Calculus 3 – Complex Number and Multivariable Calculus

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0 Overview

In the last two weeks, we mainly talked about differentiation and integration with one input variable, and one real output variable. In this week, we discuss functions with complex output variable, and those with many input variables and many output variables.

1. Complex Number: $e^{i\pi} + 1 = 0$
2. Euclidean Space: What if we have one more dimension?
3. Partial Differentiation: Approximating multivariable functions
4. Integration: Doing sums on curves and surfaces
5. Vector Calculus: It's all Stoke's theorem, we have been lied

1 Complex Number

Idea: roots of real polynomials may not be real

Definition 1.1 (Complex Number).

$$\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R}, i^2 = -1\}$$

A complex number can be visualized using a Argand diagram.

Definition 1.2. Consider $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$,

- the real part of z is $\mathrm{Re}\ z = a$
- the imaginary part of z is $\mathrm{Im}\ z = b$

The arithmetic operations of complex numbers are as follows. Given $z = a + bi, w = c + di$

$$\begin{aligned}
z + w &= (a + bi) + (c + di) = (a + c) + (b + d)i \\
z - w &= (a + bi) - (c + di) = (a - c) + (b - d)i \\
zw &= (a + bi)(c + di) = (ac - bd) + (ad + bc)i \\
\frac{z}{w} &= \frac{a + bi}{c + di} = \frac{a + bi}{c + di}\frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i
\end{aligned}$$

Definition 1.3 (Complex Conjugate and Modulus). Given $z = a + bi \in \mathbb{C}$,

- the complex conjugate of z is $\bar{z} = a - bi$, and
- the modulus of z is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$

Proposition 1.1. Given $z, w \in \mathbb{C}$,

$$\overline{z+w} = \bar{z} + \bar{w}, \overline{z-w} = \bar{z} - \bar{w}, \overline{zw} = \bar{z}\bar{w}, \overline{z/w} = \bar{z}/\bar{w}$$

$$\bar{\bar{z}} = z, |\bar{z}| = |z|, \text{Re } z = \frac{z+\bar{z}}{2}, \text{Im } z = \frac{z-\bar{z}}{2i}$$

Exercise 1.1 (Triangle Inequality). Prove that for any $z, w \in \mathbb{C}$,

$$|z+w| \leq |z| + |w|$$

Hence, prove that

$$||z| - |w|| \leq |z - w|$$

Recall the polar coordinates

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases},$$

considering the Argand diagram, every complex number can be expressed in the polar form.

Definition 1.4 (Principal Argument). Consider $z = |z|(\cos\theta + i\sin\theta) \neq 0, \text{Arg } z = \theta \in (-\pi, \pi]$ is the principal argument of z

Note that Arg $(x + yi)$ is not simply equals to atan $\frac{y}{x}$ since the range of latter is restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$. You may need extra $\pm\pi$.

Example 1.1.

$$-1 - \sqrt{3}i = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$$

so Arg $(-1 - \sqrt{3}i) = -2\pi/3$

Example 1.2. Geometrical meaning of multiplying $w \in \mathbb{C}$ in the Argand diagram: rotation by Arg w about the origin and scaling by $|w|$.

Consequently, we have

Theorem 1.1 (De Moivre's Theorem). For any $\theta \in \mathbb{R}, n \in \mathbb{Z}$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. By induction, left as an exercise to reader. □

Exercise 1.2. Using the De Moivre's Theorem, derive the triple angle formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Definition 1.5 (Complex Exponential). Let $z \in \mathbb{C}$, its complex exponential is

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Theorem 1.2 (Euler's Identity).

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Corollary 1.1.

$$e^{i\pi} + 1 = 0$$

Proposition 1.2. For any $z, w \in \mathbb{C}, e^{z+w} = e^z e^w$. Consequently,

$$(e^z)^n = e^{nz}, e^z = e^{x+iy} = e^x(\cos x + i\sin y), e^z \neq 0$$

Definition 1.6. For real number $a > 0$,

$$a^z := e^{z\ln a}$$

2 Euclidean Space

2.1 Vectors in Euclidean Space

Vectors in general are objects that live in vector spaces and obey linearity. But it would be unnecessary to talk about them too general so let's talk about vectors in an Euclidean Space.

Definition 2.1 (Vectors in Euclidean Space). A vector $\mathbf{v}$ in a n-dimensional Euclidean space $\mathbb{R}^n$ is

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}^T = (v_1, \cdots, v_n) = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n, v_1, \cdots, v_n \in \mathbb{R}.$$

The zero vector is defined as

$$\mathbf{0} = (0, \cdots, 0)$$

Notation. Henceforth we may omit the transpose symbol while typing vector. Also, some people may use the angular brackets $\langle\ \rangle$ or square brackets [ ] instead to denote vectors.
Also, while bold font is used to denote vectors in computer generated documents, we tend to use vectored hat while writing by hand e.g. $\vec{v}$.

