0 \,,$$

as expected. In the same sense that (59) generalises to (93) by using Cartesian coordinates, so equation (62) also generalises to the three equations

$$\dot{\mathcal{L}_M} = -6\frac{\dot{a}}{a}\frac{\delta_t \mathcal{L}_M}{\delta_t g^{ii}} g^{ii} \,, \qquad \text{for } i \in (x, y, z) \,. \tag{95}$$

With (92), the quotient of variations with respect to, say, g^{xx} is

$$\begin{aligned}
\frac{\delta_t \mathcal{L}_{EM}}{\delta_t g^{xx}} &= g^{tt}\Bigg[-\tfrac{1}{2}\dot{\mathcal{A}}^2 + \ddot{\mathcal{A}}\mathcal{A} + \dot{\mathcal{A}}\mathcal{A}\Bigg(\tfrac{1}{2}g^{kl}g_{kl,t} - g^{xx}g_{xx,t} \\
&\qquad + \delta_{ij}g^{ij}\frac{\delta_t}{\delta_t g^{xx}}(g^{kl}g_{kl,t}) + g^{xk}\frac{\delta_t}{\delta_t g^{xx}}(g^{xl}g_{kl,t})\Bigg)\Bigg] \\
&= g^{tt}\left[-\tfrac{1}{2}\dot{\mathcal{A}}^2 + \ddot{\mathcal{A}}\mathcal{A} + \dot{\mathcal{A}}\mathcal{A}\left(3\frac{\dot{a}}{a} - 2\frac{\ddot{a}}{\dot{a}}\right)\right] \\
&= 2g^{tt}\dot{\mathcal{A}}\mathcal{A}\left(\frac{\dot{a}}{a} - \frac{\ddot{a}}{\dot{a}}\right) ,
\end{aligned} \tag{96}$$

where in the last step we used (94). Variations with respect to the g^{yy} and g^{zz} components give the same result.

Since the EM Lagrangian vanishes, so does its time derivative. Thus, from (95) and (96), we have that

$$\dot{\mathcal{A}}\mathcal{A}\frac{\dot{a}}{a^3}\left(\frac{\dot{a}}{a} - \frac{\ddot{a}}{\dot{a}}\right) = 0 \,. \tag{97}$$

If $\dot{\mathcal{A}}{=}0$ then this is automatically satisfied, and general solutions to the metric are given by (68) and (69). However, if $\dot{\mathcal{A}}{\neq}0$, then the term in brackets in (97) must vanish, and the solutions for $a(t)$ are exponential. Now, since the function $\mathcal{A}$ is (supposedly) arbitrary, then $\dot{\mathcal{A}}$ is too, so the only option consistent with both of these types of solution is that $a(t)$ is exponential.

This solution seems reasonable for an empty universe as it could be constant, but the empty matter Lagrangian is still not gauge invariant as supposed, since $\dot{a}/a = const. = C$, so from (94)

$$\frac{\dot{a}}{a} = C = \frac{\dot{\mathcal{A}}}{2\mathcal{A}} - \frac{\ddot{\mathcal{A}}}{\dot{\mathcal{A}}} = \partial_t \ln\frac{\sqrt{\mathcal{A}}}{\dot{\mathcal{A}}} \qquad \Rightarrow \qquad \mathcal{A} = \left(c_3 + c_4 e^{Ct}\right)^2 \tag{98}$$

for constants c_3 and c_4. The gauge field is thus strongly constrained, contrary to requirements. Future work may involve analysing fully gauge invariant theories, for instance Dirac electrodynamics, and modelling inhomogeneities.

VI. Closing Remarks

It is unfortunate that we were unable to derive useful EM dynamics, although this was perhaps foreseeable given the symmetry and the nature of the EM field. We also used a source term which does not have the required property of gauge invariance and is thus not suitable as part of a theory that includes gravitation. This term should be replaced by a physically reasonable one, or omitted.

The case of sourceless EM is very interesting, however it is also unfortunate that there does not seem to be an obvious means by which to describe matter in terms of its density and pressure, which accurately reproduces the effect of the source term in the field equations. This would of course be extremely valuable for cosmology. A perturbative method, where photons are either sparse enough or arranged in some intricate way such that they never coincide, might render a modelling approach similar to using (64) as reasonable. One could in that case even treat collections of photons propagating in different directions as separate 'fields', writing the total matter Lagrangian as a sum of matter Lagrangians for each such field, to make calculations easier.

