Calculus 2

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1 Review of Last Tutorial

1.1 Limit

Definition 1.1 ((non-rigorous) Sequence Limit). Given a real sequence $(a_n) = (a_1, a_2, \cdots) \subset \mathbb{R}$, it *converges* to the limit $L \in \mathbb{R}$ if the terms get closer and closer to L i.e.

$$\lim_{n\to\infty} a_n = L, a_n \to L.$$

If we cannot find such L, the sequence *diverges*.

Definition 1.2 ((non-rigorous) Function Limit). Given a real function $f(x)$, it *converges* to the limit $L \in \mathbb{R}$ when x approaches a, if the terms get closer and closer to L when x gets closer and closer, but is not equal to a. i.e.

$$\lim_{x\to a} f(x) = L, f(x) \to L.$$

If we cannot find such L, the sequence *diverges*.

Definition 1.3 (Continuity). $f(x)$ is continuous at $x = a$ if

$$\lim_{x\to a} f(x) = f(a)$$

Proposition 1.1 (Limit Rules).

1. Arithmetic rules: assume $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M$
   - $\lim_{x\to a} f(x) \pm g(x) = L \pm M$
   - $\lim_{x\to a} f(x)g(x) = LM$
   - $M \neq 0, \lim_{x\to a} f(x)/g(x) = L/M$
   - $\lim_{x\to a} cf(x) = cL$
2. Elementary functions are continuous in their domain
3. Squeeze Theorem: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, and $f(x) \leq h(x) \leq g(x)$ for $x \approx a$, then $\lim_{x\to a} h(x) = L$
4. Composition rule: If $g(y)$ is continuous, then $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))$

Definition 1.4 (Elementary Functions). Elementary functions are functions obtained from finite sums, products, and compositions of rational, trigonometric, hyperbolic and exponential functions as well as their inverses

The elementary functions

polynomial	rational	trigonometric	hyperbolic	exponential
$a_nx^n + \cdots + a_0$	$\frac{a_nx^n+\cdots+a_0}{b_mx^m+\cdots+b_0}$	$\sin x, \cos x, \tan x$	$\sinh x, \cosh x, \tanh x$	e^x

Proposition 1.2 (Standard Limits).

$$\lim_{x\to+\infty} \frac{1}{x^p} = 0, p > 0$$
$$\lim_{x\to+\infty} x^pa^x = 0, 0 < a < 1, p \geq 0$$
$$\lim_{x\to 0^+} x^x = 1$$
$$\lim_{x\to 0}(1+x)^{1/x} = e$$
$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$
$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

Note. $\lim_{n\to\infty}$ *limits can be evaluated from* $\lim_{x\to 0}$ *limits and considering* $x = \frac{1}{n}$

1.2 Differentiation

1.2.1 Evaluating Derivatives

Definition 1.5 (Derivative). $f(x)$ is differentiable at $x = x_0 \iff$ the derivative

$$f'(x_0) = \frac{df}{dx}|_{x=x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

exists.
Higher order derivatives are also defined similarly

$$f^{(n)}(x_0) = \frac{(n-1)}{dx}|_{x=x_0}$$

Proposition 1.3 (Basic Derivatives).

$$(c)' = 0$$
$$(x^p)' = px^{p-1}$$
$$(a^x)' = a^x \ln a, a > 0$$
$$(\ln \frac{1}{x})' = \frac{1}{x}$$
$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\operatorname{atan} x)' = \frac{1}{1+x^2}$$

Proposition 1.4 (Differentiation Rules). Consider two differentiable functions f, g

- Linearity: $(cf \pm g)' = cf' \pm g'$
- Product rule: $(fg)' = f'g + g'f$
  In general, $(fg)^{(n)} = \sum_{k=0}^{n} C_k^n f^{(n-k)} g^{(k)}$
- Quotient rule: $(\frac{f}{g})' = \frac{f'g-g'f}{g^2}$

- Chain rule: $(g \circ f)'(x) = g'f((x))f'(x)$
- Inverse: $(f^{-1})'(y) = \frac{1}{f'(x)}|_{f(x)=y}$

Implicit Differentiation: given $F(x, y) = c$, differentiate by x. Treat $y = y(x)$ and use the chain rule

$$\frac{d}{dx}f(y) = \frac{dy}{dx}f'(y)$$

.

