Calculus 1

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1 Introduction to Mathematics

1.1 Logic 101

1.1.1 Boolean

In mathematics, a statement is used to describe a sentence. It is either true or false, without any ambiguity.

Example 1.1. "Today is hot." is not a statement since it is ambiguous to define what is hot. However, "Today's highest temperature is 33 °C." is a valid statement. Similarly, "Is today hot?" is not a statement.

We can use variables to represent statements.

Conditional statements are true when some conditions are satisfied, and false otherwise. For example, $2x > 6$ is true when $x > 3$, and false if $x \leq 3$.

Open statements are those that are possible to be verified, but we still do not know yet.

Example 1.2 (Collatz Conjecture a.k.a. $3n + 1$ Conjecture)**.** We have

$$f(n) = \begin{cases} n/2, & n \text{ is even} \\ 3n+1, & n \text{ is odd} \end{cases}$$

We start with a natural number n and turn it into $f(n)$, then we repeat the progress. It is still an open problem whether all natural numbers are turned into 1 with the procedures.

Statements are also Booleans, which are either true or false

Truth table is used to represent results of logic operation

A	not A
T	F
F	T

A	B	A and B	A or B	$A \Rightarrow B$	$A \iff B$

$A \Rightarrow B$ means if A then B. We can also say that B only if A, A implies B, A is the sufficient condition and B is the necessary condition etc. In common language this sentence structure mean A is equivalent to B. However, this is not the case in mathematics. Even if A is false, B still can be true.

$A \Rightarrow B$ is not the same as $B \Rightarrow A$. Similarly, $A \Rightarrow B$ is not the same as not $A \Rightarrow$ not B.

Example 1.3. $x = 4 \Rightarrow x$ is divisible by 2.

Then $x = 4$ is the sufficient condition for x is divisible by 2, since you can take other values of x but they are still divisible by 2.

Also, x is divisible by 2 is the necessary condition for $x = 4$.

Also, since the negation of $A \Rightarrow B$ is A and not B. If A is false, then $A \Rightarrow B$ must be true.

$A \iff B$ means $A \Rightarrow B$ and $B \Rightarrow A$. We say that A is equivalent to B or $A \iff B$

Example 1.4. $x = x + 1 \Rightarrow x > x$ is always true since $x = x + 1$ is always false

Example 1.5.

A: x is divisible by 3, B: x is divisible by 6

- A and B is B
- A or B is A
- $A \Rightarrow B$ is true, but $B \Rightarrow A$ is false
- A is not equivalent to B.

Usually in mathematics, we are interested in verifying "if ... then" statements, which are called propositions.

Considering the proposition $A \Rightarrow B$.

Contrapositive is not $B \Rightarrow$ not A, which is an equivalent expression.

Saying like you want to prove that a statement is false. You can start by asserting it is true, and derive contradictions (you know what is B at first but you only know A at last)

Converse is $B \Rightarrow A$. It is not equivalent to the original proposition.

Question 1.1. Prove that if $x < -3$, then $(x + 1)^2 + (x + 2)^2 > 5$

Solution 1.1.

Exercise 1.1. Using a truth table, verify the De Morgan's laws

- not (A and B) $\iff$ (not A) or (not B)
- not (A or B) $\iff$ (not A) and (not B)

1.1.2 Quantifier

Quantifiers are used in statements.

For all $\forall$ means you cannot further assume any conditions of the objects (e.g. a number). There exists $\exists$ mean you only need to find one object from those that satisfy the conditions

Example 1.6. $\forall$ real number x, $x^2 + 2x - 9 > 0$ is false since $0^2 - 2 \cdot 0 - 9 < 0$, but we can say that $\exists x = 3, x^2 + 2x - 9 > 0$

It is fine to exchange orders of same quantifiers, but not different quantifiers.

$$\forall 0 \leq x < 1, \exists 0 \leq y < 1, y \geq x$$

is true, but

$$\exists 0 \leq x < 1, \forall 0 \leq y < 1, y \geq x$$

is false. Since in the first statement, your y can flexibly change as what you require x. But in the second statement, your y is fixed for all x

Note. *You may think that $\forall$ is always a more restrictive condition than $\exists$, but note when the pre-condition is not satisfied*

1.1.3 Induction

Induction is one of the ways to prove a statement involving natural number. Here are the steps for single induction

1. set up the proposition $P(n)$
2. prove $P(1)$
3. - weak: prove $P(n) \Rightarrow P(n+1)$
   - strong: prove $P(1), \cdots,$ and $P(n) \Rightarrow P(n+1)$

Idea: try to prove the propositions are true for each n, just in different steps.

1.2 Set and Map

1.2.1 Set

A set is one of the fundamental object in mathematics. You can view a set as a bag, which contains elements. Elements can be anything, from numbers, variables ... Here are some axioms of a set

- Two sets are same iff they contain same elements i.e. a set is only defined based on the content inside, but not the label
- A set cannot contain itself

A set can be denoted in many ways

- finite set $S_1 = \{1, 2, 3\}$
- set of colors $S_2 = \{\text{red, orange, yellow, ...}\}$
- condition $S_3 = \{x : 0 \leq x \leq 10\}$

If an element x is in the set S, we say that $x \in S$, or S contains x. Otherwise, we denote as $x \notin S$.

