An Introduction to Julia Sets

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1 Introduction and Definitons

Julia sets, named after Gaston Julia (1893-1978), arise from analysing the dynamics of complex functions. We first fix a function $f : \mathbb{C} \to \mathbb{C}$ and then consider the local behaviour of f around points in the complex plane. A function f is said to display *sensitive dependence* at a point $z \in \mathbb{C}$ if, roughly speaking, there exists a constant $\delta > 0$, such that no matter how small a neighbourhood of z we consider there will be a point w in that neighbourhood such that the respective images of z and w under f are at least δ apart.

Devaney (1994: 5-7) gives four characterisations of what we might term a *Julia set*. The first being that the *Julia set* $J(f)$ of a function f is the set of all points z in $\mathbb{C}$ such that f displays sensitve dependence at z. In other words, $J(f)$ is the set of all *chaotic* points of f. This definiton could be used in the study of arbitrary functions $f : \mathbb{C} \to \mathbb{C}$ but for the purposes of this report we will only be concerned with functions which are *holomorphic* i.e functions which are complex-differentiable for all $z \in \mathbb{C}$.

An amazing aspect in the theory of Julia sets is that one does not have to make f a complicated function to find examples where $J(f)$ is an interesting and complicated set. For this reason we will be primarily interested in the Julia sets of functions of the form $f_c(z) = z^2 + c$ for some complex constant c. This apparently simple family of polynomial maps appears at first to be a big restriction but, in fact, the Julia sets which arise from these maps show most of, if not all, the properties that we require to give a full introduction to the subject.

Devaney's fourth characterisation of a Julia set is based on classifying points in the complex plane according to their long term behaviour under repeated application of f. A distinction is made between points z for which $f^n(z)$ remains bounded as $n \to \infty$ and points for which $f^n(z)$ diverges. The *Julia set* is then defined to be the boundary between these two types of behaviour. What makes Julia sets interesting to study is that, despite being borne out of apparently simple iterative processes they can be very inticate and often fractal in nature. More formally we define the *filled in Julia set*, the *Julia set* and the *Fatou set*, named after Pierre Fatou (1878-1929), as follows: (Falconer, 2003: 215-216)

Definition 1: The *filled in Julia set* of the function f is defined as

$$K(f) = \{z \in \mathbb{C} : f^k(z) \to \infty\}$$

Definition 2: The *Julia set* of the function f is defined to be the boundary of $K(f)$ i.e.

$$J(f) = \partial K(f)$$

Definition 3: The *Fatou set* of f is defined to be the compliment of $K(f)$ i.e.

$$F(f) = \mathbb{C} \setminus K(f)$$

Devaney points out that this fourth characterisation is only valid for polynomial maps. The reason for this is that for arbitrary *meromorphic* maps (functions on $\mathbb{C}$ which are holomorphic at all but an isolated set of points) the 'point at infinity' is an *essential singularity* and may not be *superattracting* (a point with derivative 0). (Devaney 1994: 3, 7)

2 History

The modern day interest in Julia sets and related mathematics began in the 1920's with Gaston Julia. Julia was born in 1893 to Delorés Delavent and Joseph Julia. His extraordinary talents were recognised from an early age and although he excelled in every subject, he was always most passionate about mathematics. He completed his studies at l'École Normale Supériore in 1914 but was not allowed to continue with his work immediately as he was called up to fight in the French army near the start of the First World War. During a dramatic conflict on January 25, 1915 he sustained a tragic injury to his face which meant he had to wear a mask for the remainder of his life. After returning from war he quickly returned to his studies and with the publication of his best known paper *Mémoire sur l'iteration des fonctions rationelles* (*A Note on the Iteration of Rational Functions*) in 1918 he rose to immediate fame in the mathematical community. It was in this paper that Julia first introduced the modern idea of a *Julia set*.