Parametric Differentiation: given $x(t)$ and $y(t)$ in parametric form,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Interpretation: the tangent of the trajectory is determined by the instantaneous velocity

1.2.2 Applications

Definition 1.6 (Extremum)**.** Consider a function $f(x)$ with domain D,

- the global maximum is $f(x^*)$ iff $\forall x \in D, f(x) \leq f(x^*)$
- a local maximum is $f(x_i)$ so $f(x_i)$ is larger than $f(x)$ for sufficiently close $x \approx x_i$

Global minimum and local minimum are defined similarly.

Proposition 1.5. The global maximum / minimum is also a local maximum / minimum.

Proposition 1.6 (First Derivative Test)**.** If $f'(a)$ exists and $x = a$ is a local extremum, then $f'(a) = 0$

Theorem 1.1 (2nd Derivative Test)**.** Suppose $f'(x_0) = 0, f''(x_0)$ exists

- $f''(x_0) > 0 \Rightarrow x_0$ is a local minimum
- $f''(x_0) < 0 \Rightarrow x_0$ is a local maximum

Theorem 1.2 (Higher Order Derivative Test)**.** If $f(x) = f(a) + c(x-a)^n + o(x-a)^n$ i.e.

$$\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)^n} = c,$$

then

- n is odd, $x = a$ is not a local max/min
- n is even, $c > 0, x = a$ is a local min
- n is even, $c < 0, x = a$ is a local max

Theorem 1.3 (Mean Value Theorem (MVT))**.** f is differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 1.4 (Comparing Functions)**.** Let $f(x), g(x)$ be continuous for $x \geq a$ and differentiable for $x > a$. If $f(a) \geq g(a)$ and $f'(x) > g'(x)$ for $x > a$, then $f(x) > g(x)$ for $x \geq a$

Definition 1.7 (Convexity). $f(x)$ is convex on an interval I if $\forall x < y \in I$

$$\forall \lambda \in [0,1], f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Concave is defined oppositely.

Proposition 1.7. Considering an interval I where f'' exists,

- f is convex if $f'' \geq 0$
- f is concave if $f'' \leq 0$

Definition 1.8 (Inflection Point). $x = a$ is an inflection point for $f(x)$ if the convexity of $f(x)$ changes near $x \approx a$

Proposition 1.8. If $x = a$ is an inflection point and $f''(x)$ is continuous near $x = a$, then $f''(a) = 0$

1.3 Practice

2 Integration

We should not discuss whether a function is integrable here because

- integrability is much more difficult to understand then differentiability
- you don't need to face "bad" functions in physics

2.1 Definite Integral

idea: we have a function $f(x)$, we want to find the signed area $A(x) = \int_a^x f(x)dx$

Definition 2.1 (Signed Area). The signed area $A(x)$ is a function that gives the signed area of the region bounded by $y = f(x)$ and $y = 0$ with the convention

- Positive: $f(x) > 0$
- Negative: $f(x) < 0$

Question 2.1.

$$\int_{-1}^{2} x dx$$

Solution 2.1. The region consists of two triangles, one in the $-$ region with base 1 and height 1, one in the $+$ region with base 2 and height 2

2.1.1 Riemann Sum

Definition 2.2 (Riemann Integral). $f(x)$ is Riemann Integrable on $[a, b]$ iff

$$\lim_{\|P\| = \max(x_i - x_{i-1}) \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

exists.

It would be difficult to understand the statement, but we can use a good result to skip that.