Example 1.7. Consider the set of all prime numbers S

- $47, 97, \cdots \in S$
- $1, \pi, \text{ Hong Kong }, \cdots \notin S$

Example 1.8. Commmon sets of numbers in mathematics

- Natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$
- Integers $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$
- Rational numbers $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0, \gcd(p, q) = 1\}$
- Real numbers $\mathbb{R}$
- Complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$
- Intervals: assume $a \leq b$

$x \in$	$[a,b]$	$(a,b]$	$[a,b)$	(a,b)
Condition	$a \leq x \leq b$	$a < x \leq b$	$a \leq x < b$	$a < x < b$

Note $\mathbb{R} = (-\infty, +\infty)$

Note. *It would take very long time to rigorously construct the set of real numbers $\mathbb{R}$, so we gotta skip this*

1.2.2 Set Operation

intersection, union, complement, subset, partition, product

Many set operations are motivated by logic operations. Here we assume A, B are two sets.

- null set $\varnothing : \forall x$
- intersection $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- union $A \cap B = \{x : x \in A \text{ or } x \in B\}$
- complement $A^{\complement} = \{x : x \notin A\}$
- difference $A \backslash B = \{x : x \in A \text{ and } x \notin B\}$
- subset $A \subset B \iff (x \in A \Rightarrow x \in B)$
- disjoint $A \cap B = \varnothing$
- partition $C = A \sqcup B \iff C = A \cup B \text{ and } A \cap B = \varnothing$
- product $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

Proposition 1.1.

- $\forall$ set $S, \varnothing \subset S$
- $A = B \iff A \subset B$ and $B \subset A$

Note.

- *The null set itself is an object, so expression like $\{\varnothing\}$ is valid.*
- *In contrary to the null set, we fail to construct a universal set which contains all objects. The reason is that it contains itself, leading to a contradiction. This can be solved if we use a more general framework called category theory.*

The operations can be visualized using Venn diagrams

However, we usually deal with many sets, or even infinitely many of them. The Venn diagrams are not useful in these cases, and we have to use logic statements to find the relationships.

Recall the summation and multiplication notations

$$\Sigma_{i=1}^{n} a_i = a_1 + \cdots + a_n, \Pi_{i=1}^{n} a_i = a_1 \cdots a_n$$

We can define similar notations for union and intersection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \cdots \cup A_n, \bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n$$

1.2.3 Map

Maps are relations between sets.

Definition 1.1 (Map). A map

$$f: \quad X \longrightarrow Y$$
$$x \longmapsto f(x)$$

receives an input $x \in X$, and returns exactly one output $f(x) \in Y$.

- X is the domain
- Y is the codomain

You can think map as a processor. For example, washing machine is a map that inputs dirty clothes and return clean clothes.

You can denote a set by many ways

- Explicit form: $f(x) = 69x + 420$
- Conditional:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- f : students $\to$ how they get to UST

The notation of map can also be applied to set.

$$f(A) = \{f(x) : x \in A\}$$

Definition 1.2 (Map, continued).
- $f(X) \subset Y$ is the range of f

Reversely, we can define preimage, which is what maps to the target set

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Note the preimage is not a map since it may not return one output from one input

You can perform operations on functions, consider $f_1, f_2 : X \to Y, g : Y \to Z$

- addition / subtraction: $(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$
- constant multiplication $(cf)(x) = c(f(x))$
- pointwise multiplication $(f_1 \cdot f_2)(x) = f_1(x)f_2(x)$
- composition $f \circ g : X \to Z, (f \circ g)(x) = f(g(x))$

Note the operations is alternative, i.e. $(f \circ g) \circ h = f \circ (g \circ h)$

For a given $y \in Y$, we would like to know if there are some x that maps to y, and if such x is unique. This gives the definition of injective and surjective

Definition 1.3. A function $f : X \to Y$ is

- injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ i.e. if $x_1 \neq x_2$, then they maps to different values
- surjective if $\forall y \in Y, \exists x \in X, f(x) = y$, equivalently $f(X) = Y$
- bijective if f is both injective and bijective

Example 1.9.

- f : date $\to$ weekday is surjective but not injective
- $\mathbb{Q} \hookrightarrow \mathbb{R}$ is injective but not surjective
- $\mathbb{R} \twoheadrightarrow \mathbb{Q}$ is surjective but not injective

These properties may make the preimage of f behave "good"

Definition 1.4 (Inverse). Given $f : X \to Y$,

- a left inverse is g such that $g \circ f = id_X : x \mapsto x$
- a right inverse is h such that $f \circ h = id_Y : y \mapsto y$
- an inverse is both a left inverse and a right inverse

