Mechanics 1

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falling with air resistance

bv $\quad \vec{F} = m\vec{a}$

$+ \downarrow$ $\quad mg$

$m\ddot{y} = mg - b\dot{y}$

1 Basics of Physics

"All models are wrong, some are useful." George Box

1.1 Systems of Units

- Système international (SI): most common, to be used in this course
- Centimetre-Gram-Second Unit (CGS): widely used in the past, now used in advanced electromagnetism
- Natural Unit: $\hbar, c, G, k_B = 1$, used in advanced physics as the quantities in quantum mechanics, special relativity, general relativity and statistical mechanics are simplified

Seven fundamental SI units (table pls)

$\frac{1}{2}mv^2$

mv ✓

$-\frac{1}{2}mv^2$ ✓ ✗

mv ✗

mechanics

thermo

EM

Quantity	Dimension Symbol	Unit
length	L	meter (m)
time	T	second (s)
mass	M	kilogram (kg)
temperature	Θ	Kelvin (K)
current	I	Ampere (A)
amount	N	mole (mol)
luminosity	J	candela (cd)

1.2 Dimensional Analysis

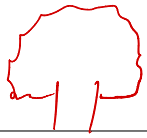
We denote the dimension of quantity X as $[X]$. For example, we denote $[\text{Area}] = L^2, [\text{Velocity}] = LT^{-1}$ etc.

Question 1.1 (Atomic Bomb). Recently a country tested an atomic bomb. As a scientist, you would like to know how much energy E is released. However, it is kept confidential, so you decide to use other ways to guess it.

a) Using dimensional analysis, guess the energy released E from air density ρ, shock radius $R(t)$ and time t.

b) Hence, describe how $R(t)$ varies as t.

Figure 1: Wrong pre-factor is fine, wrong unit is not



Solution 1.1. ρ R t Want to know E

$[\rho] = \left[\frac{M}{V}\right] = L^{-3}M$

$[E] = \left[\frac{1}{2}mv^2\right] = L^2MT^{-2}$

$\alpha = 1, \beta = 5, \gamma = -2$

$E = C\rho\frac{R^5}{t^2}$

$[R] = L$

$E = C\rho^\alpha R^\beta t^\gamma$

$L: 2 = -3\alpha + \beta$

$M: 1 = \alpha$

$T: -2 = \gamma$

$[t] = T$

$R^5 = \frac{E}{\rho C}t^2$

$R(t) \sim t^{\frac{2}{5}}$

Exercise 1.1 (Quantum Gravity). One of the frontier research topic in physics is quantum gravity, which arises from the contradiction between quantum mechanics and general relativity. We would like to discuss at which physical scale do we need to consider both quantum mechanics and general relativity. We use three fundamental constants, reduced Planck's constant $\hbar$, speed of light in vacuum c, and gravitational constant G.

2 Kinematics

Length & Time

How do we describe motion?

particle size 0

2.1 Basic Terms

- Position: Coordinate with time t as a parameter $\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z$
- Displacement: Change of position $\mathbf{s}(t) = \mathbf{r}(t) - \mathbf{r}(t_0)$
- Velocity: Rate of change of position $\dot{\mathbf{r}}(t) = \frac{d}{dt}\mathbf{r}(t)$
- Acceleration: Rate of change of velocity $\ddot{\mathbf{r}}(t) = \frac{d^2}{dt^2}\mathbf{r}(t)$

$$\begin{tikzcd}
\mathbf{r}(t) \arrow[r, bend left, "\frac{d}{dt}"] & \dot{\mathbf{r}}(t) = \mathbf{v}(t) \arrow[l, bend left, "\int \square dt"] \arrow[r, bend left, "\frac{d}{dt}"] & \ddot{\mathbf{r}}(t) = \dot{\mathbf{v}}(t) = \mathbf{a}(t) \arrow[l, bend left, "\int \square dt"]
\end{tikzcd}$$

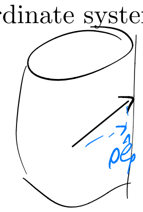
In Cartesian coordinates, the quantities are simple as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \tag{2.1}$$
$$\dot{\mathbf{r}} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z \tag{2.2}$$
$$\ddot{\mathbf{r}} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y + \ddot{z}\mathbf{e}_z \tag{2.3}$$

However, note that the terms are not so tidy in other coordinate system.

2.1.1 Cylindrical Coordinates

A 3D coordinate system obtained by adding the z-axis to the polar coordinate (ρ, ϕ).



$\dot{\vec{r}} = \frac{d}{dt}\vec{r} = \frac{d}{dt}(\rho \hat{e}_\rho + z\hat{e}_z)$

$= \dot{\rho}\hat{e}_\rho + \rho\dot{\hat{e}}_\rho + \dot{z}\hat{e}_z$

$$\mathbf{r} = \rho \mathbf{e}_\rho + z\mathbf{e}_z \tag{2.4}$$

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \tag{2.5}$$

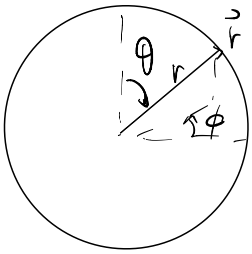
$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\mathbf{e}_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z \tag{2.6}$$

Note. *ρ-axis is different for different vectors, so you cannot simply add them up i.e. if we have* $\mathbf{r}_1 = \rho_1\mathbf{e}_{\rho_1} + z_1\mathbf{e}_{z_1}, \mathbf{r}_2 = \rho_2\mathbf{e}_{\rho_2} + z_2\mathbf{e}_{z_2}$, *we cannot say that* $\mathbf{r}_1 + \mathbf{r}_2 = (\rho_1 + \rho_2)\mathbf{e}_\rho + \cdots$.

