Mechanics 2

Jett Cheung

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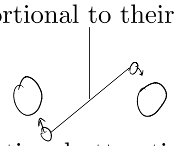
1 Gravitation

1.1 Newton's Law of Gravitation

The *Newton's law of gravitation* states that the attraction between two objects (precisely point masses i.e. particles) are proportional to their masses and the square of inverse of the distance. Mathematically



$$F_G = \frac{Gm_1m_2}{r^2}, \tag{1.1}$$

where F_G is the gravitational attractive force between the objects, $G \approx 6.674 \times 10^{-11}\,\mathrm{N\,m^{-2}\,kg^{-2}}$ is the gravitational constant, and m_1, m_2 are the masses of the objects. In vector notation,

$$\mathbf{F}_G = -\frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1) \tag{1.2}$$

where $\mathbf{r}_2 - \mathbf{r}_1$ is the separation between the two objects. From this expression, we see that gravitational force is a central force.

Exercise 1.1. Find the dimension of the gravitational constant G.

According to Einstein's postulate, information cannot travel faster than light. Therefore, forces, including gravitational one, are transmitted by fields. On one hand, particles create fields; on the other hand, fields affect motion of the particles. $\vec{F_G} = m\vec{g}$

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Physical properties of the field is only affected by the particle which creates it, but not the one which feels it. Hence, the *gravitational field* created by a particle is

$$\mathbf{g} = -\frac{Gm}{r^3}\mathbf{r} = -\frac{Gm}{r^2}\mathbf{e}_r, \tag{1.3}$$

where G is the gravitational constant, m is the mass of the particle, and $\mathbf{r}$ is the position vector from the particle.

On the Earth's surface, we assume the Earth to be spherical, so the small variation in gravitational field across space can be neglected. The gravitational field near the Earth's surface is

$$g = \frac{GM_E}{R_E^2}. \tag{1.4}$$

Gravitational force is conservative because it can be written in the form $\mathbf{F}_G = -\nabla U$, where

$$U = -\frac{Gm_1m_2}{r} \qquad r \to \infty \quad U \to 0 \tag{1.5}$$

is the *gravitational potential energy* of the two-body system.

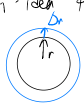
For a n-body system, the gravitational potential energy is

$$U = -\sum_{1 \le i < j \le n} \frac{Gm_im_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{1.6}$$

$n = 3$
(1, 2)
(1, 3)
(2, 3)

Question: Find energy required to break down a uniform sphere with mass M and radius R

Solution: idea find $U(r) \to U(R)$



$\rho = \frac{M}{\frac{4}{3}\pi R^3}$

$\Delta U = -\frac{G m(r) \Delta m}{r}$
$= -\frac{G\rho \frac{4}{3}\pi r^3 \cdot \rho 4\pi r^2 \Delta r}{r}$
$= -3GM^2 \frac{r^4}{R^6} \Delta r$

$U = \int_0^R \frac{\Delta U}{\Delta r} dr$
$= -\int_0^R 3GM^2 \frac{r^4}{R^6} dr$
$= -\frac{3}{5}\frac{GM^2}{R}$

$E = -U$
$= \boxed{\frac{3}{5}\frac{GM^2}{R}}$

Question 1.1. If three particles with masses m are arranged in a equilateral triangle with side length l, find the gravitational potential energy of the system.

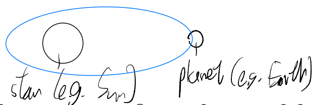
Solution 1.1.

Exercise 1.2. Hence, find the gravitational potential energy of n particles with masses m arranging in the vertexes of a regular n-gon with side length l.

Question 1.2. Find the escape velocity of a projectile i.e. the critical velocity so it can escape the Earth.

Solution 1.2.

1.2 Kelper's Laws



Kelper's Laws, which describe motions of planets around stars, are first obtained by observation. However, we can derive them using Newtonian mechanics. However, we will first state the laws and how we use them.

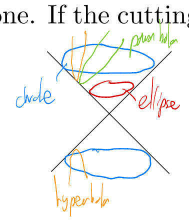
Kelper's 1st Law: The planet moves in an elliptical orbit, with the star at one of the focus.
To explain the 1st law, we would better introduce the concept of conic section first.

1.2.1 Interlude: Conic Section

A *conic section* is a curve obtained by cutting a surface of a cone. There are three cases of conic sections

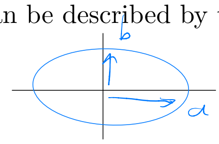
bounded
- Ellipse: bounded shape since it is less steep than the slanted edge of the cone. If the cutting surface is horizontal, we get a circle

unbounded
- Parabola: obtained by cutting the cone parallel to the slanted edge.
- Hyperbola: the cutting surface is steeper than the slanted edge.



In this section we are only concerned with ellipse.
An ellipse can be described by the equation

$f(x,y)=0$



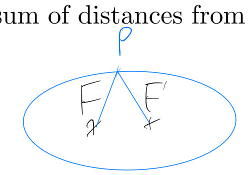
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a \geq b \tag{1.7}$$

where a, b are called the *semi-major axis* and *semi-minor axis* respectively. When $a = b$, the ellipses reduces to a circle. From how we parameterize a unit circle, an ellipse can be also described by a parametric equation

$$\mathbf{r}(t) = (a \cos t, b \sin t), 0 \leq t < 2\pi \tag{1.8}$$

Alternatively, the ellipse can be described by a locus of a point P so the sum of distances from P to two points is a constant. Mathematically, the locus is

$$\{P \in \mathbb{R}^2 : PF + PF' = 2a\} \tag{1.9}$$

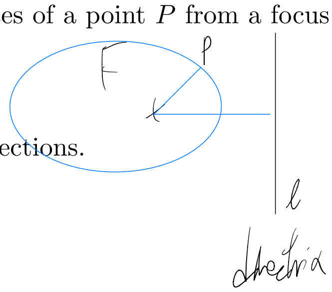


where F, F' are the plural *foci* of the ellipse.

> **Exercise 1.3.** Find the coordinates of F, F' in Eq. 1.7.