Proposition 1.2. Consider $f : X \to Y$, prove that the existence of inverse is equivalent to bijective, using the following steps

i) f is injective iff it has a left inverse

ii) f is surjective iff it has a right inverse

Proof. Left as an exercise. □

1.2.4 Real Sequence

Definition 1.5 (Real Sequence). A real sequence is a map

$$(a_n) : \mathbb{N} \to \mathbb{R}$$

from the index $\in \mathbb{N}$ to the term $\in \mathbb{R}$.
Its series is defined as the sum of the first n terms

$$(s_n) : \mathbb{N} \to \mathbb{R}, s_n = \sum_{i=1}^{n} a_i$$

A sequence is said to be alternating if the sign of any two consecutive terms are different $(+ - + - \cdots)$ or $(- + - + \cdots)$

A sequence is periodic if it repeats after a cycle

$$\exists N, \forall n, a_{n+N} = a_n$$

A geometric sequence has a fixed ratio $r \neq 0$ between the terms

$$\forall n, \frac{a_{n+1}}{a_n} = r$$

Sequence operations can be defined as term-wise

Definition 1.6 (monotonicity). A sequence (a_n) is monotonic if ,

- it is increasing $\forall m > n, a_m \geq a_n$, or
- it is decreasing $\forall m > n, a_m \leq a_n$.

We can impose the strictly condition by replacing the inequalities $\geq, \leq$ with $>, <$

Example 1.10.

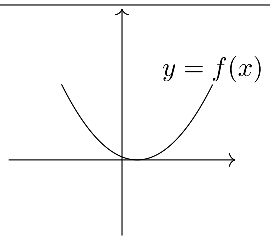
- a constant sequence $a_n = 0$ is monotonic, but not strictly monotonic
- $a_n = n$ is strictly increasing
- $a_n = \frac{1}{n}$ is strictly decreasing

1.2.5 Real Function

Here we consider $f : X \to Y, X, Y \subset \mathbb{R}$. In simple cases, f can be visualized by a graph. This allows us to visualize the transformation of function

- Horizontal translation $f(x \pm a)$
- Vertical translation $f(x) \pm b$
- Horizontal rescaling $f(ax)$
- Vertical rescaling $bf(x)$
- Horizontal reflection $f(-x)$
- Vertical reflection $-f(x)$

Definition 1.7 (Intercepts)**.** The x-intercepts are x which $f(x) = 0$ The y-intercept is $y = f(0)$, if $f(x)$ is defined at $x = 0$



Definition 1.8 (Parity)**.** a function f is

- odd if $f(x) = -f(-x)$
- even if $f(x) = f(-x)$

Exercise 1.2. Prove that for an odd function $f, f(0) = 0$

Definition 1.9 (Periodic)**.** A function f is periodic with period $P > 0$ if

$$\forall x, f(x + P) = f(x)$$

Note a function have many periods, such as $2P, 3P$. Here we take the smallest period.

Here we study some elementary functions.

Exercise 1.3. Evaluate $\sqrt{x^2}$.

Recall the m-th power of x is defined as

$$x^m = \underbrace{x \cdots x}_{m \text{ times}}$$

This extends to more related notations. For simplicity assume $x > 0$

- $x^0 = 1$

Function	$f(x)$	Graphs
constant	c	
linear	$mx + c$	
quadratic	$ax^2 + bx + c$	
polynomial	$a_n x^n + \cdots + a_0$	

- $x^{-m} = 1/x^m$
- $x^{1/n} = y \iff y^n = x$
- $x^{m/n} = z \iff z^n = x^m$

But what if the exponent is not a rational number? Here we introduce the exponential function and its inverse, the logarithm function

Function	$f(x)$	Graphs
exponential	a^x	
logarithm	$\log_a(x)$	

Here are some common values of base a

- $a = 2$: useful in computer science as binary is frequently used there
- $a = 10$: since the value is approximate the number of digits before decimal places, it would be intuitive.
- $a = e \approx 2.71828$: it appears in natural science frequently

Notation. Without specification, we refer $\log x = \ln x = \log_e x$

Trigonometric functions relate angles and sides on the unit circle $x^2 + y^2 = 1$.

Function	$f(x)$	Graphs	Domain	Range	Period
sine	$\sin x$		$\mathbb{R}$	$[-1, 1]$	2π
cosine	$\cos x$		$\mathbb{R}$	$[-1, 1]$	2π
tangent	$\tan x$		$x \neq (2n+1)\pi, n \in \mathbb{Z}$	$\mathbb{R}$	π

For example, consider a transformed trigonometric function $f(t) = A\cos(2\pi t/T + \varphi)$

- A: amplitude
- T: period
- φ: phase

The reciprocal of the trigonometric functions are

Function	$f(x)$	Graphs
cosecant	$\csc x = 1/\sin x$	
secant	$\sec x = 1/\cos x$	
cotangent	$\cot x = 1/\tan x$	

Sometimes we would like to find the angles from the sides. However, we have to restrict the input to a certain range, so the restricted function is bijective there.

Function	$f(x)$	Graphs
arcsine	$\operatorname{asin} x$	
arccosine	$\operatorname{acos} x$	
arctangent	$\operatorname{atan} x$	

Note. *Some literatures define* $\operatorname{atan} \frac{b}{a}$ *depending on the signs of* b *and* a*, so*

$$\operatorname{atan} \frac{-b}{-a} = \pi + \operatorname{atan} \frac{b}{a}, a, b > 0$$

Also, some may define $\operatorname{atan} 1/0 = \frac{\pi}{2}$

Proposition 1.3 (Properties of trigonometric function).