Take $z = 0$ reduces to the 2D polar coordinates.

2.1.2 Spherical Coordinates

Describe the system with radius r and 2 angles, polar angle θ and azimuthal angle ϕ. Very good for spherical symmetric configuration, or even just 2 of the variables. The coordinate transformation is as



$$\begin{cases} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{cases}$$

$$\mathbf{r} = r\mathbf{e}_r \tag{2.7}$$

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta\dot{\phi}\mathbf{e}_\phi \tag{2.8}$$

$$\begin{aligned}\ddot{\mathbf{r}} = &(\ddot{r} - r\ddot{\theta} - r\sin^2\theta\dot{\phi}^2)\mathbf{e}_r + \\ &(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\mathbf{e}_\theta + \\ &(2\sin\theta\dot{r}\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi})\mathbf{e}_\phi\end{aligned} \tag{2.9}$$

2.2 1D motion

$$x(t) \underset{\text{area}}{\overset{\text{slope}}{\rightleftarrows}} v(t) \underset{\text{area}}{\overset{\text{slope}}{\rightleftarrows}} a(t)$$

Proposition 2.1 (1D motion with constant acceleration). We denote $v(t) = \dot{x}(t), u = \dot{x}(0), a = \ddot{x}, s(t) = x(t) - x(0)$, we have the following formula

$$\begin{gathered} v = u + at \\ s = \frac{1}{2}(u+v)t = ut + \frac{1}{2}at^2 = vt - \frac{1}{2}at^2 \\ v^2 - u^2 = 2as \end{gathered} \tag{2.10}$$

Proposition 2.2.

$v = \frac{dx}{dt}$ $\quad dt = \frac{dx}{v}$ $\quad a = \frac{dv}{dt}$

$$a = \frac{vdv}{dx} \tag{2.11}$$

which is particularly useful if the acceleration a is a function of the position x.

$$t = \int \frac{dx}{v} = \int \frac{dv}{a} \tag{2.12}$$

2.3 Projectile Motion

2D motion with acceleration in one axis. In reality, we assume the particle experiences acceleration $\ddot{\mathbf{r}} = -g\mathbf{e}_y$.

gravity

$u_x = u\cos\theta$ $u_y = u\sin\theta$

Find $y(x)$ eliminate t

Proposition 2.3. The trajectory of 2D projectile motion is $t = \frac{x}{u\cos\theta}$

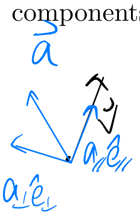
$$
\begin{aligned}
x &= u\cos\theta\ t \\
y &= u\sin\theta\ t - \frac{1}{2}gt^2, \quad y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}
\end{aligned}
\tag{2.13}
$$

where u is the initial speed, θ is the angle of initial velocity and x-axis, g is gravitational acceleration, t is time.

2.4 Circular Motion

The acceleration $\mathbf{a}$ can be decomposed into the tangential and normal components (w.r.t. velocity) $\mathbf{a} = a_{\parallel}\mathbf{e}_{\parallel} + a_{\perp}\mathbf{e}_{\perp}$

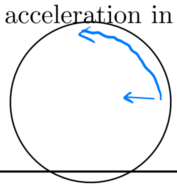
- The parallel component changes the magnitude of $\mathbf{v}$
- The perpendicular component changes the direction of $\mathbf{v}$



Now we consider a case which a particle travels in a circular path.

Proposition 2.4 (Acceleration in circular motion). The perpendicular acceleration in circular motion is

$$a_{\perp} = \frac{v^2}{R} = R\omega^2, \tag{2.14}$$



where v is the speed, R is the radius of curvature, and ω

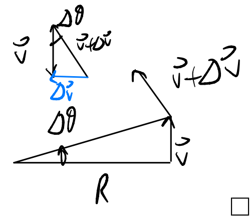
Proof.

1. Consider small changes of quantities within Δt.
2. Find the parametric equation $x(t), y(t)$.
3. Find the acceleration in polar coordinates (in Calculus 3 Notes). □

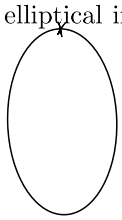
Δt

$|\Delta\vec{v}| \approx |\vec{v}|\Delta\theta = \frac{v^2 \Delta t}{R}$

$a = v^2/R$



This is also useful beyond circular motion. Saying like the trajectory is elliptical instead and you want to find the acceleration at the vertex, you can still use the equation.



$x(t)$ $y(t)$

radius of curvature $\Rightarrow \simeq$ circular motion

$a = \frac{v^2}{R}$

3 Newton's Laws

3.1 1st Law

3.1.1 Reference frames

Definition 3.1 (Reference Frame). A reference frame is a set of spacetime coordinates $(\mathbf{r}, t)$ that maps the particles into their trajectories.

You can think reference frames as cameras. You are free to place, face or move it anywhere as you want. Also, you can scale the coordinates. You can also start recording at any time.
For two frames S, S' with constant orientation (no rotation), the relative velocity is

$$\mathbf{v}_{A/S} = \mathbf{v}_{A/S'} + \mathbf{v}_{S'/S} \tag{3.1}$$

Note. *In scenarios related to electromagnetism i.e. with prescence of electric or magnetic field, be cautious with changing reference frames because the E, B-fields change according to the Lorentz transformation. For safe you may stay in the original frame.*

The genral statement of the Newton's 1st Law is that "A particle moves at constant velocity iff it receives no net force."

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \sum_i \mathbf{F}_i = \mathbf{0}$$

However, this is not true in any frames.

Definition 3.2 (Inertial Frames). Inertial frames are reference frames in which particles satisfy Newton's Laws.