Another way to describe the ellipse by a locus is the ratio of distances of a point P from a focus F and a line *directrix* l

$$\{P \in \mathbb{R}^2 : \frac{PF}{d(P,l)} = e\} \tag{1.10}$$



where e is the *eccentricity*. This description is valid for other conic sections.

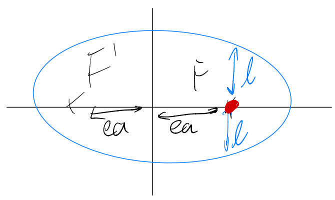
- $e = 0$: circle
- $0 < e < 1$: ellipse
- $e = 1$: parabola
- $e > 1$: hyperbola

Coincidentally, the distance of focus to center is $c = ea$.

> **Exercise 1.4.** Show that $e = \sqrt{1 - \frac{b^2}{a^2}}$.

We can also express the ellipse in polar coordinates. However, we use a focus (position of the star) as the origin, instead of that in Eq. 1.7.

$$r(\theta) = \frac{l}{1 + e\cos\theta}, \tag{1.11}$$



where $l = a(1 - e^2)$ is the *semi-latus rectum*.
There are two more terms which describe points in the orbit.

- *aphelion (apogee)*: farthest point from the star $(1+e)a$
- *perihelion (perigee)*: closest point to the star $(1-e)a$

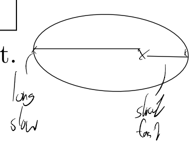
It can be shown that the total mechanical energy of the system is

$$E = K + U = -\frac{GMm}{\underbrace{2a}_{\text{semi-major axis}}}. \tag{1.12}$$

This expression is valid even for non-circular but bounded orbits.

> **Exercise 1.5.** Prove the equation above for the case of circular orbits.

Kelper's 2nd Law: The rate of area swept by the segment joining the star and the planet is constant.
Kelper's 3rd Law: Let T be the period of the orbit, and a be the semi-major axis. Then



$$T^2 = \frac{4\pi^2}{GM} a^3, \tag{1.13}$$

i.e. $T^2 \sim a^3$.

> **Question 1.3.** Comet Halley moves in an elliptical orbit around the sun. If its distances from the sun at perihelion and aphelion are 8.80×10^7km and 5.25×10^9km respectively, find the period.

Solution 1.3.

$G = 6.674 \times 10^{-11}\ \mathrm{N m^2 kg^{-2}}$
$M_\odot = 2 \times 10^{30}\ \mathrm{kg}$

$$a = \frac{8.80\times10^7 + 5.25\times10^9}{2} = 2.669\times10^9\ \mathrm{km} = 2.669\times10^{12}\ \mathrm{m}$$

$$T = \frac{2\pi}{\sqrt{GM_\odot}} \cdot a^{\frac{3}{2}} = 2.37\times10^9\ \mathrm{s} = \boxed{75.2\ \mathrm{y}}$$

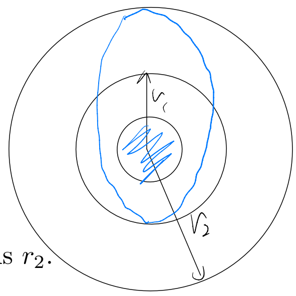
Question 1.4 (PEP 2022 Phase 1 Test 1). One of the brightest comets of the 20th century was Comet Hyakutake, which passed close to the sun in early 1996. The orbital period of this comet is estimated to be about 30,000 years.

(a) Find the semi-major axis of this comet's orbit. Express your answer in term of the average Earth-Sun distance AU. You can assume the Earth is in a circular orbit around the Sun.

(b) It is given that the eccentricity of the Comet Hyakutake is $e \approx 0.9999$. Estimate its closest distance to the Sun. Express your answer in term of the average Earth-Sun distance AU.

Solution 1.4.

1.3 Hohmann Transfer



Transfer a spacecraft across two orbits of a star.

1. Start with the spacecraft in a low circular orbit with radius r_1.
2. Then accelerate so the spacecraft is in an elliptical orbit.
3. At last accelerate again so the spacecraft is in a high circular orbit with radius r_2.

Although this method takes long time, it only requires little fuel.

Question 1.5. In rocket science, the propulsion required is measured in Delta-v (Δv), which is defined as sum of absolute of change in velocity by the propulsion engine. i.e.

$$\Delta v = |\Delta \mathbf{v}_1| + \cdots + |\Delta \mathbf{v}_n| \tag{1.14}$$

If the mass of the body which the spacecraft is orbiting around is M, find Δv in Hohmann transfer.

Solution 1.5. $m\frac{v^2}{r} = \frac{GMm}{r^2} \Rightarrow v = \sqrt{\frac{GM}{r}}$ $\quad v_1 = \sqrt{\frac{GM}{r_1}}$

$$-\frac{GMm}{r_1+r_2} = -\frac{GMm}{r_1} + \frac{1}{2}mv_i^2$$

$$v_i^2 = 2GM\left(\frac{1}{r_1} - \frac{1}{r_1+r_2}\right) \quad v_i = \sqrt{2GM\frac{r_2}{r_1(r_1+r_2)}}$$

$$\Delta v_1 = |v_i - v_1| = \sqrt{GM}\left(\sqrt{\frac{2r_2}{r_1(r_1+r_2)}} - \sqrt{\frac{1}{r_1}}\right) = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)$$

1.4 Binary Star

Previously, we assume that one mass is much larger than the other i.e. $M \gg m$. What if this assumption breaks down? Then we have *binary star*, which is a two-body system. We assume that the center of mass is at rest, so that we can reduce the problem into one-body.