- $\sin \theta^2 + \cos \theta^2 = 1$
- $\sin \theta = \cos(\pi/2 - \theta), \cos x = \sin(\pi/2 - \theta)$
- $\tan \theta = 1/\tan(\pi/2 - \theta)$

Proposition 1.4 (Compound angle formula). The compound angle formulas are given by:

- $\sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$
- $\cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$
- $\tan(\theta + \varphi) = \dfrac{\tan \theta + \tan \varphi}{1 - \tan \theta \tan \varphi}$

Exercise 1.4. From the compound angle formula, derive the sum to product formula and product to sum formula i.e. find the following terms

$$\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \cos \varphi, \sin \alpha + \sin \beta, \sin \alpha + \cos \beta, \cos \alpha + \cos \beta.$$

You can define a function based on different pieces of the domain. This way a function is piecewise.

Example 1.11 (Floor function).

$$\lfloor x \rfloor = \text{ integer part of } x$$

is a piece-wise constant function with

$$\lfloor x \rfloor = \{n, n \leq x < n + 1, n \in \mathbb{Z}$$

Example 1.12 (Step function).

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

- There is freedom in choosing the value of $\Theta(0)$, that the most common choices are 0 (left), 1/2 (middle), 1 (right).
- The function is "turned on" when x exceeds 0

Exercise 1.5. Assume we would like to use a function $f(x)$ to describe the strength of a signal, express it in terms of step functions, if it the strength is 0.5 when $3 < x < 7$.

2 Limit

2.1 Sequence Limit

We would like to study the long-term behavior of a sequence (bunch of numbers).

Definition 2.1 (non-rigorous Sequence Limit). (a_n) converges to L means when n gets larger and larger, a_n gets closer to L.

$$\lim_{n\to\infty} a_n = L$$

or in short, $(a_n) \to L$.
Otherwise, (a_n) diverges.

If we cannot find a number to bound (a_n) above / below, we say that (a_n) diverges to $+\infty/-\infty$ or $(a_n) \to +\infty/-\infty$. Note it is not the same as plugging in $n = \infty$ into calculation.

Example 2.1. The sequence

$$a_n = \begin{cases} 1, n = & 2k+1 \\ 0, n = & 2k \end{cases}, k \in \mathbb{Z}$$

diverges since it is impossible for the sequence to get closer to 0 or 1 consistently.

Proposition 2.1.

1. $\lim_{n\to\infty} c = c$
2. $$n^p \to \begin{cases} 0, & p < 0 \\ 1, & p = 0 \\ +\infty, & p > 0 \end{cases}$$
3. $\sqrt[n]{a} \to 1, a > 0$
4. $\sqrt[n]{n} \to 1$
5. $1/a^n \to 0$
6. Arithmetic Rules: assume $(a_n) \to L, (b_n) \to M$
   - $(a_n \pm b_n) \to L \pm M$
   - $(a_n b_n) \to LM$
   - $M \neq 0, (a_n/b_n) \to L/M$
   - $c \neq 0, (ca_n) \to cL$

7. $a^n \to 0$
8. $\forall$ polynomial $p(n), p(n)a^n \to 0, |a| < 1$

Sometimes we cannot use

Theorem 2.1 (Squeeze Theorem). If $(x_n) \to L, (z_n) \to L$ and $x_n \le y_n \le z_n$ for sufficiently large n, i.e. $\exists N, \forall n > N$, then $(y_n) \to L$

To rigorously prove them, we need to use the $\varepsilon - \delta$ definition, which we should not talk here. Real number is a very special set, which has some very good properties

Theorem 2.2 (Completeness of $\mathbb{R}$).

$$(a_n) \subset \mathbb{R}, (a_n) \to L \Rightarrow L \in \mathbb{R}$$

Exercise 2.1. Give a counter-example to explain why we cannot replace $\mathbb{R}$ with $\mathbb{Q}$.

Proposition 2.2.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \text{ converges}$$

Proof. We would like to use an interesting result called Monotonic Convergence Theorem: if a sequence is monotonic and bounded, then it converges. The sketch of the proof is as follows

1. $\forall n, \left(1 + \frac{1}{n}\right)^n < 3$
2. $\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n$

□

We define $e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71827\cdots$, which is called the Euler's numbers.

Sometimes we may encounter limits in indeterminate form e.g. $\frac{\infty}{\infty}, 1^\infty \infty - \infty, 0/0, 0 \times \infty$ etc., then we need to simplify the definition first

Question 2.1.

$$\lim_{n\to\infty} \frac{6n^2 + 9n}{4n^2 + 20n}$$

Solution 2.1. Direct evaluation gives ∞/∞, so we have to divide both sides by n^2 to get

$$\frac{6 + 9/n}{4 + 20/n} \to 3/2$$

Definition 2.2 (Series Limit). $(s_n) \to L$ is defined similarly as sequence limit. We also denote it by

$$\lim_{n\ to\infty} s_n = \sum_{n=1}^{\infty} a_n = L$$

Proposition 2.3.

$$\sum_{n=1}^{\infty} a_n = L \Rightarrow (a_n) \to 0$$

Proof.