Proposition 2.1. Continuous and bounded functions are Riemann integrable

Idea:

1. Divide the interval $[a, b]$ into n sub-interval. For simplicity we can assume each sub-interval have equal length $\frac{b-a}{n}$.
2. For each sub-interval $[x_{i-1}, x_i]$, find a tag $x_i^* \in [x_{i-1}, x_i]$. For simplicity you can choose the left, middle or right point $x_{i-1}, \frac{x_{i-1}+x_i}{2}, x_i$.
3. The area of $f(x)$ can be approximated as the sum
$$\sum_{i=1}^{n} \frac{b-a}{n} f(x_i^*)$$
4. Take $n \to \infty$ and get
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(x_i^*)$$

Question 2.2. Evaluate the following integrals using Riemann sums

1. $\displaystyle\int_0^1 x^2 dx$
2. $\displaystyle\int_0^1 2^x dx$
3. $\displaystyle\int_0^\pi \sin x dx$

Solution 2.2.

Exercise 2.1. Evaluate
$$\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)\cdots(n+n)}}{n}$$

2.2 Fundamental Theorem of Calculus

Relating sums and inverses of minuses

Theorem 2.1 (Fundamental Theorem of Calculus).

1. If $f(x)$ is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is an anti-derivative of $f(x)$ on $[a, b]$ i.e.
$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$
2. (Newton-Leibniz Formula) If f is integrable on $[a, b]$, and $F'(x) = f(x)$, then
$$\int_a^b f(x)dx = F(b) - f(a)$$

Question 2.3 (HKALE PM 00IIQ5). Let $k \in \mathbb{N}$, evaluate

i) $\displaystyle\frac{d}{dx}\int_0^x \cos t^2 dt$

ii) $\dfrac{d}{dy}\displaystyle\int_0^y \cos t^2 dt$

Solution 2.3.

Proposition 2.2 (Properties of Definite Integral).

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\text{Linearity: } \int_a^b (cf(x)+g(x))dx = c\int_a^b f(x)dx + \int g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\text{Comparison: } a<b, f(x) \le (<) g(x) \Rightarrow \int_a^b f(x)dx \le (<) \int_a^b g(x)dx,$$

At last, a simple rule to do eliminate unnecessary calculation

Proposition 2.3 (Integration of Odd and Even Functions). Let $f(x)$ be an odd function, and $g(x)$ be an even function. We have

$$\int_{-a}^a f(x)dx = 0$$

$$\int_{-a}^a g(x)dx = 2\int_0^a g(x)dx$$

Proof. Easily follows from the property of definite integral. □

2.3 Indefinite Integral

Definition 2.3 (Indefinite Integral). The indefinite integral of $f(x)$ is

$$\int f(x)dx = F(x) + C, C \in \mathbb{R}$$

so that $F'(x) = f(x)$

Strictly speaking, the indefinite integral is an equivalence class. If you get more conditions (restrictions), you can treat C as an unkonwn and find it.

Proposition 2.4 (Linearity of Indefinite Integral).

$$\int (cf(x)+g(x))dx = c\int f(x)dx + \int g(x)dx$$

2.3.1 Integration by Substitution

Theorem 2.2 (Integration by Substitution). If f is continuous, $u = g(x)$ is differentiable and Range(g) $\subset$ Domain (f), then

$$\int f(g(x))g'(x)dx = \int f(u)du = F(g(x)) + C \int_a^b f(g(x))g'(x)dx \qquad = \int_{g(a)}^{g(b)} f(u)du$$

Proposition 2.5 (Standard Indefinite Integrals).

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C, p \neq -1 \int \frac{dx}{x} = \ln|x| + C \int a^x dx = \frac{a^x}{\ln a} + C, a > 0, a \neq 1$$
$$\int \sin x dx = -\cos x + C$$
$$\int \cos x dx = \sin x + C$$

Corollary 2.1.

$$\int f(ax+b)dx = \frac{F(ax+b)}{a} + C$$

Example 2.1.