In this sense, even things like "(1 apple, 3 oranges, 6 bananas)" are vectors. However, in physics, vectors have meaningful coordinate transformations (e.g. rotation).

Example 2.1 (Vectors in physics). position, velocity, momentum etc.

With an origin, a vector can be interpreted as an arrow pointing from the origin to a point.

Example 2.2 (Vectors are just points). Suppose we have a point $P = (3, 4)$, the corresponding vector is $\overrightarrow{OP} = 3\mathbf{e}_x + 4\mathbf{e}_y$

Definition 2.2 (Vector Operations). Let $\mathbf{u} = (u_1, \cdots, u_n), \mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$ be two vectors, and $c \in \mathbb{R}$ be a number (scalar).

The following operations yield a vector.

- Vector addition: $\mathbf{u}+\mathbf{v}=(u_1+v_1,\cdots,u_n+v_n)$
- Scalar multiplication: $c\mathbf{u}=(cu_1,\cdots,cu_n)$
- Vector subtraction: $\mathbf{u}-\mathbf{v}=(u_1-v_1,\cdots,u_n-v_n)$

Definition 2.3 (Parallel Vectors). Two vectors $\mathbf{u},\mathbf{v}$ are parallel if one is the scalar multiple of another i.e.

$$\exists c\in\mathbb{R}, \mathbf{u}=c\mathbf{v} \text{ or } \mathbf{v}=c\mathbf{u}$$

Proposition 2.1. The zero vector $\mathbf{0}$ is parallel to any vector.

Proof. Take $c=0$. □

Our next goal is to formulate the concept of perpendicularity.

Definition 2.4 (Dot Product). The dot product of $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ is

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+\cdots+u_nv_n$$

Geometric interpretation: project $\mathbf{u}$ onto $\mathbf{v}$, the dot product is the length of $\mathbf{v}$ times the projected length of $\mathbf{u}$

Definition 2.5 (Length, Unit Vector, Normalization). The length / magnitude of $\mathbf{u}$ is

$$|\mathbf{u}|=u=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{u_1^2+\cdots+u_n^2},$$

which resembles the Pythagora's Theorem. A unit vector has legnth 1.
For any non-zero vector $\mathbf{u}\neq\mathbf{0}$, the normalized unit vector is

$$\hat{\mathbf{u}}=\mathbf{u}/|\mathbf{u}|,$$

which indicates the direction of the vector.

Theorem 2.1 (Cauchy-Schwarz Inequality).

$$|\mathbf{u}||\mathbf{v}|\geq(\mathbf{u}\cdot\mathbf{v}) \text{ i.e. } (u_1^2+\cdots+u_n^2)(v_1^2+\cdots+v_n^2)\geq(u_1v_1+\cdots+u_nv_n)^2$$

The inequality holds $\iff$ $\mathbf{u},\mathbf{v}$ are parallel.

Proof. Leave as exercise. Tips: Consider $|c\mathbf{u}-\mathbf{v}|^2$ □

Definition 2.6 (Angle). The angle $\theta\in[0,\pi]$ between two vectors $\mathbf{u},\mathbf{v}$ is

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Exercise 2.1 (Cosine formula).

$$|\mathbf{u}-\mathbf{v}|^2=|\mathbf{u}|^2+|\mathbf{v}|^2-2|\mathbf{u}||\mathbf{v}|\cos\theta$$

This imply that the angle from the previous discussion is same as the geometric angle.

Proposition 2.2 (Projection). Consider two vectors $\mathbf{u}, \mathbf{v}$, the vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v},$$

the scalar projection is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

Definition 2.7 (Perpendicular). $\mathbf{u}, \mathbf{v}$ are perpendicular / orthogonal if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

i.e. the angle between them is $\pi/2$ or $90°$

Proposition 2.3. The zero vector $\mathbf{0}$ is perpendicular to any vector.

From now on we would restrict ourselves to at most 3 dimensions as our space is 3 dimensional.

Notation. For $\mathbf{r} \in \mathbb{R}^3$, we can denote it as

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = x\hat{i} + y\hat{j} + z\hat{k} = (r_1, r_2, r_3) = (x, y, z)$$

Definition 2.8 (Cross product). The cross product $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is $\mathbf{u} \times \mathbf{v} \in \mathbb{R}^3$ such that

- the direction is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and given by the right-hand rule, and
- the length is $|\mathbf{u}||\mathbf{v}|\sin\theta$.