Example 3.1.

- Near the Earth's surface, the rest frame is inertial, but the frame falling at acceleration g is non-inertial.

From this, we may rephrase the Newton's Laws.

Proposition 3.1 (Newton's 1st Law). Inertial frames exist.

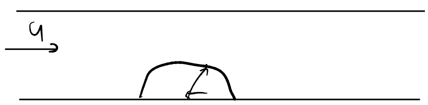
The following operations also give inertial frames

- Position translation $(\mathbf{r}, t) \mapsto (\mathbf{r} + \mathbf{r}', t)$
- Velocity translation $(\mathbf{r}, t) \mapsto (\mathbf{r} + \mathbf{v}t, t)$
- Time translation $(\mathbf{r}, t) \mapsto (\mathbf{r}, t + t')$
- Rotation $(\mathbf{r}, t) \mapsto (\mathbf{R}\mathbf{r}, t), \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$

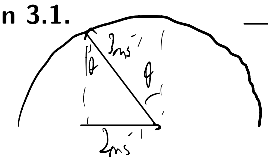
These are the Galilean transformations. This imply that there are no preferred position, velocity, orientation and time in physics. However, this is not true for the case in our universe.

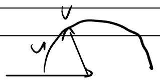
Question 3.1. A boat travels at 3ms^{-1} on still water. To minimize the distance travelled, in what direction should the boat travel across a straight river if the speed water is

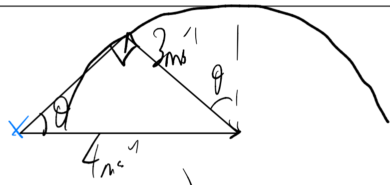
- 2ms^{-1}?
- 4ms^{-1}?



Solution 3.1.







$\theta = \text{asin}\left(\frac{2}{3}\right)$

$\theta = \text{asin}\left(\frac{3}{4}\right)$

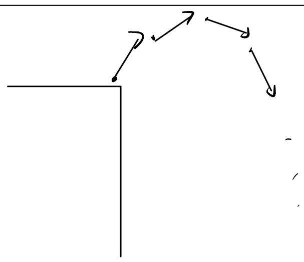
Question 3.2. A projectile is launched on a slope with inclination θ. If the projectile is launched at initial velocity u with the angle α to the slope.

a) Find the range of the projectile.

b) Find α such that the range is the largest.

Solution 3.2.

3.2 2nd Law



> **Proposition 3.2** (Newton's 2nd Law). The equation of motion of a particle is
>
> $$\frac{d}{dt}(\mathbf{p}) = \sum_i \mathbf{F}_i, \tag{3.2}$$
>
> where $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum.

If the mass m is constant, we have

$$\sum_i \mathbf{F}_i = m\ddot{\mathbf{r}} \tag{3.3}$$

3.3 3rd Law

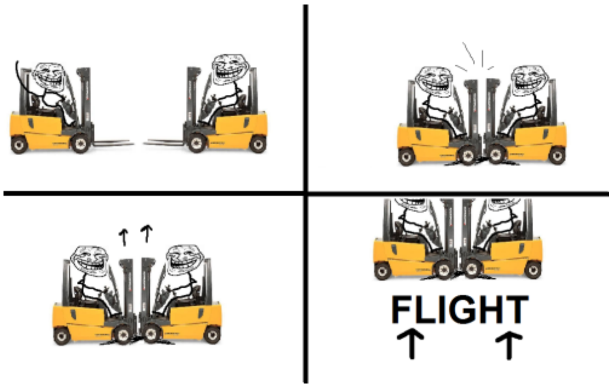


Figure 2: What if we break laws in physics

> **Proposition 3.3** (Newton's 3rd Law). Force comes in a pair: For any two objects A, B, the following relations apply:
>
> $$\mathbf{F}_{A/B} + \mathbf{F}_{B/A} = \mathbf{0}, \tag{3.4}$$
>
> where $\mathbf{F}_{A/B}$ is the force acting on B by A, and $\mathbf{F}_{B/A}$ is the force acting on A by B.

This imply that the forces

- have same magnitude
- point at opposite direction
- act on different objects

This indeed come from the conservation of momentum, as the result of translational invariance of both position and velocity.

Later you may see a paradox in electromagnetism, which can be explained by the presence of electromagnetic momentum, hence we cannot solely consider the two particles (as what we do in the paradox).

4 Forces

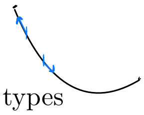
- Weight $\mathbf{F}_G$: Gravitational attraction between two objects is

  $$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{r}, \tag{4.1}$$

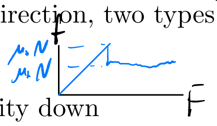
  where $G \approx$ is the gravitational constant, M, m are the masses of the objects, and $\mathbf{r}$ is the separation between them. Near the Earth's surface, the weight is

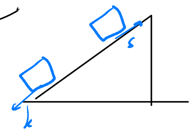
  $$\mathbf{F}_G = -mg\mathbf{e}_y, g = \frac{GM_E}{R_E^2}. \tag{4.2}$$

- Normal Force N: a force that constrain an object onto a surface, curve or point. Normal means to the surface (c.f. tangent).
- Tension T: a constraint force that restricts the coordinates of an object along a rod, rope etc.
  - Assumption: the rope is light enough (so T is constant along the rope) $\lambda l g \ll T$
  - If not, then only the horizontal component is constant

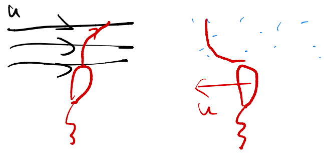


- Friction f: microscopically due to rough surfaces, only in tangential direction, two types
  - Static $f_s \le \mu_s N$: prevents two surfaces from sliding relatively. $\mu_s > \mu_k$
  - Kinetic $f_k = \mu_k N$: dissipative, trying to slow the relative velocity down