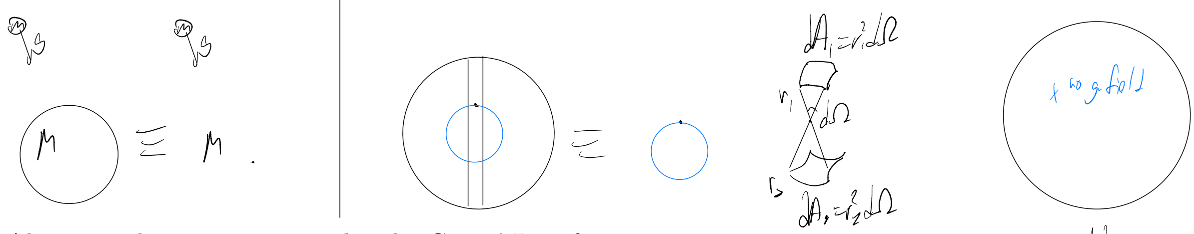
$\vec{g} = -\int \frac{G\rho(r)\,dV}{r^2}\hat{e}_r$

1.5 Spherical Mass Distribution

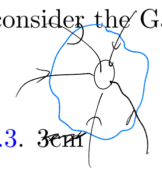
In this section, we assume the object is isotropic i.e. density only depends on the radius $\rho(r)$, so the object is made of spherical shells. We get two interesting results:

- Gravitational effect outside the object is same as if the mass is concentrated at the center.
- Gravitational effect inside the spherical object is same as if the mass of the interior is concentrated at the center.

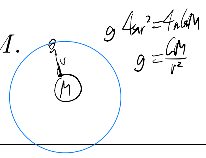
To show the first result, we can do the integration on a mass shell. Then we extend the result to even solid object. To show the second result i.e. mass at position further than the point do not contribute to the gravity, we can consider a small solid angle $d\Omega$ and gravity exerted by two pieces of masses.

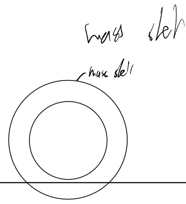


Alternatively, one may consider the Gauss' Law for gravity



$$\int \mathbf{g} \cdot d\mathbf{A} = -4\pi GM. \tag{1.15}$$





It is equivalent to Eq. 1.3. ~~3cm~~

Question 1.6. Assume the Earth is a uniform sphere with mass M and radius R. Now we penetrate a straight line across the Earth and let an object fall into it. Describe the motion of the object.

Solution 1.6.

Question 1.7. There is a uniform sphere of mass M and radius R. Now we carve out a spherical cavity in the uniform sphere, so the cavity has radius $R/2$, and it touches the surface and center of the original sphere. Find gravitational field inside the cavity.

Solution 1.7.

2 Simple Harmonic Motion

2.1 Dynamics

A system is said to be performing *simple harmonic motion* (SHM) if a quantity q is governed by a differential equation

$$\ddot{q} + \omega^2 q = 0, \tag{2.1}$$

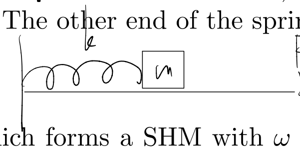
$f=\frac{\omega}{2\pi}$ $T=\frac{2\pi}{\omega}$

where $\omega \in \mathbb{R}^+$ is the angular frequency. The quanitity q is called the generalized coordinates. Using trial solution, the equation has the general solution

$$q(t) = A\cos(\omega t + \varphi) = \mathrm{Re}(Qe^{i\omega t}), Q = Ae^{i\varphi}. \tag{2.2}$$

Note. *SHM, or sinusoidal waves in general, are just projection of uniform circular motion onto an axis.*

Example 2.1. On a smooth, horizontal table, a block of mass m is tied to a light spring with constant k. The other end of the spring is a fixed wall. The equation of motion is



$F=-kx$ $m\ddot{x}=-kx$

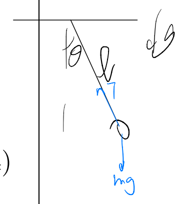
$F=m\ddot{x}$

$$m\ddot{x} + kx = 0, \tag{2.3}$$

$\ddot{x}+\frac{k}{m}x=0$

which forms a SHM with $\omega = \sqrt{k/m}$. This is a common example which we will frequently use in this section.

Example 2.2. A bead is tied by a string of length l. The other side of the string ties to a fix point. The equation of motion is



$$l\ddot{\theta} + g\sin\theta = 0.$$

$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \dots$

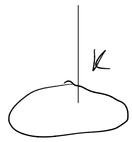
At small angle limit $|\theta| \ll 1$, the approximation $\sin\theta \approx \theta$ is valid. Hence, we arrive at

$$l\ddot{\theta} + g\theta = 0. \tag{2.4}$$

which is a SHM with $\omega = \sqrt{g/l}$.

Example 2.3. We have a rigid body rotating around an elastic wire with torsion coefficient κ. The restoring torque is $\tau = -\kappa\theta$. Hence, the equation of motion is

$$I\ddot{\theta} + \kappa\theta = 0, \tag{2.5}$$



which is a SHM with $\omega = \sqrt{\kappa/I}$. $\tau = I\ddot{\theta}$

- Frequency $f = \frac{\omega}{2\pi}$
- Period $T = \frac{1}{f} = \frac{2\pi}{\omega}$

Consider the SHM

$$x(t) = A\cos(\omega t + \varphi),$$

- velocity

$$\dot{x}(t) = -A\omega\sin(\omega t + \varphi) \tag{2.6}$$

- acceleration

$$\ddot{x}(t) = -A\omega^2\cos(\omega t + \varphi) \tag{2.7}$$

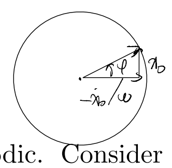
How to find the amplitude A and phase angle φ from initial conditions $x_0 := x(t = 0), \dot{x}_0 := \dot{x}(t = 0)$? Consider the identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{x}{A}\right)^2 + \left(\frac{\dot{x}}{A\omega}\right)^2 = 1$$

i.e.

$$A = \sqrt{x_0^2 + (\dot{x}_0/\omega)^2}. \tag{2.8}$$

Dividing the two expressions, we have

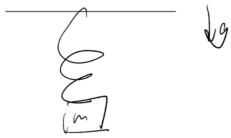


$$\varphi = \operatorname{atan}\frac{-\dot{x}_0/\omega}{x_0} \tag{2.9}$$

In a SHM, mechanical energy is conserved as the motion is periodic. Consider example 2.1, while the kinetic energy is $K = \frac{1}{2}m\dot{x}^2$, the potential energy is $U = \frac{1}{2}kx^2$. Hence, the total energy is

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \varphi) + \frac{1}{2}kA^2\cos^2(\omega t + \varphi) = \frac{1}{2}kA^2 \tag{2.10}$$

2.2 SHM systems



For more complicated systems, how can we find ω if they perform SHM? Consider an object hanging on a spring, the equation of motion becomes

$$m\ddot{y} + ky + mg = 0. \tag{2.11}$$

This can be resolved by shifting the coordinate $y' = y - \frac{mg}{k}$. Since the shift is constant, $\ddot{y'} = \ddot{y}$. However, we have

$$m\ddot{y'} + ky' = 0, \tag{2.12}$$

which is still a SHM.