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \to L - L = 0$$

□

Note. *Note the converse is not true.* $1/n \to 0$ *but* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges*

Telescoping series

Question 2.2. Evaluate $\sum_{n=1}^{\infty} \frac{1}{k(k+1)}$

Solution 2.2.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \lim_{n\to\infty} \frac{1}{n+1} \\ &= 1\end{aligned}$$

Exercise 2.2. Evaluate $\sum_{n=1}^{\infty} \frac{1}{k(k+1)(k+2)}$

Geometric series

Exercise 2.3. Evaluate $\sum_{n=0}^{\infty} ar^n, |r| < 1$. Explain why it does not work for $|r| \geq 1$. For example, using the formula, we get

$$1 + 2 + 4 + \cdots = -1$$

Alternating Series

Exercise 2.4. Using the Monotone Convergence Theorem, prove that if (a_n) is alternating, but $(|a_n|)$ is decreasing, then (a_n) converges
Tips: WLOG you can assume $a_1 > 0$

2.2 Function Limit

Definition 2.3 (Limit of a function)**.** The limit of $f(x)$ at $x \approx a$ is L means

$$\lim_{x\to a} f(x) = L$$

When x approaches a (but not equals to a), $f(x)$ approaches L The one sided limit, replacing a by $a^{\pm}$, specifies which side x is approaching to a.

Definition 2.4 (Continuity)**.** $f(x)$ is continuous at $x = a$ if

$$\lim_{x\to a} f(x) = f(a)$$

$f(x)$ is continuous at a set $S \subset \mathbb{R}$ if it is continuous at $x = s \forall s \in S$.

Proposition 2.4.

1. Elementary functions, including polynomials, exponential, logarithm, trigonometric, inverse trigonometric etc. are continuous within their domains

2.
$$\lim_{x\to 0} x^p = \begin{cases} 0, & p > 0 \\ 1, & p = 0 \\ +\infty, & p < 0 \end{cases}$$

3. Arithmetic rules: assume $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M$
   - $\lim_{x\to a} f(x) \pm g(x) = L \pm M$
   - $\lim_{x\to a} f(x)g(x) = LM$
   - $M \neq 0, \lim_{x\to a} f(x)/g(x) = L/M$
   - $\lim_{x\to a} cf(x) = cL$
4. Squeeze Theorem: assume $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, if $f(x) \leq h(x) \leq g(x)$ for $x \approx a$, then $\lim_{x\to a} h(x) = L$

Proposition 2.5.
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Proof. Consider the figure

□

Note that $\lim_{n\to\infty} f(n)$ as a sequence limit can be different from $\lim_{x\to+\infty} f(x)$ as a function limit.

Example 2.2. Consider
$$f(x) = \begin{cases} 1, & x + \frac{1}{2} \in \mathbb{Z}, \\ 0, & x + \frac{1}{2} \notin \mathbb{Z} \end{cases},$$
then $\lim_{n\to\infty} f(n) = 0$ but $\lim_{x\to+\infty} f(x)$ diverges.

Luckily, under some conditions, we can conclude that they are the same

Theorem 2.3 (Extended Squeeze Theorem). $n \leq x < n+1, y_n \leq f(x) \leq z_n$
$$(y_n), (z_n) \to L \Rightarrow \lim_{x\to+\infty} f(x) = L$$

i.e. bound the function by two step functions (which terms come from sequence.

Proposition 2.6.
$$\lim_{x\to+\infty} a^x = \lim_{x\to\infty} x^L a^x = 0$$
$$\lim_{x\to\pm\infty} (1 + \frac{1}{x})^x = e$$

Proposition 2.7.
$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

Proof.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\ln(1+x)^{1/x}=\ln\Big(\lim_{x\to 0}(1+x)^{1/x}\Big)=\ln e=1$$

□

Proposition 2.8.

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

Proof.

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{y=e^x-1\to 0}\frac{y}{\ln(1+y)}=1$$

□

Proposition 2.9.

$$\lim_{x\to 0}\frac{(x+1)^p-1}{x}=p$$

Proof.

$$\lim_{x\to 0}\frac{(x+1)^p-1}{x}=\lim_{x\to 0}\frac{e^{p\ln(1+x)}-1}{x}=\lim_{y=p\log(1+x)\to 0}\frac{e^y-1}{y}\lim_{x\to 0}\frac{p\ln(1+x)}{x}=p$$

□

2.3 Continuity

Theorem 2.4 (Limit Composition). If $f(x)$ is continuous at $x=a$, and $g(y)$ is continuous at $y=f(a)$, then $g\circ f$ is continuous at $x=a$ i.e.

$$\lim_{x\to a}g(f(x))=g(f(a))$$

Example 2.3.

$$g(x)=\begin{cases}x^2, & x\neq 0\\ 1, & x=0\end{cases}, f(x)=x$$

$$\lim_{x\to 0}g(f(x))=0\neq g(\lim_{x\to 0}f(x))=1$$

Definition 2.5 (Different types of discontinuity).