Interest in the subject flourished over the next 10 years and many other well-known mathematicians began to study Julia sets. Despite the lack of computing power available at the time, Harald Cramér was able to become the first man to approximate an image of a Julia set. The obvious disadvantages of not having any reasonable computing power available perhaps contributed to the ensuing lull in interest in Julia sets. It was not until Benoit Mandelbrot began studying iteration in the 1970's that Julia sets re-emerged. By then computing facilities were available and much more detailed images could be produced. Julia sets remain a topic of current research with much interest being in describing the intricate structure of Julia sets and in calculating their 'fractal dimension'. Also, some recent work has been done on the link between Julia sets and the limit sets of Kleinian groups in hyperbolic geometry.

3 Some Geometry of Julia Sets

In this section we will prove some general results relating to the structure of Julia sets. We will first show that they are compact subsets of $\mathbb{C}$.

Theorem 3.1. *The Julia set J of a function $f_c(z) = z^2 + c$ is compact for all $c \in \mathbb{C}$.*

Proof. We first show that J is bounded.

Choose $r = max(|c|, 3)$ and let $|z| \geq r$. Now we have:

$$|z^2| = |f_c(z) - c| \leq |f_c(z)| + |c|$$

and so we have that:

$$|f_c(z)| \geq |z^2| - |c| \geq 3|z| - |z| = 2|z|$$

Now we can deduce that if $|z| \geq r$ then

$$f_c^n(z) \geq 2^n|z| \to \infty$$

and so $J(f_c) \subseteq B(0, r)$

Now it remains to show that J is closed. Let $z \in \mathbb{C} \setminus K(f_c)$. Hence, there exists an $m \in \mathbb{N}$ such that $f_c^m(z) \geq r$ and using the continuity of f_c^m we have that there exists a ball $B(z, \delta)$ for some small δ such that $f_c^m(z) \geq r$ for all $z \in B(z, \delta)$. Hence, $\mathbb{C} \setminus K(f_c)$ is open and hence $K(f_c)$ is closed. Since, $J(f_c) = \partial K(f_c)$ we have that $J(f_c)$ is closed.

And hence, J is compact. $\square$

The fact that Julia sets are closed and bounded (compact) is a useful property. For example, in determining whether or not a particular point z is in $K(f_c)$ we may conclude that $z \notin K(f_c)$ if $f_c^m(z) \notin B(0, r)$ for some m. This is particularly useful when using a computer to plot Julia sets.

We will now show that $J(f_c)$ is symmetric about the origin.

Theorem 3.2. $z \in J(f_c) \iff -z \in J(f_c)$

Proof. First note that we have $f_c(z) = z^2 + c = (-z)^2 + c = f_c(-z)$. Hence, $f_c^k(z) = f_c^{k-1}(f_c(z)) = z^2 + c = (-z)^2 + c = f_c^{k-1}(f_c(-z)) = f_c^k(-z)$. And we have that $f_c^k(z) \to \infty \iff f_c^k(-z) \to \infty$.

Hence, we can deduce that $K(f_c)$ and hence $J(f_c)$ is symmetric about the origin. □

It can easily be shown that Julia sets are invariant under f and f^{-1} (Falconer, 2003: 218). This is a useful property and demonstrates the fact that Julia sets are *repellers*, in that they themselves are invariant under f but points just outside J will be mapped away from J either to ∞, if in the Fatou set, or towards some fixed point of $f_c(z)$ in the filled in Julia set.

4 The Mandelbrot Set

A given complex constant c will determine the structure of $J(f_c)$. It turns out that the location of c in the *Mandelbrot set* tells us a lot about this structure. Named after Benoit Mandelbrot (1924-), the *Mandelbrot set* is an extremely complicated set. It is defined as follows:

Definiton 4: The Mandelbrot set M is defined as:

$$M = \{c \in C : J(f_c) \text{ is connected}\}$$

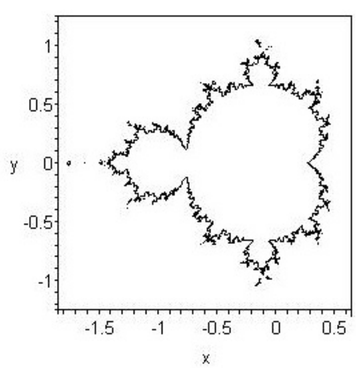

Figure 1: A plot of the Mandelbrot set using Maple

Given the link to the connectedness of Julia sets it is hardly surprising that the location of c in M is linked to the structure of $J(f_c)$. It is not immediately obvious, however, just *how* strongly the two are linked. The definiton given above is usually awkward to use directly and so the following theorem is useful:

Theorem 4.1. $c \in M \iff f_c^n(0) \nrightarrow \infty$

Proof. Omitted. See (Falconer, 2003:225-227) $\square$

This theorem is commonly referred to as *the Fundamental Theroem of the Mandelbrot set.*

From the definiton of the Mandelbrot set we see that the Julia set corresponding to a point outside M will be disconnected and inside M will be connected. The link goes much deeper than that. The Mandelbrot set is split into obvious *chambers* (see figure 1). The large heart shaped region containing the origin is referred to as the *main cardioid* with the smaller circular regions commonly referred to as *buds*. Each of these *buds* has further smaller buds growing out from them and then at the far reaches of M we see tiny *hairs* extending out into the complex plane. Julia sets corresponding to points in each of these distinctive areas have very different structures.

If c lies in the main cardioid of M then it can be shown that $J(f_c)$ is a simple closed curve (Falconer, 2003: 230-231). For example, the following is a plot of the Julia set corresponding to the function defined by $f_c(z) = z^2 - 0.2 + 0.2i$ and a plot of the Mandelbrot set with a dot showing the location of $-0.2 + 0.2i$.

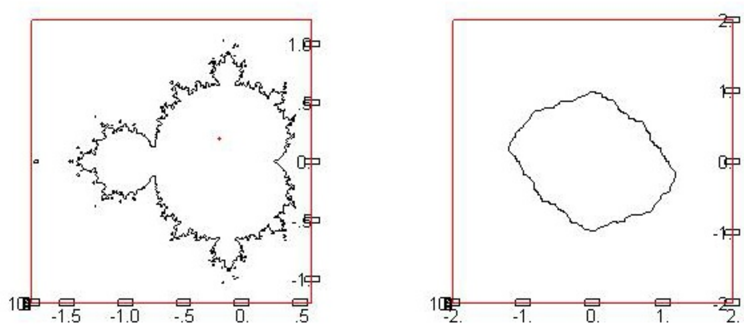

Figure 2: A plot of the Julia set corresponding to $c = -0.2 + 0.2i$ in the Mandelbrot set

Whereas, if c lies in one of the buds it will not be simple and will in fact consist of several (a number depending on *which* bud) loops. For example, the following is a plot of the Julia set corresponding to the function defined by $f_c(z) = z^2 - 0.2 + 0.75i$ and a plot of the Mandelbrot set with a dot showing the location of $-0.2 + 0.75i$.

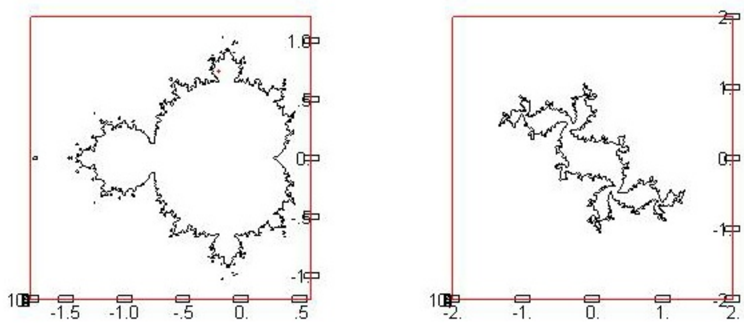

Figure 3: A plot of the Julia set corresponding to $c = -0.2 + 0.75i$ in the Mandelbrot set

If c lies on one of the *hairs* then it will be equal to $F(f_c)$ and so delicate in structure that the removal of a single point would make it *disconnected*. In this situation $J(f_c)$ is called *dendrite*. We do not provide an example plot here because the programme used for plotting is not precise enough to pick out the fine structure of a dendrite Julia set (see section 7).