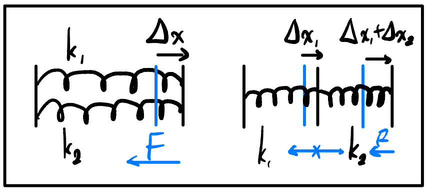
The restoration force by a spring is $F = -kx$. For a system consisting of more springs, we can find the equivalent spring constant by considering

extension = length - natural length

- forces at where springs are attached to walls or each other, and
- extension of different parts.

$F = kx$

(Figure needed)



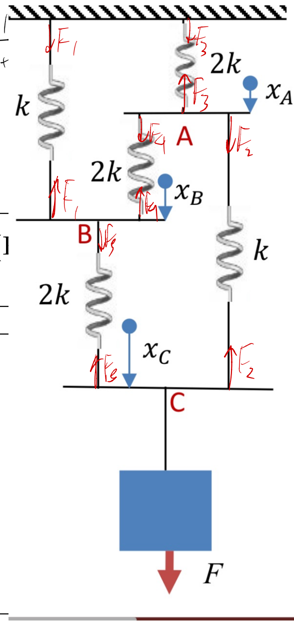
	k_{eff}	F	x
Series	$(k_1^{-1} + k_2^{-1})^{-1}$	same	Σ
Parallel	$k_1 + k_2$	Σ	same

$F = kx_1 = k_2x_2$

$k_{eff} = \frac{F}{x_1 + x_2} = \frac{1}{\frac{1}{k_1} + \cdots}$

$F_1 = k_1x \quad F_2 = k_2x$

$k_{eff} = \frac{F_1 + F_2}{x} = k_1 + k_2$



Question 2.1 (HKPhO 2020). Find the effective spring constant of the following system. (Figure needed)

$k_{eff} = \boxed{\frac{10}{3}k}$

Solution 2.1. $x = x_A + x_B + x_C$

$F_1 = k(x_A + x_B)$
$F_2 = k(x_B + x_C)$
$F_3 = 2k\, x_A$
$F_4 = 2k\, x_B$
$F_5 = 2k\, x_C$

$F = F_5 + F_2 = F_1 + F_3$
$F_3 = F_4 + F_2$
$F_4 + F_1 = F_5$

$2kx_C + k(x_B + x_C) = k(x_A + x_B) + 2kx_A$

$x_A = x_C$

$2kx_A = 2kx_B + k(x_B + x_C)$

$x_A = 3x_B$

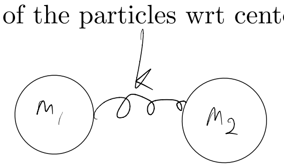
$2x_B + x_A + x_B = 2x_C$

$F = 3kx_A + x_B$
$= \frac{10}{3}kx_A$
$x = \frac{?}{3}x_A$

Now we consider motion involving many bodies. Two particles with masses m_1, m_2 respectively are tied by a spring with constant k. We would like to find the subsequent motion. For simplicity, assume there are no external force.

Note that the restoration force is an internal force in this case, so we can split the motion into

- pure translation of the center of mass, which is not affected by the spring
- oscillation of the particles wrt center of mass



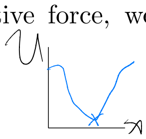
$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$

Exercise 2.1. Consider the scenario where a mass M is tied to a spring with constant k, which the other side is attached to a wall. The mass is allowed to move freely on the horizontal, smooth table. Now we also consider the mass of the spring m. Find the angular frequency ω in this case.

2.3 Oscillation in general

$F=-kx$

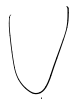
Find period of motion of a conservative force of a particle.
For a conservative force, we have a potential landscape $U(x)$. The particle with mass m is at the equilibrium if



$$\frac{dU}{dx} = 0 \qquad x=x_0 \text{ is stationary pt} \tag{2.13}$$

since $F = -\frac{dU}{dx}$. Let us call this point x_0. The equilibrium is stable iff $U(x)$ attains a local minimum at x_0. One sufficient condition is that

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0} > 0. \qquad U(x)=\beta x^4 \tag{2.14}$$



Although it is possible that x_0 is a local minimum with $\left.\frac{d^2U}{dx^2}\right|_{x=x_0} = 0$ e.g. quartic potential, we only consider cases where the second order derivative does not vanish. Expanding $U(x)$ around x_0 gives

$$U(x) = \underbrace{U(x_0)}_{\text{P.E. at eqm}} + \underbrace{\frac{1}{2}U''(x)(x-x_0)^2}_{\text{parabolic approx.}} + \underbrace{o(x-x_0)^2}_{\text{higher ord. term}} \tag{2.15}$$

The elastic potential energy of a spring is $\frac{1}{2}kx^2$. Hence, the effective spring constant in this case is

$$k_{\text{eff}} = \left.\frac{d^2U}{dx^2}\right|_{x=x_0}. \qquad \omega=\sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} \tag{2.16}$$

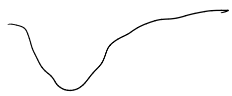
$K=\frac{1}{2}m_{\text{eff}}\dot{x}^2$

Question 2.2. The Morse potential is a model to explain interaction of a diatomic molecule. The Morse potential is of the form

$$U(r) = D_e(1-e^{-a(r-r_e)})^2 \qquad r>0 \tag{2.17}$$

(a) Find the equilibrium position.

(b) If the two atoms have same mass m, find the period of oscillation.



a $\frac{dU}{dr}=0 \Rightarrow \frac{d}{dr}(1-e^{-a(r-r_e)})=0$ or $1-e^{-a(r-r_e)}=0$

$a e^{-a(r-r_e)}=0$ $\qquad r=r_e$

$U(r=r_e)=0$

b $\frac{dU}{dr}=2D_e(1-e^{-a(r-r_e)})a e^{-a(r-r_e)}$

$$\frac{d^2U}{dr^2}=2aD_e\left((a e^{-a(r-r_e)})e^{-a(r-r_e)}+(1-e^{-a(r-r_e)})(-a e^{-a(r-r_e)})\right)$$
$$=2a^2D_e\left(2e^{-2a(r-r_e)}-e^{-a(r-r_e)}\right)$$
$$=2a^2D_e \text{ at } r=r_e$$

Solution 2.2.