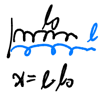
- Fluid Resistance: dissipation due to fluctuation in fluid, as linear $\mathbf{F} = -b_1\mathbf{v}$ or quadratic model $\mathbf{F} = -b_2 v^2 \mathbf{e_v}$, where $\mathbf{v}$ is the relative velocity of the object to the fluid i.e. in fluid's frame



- Restoration Force: Hooke's Law $F = -kx$, not dissipative

natural elongation

spring constant



Some questions? (WIP)

Question 4.1 (1D motion with drag). A particle with mass m falls with gravity g while experiencing air resistance with magnitude $F = bv$, where v is the speed. Find the trajectory of the particle.

Solution 4.1.

$\downarrow +y$ $\uparrow bv$ $\downarrow mg$

Newton's 2nd Law

$m\ddot{y} = mg - b\dot{y}$

$m\frac{d\dot{y}}{dt} = mg - b\dot{y}$

$\frac{d\dot{y}}{g - \frac{b}{m}\dot{y}} = dt$

$-\frac{m}{b}\ln\left(g - \frac{b}{m}\dot{y}\right) = t + C$

$C = -\frac{m}{b}\ln g$

$t = \frac{m}{b}\ln\left(\frac{g}{g - \frac{b}{m}\dot{y}}\right)$

$\frac{g}{g - \frac{b}{m}\dot{y}} = e^{\frac{bt}{m}}$

boundary condition

$g e^{-\frac{bt}{m}} = g - \frac{b}{m}\dot{y}$

$\dot{y} = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)$

$y = \frac{mg}{b}t + \frac{mg}{b}\frac{m}{b}e^{-\frac{bt}{m}} + C'$

$= \frac{mg}{b}\left(t + \frac{m}{b}\left(e^{-\frac{bt}{m}} - 1\right)\right)$

terminal velocity $v_\infty = \frac{mg}{b}$

time factor $\frac{bt}{m} = \frac{t}{\tau}$ - time factor

$\tau = \frac{m}{b}$

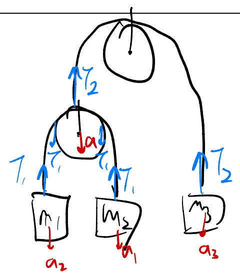
General idea of solving problems with force

- Find all unknowns (e.g. N, a)
- Find all constraints (e.g. length, surface)
- Find all equations of motions (Newton's 2nd Law for different objects and axes)
- # of EoM - # of constraints = # of unknowns

Question 4.2 (Triple Pulley). Find the acceleration of the three ~~objectrs.~~ objects Assume the pulleys have no friction, and the ropes are light.

Solution 4.2.

$\downarrow +$



$m_1 g - T_1 = m_1 a_1$

$m_2 g - T_1 = m_2 a_2$

$m_3 g - T_2 = m_3 a_3$

$T_2 = 2T_1$

$x + x_3 = 0 \Rightarrow a + a_3 = 0$

$(x_1 - x) + (x_2 - x) = 0 \Rightarrow a_1 + a_2 = 2a$

Question 4.3. A ring of mass M hangs from a thread. Two beads, each of mass m, slide on it without friction. The beads are released from rest simultaneously from the top of the ring and slide down opposite sides. Find the condition when the ring will start to rise, and the angle at which this occurs.

Solution 4.3.

5 Energy

5.1 Work

Idea: Result = Effort * Progress

- In 1D, $dW = F dx$
- In 3D, $dW = \mathbf{F} \cdot d\mathbf{r}$
- Total work is $W = \int \mathbf{F} \cdot d\mathbf{r}$
- Power $P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$

$\Delta W = |\vec{F}| |\Delta\vec{r}| \cos\theta$
$= \vec{F} \cdot \Delta\vec{r}$

Example 5.1. do negative work $W<0$

- Friction is dissipative (losing energy) since the force is against velocity

Kinetic

5.2 Kinetic Energy

Idea: Work makes an object move
The kinetic energy of an object is

$$K = \frac{1}{2} m |\dot{\mathbf{r}}|^2 \tag{5.1}$$

where m is the mass of the object, and $\dot{\mathbf{r}}$ is the velocity of the object.

Theorem 5.1 (Work-Energy Theorem)**.** Work is the change in kinetic energy. Mathematically,

$$W = \int_i^f \mathbf{F} \cdot d\mathbf{r} = \Delta K = \frac{1}{2} m |\dot{\mathbf{r}_f}|^2 - \frac{1}{2} m |\dot{\mathbf{r}_i}|^2 \tag{5.2}$$

Proof. Left as an exercise to readers. □

5.3 Potential Energy

Idea: an energy due to position

Example 5.2.

- Gravitational PE near Earth's surface $U = mgh$
- Gravitation PE in general $U = -\frac{GMm}{r}$
- Elastic PE in spring $U = \frac{1}{2} k x^2$

Explanation: There is a force trying to pull the object to a lower potential. Which forces can do so?

Definition 5.1 (Conservative Force). A force $\mathbf{F}(\mathbf{r})$ is conservative iff it can be expressed as

$$\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r}), \tag{5.3}$$

where $U(\mathbf{r})$ is the potential energy.

Proposition 5.1. It is equivalent to the following statements

- The path integral $\displaystyle\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r}$ is path-independent for any points $\mathbf{r}_i, \mathbf{r}_f$.
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C
- $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ in a simply connected region (i.e. where any closed loop can contract into a point continuously)
- (IMPORTANT) The total energy $E = K + U$ is conserved.