$f(x)$ has a removable discontinuity at $x=a$ if $\exists$ continuous $g(x),\forall x\neq a, g(x)=f(x)$ i.e. $g(x)$ fixes the discontinuity of $f(x)$

$f(x)$ has a jump discontinuity at $x=a$ if $\lim_{x\to a^-}f(x)\neq\lim_{x\to a^+}f(x)$, though both one-sided limits exist.

$f(x)$ has an essential discontinuity at $x=a$ if either of the one-sided limits do not exist.

Example 2.4.

$\sin x/x$ is discontinuous at $x=0$. However, since $\lim_{x\to 0}\frac{\sin x}{x}=1$, we can remove the discontinuity by

$$\text{sinc}x=\begin{cases}1, & x=0\\ \frac{\sin x}{x}, & x\neq 0\end{cases}$$

$$f(x) = \begin{cases} x^2, & x < 0 \\ 1, & x = 0 \\ x+2, & x > 0 \end{cases}$$

has a jump discontinuity at $x = 0$

$\sin \frac{1}{x}$ and $1/x$ have an essential discontinuity at $x = 0$

Here are some consequences of continuity

Theorem 2.5 (Extreme Value Theorem)**.** If $f(x)$ is continuous on $[a, b]$, then $f(x)$ attains a maximum and minimum there i.e.

$$\exists x_-, x_+ \in [a, b], \forall x \in [a, b], f(x_-) \leq f(x) \leq f(x_+)$$

Why this is not as trivial as you thought?

Example 2.5.

- $1/x$ does not attain a maximum in $(0, 1]$ since it is unbounded above
- x^2 does not attain a maximum in $(0, 1)$
- $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ does not attain maximum or minimum in $\mathbb{R}$

Theorem 2.6 (Intermediate Value Theorem)**.** $f(x)$ is continuous on $[a, b], \forall \gamma$ strictly in between $f(a)$ and $f(b), \exists c \in [a, b], f(c) = \gamma$

3 Differentiation

Differentiation is about approximating functions with much simpler ones. As sometimes we only care about a region near a point, we don't need to know the full behavior of the function, just simple approximation near it is sufficient.

0th order approximation is just evaluating the function at some point

3.1 1st Order

1st order approximation is approximating the function with a straight line. Saying like we want to approximate $f(x)$ near x_0, then we say that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$$

, where

$$\lim_{x - x_0 \to 0} \frac{o(x - x_0)}{x - x_0} = 0$$

i.e. the error just even smaller than a linear term.

3.1.1 Basic Techniques

We would like to find the slope of $f(x)$ at $x = x_0$. However, as $f(x)$ may not be a straight line, the slope = rise / run vary. We take the limit $x \to x_0$ to find the instataneous slope of $f(x)$

Definition 3.1 (Differentiation)**.**

$$f'(x_0) = \frac{df}{dx}|_{x=x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Proposition 3.1. A differentiable function is continuous, but not the converse.

Proof.

$$\lim_{h\to 0} |f(x_0+h) - f(x_0)| = \lim_{h\to 0} h \cdot |f'(x_0)| = 0$$

□

Notation. We will use $f'(x)$ or f' to denote the derivative as a function, instead of a fixed value.

Proposition 3.2. Consider two differentiable functions f, g

- $(f \pm g)' = f' \pm g'$
- Product rule: $(fg)' = f'g + g'f$
- Quotient rule: $(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$

Proposition 3.3. The derivative of the inverse function is

$$(f^{-1}(y))' = \frac{1}{f'(f^{-1}(y))}$$

Proof. Use $f \circ f^{-1}(y) = y$ and use the chain rule □

This is the list of derivatives of common functions

$f(x)$	$f'(x)$
$a_n x^n + \cdots + a_0$	$n a_n x^{n-1} + \cdots + a_1$
x^p	$p x^{p-1}$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\operatorname{asin} x$	$1/\sqrt{1-x^2}$
$\operatorname{acos} x$	$-1/\sqrt{1-x^2}$
$\operatorname{atan} x$	$1/(1+x^2)$

Exercise 3.1. As an exercise, use the definition (first principle)

Suppose we have an equation $f(x, y) = c$ relating two variables x, y. In 2D, the equation looks like a curve. We are interested in the derivative of the curve. We can use implicit differentiation

Question 3.1. Find $\frac{dy}{dx}$ of $x^2 + y^2 = 1$

Solution 3.1.

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx} : 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} = \pm\frac{x}{\sqrt{1-x^2}} \end{aligned}$$

It is fine to keep the answer with some terms in y since usually you do not know how to express y in terms of x directly.

Note this is still single variable calculus since we do not discuss how a variable changes wrt many variables.

Parametric Equation

$$\begin{cases} x &= f(t) \\ y &= g(t) \end{cases}$$

For example, position of a particle in space can be described by parametric equation, that t is time. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

Here $\frac{dy}{dx}$ is pointing in the direction tangent to which the particle moves at.