Proposition 2.4. The cross product is given by

$$\mathbf{u} \times \mathbf{v} = \det\begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{pmatrix} = (u_yv_z - u_zv_y)\mathbf{e}_x + (u_zv_x - u_xv_z)\mathbf{e}_y + (u_xv_y - u_yv_x)\mathbf{e}_z$$

For two dimensional vectors, we can simply treat the z-component to be 0, so the cross product yields a scalar.

Proposition 2.5. Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are parallel if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$

Parallelogram spanned by two vectors, Parallelopipe

2.2 Lines

Definition 2.9 (Parametric Equation). A parametric equation is a vector-valued function. A equation with one parameter is a curve

$$\mathbf{r}(t) = (f(t), g(t), h(t))$$

In physics, a common parameterization is time, and the parametric equation is the position of a particle.

Example 2.3. Consider projectile motion in 2D

Definition 2.10 (Line). A line is a parametric equation in the form

$$\mathbf{r}(t) = \mathbf{u} + \mathbf{v}t = (u_x + v_xt, u_y + v_yt, u_z + v_zt).$$

Alternatively, it can be written as equalities

$$\frac{x-u_x}{v_x}=\frac{y-u_y}{v_y}=\frac{z-u_z}{v_z}.$$

or understood as a set of points (i.e. subset of $\mathbb{R}^n$)

$$L=\{\mathbf{r}(t)=\mathbf{u}+\mathbf{v}t : t\in\mathbb{R}\}$$

Note. *Here we assume t is infinite on both sides. If t is finite on one side or both sides, then the equation is a ray or a segment respectively.*

Question 2.1. Find a line joining two points $\mathbf{p}$ and $\mathbf{q}$.

Solution 2.1.

Relationship between two lines $\mathbf{r}_1(t), \mathbf{r}_2(t)$

Relationship	Direction	Share a point?
Same	Parallel	T
Parallel	Parallel	F
Intersect	Different	T
Skew	Different	F

Note that in 2D, if two lines are not parallel, then they meet at exactly one point, but this is not the case in 3D.

2.3 Planes

Definition 2.11 (Plane)**.** A plane Π is a set of points $\mathbf{r}$ which the displacement from a point $\mathbf{p}$ is perpendicular to a normal vector $\mathbf{n}$.

$$\Pi : (\mathbf{r}-\mathbf{p})\cdot\mathbf{n}=0$$

2.4 Curves

Definition 2.12 (Tangent, Regular, Arclength)**.** Consider a curve $\mathbf{r}(t)=(f(t),g(t),h(t))$, the *tangent* is

$$\mathbf{r}'(t)=(f'(t),g'(t),h'(t)).$$

$\mathbf{r}(t)$ is *regular* on an interval $I\subset\mathbb{R}$ if its tangent does not vanish on I i.e.

$$\forall t\in I, \mathbf{r}'(t)\neq 0$$

The *arclength* of $\mathbf{r}(t)$ from $t=a$ to $t=b$ is

$$\int_a^b |\mathbf{r}'(t)|dt=\int_a^b \sqrt{(f'(t))^2+(g'(t))^2+(h'(t))^2}$$

Example 2.4. In physics, if we use time t as a parameter, then

- the tangent is the velocity,
- a particle is never at rest iff its path is regular, and
- the arclength is the distance.

Definition 2.13 (Arc-length Parametrization (ALP))**.** $\mathbf{r}(s)$ is an ALP if $|\mathbf{r}'(s)| = 1$.

For any regular curve $\mathbf{r}(t)$, we can reparametrize it into arc-length parametization $\mathbf{r}(s)$ by the procedures

1. Evaluate the arclength
$$s(t) = \int_a^t |\mathbf{r}'(\tau)| d\tau$$
2. Express t in terms of s i.e. find $t(s)$
3. Replace $t \mapsto t(s)$ in the curve $\mathbf{r}(t) \mapsto \mathbf{r}(s)$

Proposition 2.6. $\mathbf{r}(s)$ found from the above procedure satisfies $|\mathbf{r}'(s)| = 1$

Exercise 2.2 (Projectile motion)**.** Consider the projectile curve $\mathbf{r}(t) = (x(t), y(t)) = (u_x t, u_y t - \frac{1}{2}gt^2), u_x, u_y > 0$. We only consider when $y(t) \geq 0$.

(i) Find the arc-length of the whole curve.

(ii) Find the arc-length parametization.

The arc-length parametization gives a fair comparison of how a curve behaves, independent of the parametization used.