1. $$\int \frac{x}{x^2+1}dx \underset{du=2xdx}{\overset{u=x^2+1}{=}} \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$
2. $$\int e^x \cos e^x dx \underset{du=e^x dx}{\overset{u=e^x}{=}} \int \cos u du = \sin u + C = \sin e^x + c$$
3. $$\int_0^1 e^x \cos e^x dx = \int_1^e \cos u du = \sin e - \sin 1$$
4. $$\tan x dx = \int \frac{\sin x}{\cos x}dx \underset{du=-\sin x dx}{\overset{u=\cos x}{=}} -\int \frac{du}{u} = -\ln|u| + C = \ln|\sec x| + C$$

Trigonometric Substitutions

Terms in integrand	x	dx
$a^2 - x^2$	$a\sin\theta$	$a\cos\theta d\theta$
$a^2 + x^2$	$a\tan\theta$	$a\sec^2\theta d\theta$

Example 2.2. As a common example of trigonometric substitution

$$\int_0^1 \frac{dx}{1+x^2} \underset{dx=\sec^2\theta d\theta}{\overset{x=\tan\theta}{=}} \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{1+\tan^2\theta} = \int_0^{\pi/4} d\theta = \theta|_0^{\pi/4} = \frac{\pi}{4}$$

At one of the step, we use $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$. But using the same logic,

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^{9\pi/4} \frac{\sec^2\theta d\theta}{1+\tan^2\theta} = \int_0^{9\pi/4} d\theta = \theta|_0^{9\pi/4} = \frac{9\pi}{4}$$

which is wrong since $\tan\theta$ is not differentiable in $[0, 9\pi/4]$.

Question 2.4.

$$\int \frac{31^x + 31^{-x}}{961^x + 961^{-x} + 7}dx$$

Solution 2.4.

2.3.2 Integration by Parts

Theorem 2.3 (Integration by-Part)**.** Let f, g be invertible with continuous antiderivatives F, G respectively, then

$$\int F(x)g(x)dx = F(x)G(x) - \int f(x)G(x)dx.$$

Let $u = F(x), v = G(x)$, we have the differential form of the rule

$$\int udv = uv - \int vdu$$

Proof. $(FG)' = Fg + fG$, then integrate both sides □

The key is to find an F easy to differentiate, and a g easy to integrate.

Example 2.3.

1. $$\int \ln|x|dx = x\ln|x| - \int x \cdot \frac{1}{x}dx = x\ln|x| - x + C$$
2. $$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

In general, the procedure can be repeated as many times as you need

$$\int fgdx = \int f^{(0)}g^{(0)}dx = \int f^{(0)}g^{(-1)} - \int f^{(1)}g^{(-1)}dx = \cdots = f^{(0)}g^{(-1)} - f^{(1)}g^{(-2)} + \cdots + (-1)^n \int f^{(n)}g^{(-n)}dx$$

This method can be visualized by a DI table

sign	D	I
+	f	g
-	f'	$\int g$
+	f''	$\iint g$
$\vdots$	$\vdots$	$\vdots$

Question 2.5.

$$\int x^3 e^x dx$$

Solution 2.5.

2.3.3 Partial Fraction

Rational Functions are fractions of two polynomials in $\mathbb{R}$ i.e. $\frac{P(x)}{Q(x)}$. By expressing a rational function in terms of partial fraction, it can be integrated in closed form.

Theorem 2.4 (Partial Fraction). Every rational function in the form $P(x)/Q(x)$ can be written in the form

$$\frac{P(x)}{Q(x)} = q(x) + \frac{R(x)}{Q(x)}, \deg R < \deg Q$$
$$\frac{R(x)}{Q(x)} = \sum_{s,n} \frac{r(x)}{s(x)^n}, \deg r < \deg s$$

which $s(x)$ are irreducible factors of $Q(x)$

Proof. Calm down this is not an abstract algebra class □

This shows the existence of such partial fraction, but how simple are the denominators

Theorem 2.5 (Fundamental Theorem of Algebra). $\forall$ real polynomials $p(x)$ with degree $n, \exists n$ (possibly degenerate/repeated) roots in $\mathbb{C}$ such that

•

i.e. all real polynomials are products of linear factors and quadratic factors.

Question 2.6.

$$\int \frac{7x^2 - 3x + 26}{(x-2)(x^2+2x+4)}$$

Solution 2.6.