$\mu=\frac{1}{\frac{1}{m}+\frac{1}{m}}=\frac{m}{2}$ $\qquad T=\frac{2\pi}{\omega}=\boxed{\frac{2\pi}{\sqrt{2a^2D_e/\frac{m}{2}}}}$

3 Waves

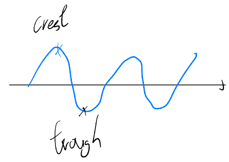
3.1 Introduction

Classical interpretation: transfer of information without material transfer.

Two types of waves

Why SHM
- (a) linear restoration
- Quantum $E_n=(n+\frac{1}{2})\hbar\omega$
  $\hat{a}, \hat{a}^\dagger$
  second quantization

- Transverse: oscillation is perpendicular to propagation
- Longitudinal: oscillation is parallel to propagation



Most simple example: sinusoidal wave

amplitude wave # angular frequency

$$y(x,t) = A\cos(kx - \omega t) \tag{3.1}$$

position

This wave is traversing to $+x$ direction since

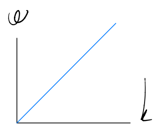
$$y(x,t) = A\cos(kx - \omega t) = A\cos\Big(\omega(t - \frac{x}{v})\Big).$$

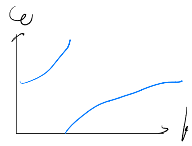
However, this wave

$$y(x,t) = A\cos(kx + \omega t) \tag{3.2}$$

represents a wave traversing to $-x$ direction.

- Wavelength λ: after how much length does the wave repeat
- Wavenumber $k[\mathrm{m}^{-1}] = 2\pi/\lambda$
- Period T[s] after how much time does the wave repeat
- Angular frequency $\omega[\mathrm{s}^{-1}] = 2\pi/T$
- Frequency $f[\mathrm{Hz}] = \omega/2\pi$
- Phase velocity $v = \omega/k = f\lambda$: how fast the peaks move





Note. *The relation between ω and k is called the dispersion relation. In some cases, it may not be linear, that non-linear dispersion occurs. Waves with different wavelength (or wavenumber) travels at different speed.*

Note that $y(x,t)$ depends on both the position x and time t.
If we fix x i.e. record the displacement at a point, we will get a $y - t$ graph. The derivative of the graph is $\frac{\partial y}{\partial t}$ then.
If we fix t i.e. take a picture of the graph at some time, we will get a $y - x$ graph. The derivative of the graph is $\frac{\partial y}{\partial x}$ then.

Question 3.1. $A = 0.2$m $\omega = 5\,\mathrm{s}^{-1}$ $f = \frac{\omega}{2\pi} = .796$Hz $\lambda = \frac{2\pi}{k} = 6.28$m $v = \frac{\omega}{k} = \frac{5}{2} = 2.5\,\mathrm{ms}^{-1}$

1. Find the amplitude, frequency, wavelength, and speed of propagation of the wave described by the equation

   $5t - 2x$

   $$y = 0.2\cos((\cancel{5t2x})). \quad = 0.2\cos(2x - 5t)$$

   Here, the units of length and time are taken to be meter and second, respectively. $k = \frac{2\pi f}{v}$

2. When a sinusoidal wave of amplitude 0.1 m and frequency 2 Hz travels at a speed of 2 m/s in the x direction, derive the expression for the displacement y at position x at time t. Here, we assume that the displacement at the origin $x = 0$ at time $t = 0$ is zero, i.e., $y(x,t) = 0$.

$$y(x,t) = A\cos(k(x - vt) + \varphi)$$
$$= 0.1\sin(2\pi(x - 2t))$$

3.2 Wave Equation

However, in real life, we see waves with different shapes. When can we conclude if the motion is a wave? Wave motion is governed by the *wave equation*

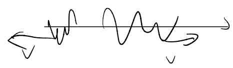
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \tag{3.3}$$

Partial D.E

> **Exercise 3.1.** Verify that this partial differential equation is linear i.e. if $y_1(x,t), y_2(x,t)$ are solution, then $c_1y_1 + c_2y_2$ is also a solution, where $c_1, c_2 \in \mathbb{R}$.

Let us skip the steps since we are not having a PDE lecture. The solution is $y(x,t) = X(x)\,T(t)$

$$y(x,t) = f(x - vt) + g(x + vt), \tag{3.4}$$



i.e. superposition of two waves traversing at opposite directions. Indeed they are the sums of the sinusoidal waves, given by the Fourier transform.

> **Exercise 3.2.** Show that this form of solution satisfies the wave equation.

> **Example 3.1** (Wave equation on a string)**.** We would like to show that wave motion occurs on a string, and find the wave speed. Let T be the equilibrium tension of the string, and μ be the mass per unit length of the string. Using dimensional analysis, we have $v \sim \sqrt{T/\mu}$. However, we need further derivation to find the constant.
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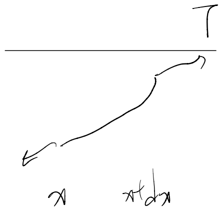
> 
>
>
> $T\frac{dy}{dx}(x+dx) - T\frac{dy}{dx}(x) = \mu\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \cdot \frac{\partial^2 y}{\partial t^2}$
>
> ard. ②
>
> $T\frac{\partial^2 y}{\partial x^2}\cdot dx = \mu\, dx \frac{\partial^2 y}{\partial t^2}$
>
> $\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$
>
> $v = \sqrt{T/\mu}$

3.3 Energy of Waves

Power delivered by a string wave is

$$\begin{aligned} P(x,t) &= F_y\dot{y} = -T\partial_x y \partial_t y \\ &= Tk\omega A^2 \sin^2(kx - \omega t) \end{aligned}$$

i.e.