Definition 2.14. For an ALP $\mathbf{r}(s)$, the curvature is
$$\kappa(s) := |\mathbf{r}''(s)|,$$
and the radius of curvature is
$$R(s) = 1/\kappa(s)$$

Note. *You may have seen that in physics, the radius of curvature is given by*
$$R = \frac{v^2}{a_\perp}$$
These are indeed the same quantity (with different parametization).

2.5 Surfaces

Previously we discussed functions with only 1 input variable. However, functions can have multiple input variable, same as you can have different combinations for a set meal.

Example 2.5. $f(x, y, z) = x^2 + y^2 + z^2 : \mathbb{R}^3 \to \mathbb{R}$ is a multivariable function that maps real variables x, y, z to a real output

A two-variable function $f(x, y)$ can be visualized by a surface in 3D so the points on the surface are $(x, y, f(x, y))$

Example 2.6. The graph of $z = x^2 - y^2$ is

Definition 2.15 (Level Set). The level set of a function $f(x_1, \cdots, x_n) : \mathbb{R}^n \to \mathbb{R}$ is the set of points $(x_1, \cdots, x_n)$ satisfying

$$f(x_1, \cdots, x_n) = c, c \in \mathbb{R}$$

Below shows how the graph and level sets look like

n	Graph	Level Set
1	curve $\subset \mathbb{R}^2$	point $\subset \mathbb{R}$
2	surface $\subset \mathbb{R}^3$	usually a curve $\subset \mathbb{R}^2$
3	hypersurface $\subset \mathbb{R}^4$	usually surface $\subset \mathbb{R}^3$

Example 2.7 (Level Set of a n-variable function is not necessarily a local $n-1$-variable graph). Consider $f(x, y) = x^2 + y^2$, the level set $f(x, y) = 0$ is just a point $\{(0, 0)\}$, but not a curve

3 Differentiation

3.1 Limit

Omitted, it is fine without understanding that rn

3.2 Derivatives

3.2.1 Partial Derivatives

Definition 3.1 (Partial Derivative). For a multivariable function $f(x_1, \cdots, x_n)$, the partial derivatives are

$$\frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) = \lim_{h_i \to 0} \frac{f(x_1, \cdots, x_i + h_i, \cdots, x_n) - f(x_1, \cdots, x_n)}{h_i}, 1 \leq i \leq n$$

Notation. $\frac{\partial f}{\partial x}$ is also denoted as f_x and $\partial_x f$ etc.

In actual computation, we differentiate w.r.t. the target variable, while keeping other variables constant. Morewover, rules from single-variable differentiation apply.

Question 3.1. Find the partial derivatives of $f(x, y) = x^2 \sin xy$ w.r.t. x and y at $(1, pi)$.

Solution 3.1.

Geometrically, consider the graph $z = f(x, y)$, the partial derivatives are just finding the slope wrt a direction.

For functions with more variables, the rule is still the same

Example 3.1.

$$\frac{\partial}{\partial z}e^{x^2+y^3+xyz} = e^{x^2+y^3+xyz}\frac{\partial}{\partial z}x^2 + y^3 + xyz = xye^{x^2+y^3+xyz}$$

Example 3.2 (Second Partial Derivatives). Consider $f(x,y) = 3x^4y - 2xy + 5xy^3$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 12x^3y - 2y + 5y^3 \\
\frac{\partial f}{\partial y} &= 3x^4 - 2x + 15xy^2 \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}(12x^3y - 2y + 5y^3) = 36x^2y \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial y}(12x^3y - 2y + 5y^3) = 12x^3 - 2 + 15y^2 \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial x}(3x^4 - 2x + 15xy^2) = 12x^3 - 2 + 15y^2 \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial y}(3x^4 - 2x + 15xy^2) = 30xy
\end{aligned}$$

Theorem 3.1 (Mixed Partial Theorem / Clairaut's Theorem). Consider $f(x,y)$, if at least one of f_{xy} or f_{yx} exists and is continuous, then $f_{xy} = f_{yx}$.

Note. *It is used in deriving the Maxwell's relations in thermodynamics.*

3.2.2 Chain Rule

In single-variable differentiation, the chain rule is named from visualized the procedures of doing chained differentiation.

In multivariable calculus, suppose we have $u(x(t), y(t), z(t), t)$, then chain rule gives

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

$\frac{du}{dt}$ is the total derivative of u w.r.t. t, and $\frac{\partial u}{\partial t}$ is the partial derivative of u w.r.t. t

Example 3.3. If u is the temperature of a fluid, (x,y,z) is the position and t is time, then $\frac{du}{dt}$ is the rate of temperature change of a particle in a fluid (which is also influenced by the velocity), while $\frac{\partial u}{\partial t}$ is the rate of temperature change at the region (suppose you put a thermometer there).