2.3.4 Trigonometric Polynomial Function

Now consider $\int P(\sin\theta, \cos\theta) d\theta = \int \sum_{m,n} c_{m,n} \cos^n\theta \sin^n\theta d\theta$

1. If either m or n is odd, use substitution and $\sin^2\theta + \cos^2\theta = 1$ to reduce the term into a polynomial. This also works for $m, n < 0$
2. If both m and n is even, use double angle formula, or even product to sum formula

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Example 2.4.

$$\int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2} = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

2.3.5 t-Substitution

If you have a trigonometric rational function, you can use the t-substitution $t = \tan\frac{\theta}{2}, dx = \frac{2}{1+t^2}dt, \sin\theta = \frac{2t}{1-t^2}, \cos\theta = \frac{2t}{1+t^2}, \tan\theta = \frac{1-t^2}{1+t^2}$

2.3.6 Reduction Formula

Idea: find a relationship between different integrals first

Question 2.7.
$$\int_0^1 x^n e^x dx$$

Solution 2.7.

2.4 Improper Integral

We cannot use Riemann Sum if either the domain or range is unbounded

Definition 2.4 (Improper Integral)**.** Unbounded domain

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_a^t f(x)dx \int_{-\infty}^b f(x)dx \qquad = \lim_{t\to-\infty}\int_t^b f(x)dx$$

Unbounded discontinuity Assume $f(x)$ is discontinuous and unbounded at $x = a$ (as an auxilliary condition),

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$
$$\int_c^a f(x)dx = \lim_{t\to a^-}\int_c^t f(x)dx$$

Example 2.5.

$$\int_1^{+\infty}\frac{dx}{x^p} = \begin{cases}\lim_{t\to+\infty}\frac{1}{p-1}(1-\frac{1}{t^{p-1}}), & p\neq 1\\ \lim_{t\to+\infty}\ln t, & p=1\end{cases} = \begin{cases}\frac{1}{p-1}, & p>1\\ \text{diverge}\ , & p\leq 1\end{cases}$$

Example 2.6.

$$\int_0^1 \ln x dx = [x\ln x - x]_0^1 \;= -1 - \lim_{x\to 0^+} x\ln x - x = -1$$

Example 2.7 (What?)**.** One common conception is mistreating the limit

$$\int_{-1}^2 \frac{dx}{x^2} = [-\frac{1}{x}]_{-1}^2 = \frac{1}{2}$$

In fact the integral diverges. FTC cannot be used here since $-\frac{1}{x}$ is not an antiderivative of $\frac{1}{x^2}$ at $x = 0$.
Instead, note that

$$\int_{-1}^2 \frac{dx}{x^2} = \lim_{s\to0^+}\int_s^2\frac{dx}{x^2} + \lim_{t\to0^-}\int_{-1}^t\frac{dx}{x^2}$$

As the limit is different, you cannot just cancel them. However, the Cauchy Principal Value of the integral exists as it treat both limit as the same one.

2.5 Applications

2.5.1 (Review/Preview?) Polar Coordinates

In 2D, the Cartesian coordinates is (x, y), which can be transformed to the polar coordinates (r, θ) (or back) by

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases} \iff \begin{cases} r &= \sqrt{x^2+y^2} \\ \theta &= \operatorname{atan}\frac{y}{x} \end{cases}$$

While the Cartesian coordinate has a fixed orthogonal basis, which is very convenient for describing directions, the polar coordinate is helpful in case of radially symmetric cases (e.g. cylinder)

2.5.2 Arc Length

Definition 2.5 (2D Parametric Curve).

$$\mathbf{r}(t) = (f(t), g(t))$$

The arc length from $t = a$ to $t = b$ is

$$\int_a^b |\mathbf{r}'(t)|dt = \int_a^b \sqrt{(f')^2 + (g')^2}dt$$

If y is a function of x, we can further treat $f(t) = t$ and simplify it into

$$\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx$$

Example 2.8. A parametization of a unit circle is $(\cos t, \sin t), 0 \le t < 2\pi$, so the total length of the circle is

$$\int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t}dt = 2\pi$$

Alternatively, one can use other parametization such as $(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$, but this may not be the best one

2.5.3 Area

The signed area of the graph $y(x)$ in $[a, b]$ is $\int_a^b y(x)dx$. Graphically, the area between two continuous functions f and g is $\int_a^b |f - g|dx$, where a and b are where the functions intersect.