Note. *Since U depends linearly on the mass of the object m, sometimes we use the potential $\varphi := U/m$ instead.*
Moreover, conservation of energy is a consequence of time invariance.

Question 5.1. A glider of mass m is attached to a spring with spring constant k. Initially the spring is at the natural length, and the glider moves at v. What is the maximum displacement d to the right if friction~~al~~ coefficient is μ_k?

$ax^2+bx+c=0$
$\Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Solution 5.1. $\frac{1}{2}mv^2 = \frac{1}{2}kd^2 + \mu_k mg\, d$

$$\frac{k}{m}d^2 + 2\mu_k g\, d - v^2 = 0$$

$$d = \frac{2\mu_k g \pm \sqrt{4\mu_k^2 g^2 + 4\frac{k}{m}v^2}}{2\frac{k}{m}} = \frac{m}{k}\left(\mu_k g \pm \sqrt{\mu_k^2 g^2 + \frac{k}{m}v^2}\right)$$
$$= \frac{m}{k}\left(\mu_k g + \sqrt{\mu_k^2 g^2 + \frac{k}{m}v^2}\right)$$

6 Momentum

Definition 6.1 (Momentum). The momentum of an object is

$$\mathbf{p} = m\dot{\mathbf{r}}. \tag{6.1}$$

Hence, the kinetic energy can be also written as

$$K = \frac{p^2}{2m}. \tag{6.2}$$

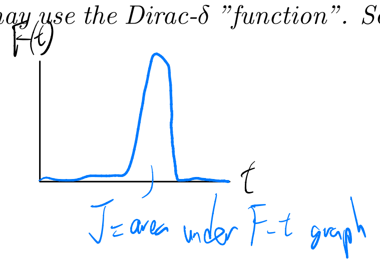
Note. *These are only valid for classical physics. Don't use them in special relativity. Also, in advanced physics, momentum is used more frequently than velocity.*

Definition 6.2 (Impulse).

$$\mathbf{J} = \int \mathbf{F}(t)dt \tag{6.3}$$

Graphical interpretation: area under $F - t$ graph.

Note. *To describe a very short duration of impulse, we may use the Dirac-δ "function". See the Appendix for more details.*



Proposition 6.1 (Newton's 2nd Law). The equation of motion of a particle is

$$\frac{d}{dt}(\mathbf{p}) = \sum_i \mathbf{F}_i, \tag{6.4}$$

where $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum.

Hence, we have

$$\mathbf{J} = \Delta \mathbf{p}$$

Proposition 6.2 (Conservation of Momentum). For an isolated system i.e. a collection of objects without external force, the momentum is conserved

$$\sum_i \mathbf{p}_i = \sum_f \mathbf{p}_f \tag{6.5}$$

which is a consequence of space invariance.

Proposition 6.3. The conservation of momentum is equivalent to the Newton's 3rd Law.

Question 6.1. A hose ejects water at the mass rate of b and at speed v

a) Find the force exerted by the hose.

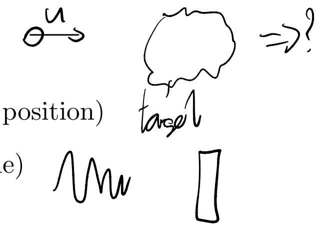
b) Find the power delivered by the hose.

Solution 6.1. Consider Δt

a) $\Delta p = (\Delta m) v = b \Delta t v$ $\quad F = \frac{\Delta p}{\Delta t} = bv$

b) $\Delta E = \frac{1}{2}(\Delta m) v^2 = \frac{1}{2} b \Delta t v^2$ $\quad P = \frac{\Delta E}{\Delta t} = \frac{1}{2} b v^2$

6.1 Scattering



1. Start with a particle and its state (e.g. velocity, position)
2. Let it bombard some target (e.g. another particle)
3. Measure its state afterward

For simplicity, we only discuss 2 particles collision here. The conservation of momentum gives

$$m_A \mathbf{u}_A + m_B \mathbf{u}_B = m_A \mathbf{v}_A + m_B \mathbf{v}_B \tag{6.6}$$

The collision is said to be

- elastic if energy is conserved i.e.

$$\frac{1}{2} m_A |\mathbf{u}_A|^2 + \frac{1}{2} m_B |\mathbf{u}_B|^2 = \frac{1}{2} m_A |\mathbf{v}_A|^2 + \frac{1}{2} m_B |\mathbf{v}_B|^2 \tag{6.7}$$

- inelastic if energy is not conserved (gain/dissipate)
  - dissipate e.g. deformation
  - gain e.g. explosion
- perfectly inelastic if the two objects stick together afterward i.e. $\mathbf{v}_A = \mathbf{v}_B$

Exercise 6.1. Show that the kinetic energy is minimum iff the collision is perfectly inealstic.

For elastic collision, we do some simplification.

$$\begin{cases} m_A(|\mathbf{v}_A|^2 - |\mathbf{u}_A|^2) & = m_B(|\mathbf{u}_B|^2 - |\mathbf{v}_B|^2) \\ m_A(\mathbf{v}_A - \mathbf{u}_A) & = m_B(\mathbf{u}_B - \mathbf{v}_B) \end{cases}$$

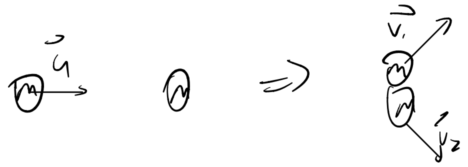
$$m_A(\mathbf{v}_A - \mathbf{u}_A)(\mathbf{v}_A + \mathbf{u}_A) = m_B(\mathbf{u}_B - \mathbf{v}_B)(\mathbf{v}_B + \mathbf{u}_B)$$

$$\mathbf{v}_A + \mathbf{u}_A = \mathbf{v}_B + \mathbf{u}_B$$

$$\mathbf{v}_A - \mathbf{v}_B = \mathbf{u}_B - \mathbf{u}_A$$

i.e. the relative velocity becomes the opposite after collision.