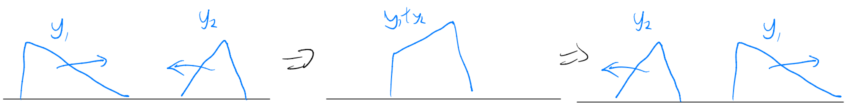
$$P(x,t) = \sqrt{\mu F}\omega^2 A^2 \sin^2(kx - \omega t) \tag{3.5}$$

- $P_{\max} = \sqrt{\mu F}\omega^2 A^2$
- $\langle P \rangle = \frac{1}{2}P_{\max}$

at $y=0$ P is largest

y is crest/trough, P is smallest

It can be shown that half of the energy is delivered in form of kinetic energy, and the other half is in form of elastic potential energy.

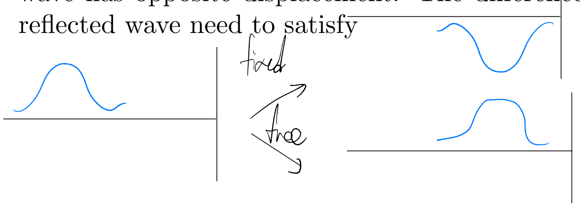


3.4 Superposition Principle

As a direct result of linearity, we can add up many waves $y = y_1 + y_2 + \cdots + y_n$. Although they interfere the overall waveform, they do not interfere each individual's waveform.

If the waves have the same phase and add-up each other, we call this case constructive interference. On the other hand, if the waves have opposite phase and cancel each other, we call this case destructive interference.

Consider now we tie a string to a wall and start a wave. If the end is completely free, then the reflected wave has the same direction of displacement. However, if the end is completely fixed, then the reflected wave has opposite displacement. The difference is due to difference in boundary condition, which the reflected wave need to satisfy



Mathematically, let $x = 0$ be where the end is located at. The fixed end requires $y(0, t) = 0$. Hence, the reflected wave has opposite displacement. Meanwhile, the free end requires $\partial_x y(0, t) = 0$, which can be satisfied by having a reflected wave with same direction of displacement, but opposite direction of propagation.
But if the end is neither completely free or completely fixed e.g. tied to a finite mass m, this case would be more complicated.

$\overrightarrow{A\cos(kx-\omega t)}$ $\overleftarrow{A_r\cos(kx+\omega t+\varphi)}$

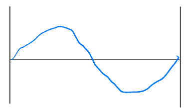
3.5 Standing Waves

Consider two waves travelling at opposite direction, the waveform is

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) + A\cos(kx + \omega t) \\ &= 2A\cos kx \cos \omega t \end{aligned} \tag{3.6}$$

for open boundary conditon, and

$$\begin{aligned} y(x,t) &= A\cos(kx - \omega t) - A\cos(kx + \omega t) \\ &= 2A \underbrace{\sin kx}\, \underbrace{\sin \omega t} \end{aligned} \tag{3.7}$$


$x=0$ $x=L$

for closed boundary condition.
Normal modes are standing waves within some length. For both ends being closed, this occurs at

$$L = n\frac{\lambda}{2}, \lambda_n = \frac{2L}{n}, n \in \mathbb{N} \tag{3.8}$$

$\sin kL = 0$
$kL = n\pi, n \in \mathbb{N}$
$\frac{2\pi}{\lambda} L = n\pi$
$L = \frac{n\lambda}{2}$

There are different ways to call these special frequencies

n	
1	fundamental frequency, first harmonic
2	first overtone, second harmonic
3	second harmonic, third overtone

However, if one end is open, then the boundary condition is $\partial_x y(x = L) = 0$ instead.

Question 3.2. A 5.00m, 0.732kg wire is used to support two uniform 235N posts of equal length . Assume that the wire is essentially horizontal and that the speed of sound is $344\,\mathrm{m\,s^{-1}}$. A strong wind is blowing, causing the wire to vibrate in its 5th overtone. What are the frequency and wavelength of the sound this wire produces?

Solution 3.1.

Exercise 3.3. Estimate the power while a guitar string is vibrating.

3.6 Beats

We have two waves with same amplitude but slightly different frequencies $f_1 \approx f_2$. At $x = 0$, the waves are

$$y_1 = A\cos(2\pi f_1 t + \varphi_1)$$
$$y_2 = A\cos(2\pi f_2 t + \varphi_2)$$

which adds up to

$$y = y_1 + y + 2 = 2A\cos\left(2\pi\frac{f_1+f_2}{2}t + \frac{\varphi_1+\varphi_2}{2}\right)\cos\left(2\pi\frac{f_1-f_2}{2}t + \frac{\varphi_1-\varphi_2}{2}\right).$$

- The first term has the similar frequency as f_1, f_2, representing the pitch of the sound
- The latter term has a much slower frequency, which is how frequent the amplitude of the sound change.

The *beat frequency* is $f_{\text{beat}} = |f_1 - f_2|$.

3.7 Doppler Effect

V_s

The frequency which a receiver perceive may perceive if either the receiver or the source is moving. Assume the wave is a mechanical wave (e.g. sound). The frequency perceived by the listener is

$$f_L = \frac{c + v_L}{c + v_S} f_S, \tag{3.9}$$

$\Delta t = \frac{1}{f_s}$ $\quad \frac{1}{f_L} = \frac{1}{f_s} + \frac{V_s \frac{1}{f_s}}{c}$ $\quad f_L = \frac{c}{c+v_s} f_s$

$\frac{1}{f_L} = \frac{1}{f_s}\frac{c}{c+v_L}$ $\quad f_L = \frac{c+v_L}{c} f_s$

where

- v_L is the velocity of the listener parallel to the separation (between listener and speaker)
- v_S is the velocity of the speaker parallel to the separation
- c is the speed of wave in the medium
- f_S is the frequency emitted by the source

Note that the velocities are the one relative to the medium. For example, if there is wind, then we need to think in wind's frame.

speed of sound $c = 340\,\text{m/s}$

Question 3.3. Two train whistles, A and B, each have a frequency of 392Hz. A is stationary and B is moving toward the right (away from A) at a speed of $35.0\,\text{m}\,\text{s}^{-1}$. A listener is between the two whistles and is moving toward the right with a speed of $15.0\,\text{m}\,\text{s}^{-1}$. No wind is blowing.