Logarithmetric Differentiation

Question 3.2. Find $\frac{d}{dx}x^x$

Solution 3.2.

$$\begin{aligned} y &= x^x \\ \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\ \frac{dy}{dx} &= (\ln x + 1)y = x^x(\ln x + 1) \end{aligned}$$

Equivalently, you can use $x^x = e^{x \ln x}$, idea is to resolve the term into one with constant base, at the cost of a more complicated exponent

3.1.2 Extrema

Definition 3.2 (Extrema). Consider a function $f(x)$ with domain D,

- the global maximum is $f(x^*)$ iff $\forall x \in D, f(x) \leq f(x^*)$
- a local maximum is $f(x_i)$ so $f(x_i)$ is larger than $f(x)$ for sufficiently close $x \approx x_i$

Global minimum and local minimum are defined similarly.

Example 3.1.

- x^2 has no global maximum since it is unbounded above. But it has a global minimum of 0 at $x = 0$.
- $\cos x$ has many global maximums at $x = 2n\pi, n \in \mathbb{Z}$, and many global minimums at $x = (2n+1)\pi, n \in \mathbb{Z}$

Theorem 3.1. Let f be a differentiable function on (a, b). We can infer the monotonicity of f from its derivative

Theorem 3.2 (First Derivative Test)**.** Suppose $f(x)$ is differentiable at $x = a$. $x = a$ is local extrenum $\Rightarrow f'(a) = 0$

Note. *The converse is not true. e.g. for $f(x) = x^3, f'(0) = 0$ but $x = 0$ is not a local extrenum.*

Finding local extrenum

- $f'(x) = 0$
- $f'(x)$ is undefined
- At the boundaries of the interval
- At the boundaries of piecewise function

The largest/smallest of them is the global extrenum.

Question 3.3. Find the global maximum of $f(x) = x^2e^{-x}$ for $x > 0$

Solution 3.3.

$$\begin{aligned}\frac{df}{dx} &= 2xe^{-x} - x^2e^{-x} = 0\\ x &= 0, 2\\ x = 0, f(x) &= 0,\\ x = 2, f(x) &= 4e^{-2} \text{ is the global maximum}\end{aligned}$$

3.1.3 Mean Value Theorem

Theorem 3.3 (Mean Value Theorem (MVT))**.** f is differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 3.2. MVT does not work for $|x|$.

$$\frac{f(a) - f(-a)}{a - (-a)} = 0 \neq f'(c) \forall c \in (-a, a)$$

One application of the MVT is Section Control. To combat speeding, speeding cameras are installed roadside to catch drivers speeding. However, the drivers are not that dumb. They would slow down right in front of the camera. Meanwhile, section control records the time for the vehicle t to travel along a distance d. As long as $v = d/t$ is beyond the speed limit, the driver must have been speeded while in the section, even though he may not do so while he was leaving the section.

3.1.4 Comparing Functions

Theorem 3.4. Let $f(x), g(x)$ be continuous for $x \geq a$ and differentiable for $x > a$. If $f(a) \geq g(a)$ and $f'(x) > g'(x)$ for $x > a$, then $f(x) > g(x)$ for $x \geq a$

Proof. Left as an exercise, WLOG you can prove that a simpler version: if $f(0) \geq 0, f'(x) > 0$ for x, then $f(x) > 0$ for $x \geq 0$ $\square$

Question 3.4. Prove that $e^x \geq 1 + x$ for $x \neq 0$

Solution 3.4.

$$
\begin{aligned}
e^0 &= 1 \\
\frac{d}{dx}e^x &> 1 \text{ for } x > 0 \\
e^x &> 1 + x, x > 0 \\
\frac{d}{dx}e^x &< 1 \text{ for} x < 0 \\
e^x &> 1 + x, x < 0
\end{aligned}
$$

3.1.5 Optimization

Since the first derivative test can be used to find local extrenum, we can use it to find optimized value of a function.

Question 3.5 (Snell's Law)**.** By the Fermat's Principle, light takes a path with least travelling time. Suppose we have two points A, B on two media with a straight interface. The speed of light in the media are u, v respectively. The perpendicular distances of A and B to the interface are a and b respectively. The parallel distance of A, B is l. Derive the Snell's Law.

Solution 3.5. The time function is

$$t = \frac{a \sec x}{u} + \frac{b \sec y}{v},$$

and the restriction is

$$l = a \tan x + b \tan y$$

We have $\frac{dl}{dx} = 0$, then use $\frac{dt}{dx} = 0$

3.1.6 L'Hôpital's rule

Theorem 3.5 (L'Hôpital's Rule)**.**

- $f(x), g(x)$ are differentiable on (a, b)

- $g'(a) \neq 0$ for $x \neq a$
- $\lim_{x\to a^\pm} f(x) = \lim_{x\to a^\pm} g(x) = 0$ or $\lim_{x\to a^\pm} |g(x)| = +\infty$
- $\lim_{x\to a^\pm} \frac{f'(x)}{g'(x)} = L$

Then

$$\lim_{x\to a^\pm} \frac{f(x)}{g(x)} = L$$

Note. *It would lead to circular argument using L'Hôpital's Rule to conclude that* $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x = 1$ *since we use this fact while differentiating* $\sin x$

Question 3.6. Evaluate $\lim_{x\to 0^+} x^x$

Solution 3.6.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = \exp\left(\lim_{x\to 0^+} \frac{\ln x}{1/x}\right) = \exp\left(\lim_{x\to 0^+} x\frac{1/x}{-1/x^2}\right) = \exp\left(\lim_{x\to 0^+} -x\right) = 1$$

3.1.7 Root Finding

Usually we would like to find the root of a function $f(x)$ i.e. x such that $f(x) = 0$. While $f(x)$ is simple enough, we may have a closed form expression of the roots. However, in most cases, we have to numerically evaluate the roots. Here are a few methods to do so. Assume $f(x)$ is a continuous function on $[a, b]$.