5 Fractal Dimension

Calculating the 'fractal dimension' of Julia sets in general is a difficult problem. Japanese mathematician Mitsuhiro Shishikura proved in 1994 that the boundary of the Mandelbrot set has Hausdorff dimension 2 (Weisstein). This had been an open problem for a long time and gives a good indication of both the intricacy of the Mandelbrot set itself and the difficulty involved in the calculation of the Hausdorff dimension of this and similar sets.

Despite this, in certain cirumstances, we can obtain good estimates for the Hausdorff dimension of Julia sets quite easily. As an example we will consider the case when $|c|$ is large i.e. we will look to estimate the *box* and *Hausdorff* dimensions of $J(f_c)$ when $|c|$ is large and $J(f_c)$ is consequently a disconnected set.

We will first recall some powerful results from fractal geometry:

Lemma 5.1. *Given a set of contractions $\{S_1 \dots S_m\}$ on $D \subset \mathbb{R}^n$ satisfying:*

$$|S_i(x) - S_i(y)| \leq c_i|x - y|$$

for all $x, y \in D$ and with $c_i < 1$ for all i. Then there exists a unique, non-empty, compact attractor F such that:

$$F = \bigcup_{i=1}^{m} S_i(F)$$

For a proof of this lemma see Falconer (2003: 124-125).

Furthermore, the *box* and *Hausdorff* dimensions of F can be estimated using the following lemma:

Lemma 5.2. *If we have*

$$b_i|x - y| \leq |S_i(x) - S_i(y)| \leq c_i|x - y|$$

for all $x, y \in D$ and with $b_i, c_i < 1$ for all i then

$$t \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F \leq s$$

where t, s satisfy $\sum_{i=1}^{m} b_i^t$ and $\sum_{i=1}^{m} c_i^s$ respectively.

For a proof of this lemma see Falconer (2003: 135-136).

We can now state and prove our main theorem. The following proof is adapted from (Falconer, 2003: 228-230).

Theorem 5.3. *For $|c| > \frac{1}{4}(5 + 2\sqrt{6}) \simeq 2.475$ we have that:*

$$\frac{2\log 2}{\log 4(|c| + |2c|^{1/2})} \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F \leq \frac{2\log 2}{\log 4(|c| - |2c|^{1/2})}$$

and hence, asymptotically we have:

$$dim_B J(f_c) = dim_H J(f_c) \simeq \frac{2\log 2}{\log 4|c|}$$

Proof. We will first aim to find a set of contractions on $\mathbb{C}$ with $J(f_c)$ as the attractor.

Note that $J(f_c)$ is a *repeller* as described in section 3 but in the context of this proof we will find $J(f_c)$ to be the *attractor* of a set of contractions, seperate from the map f_c itself. So we see that $J(f_c)$ can be both an *attractor* and a *repeller* but in different contexts.

Let $|c| > \frac{1}{4}(5 + 2\sqrt{6})$. The reason for choosing this lower bound for $|c|$ will soon become apparent. Now let C be the circle with centre 0 and radius $|c|$, and D be the interior of C. We have that $f_c^{-1}(C)$ is a figure of eight with its point of self-intersection at the origin (Falconer, 2003: 224). Now, since f_c must map $f_c^{-1}(D)$ bijectively back to C we can define S_1 and S_2 to be the parts of f_c^{-1} which map D into the interior of each loop of $f_c^{-1}(C)$.

Now we choose V to be a filled in circle, centred at the origin, with the minimum radius r required to contain the figure of eight $f_c^{-1}(C)$. Falconer (2003: 229) shows that $r = |2c|^{1/2}$. Now, since $V \subsetneq D$, we have that $S_1(V)$ and $S_2(V)$ are disjoint and are contained in each of the separate loops of $f_c^{-1}(C)$. We now claim that S_1 and S_2 are contractions on V.