In general, if we have $u = u(x_1, \cdots, x_n), x_i = x_i(t_1, \cdots, t_m)$, then

$$\frac{du}{dt_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t_j} + \frac{\partial u}{\partial t_j}$$

Using chain rule, we can interpret implicit differentiation in another way

3.2.3 Directional Derivatives

Idea: slope w.r.t. an arbitary direction

Definition 3.2 (Directional Derivative). Given a unit vector $\mathbf{u}$ and a function $f(\mathbf{x}) = f(x_1, \cdots, x_n)$, the directional derivative of f in direction $\mathbf{u}$ evaluated at point $\mathbf{x}$ is

$$D_{\mathbf{u}}f(\mathbf{x}) = \frac{d}{dt}f(x_1 + tu_1, \cdots, x_n + tu_n)|_{t=0} = \lim_{h\to 0}\frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}$$

Definition 3.3 (Gradient). Given $f(\mathbf{x})$, the gradient vector of f at $\mathbf{x}$ is

$$\boldsymbol{\nabla} f(\mathbf{x}) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\mathbf{x})\mathbf{e}_i \in \mathbb{R}^n$$

Example 3.4. For two variable, $\boldsymbol{\nabla} f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{e}_x + \frac{\partial f}{\partial y}(x, y)\mathbf{e}_y$

An easy way to compute directional derivative

Proposition 3.1. If $f(\mathbf{x})$ in C^1 i.e. all of $\frac{\partial f}{\partial x_i}$ are continuous, then

$$D_{\mathbf{u}}f(\mathbf{x}) = \boldsymbol{\nabla} f(\mathbf{x}) \cdot \mathbf{u}$$

Proof. Use chain rule □

Interpretation of $\boldsymbol{\nabla} f$:

- Points in the direction which f increases the most rapidly.
- At the opposite direction, f decreases the most rapidly
- It is perpendicular to the level set

Exercise 3.1. Prove the last statement for 2-variable case. Given $f(x, y) \in C^1$, and a point (a, b) on the level set curve $f(x, y) = c$. Prove that the gradient vector $\boldsymbol{\nabla} f(a, b)$ is perpendicular to the level curve $f(x, y) = c$ at (a, b).

3.2.4 Tangent Planes

Idea: approximate a surface with a plane.
For a level set $g(\mathbf{x}) = c$, $\boldsymbol{\nabla} g$ is normal to the level set, so the tangent plane at $\mathbf{x}_0$ is the set of points $\mathbf{x}$ which displacement is perpendicular to $\boldsymbol{\nabla} g$ i.e.

$$(x - x_0) \cdot \boldsymbol{\nabla} g(x_0) = 0$$

Note the points on the tangent plane need not to be in the level set.

Example 3.5. The graph $z = f(x, y)$ can be interpreted as a level set

$$g(x, y, z) = z - f(x, y) = 0,$$

so the tangent plane at $(x_0, y_0, f(x_0, y_0)$ is

$$-\frac{\partial f}{\partial x}(x - x_0) - \frac{\partial f}{\partial y}(y - y_0) + (z - f(x_0, y_0) = 0$$

3.3 Optimization

3.3.1 Critical Point

Definition 3.4. Given $f(\mathbf{x}) \in \mathbb{C}^1$, $\mathbf{a}$ is a critical point $\iff$

$$\boldsymbol{\nabla} f(\mathbf{a}) = \mathbf{0}$$

i.e. all partial derivatives vanish.

However, we still need more information to deduce if the point is a local extremum or not.

3.3.2 2nd Derivative Test

Theorem 3.2 (2nd Derivative Test for Two-variable Functions). Let $f(x, y) \in C^2$ i.e. all 2nd derivatives exist and are continuous, and (x_0, y_0) be a critical point of f. The nature of the point depends on the determinant of the Hessian matrix $\det H = f_{xx}f_{yy} - f_{xy}^2$.

$f_{xx}f_{yy} - (f_{xy})^2$	f_{xx}	(x_0, y_0) is a
+	+	maximum point
+	-	minimum point
-		saddle point

Other cases are inconclusive.

Exercise 3.2. Show that

$$\frac{y(1-y)x(1-x)}{1-xy} \leq \left(\frac{\sqrt{5}-1}{2}\right)^5, 0 < x, y < 1$$

and the maximum occurs at $x = y$.

3.3.3 Global extrema

Extrema can be found at

- critical point
- where partial derivative is undefined
- boundary

3.3.4 Lagrange Multiplier

Previously we talked about unconstrained optimization, that the only restriction is a region. However, what if we have an extra constrain? Saying like we want to maximize / minimize $f(\mathbf{x})$ under the constrain $g(\mathbf{x}) = c$. This can be done by the Lagrange's multiplier method (Yes, same person as the one who discovered Lagrangian mechanics).