In the absence of such devices, progress even in supposedly simple situations like cosmology seems difficult. However, the arguments presented here do point to a flaw in the derivation of the EFEs from the EH action, and some convincing results arise from the framework put forth here, such as the zero-energy universe (20), the potential spatial flatness from the matter Lagrangian alone (58), and the concise form of the field equations, which may be written

$$\delta_\gamma^{g^{\mu\nu}}\big(R + 2\kappa\mathcal{L}_M\big) = 0\,. \tag{99}$$

There is also the fact that, from the EH action alone, GR can only be derived when the spacetime manifold is without boundary. In contrast, the variation of the action in (15) not only allows for potential boundary term contributions to the equations of motion, but will in fact provide different behaviours depending on whether or not the universe has a boundary, and so a means to detect one.

In (15), the term $\delta_c R_{\rho\sigma}$ appears in the integral. This can be rewritten, using the so-called *Palatini identity*[28] as

$$\delta_c R_{\rho\sigma} = \nabla_\alpha(\delta_c \Gamma^\alpha{}_{\sigma\rho}) - \nabla_\sigma(\delta_c \Gamma^\alpha{}_{\rho\alpha})\,,$$

so that contraction with $g^{\rho\sigma}$ can be written

$$g^{\rho\sigma}\delta_c R_{\rho\sigma} = \nabla_\alpha(g^{\rho\sigma}\delta_c\Gamma^\alpha{}_{\sigma\rho} - g^{\rho\alpha}\delta_c\Gamma^\sigma{}_{\rho\sigma})\,.$$

The integral of this in (15) can then be written

$$\begin{aligned}\frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,g^{\rho\sigma}\,\delta_c R_{\rho\sigma} &= \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\alpha(g^{\rho\sigma}\delta_c\Gamma^\alpha{}_{\sigma\rho} - g^{\rho\alpha}\delta_c\Gamma^\sigma{}_{\rho\sigma})\\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\partial_\alpha\left[\sqrt{-g}\,(g^{\rho\sigma}\delta_c\Gamma^\alpha{}_{\sigma\rho} - g^{\rho\alpha}\delta_c\Gamma^\sigma{}_{\rho\sigma})\right]\end{aligned}$$

which, by Stokes' theorem, is equivalent to the integral over a boundary since it is the integral of a total derivative. If the universe does not have a boundary, this will vanish. In this case, note that we can write

$$0 = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, g^{\rho\sigma}\, \delta_c R_{\rho\sigma} = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \delta_i g^{ii} g^{\rho\sigma}\, \frac{\delta_i R_{\rho\sigma}}{\delta_i g^{ii}}\,,$$

where we have set $c = i$, a spatial component, and chosen to vary g^{ii}. Then for arbitrary $\delta_i g^{ii} = \epsilon V^i(x) \partial_i g^{ii}$, the fundamental lemma of the variational calculus implies that

$$g^{\rho\sigma}\, \frac{\delta_i R_{\rho\sigma}}{\delta_i g^{ii}} = 0\,.$$

However, we already calculated this in (49) for the case of a cosmologically symmetric distribution of matter and vanishing spatial curvatuve. Using this, if the boundary term vanishes, we find from these equations that $2\dot{a}^2 = -a\ddot{a}$. That is, the scale of the universe would have a negative acceleration. In reality, the universe appears to be spatially flat with *positively* accelerating scale[29], which indicates that the universe does indeed have a boundary.

In terms of other cosmological phenomena, since it is possible that the universe has a boundary, it is my hope that boundary contributions to the equations of motion may help explain the dark matter problem. This is rather speculative, but note that in vacuum, the field equations (99) reduce to

$$0 = \delta_c^{g^{\mu\nu}} R = R_{\mu\nu} + g^{\rho\sigma}\, \frac{\delta_c R_{\rho\sigma}}{\delta_c g^{\mu\nu}}\,,$$

which suggest initial boundary condition 'imprints' on the geometry, in addition to contributions of nearby matter, as understood from the vacuum EFEs.

In terms of the possible presence of singularities in the new field equations, note that the Hawking-Penrose singularity theorems apply to GR and might not be transferrable to the equations presented here: it remains to be seen whether (99) are susceptible to similar problems.