Question 6.2. Two particles have mass m. Now one of them is travelling at $\mathbf{u}$ to bombard at the other particle at rest. What are the velocities of them afterward?



$\vec{u} = \vec{v}_1 + \vec{v}_2 \Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0$

$|\vec{u}|^2 = |\vec{v}_1|^2 + |\vec{v}_2|^2$

$\vec{u}\cdot\vec{u} = (\vec{v}_1 + \vec{v}_2)\cdot(\vec{v}_1 + \vec{v}_2)$
$= |\vec{v}_1|^2 + |\vec{v}_2|^2 + 2\vec{v}_1\cdot\vec{v}_2$

Solution 6.2.

Note. *This is an evidence why α-radiation is Helium-4 in nature. Scientists tried to put Helium-4 atoms into a cloud chamber with α-radiation. they found that trajectories forming right angles were formed.*

Exercise 6.2. You can bounce a light object up to very high using a trick – put a heavier object below. Assume the collision time is sufficiently short so you can neglect their weight during collision.

6.2 System of Particles

Idea: how to describe the dynamics many particles (possibly infinitely many)?
Suppose we have particles $1, \cdots$ with masses $m_1, \cdots$ and positions $\mathbf{r}_1. \cdots$ respectively. The (position of the) center of mass (cm) is the weighted sum of position

$$\mathbf{r}_{\text{cm}} = \mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \tag{6.8}$$

Note. *You may have come up with the terms "center of gravity". The two terms are same iff gravity is uniform e.g. on Earth's surface. Else, the center of gravity refers to where is the whole system viewed as a particle, so that the gravitational force is same.*

Consequently, we can also derive the velocity and acceleration of the cm.

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{\sum_i m_i \dot{\mathbf{r}}_i}{\sum_i m_i} = \frac{\sum_i \vec{p}_i}{M} \tag{6.9}$$

$$\mathbf{A} = \ddot{\mathbf{R}} = \frac{\sum_i m_i \ddot{\mathbf{r}}_i}{\sum_i m_i} \tag{6.10}$$

For a continuum object (e.g. a solid cube), we replace the summation by integral

$$\mathbf{R} = \frac{\int \mathbf{r}\ dm}{\int dm} \tag{6.11}$$

The form of dm depends on the nature of the object.

Nature of the object	dm
curve	$\lambda\ dl$
surface	$\sigma\ dA$
solid	$\rho\ dV$

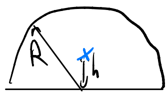
Question 6.3. Find the position of the center of mass of a uniform

a) semicircular disk

b) hemisphere

with radius R.

Solution 6.3.



$$h = \frac{\int y\,dm}{M} = \frac{\int \sigma y\,da}{M} = \frac{2}{\pi R^2}\int_0^R \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} y\,dx\,dy$$

$$= \frac{2}{\pi R^2}\int_0^R 2y\sqrt{R^2-y^2}\,dy = \frac{2}{\pi R^2}\left(-\int_0^R \sqrt{R^2-y^2}\,d(R^2-y^2)\right)$$

$$= \frac{2}{\pi R^2}\left[-\frac{(R^2-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^R = \frac{4}{3\pi}R \quad //$$

Proposition 6.4. The total momentum measured in a reference frame is

$$\mathbf{P} = M\mathbf{V} \tag{6.12}$$

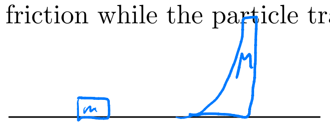
where $M = \sum_i m_i$ is the total mass and $\mathbf{V}$ is the velocity of cm in the reference frame.

Exercise 6.3. Show that the total kinetic energy measured in a reference frame is

$$K = \underbrace{\frac{1}{2}M|\mathbf{V}|^2}_{\text{C.m. energy}} + \underbrace{\sum_i \frac{1}{2}m_i|\mathbf{v}_i'|^2}_{\text{energy w.r.t. C.m.}}, \tag{6.13}$$

where $\mathbf{v}_i'$ is the relative velocity of the i-th particle to cm.

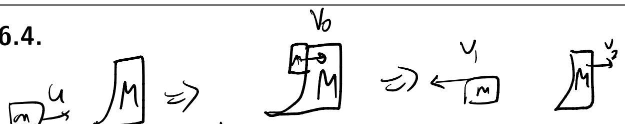
Question 6.4. A particle with mass m travels on a smooth, horizontal surface at velocity u before it goes onto a movable ramp with mass M. Assume there is no friction while the particle travels on the ramp.



a) Find the maximum height attained by the particle

b) Find the velocities of both the particle and the ramp after the particle leaves the ramp.

Solution 6.4.



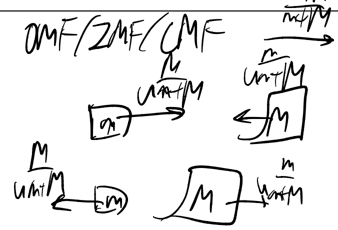
$$\frac{1}{2}mu^2 = \frac{1}{2}(m+M)v_0^2 + mgh \Rightarrow \frac{1}{2}mu^2 = \frac{1}{2}(m+M)\left(\frac{mu}{m+M}\right)^2 + mgh$$

$$mu = (m+M)v_0 \qquad = \frac{1}{2}\frac{m u^2}{m+M} + mgh$$

$$\frac{Mu^2}{2g(m+M)} = h$$

$$h = \frac{u^2}{2g}\frac{M}{m+M}$$

0MF/ZMF/CMF



$$v_1 = u\frac{M-m}{m+M} \qquad v_2 = u\frac{2m}{m+M}$$

7 Rotation

If you need a fancy name for this, call it $\cancel{S}O(3)$.
In this section, we deal with rigid body i.e. the distances between each pair of particles remain unchanged.

$$\forall i, j, |\mathbf{r}_i - \mathbf{r}_j| = \text{const.} \tag{7.1}$$

7.1 Kinematics

Use radian $2\pi\text{rad} = 360°$, so the angle is $\theta = s/r$.