1. Solve
$$\begin{cases} \boldsymbol{\nabla} f(\mathbf{x}) = \lambda \boldsymbol{\nabla} g(\mathbf{x}) \\ g(\mathbf{x}) \qquad\qquad = c \end{cases}$$
for $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$
2. The solutions $\mathbf{x}$ are boundary critical points

3. Evaluate each $f(\mathbf{x})$ to deduce whether they are maximum or minimum.

Geometrically, f attain maximum / minimum when the level set of f is tangent with the constraint level set. If not, then there are two cases

- There are no points on the level set of f matching the constraint
- There are many points on the level set of f matching the constraint, but there is a more extreme value of f that still satisfy the constraint.

Optional: What if there are many constraints e.g. $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2$? Set up multiple Lagrange multiplier

$$\begin{cases} \boldsymbol{\nabla} f(\mathbf{x}) &= \lambda_1 \boldsymbol{\nabla} g_1(\mathbf{x}) + \lambda_2 \boldsymbol{\nabla} g_2(\mathbf{x}) \\ g_1(\mathbf{x}) &= c_1 \\ g_2(\mathbf{x}) &= c_2 \end{cases}$$

4 Integration

4.1 Coordinate Systems in 3D

So far we have been using the Cartesian coordinate system. However, in some scenarios, other coordinate systems may be more useful.

Definition 4.1 (Coordinate System)**.** Coordinate System is a set of orthonormal basis (unit vectors, axes) used to describe any vector in the Euclidean space. Mathematically,

$$\{\mathbf{e}_i\}_{i=1}^n, \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

4.1.1 Cartesian Coordinate

The basis vectors are fixed x, y, z axes. This system is good at describing rectangular configurations. The velocity and acceleration are very trivial because the basis vectors are fixed

$$\begin{aligned} \mathbf{r} &= x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \\ \dot{\mathbf{r}} &= \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z \\ \ddot{\mathbf{r}} &= \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y + \ddot{z}\mathbf{e}_z \end{aligned}$$

4.1.2 Cylindrical Coordinates

A 3D coordinate system obtained by adding the z-axis to the polar coordinate (ρ, ϕ).

$$\begin{aligned} \mathbf{r} &= \rho\mathbf{e}_\rho + z\mathbf{e}_z \\ \dot{\mathbf{r}} &= \dot{\rho}\mathbf{e}_\rho + \rho\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \\ \ddot{\mathbf{r}} &= (\ddot{\rho} - \rho\dot{\phi}^2)\mathbf{e}_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z \end{aligned}$$

Note the ρ-axis is different for different vectors, so you cannot simply add them up i.e. if we have $\mathbf{r}_1 = \rho_1\mathbf{e}_{\rho_1} + z_1\mathbf{e}_{z_1}, \mathbf{r}_2 = \rho_2\mathbf{e}_{\rho_2} + z_2\mathbf{e}_{z_2}$, we cannot say that $\mathbf{r}_1 + \mathbf{r}_2 = (\rho_1 + \rho_2)\mathbf{e}_\rho + \cdots$

4.2 Spherical Coordinates

Describe the system with radius r and 2 angles, polar angle θ and azimuthal angle ϕ. Very good for spherical symmetric configuration, or even just 2 of the variables. The coordinate transformation is as

$$\begin{cases} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{cases}$$

$$\begin{aligned}
&\mathbf{r} = r\mathbf{e}_r \\
&\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta\dot{\phi}\mathbf{e}_\phi \\
&\ddot{\mathbf{r}} = (\ddot{r} - r\ddot{\theta} - r\sin^2\theta\dot{\phi}^2)\mathbf{e}_r + \\
&\quad (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\mathbf{e}_\theta + \\
&\quad (2\sin\theta\dot{r}\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi})\mathbf{e}_\phi
\end{aligned}$$

4.3 Double Integrals

Consider a function $f(x, y)$, we would like to find its double integral over a region $R \subset \mathbb{R}^2$ i.e. $\iint_R f dA$

Question 4.1.

$$\int_{y=1}^{y=2} \int_{x=0}^{x=1} (4 - x - y^2 x) dx dy$$

Solution 4.1. While evaluating the inner integral, we treat other variables as constants.

Note. *If $f(x, y) = 1$, then the integral evaluates the area of region R. If $f(x, y)$ is height, the integral evaluates the volume of the solid.*

Another interpretation:

$$\iint_R f dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx$$

Theorem 4.1 (Fubini's Theorem). Let $f(x, y)$ be a continuous function, then

$$\iint (x, y) dx dy = \iint f(x, y) dy dx$$

i.e. exchanging order of integration is allowed

Tips: draw the integration region in 2D

Question 4.2.