(a) What is the frequency from A as heard by the listener?

(b) What is the frequency from B as heard by the listener?

(c) What is the beat frequency detected by the listener?

A   B

Solution 3.2.

a $\quad f_A = \frac{c-v}{c} f = 374.7\,\text{Hz}$

b $\quad f_B = \frac{c+v}{c+v_B} f = 371.1\,\text{Hz}$

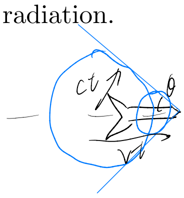
$f_{\text{beat}} = |f_A - f_B| = 3.6\,\text{Hz}$

For light it is different. The Doppler's effect only depends on the relative velocity of the listener and source, since the speed of light is constant in any reference frames.

$$f_L = \sqrt{\frac{c-v}{c+v}} f_S \tag{3.10}$$

What if the source moves faster than the medium?
For sound, an objects travelling at supersonic speed generates shock waves. This phenomenon is called the sonic boom. The waves emitted at different time undergo constructive interference at a cone.
For light, objects can travel faster than light in a medium, where light is slowed down. This is called the Cherenkov radiation.

$\theta = \sin^{-1}\left(\frac{c}{v}\right)$



3.8 Spherical Waves

Waves in 3 dimensions.
We can generalize the wave equation into 3 dimensions

$$\boldsymbol{\nabla}^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{3.11}$$

Here we use ψ instead of y to denote the wave. We assmue that the point source generates wave in a homogeneous (same in all position) and isotropic (same in all direction) medium. Hence, we expect the wave only depends on the radius r. In spherical coordinates, the Laplacian is

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial}{\partial \phi}\right) \tag{3.12}$$

The assumption leads to the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\psi\right) = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{3.13}$$

Using the substitution $u = r\psi$, we reduce to the 1 dimensional wave equation

$$\frac{\partial^2 u}{\partial r^2}\psi = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}. \tag{3.14}$$

Hence, the solution is

$$\psi(r) = \frac{A}{r}\cos(kr - \omega t). \tag{3.15}$$

The inverse relation can be explained as follows. As the wave travels to farther r, the area covered $\sim r^2$. Conservation of energy requires intensity $I \sim r^{-2}$. As $I \sim \psi^2, \psi \sim 1/r$.

4 Fluid Mechanics

4.1 Pressure

Density is the mass over a volume. For a homogeneous material, the *density* of the material is

$$\rho = m/V. \tag{4.1}$$

mass volume

If the material is not homogeneous, we can still define the density at a position

$$\rho(\mathbf{r}) = \left.\frac{dm}{dV}\right|_{\mathbf{r}}. \tag{4.2}$$

Hence, the mass of an object is

$$m = \int \rho dV. \tag{4.3}$$

Pressure is defined as normal force over unit area

$$p = \frac{dF_\perp}{dA}, \tag{4.4}$$

LMT^{-2}

L^2

which is a scalar quantity. The unit of p is Pascal [Pa]. If there is only normal force (no tangent force), it can be found by

$L^{-1}MT^{-2}$

$$\mathbf{F} = \int p d\mathbf{A}. \tag{4.5}$$

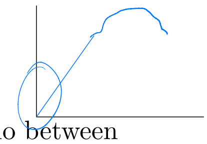
For example, $F_x = \int p dA_x$.

Total pressure = Atmosphere pressure + Gauge pressure

extra pressure

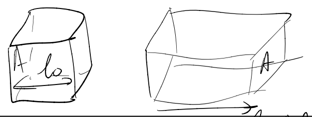
4.2 Soft Body

elastic modulus $= \frac{\text{stress}}{\text{strain}}$



In contrast to a rigid body, a soft body can deform and restore. An elastic modulus is the ratio between stress and strain of an object. We assume that strain is proportional to stress, so elastic modulus is a constant within the linear region.

Young's modulus is used for elongation and compression of an object.

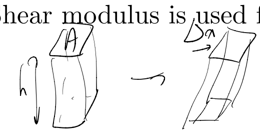


$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} \tag{4.6}$$

unit [Pa]

> **Exercise 4.1.** If a long rod has length l_0, cross-sectional area A and Young's modulus Y, find its spring constant k.

Shear modulus is used for shearing i.e. displacement perpendicular to the orientation of the object.



$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F_\parallel/A}{\Delta x/h} \tag{4.7}$$

Bulk modulus describes how object respond to change in pressure.

fluid

$$B = \frac{\text{bulk stress}}{\text{bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \tag{4.8}$$

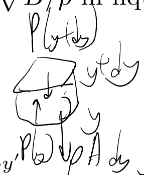
We know how fast sound transmit in a solid or liquid.

oscillation of particles

$$v = \sqrt{Y/\rho} \text{ in solid} \tag{4.9}$$

$$v = \sqrt{B/\rho} \text{ in liquid} \tag{4.10}$$

4.3 Hydrostatic Pressure



$A(P(y+dy) - P(y)) = -\rho A dy g$

The pressure in a fluid is

$$P(y) = P(0) - g\int_0^{y'} \rho(y')dy' \text{ in general;} \tag{4.11}$$

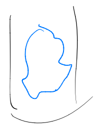
$\frac{dP}{dy} = -\rho g$

$$= P(0) - \rho g y \text{ if density } \rho \text{ is a constant of} y. \tag{4.12}$$

pressure at $y=0$ e.g. surface

Archimedes Principle: when a body is at least partly immersed into a fluid, the fluid exerts an upward buoyancy force onto the body, which the magnitude is the weight of the fluid displaced by the body. Mathematically, the buoyancy force is

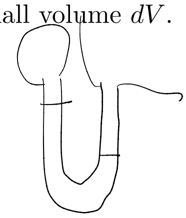
$$F = \int \rho g dV \text{ in general;} \tag{4.13}$$



$$= \rho g V \text{ if density } \rho \text{ is a constant.} \tag{4.14}$$

This can be shown by considering the overall pressure acting on a small volume dV.

One application is mercury pressure gauge.