The bisection method is as follows: choose two points $c_1, c_2 \in [a, b]$ so $f(c_1) < 0, f(c_2) > 0$, evaluate $f(c_3 = \frac{c_1+c_2}{2})$

- If $f(c_3) = 0$, c_3 is the desired root
- If $f(c_3) > 0$, choose c_1, c_3 and repeat the procedure
- If $f(c_3) < 0$, choose c_2, c_3 and repeat the procedure

This method guarantees to succeed since the interval which a root is in is halved in each step.

The Newton's method is as follows: choose an initial point x_1, the subsequent points are

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Graphically, x_{n+1} is the x-intercept of the tangent of $f(x)$ at x_n.
Though this have a much faster convergence rate, not all initial values give the correct root.

Example 3.3. While finding the root of $\sqrt[3]{x}$, x_n eventually diverge.

The recursive method is as follows: Suppose we can rewrite the equation as $g(x) = x$. We can start with an initial value x_1, and

$$x_{n+1} = g(x_n)$$

This is very convenient to perform on a calculator. However, x_n may diverge in some cases.

Exercise 3.2. Consider $f(x) = \tan x - \frac{1}{x}$

(i) Consider $x = n\pi$, show that $f(x)$ has infinitely many roots. (If you can't do this, draw a graph to convince that)

(ii) Find the first three roots of $f(x)$ at $x > 0$ numerically.

3.2 Higher Order Differentiation

Approximating functions with polynomials.

Definition 3.3 (Higher Order Derivatives). Suppose $f^{(n)}$ exists The second order derivative is

$$f''(x_0) = \frac{df'}{dx}|_{x=x_0} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0)}{h}$$

The higher order derivative is also defined similarly

$$f^{(n)}(x_0) = \lim_{h \to 0} \frac{f^{(n-1)}(x_0 + h) - f^{(n-1)}(x_0)}{h}$$

3.2.1 Taylor's Series

gotta leave that to next lecture

3.2.2 2nd Order test

Other than first derivative, we can use the 2nd order derivative to find if the point is a local maximum, local minimum.

Theorem 3.6 (2nd Derivative Test). Suppose $f'(x_0) = 0, f''(x_0)$ exists

- $f''(x_0) > 0 \Rightarrow x_0$ is a local minimum
- $f''(x_0) < 0 \Rightarrow x_0$ is a local maximum

Example 3.4. Note this is not a necessary condition. For example, $x = 0$ is local minimum of $f(x) = x^4$, but $f''(0) = 0$.

3.2.3 Convexity

Definition 3.4 (Convexity). a function $f(x)$ is convex on an interval I if $\forall x < y \in I, f$ is less than the line joining $(x, f(x)$ and $(y, f(y))$. Formally speaking

$$\forall z \in [x, y], f(z) \leq f(x) + \frac{f(y) - f(x)}{y - x}$$

This is equivalent to

$$\forall \lambda \in [0, 1], f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Concave is defined as oppositely.

Theorem 3.7. Considering an interval I where f'' exists,

- f is convex if $f'' \geq 0$
- f is concave if $f'' \leq 0$

3.2.4 Curve Sketching

You will need to include the followings

- x-intercept: x_i such that $f(x_i) = 0$
- y-intercept: $f(0)$

- disruptions: where $f(x)$ is discontinuous or not differentiable
- $f'(x) = 0$ (local maximum, local minimum or stationary point)
- convexity
- inflexion point (where convexity changes)
- asymptotes
  - vertical $x = a$: $\lim_{x\to a} |f(x)| = +\infty$
  - horizontal $y = b$: $\lim_{x\to\pm\infty} f(x) = b$
  - oblique $y = mx + c$: $\lim_{x\to\pm\infty} f(x) - (mx + c) = 0$
  - use broken line for asymptotes

Exercise 3.3 (HKALE 2004 PM II Q7).

Let

$$f(x) = \frac{|x|x^3}{x^2 - 2} \quad (x \neq \sqrt{2})$$

(a) (i) Find $f'(x)$ and $f''(x)$ for $x > 0$.
(ii) Write down $f'(x)$ and $f''(x)$ for $x < 0$.
(iii) Prove that $f'(0)$ exists.
(iv) Does $f''(0)$ exist? Explain your answer.

(b) Determine the range of values of x for each of the following cases:

(i) $f'(x) > 0$,
(ii) $f'(x) < 0$,
(iii) $f''(x) > 0$,
(iv) $f''(x) < 0$.

(c) Find the local extrema and point(s) of inflexion of $f(x)$.

(d) Find the asymptote(s) of the graph of $f(x)$.

(e) Sketch the graph of $f(x)$.