In regions of the manifold very far from the boundaries, it is usually assumed that boundary contributions to local equations of motion are negligible and can be dismissed. For tests of gravity on small scales, this corresponds to rewriting the $g^{\rho\sigma}\delta_c R_{\rho\sigma}$ term in the varied action integral as an integral over the boundary as we have just done. Over relatively small spacetime volumes in which we test gravity, this is approximately constant and makes very little contribution to the equations of motion. In that case, the equations as in (17) reduce to

$$R_{ab} - \tfrac{1}{2} g_{ab} R = \kappa g_{ab} \mathcal{L}_M - 2\kappa\, \frac{\delta_c \mathcal{L}_M}{\delta_c g^{ab}} = \frac{-2\kappa}{\sqrt{-g}}\, \frac{\delta_c(\sqrt{-g}\mathcal{L}_M)}{\delta_c g^{\mu\nu}}\,.$$

The right-hand-side is almost the stress-energy tensor as defined in (54), apart from the fact that the variation operators represent a set of terms, rather than a single one. However, these equations *are* the EFEs if one assumes the boundary term vanishes exactly, in which case, since c is arbitrary, we may define the right-hand-side as a 'special case' stress-energy tensor, applicable when the boundary terms vanish exactly. This leads to a suggested form for a general case, in which we could decide to use the usual EFE form, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$, except where

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta_c(\sqrt{-g}\mathcal{L}_M)}{\delta_c g^{\mu\nu}} - \kappa^{-1} g^{\rho\sigma} \frac{\delta_c R_{\rho\sigma}}{\delta_c g^{\alpha\beta}} \,,$$

which is the same tensor for all valid c.

Further constraints on solutions of the EFEs arise by taking their covariant divergence. If we do the same with equations (18), we get

$$\begin{aligned} 0 &= \nabla_\mu \left(R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} + g^{\mu\alpha} g^{\nu\beta} \left(g^{\rho\sigma} \frac{\delta_a R_{\rho\sigma}}{\delta_a g^{\alpha\beta}} + 2\kappa \frac{\delta_a \mathcal{L}_M}{\delta_a g^{\alpha\beta}} - \kappa g_{\alpha\beta} \mathcal{L}_M \right) \right) \\ &= g^{\mu\alpha} g^{\nu\beta} \nabla_\mu \left(g^{\rho\sigma} \frac{\delta_a R_{\rho\sigma}}{\delta_a g^{\alpha\beta}} + 2\kappa \frac{\delta_a \mathcal{L}_M}{\delta_a g^{\alpha\beta}} - \kappa g_{\alpha\beta} \mathcal{L}_M \right) \end{aligned}$$

since the first two terms vanish by the contracted *Bianchi identity*[17]. This again reduces to a conservation of the special case stress-energy tensor for vanishing boundary terms.

Finally, note that a familiar argument[1] for conservation of the stress-energy tensor used in GR may not be meticulous in a general case. Here, one considers variations of the action via infinitesimal diffeomorphism; the EH term vanishes due to its diffeomorphism invariance, and matter action variations with respect to matter diffeomorphisms always vanish (by the matter equations of motion), leaving only matter action variations with respect to metric diffeomorphisms:

$$0 = \delta^\phi \mathcal{S} = \delta^\phi \mathcal{S}_M = \int_{\mathcal{M}} \mathrm{d}^4 x \; \delta^\phi g^{\mu\nu} \frac{\delta^\phi(\sqrt{-g}\mathcal{L}_M)}{\delta^\phi g^{\mu\nu}} \,,$$

where the superscript ϕ indicates diffeomorphism variation generated by vector field V. For diffeomorphism variations of the metric, using (10) and the identity[1] $\pounds_V g_{\mu\nu} = 2\nabla_{(\mu} V_{\nu)}$, we have $\delta^\phi g_{\mu\nu} = 2\nabla_{(\mu} V_{\nu)}$, leading to

$$0 = \int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{-g} \, V_\mu \nabla_\nu \left(\frac{1}{\sqrt{-g}} \frac{\delta^\phi(\sqrt{-g}\mathcal{L}_M)}{\delta^\phi g^{\mu\nu}} \right) .$$

It is then claimed that, since V_μ is arbitrary, by the fundamental lemma of variational calculus we must have

$$\nabla_\nu \left(\frac{1}{\sqrt{-g}} \frac{\delta^\phi(\sqrt{-g}\mathcal{L}_M)}{\delta^\phi g^{\mu\nu}} \right) = 0 \,.$$

However, note that application of the fundamental lemma of variational calculus is only valid if it can be shown that the term in brackets is independent of V. Since this is not necessarily possible for *all* variations, then one must be cautious about when the term in brackets can be identified as a conserved tensor.

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