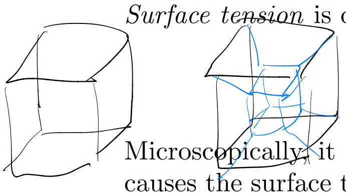
Pascal Pa

mmHg

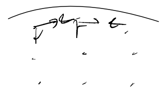
Exercise 4.2 (USAPhO 2013 A4). https://www.aapt.org/physicsteam/2014/upload/E3-1-7.pdf

4.4 Surface Tension

Surface tension is defined as the work done to change a unit of area



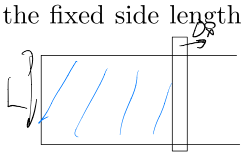
$$\gamma = \frac{dW}{dA} \quad \frac{\Delta W}{\Delta A} \tag{4.15}$$



Microscopically, it is due to the attractive force between molecules at the edge of the liquid. This force causes the surface tends to have a smaller surface area. Mathematically, the surface formed is a minimal surface as it attempts to attain a smaller surface area.

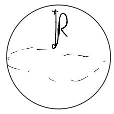
Example 4.1 (Rectangular Thin Film). Consider a case where we have a rectangular frame. Three sides of the frame is fixed, while the remaining side is movable. If the fixed side length is L, then the force required to hold the movable side is

$$F = \frac{\Delta W}{\Delta x} = \frac{\gamma \Delta A}{\Delta x} = \frac{\gamma L \Delta x}{\Delta x} = \gamma L. \tag{4.16}$$



Example 4.2 (Sphere). We would like to find the pressure exerted by a spherical soap film P. Under equilibrium,

$$P_{\text{out}} = P_{\text{in}} + P \tag{4.17}$$



First, we may consider force along the equator. Equilibrium condition requires

$$P\pi R^2 = \gamma 2\pi R \Rightarrow P = \frac{2\gamma}{R} \tag{4.18}$$

$4\pi R^2$

$\Delta A = 8\pi R \Delta R$

$\Delta W = \gamma \Delta A = 8\gamma\pi R \Delta R$

$P = \frac{F}{A} = \frac{\Delta W/\Delta R}{A} = \frac{8\gamma\pi R}{4\pi R^2} = \frac{2\gamma}{R}$

Alternatively, we may consider small change in radius of the bubble.

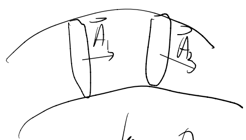
The *Young-Laplace equation* suggests pressure difference between two fluids e.g. water and air.

$$\Delta p = -\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{4.19}$$

where R_1, R_2 are radius of curvature in perpendicular directions. For a thin film, we need to multiply the result by 2 since we treat the system as three fluids.

4.5 Fluid Flow

- Laminar flow: smooth, steady flow of fluid
- Turbulent flow: rough, chaotic flow



$\rho \vec{A}_1 \cdot \vec{v}_1 - \rho \vec{A}_2 \cdot \vec{v}_2 = 0$

For an incompressible fluid, the *continuity equation* is

eqn. $\frac{dm}{dt} = 0$

$$\mathbf{A}_1 \cdot \mathbf{v}_1 = \mathbf{A}_2 \cdot \mathbf{v}_2, \tag{4.20}$$

which is a direct consequence of the conservation of mass. If the fluid is compressible, we also need to consider the density

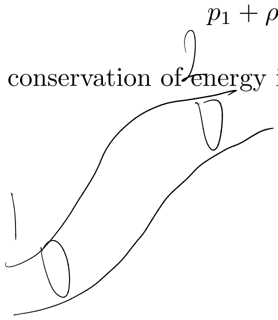
$$\rho_1 \mathbf{A}_1 \cdot \mathbf{v}_1 = \rho_2 \mathbf{A}_2 \cdot \cancel{\mathbf{A}_2}\ \mathbf{v}_2. \tag{4.21}$$

Another equation we frequently use is the *Bernoulli equation*. For an incompressible fluid, we have

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2. \tag{4.22}$$

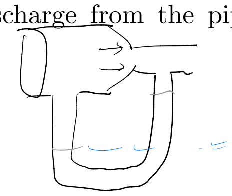
It is a result of conservation of energy i.e. work done is turned into kinetic energy and potential energy.

ΔV



Question 4.1. The horizontal pipe has a cross-sectional area of 40.0cm^2 at the wider portions and 10.0cm^2 at the constriction. Water is flowing in the pipe, and the discharge from the pipe ~~is~~ at $6.00 \times 10^{-3}\text{m}^3\,\text{s}^{-1}$. The density of water is $1000\text{kg}\,\text{m}^{-3}$. Find the

$\rho_{Hg} = 13600\,\text{kg}\,\text{m}^{-3}$



1. the flow speeds at the wide and the narrow portions;
2. the pressure difference between these portions;
3. the difference in height between the mercury columns in the U-shaped tube.

Solution 4.1.

Two parts: 1 wider, 1 narrower

$A_1 v_1 = A_2 v_2 = V$

$v_1 = 1.5\,\text{ms}^{-1}$

$v_2 = 6\,\text{ms}^{-1}$

$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$

$\Delta p = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2$

$= 1.69 \times 10^4\,\text{Pa}$

$\Delta p + \rho g h = \rho_{Hg} g h$

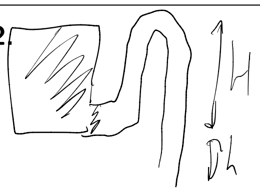
$h = 0.14\,\text{m}$

Question 4.2. ... A siphon, is a convenient device for removing liquids from containers. To establish the flow, the tube must be initially filled with fluid. Let the fluid have density ρ and let the atmospheric pressure be p. Assume that the cross-sectional area of the tube is the same at all points along it.

(a) If the lower end of the siphon is at a distance h below the surface of the liquid in the container, what is the speed of the fluid as it flows out the lower end of the siphon? (Assume that the container has a very large diameter, and ignore any effects of viscosity.)

(b) A curious feature of a siphon is that the fluid initially flows “uphill.” What is the greatest height H that the high point of the tube can have if flow is still to occur?

Solution 4.2.



$p + \rho g \cdot 0 + \frac{1}{2}\rho \cdot 0^2 = p + \rho g(-h) + \frac{1}{2}\rho v^2$

$v = \sqrt{2gh}$

$\rho g(h+H) \le p$ else pressure < 0 at the top

$H \le \frac{p}{\rho g} - h$