We have that, since S_1 and S_2 are equal to f_c^{-1} on their respective domains, for $z_1, z_2 \in V$ and $i = 1, 2$:

$$|S_i(z_1) - S_i(z_2)| = |(z_1 - c)^{1/2} - (z_2 - c)^{1/2}|$$

$$= \frac{|(z_1 - c)^{1/2} - (z_2 - c)^{1/2}||(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}$$

$$= \frac{|z_1 - c - z_2 + c|}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|} = \frac{|z_1 - z_2|}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}$$

Now, taking the maximum and minimum of $|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|^{-1}$ over all $z_1, z_2 \in V$ we have that for $i = 1, 2$:

$$\frac{1}{2}(|c| + |2c|^{1/2})^{-1/2}|z_1 - z_2| \leq |S_i(z_1) - S_i(z_2)| \leq \frac{1}{2}(|c| - |2c|^{1/2})^{-1/2}|z_1 - z_2|$$

Now, we see that S_1 and S_2 are contractions on V if $\frac{1}{2}(|c| - |2c|^{1/2})^{-1/2} < 1$ i.e.

$$(|c| - |2c|^{1/2})^{1/2} > \frac{1}{2} \iff |c| - |2c|^{1/2} - \frac{1}{4} > 0$$

Now we have a quadratic inequality in $|c|^{1/2}$. Solving this inequality using the quadratic formula gives:

$$|c|^{1/2} > \frac{\sqrt{2} + \sqrt{3}}{2}$$

And so we require: $|c| > \dfrac{5 + 2\sqrt{6}}{4}$

So we have that S_1 and S_2 are contractions. It remains to show that $J(f_c)$ is the unique attractor of S_1 and S_2. Falconer (2003) uses the fact that V must contain at least one point z in $J(f_c)$ and so $J(f_c) = \overline{\bigcup_{k=1}^{\infty} f_c^{-k}(z)} \subset (V)$ (Falconer, 2003: 222). Now, since S_1 and S_2 are equal to f_c^{-1} on their *disjoint* domains we have that $J(f_c) = S_1(J(f_c)) \cup S_2(J(f_c))$.

Now we can use lemmas 5.1 and 5.2 to estimate the dimension of $J(f_c)$ by solving: $2\left(\frac{1}{2}(|c| + |2c|^{1/2})^{-1/2}\right)^t = 1$ and $2\left(\frac{1}{2}(|c| - |2c|^{1/2})^{-1/2}\right)^s = 1$ for t and s respectively, which gives the result.

$\square$

To give an indication of how good these estimates are we can plot the upper bound and the lower bound together using Maple.

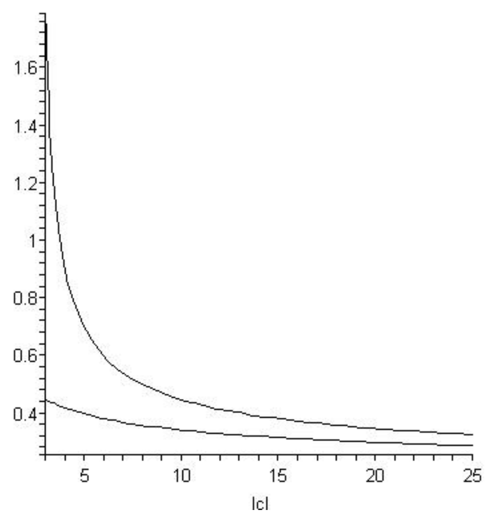

Figure 4: A plot of the lower and upper estimates

We can see from figure 2 that the upper and lower estimates are very far apart for values close to 3. The upper estimate has a singularity at $|c| = \frac{1}{4}(5+2\sqrt{6}) \simeq 2.475$. This will most likely cause upper estimates for values of $|c|$ close to this singularity to be too large. The lower estimate shows a lot less variation and is likely to be a closer approximation to the Hausdorff dimension of the Julia set. The estimates become closer as $|c|$ increases and for values of $|c|$ greater than around 15 the estimates become less 0.1 apart.

For example, if we consider the Julia set for $f(z) = z^2 - 3 + 8i$ then we observe that $|c| = \sqrt{((-3)^2 + 8^2)} \simeq 8.544$ and so, using theorem 5.3 we would estimate the dimensions of $J(f)$ to three decimal places as:

$$0.353 \leq dim_H J(f) \leq \underline{dim}_B J(f) \leq \overline{dim}_B J(f) \leq 0.483$$

6 Link to Hyperbolic Geometry

Recently a number of results have appeared on the link between *Julia sets* and the *limit sets* of Kleinian groups. Although the theory is by no means concrete these two objects have notable similarities. Firstly we recall that a group is called *Kleinian* if it is a *discrete* subgroup of the group of isometries in hyperbolic 3-space. By *discrete* we mean that when we view the group of isometries as a topological space, a Kleinian group is a discrete subset from a topological point of view.