$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$$

Solution 4.2.

However, sometimes the region would be more convenient to express in other coordinate systems (u, v) instead of Cartesian coordinates (x, y).

Proposition 4.1 (Change of Variables).

$$\iint_R f(x,y)dxdy = \iint_R f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

where the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example 4.1. If the region is a circle e.g. $R = \{(x,y) : x^2 + y^2 < 1\}$, then it would be more convenient to use the polar coordinates (ρ, ϕ). The coordinate transformation is like

$$\begin{cases} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{cases}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(\rho,\phi)} = \det \begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix}$$

The integral becomes

$$\iint_R f dA = \iint_R f(\rho\cos\phi, \rho\sin\phi)\rho d\rho d\phi$$

Question 4.3 (Solid of Revolution). We would like to come up with two methods in finding the solid of revolution. Consider $y = f(x) \geq 0$ from $x = 0$ to $x = b$, find the volume of the solid obtained by revolving $y = f(x)$ w.r.t. y-axis using the disc method and shell method separately.

Solution 4.3.

Exercise 4.1. Evaluate $\int_{-\infty}^{+\infty} e^{-x^2} dx$.

For a surface $z = f(x,y)$, the surface area in region R is

$$\iint_R \sqrt{(f_x^2 + f_y^2 + 1)}dxdy$$

4.4 Triple Integrals

Given $f(x,y,z)$, we would like to evaluate its triple integral over a region $D \subset \mathbb{R}^3$ i.e. $\iiint_D f \ dV$ One way to evaluate is $\iiint_D f(x,y,z) \ dxdydz$. By the Fubini's theorem, as long as $f(x,y,z)$ is continuous, we can exchange the integration orders.

Question 4.4. Find the volume of the solid bounded by $y = x^2 + z^2$ and $y = 16 - 3x^2 - z^2$.

Solution 4.4.

Note. *If $f(x, y, z) = 1$, the integral evaluates the volume. If $f(x, y, z)$ is density, the integral evaluates the mass.*

Example 4.2 (Moment of Inertia). Consider an object obeying 2-fold rotation symmetry w.r.t. z-axis, it's moment of intertia w.r.t. z-axis is

$$I_z = \iiint (x^2 + y^2) dm = \iiint \rho(x, y, z)(x^2 + y^2) dxdydz$$

Proposition 4.2 (Change of Variables).

$$\iiint_D f(x, y, z) dxdxdz = \iiint_D f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw,$$

which the Jacobian is

Example 4.3. Consider the transformation from Cartesian coordinates (x, y, z) to

- cylindrical coordinates (ρ, ϕ, z), the Jacobian is ρ
- spherical coordinates (r, θ, ϕ), the Jacobian is $r^2 \sin\theta$

Example 4.4. Find the following quantities of a solid sphere with mass M and radius R

- volume
- surface area
- moment of inertia

5 Vector Calculus

5.1 Vector Fields

Definition 5.1 (Vector Field). Vector Field is a multivariable vector-valued function $\mathbf{F}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$

Previously, each point in $\mathbb{R}^n$ is assigned with a value. But now each point is assigned with a vector in $\mathbb{R}^n$ instead.

Example 5.1. graviational field, electric field, magnetic field

5.2 Line Integrals

Definition 5.2 (Line Integrals of Vector Fields). Given a continuous $\mathbf{F}(x, y, z)$ and a path C parametized by $\mathbf{r}(t), a \le t \le b$, the line integral of $\mathbf{F}$ over C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt.$$

If C is a closed path i.e. $\mathbf{r}(a) = \mathbf{r}(b)$, then the integral is denoted as

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Note. *We can break C into piecewise segments $C = C_1 \sqcup C_2 \sqcup \cdots$*

Geometrical meaning:

1. Break down the path $\mathbf{r}(t), a \leq t \leq b$ into n parts.
2. As $n \to \infty$, the segments $\Delta\mathbf{r}(t_i)$ becomes approximately straight
3. For each segment, compute $\mathbf{F}(t_i) \cdot \Delta\mathbf{r}(t_i) = |\mathbf{F}(t_i)||\Delta\mathbf{r}(t_i)| \cos\theta(t_i)$
4. Sum up the dot products

Note if $\mathbf{F}$ has same direction as $\mathbf{r}$, then the integral is positive. Else, the integral is negative.

Notation. The differential form notation is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_x dx + \int_C F_y dy + \int_C F_z dz$$

Definition 5.3 (Conservative Vector Field). A vector field $\mathbf{F}$ is conservative if it is in the form $\mathbf{F} = \boldsymbol{\nabla} f$, which f is a scalar function

Example 5.2.