The quantities are defined similarly as in linear kinematics.

- Angular Displacement $\Delta\theta$: counter-clockwise is +
- Angular Velocity $\omega = \frac{d\theta}{dt} = \dot{\theta}$: actually a vector $\boldsymbol{\omega}$ with direction given by right hand rule
- Angular Acceleration $\alpha = \frac{d^2\theta}{dt} = \ddot{\theta}$: also a vector $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$

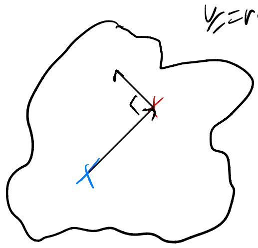
1D

Proposition 7.1 (Fixed Axis Rotation with Constant Angular Acceleration)**.** We denote $\omega(t) = \dot{\theta}(t), \omega_0 = \dot{\theta}(0), \alpha = \ddot{\theta}$,, we have the following formula

$$\begin{aligned} \omega &= \omega_0 + \alpha t \\ \theta - \theta_0 &= \frac{1}{2}(\omega_0 + \omega)t = \omega_0 t + \frac{1}{2}\alpha t^2 = \omega t - \frac{1}{2}\alpha t^2 \\ \omega^2 - \omega_0^2 &= 2\alpha(\theta - \theta_0) \end{aligned} \tag{7.2}$$

If an object is rotating w.r.t. a fixed point,

$v_{\parallel} = r\omega$



- the tangential velocity is $v_{\parallel} = r\omega$
- the tangential acceleration is $a_{\parallel} = r\alpha$
- the radial acceleration is $a_{\perp} = r\omega^2$

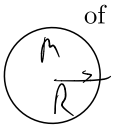
7.2 Dynamics

If mass describes the tendency of an object to move, what describes the tendency of an object to rotate
For a planar object (on the xy-plane), the moment of inertia in z-direction is

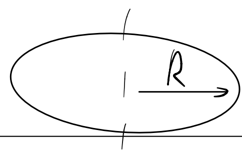
$$I = \sum_i m_i(x_i^2 + y_i^2) = \int (x^2 + y^2) dm \tag{7.3}$$

It is also valid for highly symmetric object e.g. those symmetric w.r.t. z-axis. The necessary condition of validity is that

$$\sum_i m_i x_i z_i = \sum_i y_i z_i = 0$$



$I = \int (x^2+y^2) dm = \int r^2 \sigma dA = \frac{M}{\pi R^2}\int_0^R r^2\, r\, dr \cdot 2\pi = \frac{1}{2}MR^2$



Example 7.1. Consider a thin ring of mass M and radius R. It's moment of inertia is $I = MR^2$ along the axis. But if we replace it by a circular plate with uniform density, $I = \frac{1}{2}MR^2$ instead.

Question 7.1. Suppose we have a planar object with moment of inertia I. What is the moment of inertia of a new object that is formed by

$I \sim MR^2$

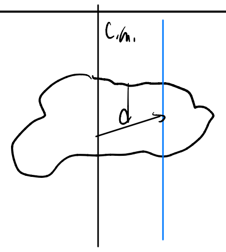
1. scaling up all axes to 2 times, with the same axis and density? $16I$
2. scaling up all axes to 2 times, with the same axis and mass? $4I$
3. rotating the original object by the axis? I

It is hard to compute the moment of inertia sometimes, luckily some theorems are helpful

Theorem 7.1 (Parallel Axis Theorem)**.** The moment of inertia about an axis is

$$I = I_{\text{cm}} + Md^2, \tag{7.4}$$

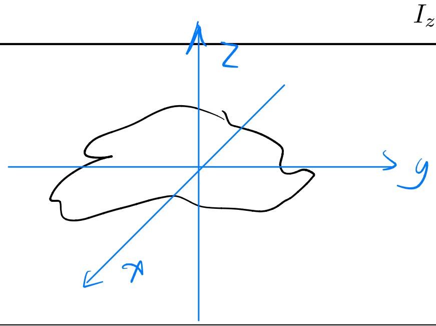
where d is the distance from the axis to cm.

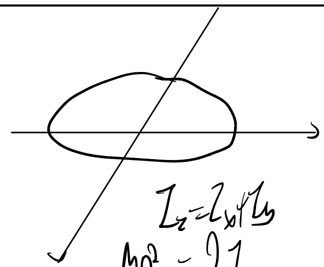


Proof. □

Theorem 7.2 (Perpendicular Axis Theorem)**.** For an planar object on the xy-plane, the following relations hold

$$I_z = I_x + I_y \tag{7.5}$$





$I_z = I_x + I_y$

$MR^2 = 2I$

$I = \frac{1}{2}MR^2$

Proof. □

Question 7.2. Find the moment of inertia of a thin spherical shell with mass M and radius R

Solution 7.1.

Torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ causes rotation. For central force i.e. $U(r)$ only depends the separation of two particles, the torque produced is 0. The equation of motion for rotation is then

$$\sum_i \tau_i = I\alpha \qquad \text{c.f.} \quad \sum_i \vec{F}_i = m\vec{a} \tag{7.6}$$

No slipping: no relative motion between two surfaces e.g. a wheel and a plane, then $s = R\theta$, so $v = R\omega, a = R\alpha$.

Theorem 7.3. Every possible motion of a rigid body is a combination of pure translation of the CM and pure rotation w.r.t. C.M.

The rotational kinetic energy is

$$K_r = \frac{1}{2}I\omega^2 \tag{7.7}$$

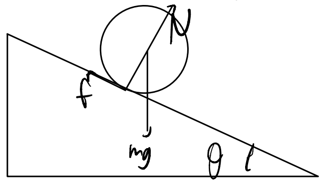
Hence the total kinetic energy is

$$K = K_t + K_r = \frac{1}{2}M|\mathbf{V}|^2 + \frac{1}{2}I\omega^2 \tag{7.8}$$

$I = \frac{2}{5}MR^2$

Question 7.3. A uniform sphere is rolling down a slope inclined at θ without slipping. Find its acceleration.

Solution 7.2. Since the sphere rolls without slipping, the point of contact is instantaneously at rest, and we can treat the system as in pure rotation.



$mg\sin\theta - f = ma$

$Rf = \frac{2}{5}mR^2\alpha$

$a = R\alpha$

$f = \frac{2}{5}ma$

$a = \frac{5}{7}g\sin\theta$

However, in reality, energy is lost due to deformation.

- Work $W = \int \tau d\theta$
- Power $P = \frac{dW}{dt} = \tau\omega$

7.3 Angular Momentum

The angular momentum of a point particle w.r.t. origin is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{7.9}$$

If the rotation axis is a symmetry axis, then $\mathbf{L} = I\boldsymbol{\omega}$. If not, then I is more complicated, said to be a tensor.
The equation of motion of a rigid body is

$$\frac{d\mathbf{L}}{dt} = \sum_i \tau_i \tag{7.10}$$

Hence, under no net external torque, which can occur while

- There is no external force on the system (isolated)
- The force exerts on the origin
- The force aligns with the position vector (e.g. central force)

the angular momentum $\mathbf{L}$ is conserved.

Exercise 7.1. Suggest how playground swing works. Make appropriate simplifications to your physics model.

For a system of particles, the angular momentum measured in lab frame is

$$\mathbf{L} = \mathbf{R} \times MV + \mathbf{L}', \tag{7.11}$$

where $\mathbf{L}'$ is the angular momentum measured in cm frame. For a rigid body, we have

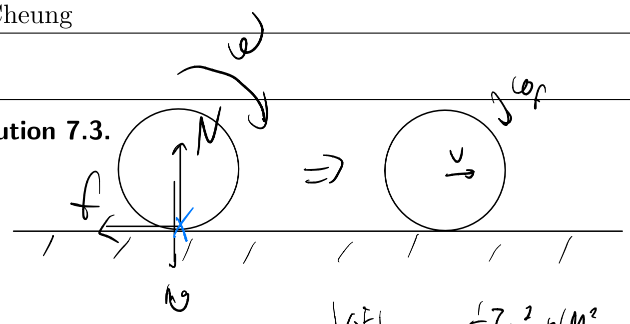
$$\mathbf{L} = \mathbf{R} \times MV + I\boldsymbol{\omega}. \tag{7.12}$$

Question 7.4. A circular object with mass M, radius R and moment of inertial $I = \beta MR^2$, where $\beta \in \mathbb{R}^+$ rolls on a rough horizontal surface. Initially, it rolls at angular velocity ω, yet with 0 linear velocity.

a) Find the final linear velocity after the coin stops slipping.

b) Find the fraction of initial kinetic energy lost.

Solution 7.3.



$v = R\omega_f$

$I\omega = I\omega_f + MvR$

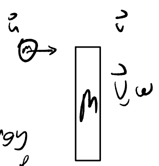
$\beta MR^2\omega = (\beta+1)MR^2\omega_f$

$\omega_f = \frac{\beta}{\beta+1}\omega \quad v = \frac{\beta}{\beta+1}R\omega$

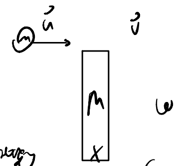
$\left|\frac{\Delta E}{K}\right| = 1 - \frac{\frac{1}{2}I\omega_f^2 + \frac{1}{2}Mv^2}{\frac{1}{2}I\omega^2} = 1 - \frac{\beta\left(\frac{\beta}{\beta+1}\right)^2 + \left(\frac{\beta}{\beta+1}\right)^2}{1} = \frac{1+\beta-\beta^2}{\beta+1}$ //

Question 7.5. A rod of length $2L$ and mass M lies on a frictionless horizontal surface. A particle of mass m hits the one end of the rod at velocity v_0 perpendicular to the orientation of the rod. Assume that mechanical energy is conserved and the ball moves along the original line of motion after the collision (if it moves). Find the final velocity v_f of the ball. What if the rod is pivoted at the other end?

Solution 7.4.



- energy
- momentum
- angular momentum



- energy
- angular momentum (w.r.t. ?)

0 Appendix

Definition 0.1 (Dirac-δ "Function").

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases} \tag{0.1}$$

so that

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

The discretized version is the Kroneck-Delta function.

Definition 0.2 (Kroneck-δ Function).

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{0.2}$$

Proposition 0.1 (Properties of Dirac-δ Function).

1. $\delta(x) = \frac{d}{dx}\Theta(x)$
2. Symmetry: $\delta(x) = \delta(-x)$
3. Scaling: $\delta(ax) = \frac{\delta(x)}{|a|}$
4. Sifting: $\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$

Summary

Wait till next week lol