The *limit set*, $L(G)$, of a Kleinian group G is then defined to be the set of all *accumulation points* of the orbit of an element ω in hyperbolic 3-space under the action of the group. By *accumulation point* we mean a point which can be approximated arbitrarily closely by other elements in the orbit. It is worth noting that the choice of the element ω is arbitrary in determining $L(G)$.

Recall that Julia sets are invariant under f and f^{-1} (see section 3). We will now show, based on an argument found in (Stratman, 2009: 36) which treats limit sets of *Fuchsian groups*, that the limit set $L(K)$ of Kleinian group K is invariant under the action of K.

Proof. Let $l \in L(K) = \{$accumulation points in the orbit of ω under $K\}$ and $k \in K$. To show that $L(K)$ is invariant under the action of K we want to show that $k(l) \in L(K)$.

Since $l \in L(K)$ we have that there exists a sequence $(k_n)_n \in K$ such that $k_n(\omega) \to l$. Hence we have:

$$(kk_nk^{-1})k(\omega) = k(k_n(\omega)) \to k(l)$$

and now, since $kk_nk^{-1} \in K$ and $k(\omega)$ is in the orbit of ω we have that $k(l)$ is a limit point of the orbit of ω and that $k(l) \in L(K)$. □

Julia sets and limit sets of non-elementary Kleinian groups have many other structural similarities. For example they both tend to be fractal sets with a very fine structure. Both are *perfect sets* i.e. closed and with no isolated points (Falconer, 2003: 220-221), (Abikoff, 2006: 1). This also means both are necessarily *uncountable*. Minsky (2007: 3-4) points out that limit sets of non-elementary groups G are the *closure* of all the fixed points of parabolic and loxodromic elements of G. Falconer (2003: 221) gives a similar result for Julia sets of polynomials by showing that are the *closure* of the set of all repelling fixed points of the polynomial.

7 Plotting Julia sets

One way to investigate Julia sets and their structure is to plot them using a computer programme. There are many ways of doing this and many programmes available capable of producing quite intricate pictures. All the pictures of the Mandelbrot set and Julia sets seen in this report were produced on Maple. A rough description of the algorithm used is as follows:

1) We first write a programme, which, given a point $z \in \mathbb{C}$, determines whether or not it is in the Julia set. This relies on the fact that Julia sets are bounded by $max(|c|, 3)$ (proved in theorem 3.1 above). We calculate the first 25 iterates of z under f_c and if at any point the iterates wander outside the ball centred at the origin with radius 3 then we deem the point *not* in the Julia set and assign it a value of '1'. If, after 25 iterates, the point remains inside the ball then we deem the point to be in the Julia set and assign it a value '0'.

2) We then split up the complex plane into a nxn grid and test each point in the grid using part (1) above. We can then plot the result in 3 dimensions. The result will be that the filled in Julia set will be projected onto the xy-plane and the Fatou set will be found at height '1' above the xy-plane. Note that increasing n increases the detail in the final plot but greatly decreases the speed at which the computer will produce the image.

3) We then look at a *contour* of height 0.5 above the xy-plane and re-orientate the plot so as to view it from above, thus picking out only the Julia set.

For example, the Maple code used to plot the Julia set corresponding to the function $f_c(z) = z^2 - 1$ on a 200x200 grid is as follows:

```
with(plots):
filled:=proc(x,y)
local in-or-out;
in-or-out:=proc(r,i)
local z;
z:=evalf(r+i*I);
to 25 while abs(z)<=3 do z := z^2 + x + y * I: end do:
if abs(z)>3 then 1 else 0 end if;
end proc: end proc:
plot3d(filled(-1,0),-2..2,-2..2,grid=[200,200],
orientation=[-90,0], style=contour, axes=box, labels=[x,y," "], contours=[0.5],
color=black);
```

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