- $\mathbf{F}(x, y, z) = (2x + y, x + z^3, 3yz^2 + 1)$ is a conservative field since $\mathbf{F} = \boldsymbol{\nabla} f, f(x, y, z) = x^2 + yx + yz^3 + z + C$
- $\mathbf{F}(x, y) = (-y, x)$ is not conservative

One property of conservative vector field is that the line integral is path-independent.

Theorem 5.1 (Fundamental Theorem of Gradients). If $\mathbf{F} = \boldsymbol{\nabla} f$ is conservative, then the line integral of $\mathbf{F}$ along C from (x_0, y_0, z_0) to (x_1, y_1, z_1) is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

If we chooose a path on the x, y or z-axes, the Fundamental Theorem for Calculus is resembled.

Corollary 5.1. If $\mathbf{F}$ is a conservative vector field, then

$$\oint_C \mathbf{F} \cdot \mathbf{r} = 0$$

Definition 5.4. The curl of the vector field is

$$\begin{aligned} \text{curl } \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z \right) \times (F_x\mathbf{e}_x + F_y\mathbf{e}_y + F_z\mathbf{e}_z) \\ &= \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \end{aligned}$$

Theorem 5.2 (Curl Test). Given $\mathbf{F} \in C^1$ on a region Ω

1. If $\mathbf{F} = \boldsymbol{\nabla} \times f$ on Ω, then $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ on Ω
2. If $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ and Ω is simply-connected i.e. every closed loop can be contracted to a point continuously without leaving Ω, then $\mathbf{F}$ is conservative.

Example 5.3. Consider

$$\mathbf{H}(x, y) = (-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}), (x, y) \neq (0, 0)$$

$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{0}$ but a loop surrounding $(0, 0)$ gives non-0 line integral.

5.3 Green's Theorem

Relates line integral with double integral

Theorem 5.3 (Green's Theorem). Let ∂R be a simple i.e. no self-intersection, closed and counter-clockwise curve in $\mathbb{R}^2$ which encloses region R, where $\mathbf{F}$ is defined.

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{e}_z dA$$

i.e.

$$\oint_{\partial R} (F_x dx + F_y dy) = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Question 5.1. Suppose points $(x_1, y_1), \cdots (x_n, y_n)$ are arranged counter-clockwise to form a n-sided polygon. Evaluate the area of the polygon.

Solution 5.1.

5.4 Parametric Surface

A curve is parametrized by 1 parameter, then a surface is parametrized by 2 parameters.

Definition 5.5 (Parametric Surface). A parametric surface is a equation with 2 parameters

$$\mathbf{r}(u, v) = x(u, v)\mathbf{e}_x + y(u, v)\mathbf{e}_y + z(u, v)\mathbf{e}_z$$

Example 5.4. A plane is a parametric surface in the form

$$\mathbf{r}(s, t) = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$$

Example 5.5. A parametrization of the cylinder with radius r_0 and z-axis as central axis is

$$\mathbf{r}(\phi, z) = r_0 \cos\phi \mathbf{e}_x + r_0 \sin\phi \mathbf{e}_y + z\mathbf{e}_z, 0 \leq \phi < 2\pi$$

Definition 5.6. Let S be a surface parameterized by $\mathbf{r}(u, v), a \leq u \leq v, c \leq v \leq d$. The surface integral of f over S is

$$\iint_S f dS = \int_{v=c}^{v=d} \int_{u=a}^{u=b} f(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv$$

5.5 Stokes Theorem

Theorem 5.4 (Stokes Theorem)**.** Let S be an orientable, simply-connected surface in $\mathbb{R}^3$, and $\mathbf{F}$ is C^1 on S, then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$$

Geometric Interpretation: sum of small vortexes is the overall vortex

5.6 Divergence Theorem

Definition 5.7 (Divergence)**.** Given $\mathbf{F} \in C^1(\mathbb{R}^3)$, the divergence is defined as

$$\begin{aligned} \text{div }(\mathbf{F}) = \boldsymbol{\nabla} \cdot \mathbf{F} &= \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \end{aligned}$$

Theorem 5.5 (Divergence Theorem)**.** Suppose $\mathbf{F}$ is a vector field C^1 in and near a solid region $D \subset \mathbb{R}^3$, then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \boldsymbol{\nabla} \cdot \mathbf{F} dV$$

5.7 What

Theorem 5.6 (Generalized Stokes theorem)**.** Let ω be a differential form and Ω be an orientable manifold, then

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

This is the generalized case for the Fundamental Theorem of Calculus, Stokes Theorem and Divergence Theorem