Proposition 2.6. Consider a simple, closed parametric closed curve $(x(t), y(t)), t \in [a, b]$ in counter-clockwise direction. The area bounded by the curve is

$$-\int_a^b y(t)x'(t)dt$$

Proof.

$$\text{Area} = \int_\alpha^\beta (y_1(x) - y_2(x))dx$$

□

2.5.4 Volume

Consider a solid with area $A(t)$ from $t = a$ to $t = b$, the volume is $\int_a^b A(t)dt$.

2.5.5 Solid of Revolution

Consider a solid constructed by revolving $y = f(x), a \leq x \leq b$ around the x-axis.
The volume of the solid is $\int_a^b \pi f^2(x)dx$

The slanted surface area of the solid is $\int_a^b 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}dx$

Exercise 2.2. Find the surface area and volume of a torus with outer radius R and inner radius r.

3 Series

3.1 Convergence Tests

Definition 3.1 (Convergence). Consider a real sequence (a_n), its series

- absolutely converges if $\sum_{n=1}^{\infty} |a_n|$ converges
- conditionally converges if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges
- diverges if $\sum_{n=1}^{\infty} a_n$ diverges

Term Test, Integral Test, p-Test, Comparison Test, Alternating Series Test, Root Test, Ratio Test

3.2 Absolute Convergence

Theorem 3.1 (Riemann Rearrangement Theorem). If a series converges absolutely, any rearrangement of it gives same limit. If a series converges conditionally, you can rearrange it to get any limit you want.

Cauchy Product Formula

3.3 Power Series

Definition 3.2 (Power Series).

$$P(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, c_n \in \mathbb{R}$$

3.3.1 Taylor's Series

Definition 3.3 (Taylor's Expansion). The n-th order Taylor expansion of f is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem 3.2 (Taylor's Theorem).

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0$$

Definition 3.4 (Taylor's Series).

$$\lim_{n \to \infty} T_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Question 3.1 (Inverse Central Potential). There are many forces in the nature exhibiting the inverse-square relationship, such as gravitational force. Their potential has an inverse relationship

$$V_0(r) = -\frac{a}{r}, a > 0$$

However, while finding the motion of an object under the force, we have to consider the centrifugal effect, which can be understood as an inverse-square centrifugal potential. Hence, the total potential is

$$V(r) = -\frac{a}{r} + \frac{b}{r^2}, a, b > 0$$

Expand $V(r)$ up to second order of r around the minimum.

Solution 3.1.

Exercise 3.1 (Lennard-Jones Potential). The Lennard-Jones potential is a model of molecular interaction. It consists of two terms to consider both attractive and repulsive force.

$$V_(r) = 4\varepsilon\left(\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right)$$

Expand $V(r)$ up to second order of r around the minimum.

3.3.2 Remainder

We want to know whether the Taylor's series can resemble the function.

Theorem 3.3 (Lagrange Remainder Formula). The remainder of the Taylor expansion is

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \exists\xi \in (x,a) \sqcup (a,x)$$

Question 3.2. By expanding e^x up to 2nd order in x, estimate $e^{1.1}$ and find the error

Solution 3.2.

Exercise 3.2. Estimate 2.024^10 using 1st order Taylor expansion of x^10 and find the error.

Example 3.1 (Finite Difference Method). In experiment papers, you may be asked to compute the

derivative from numerical data.
The forward/backward method

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

The central method

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Using the Lagrange remainder formula, we know the error in these approximation.

3.3.3 Radius of Convergence

We say the Taylor Series converges to $f(x)$ if $f(x) = \lim_{n\to\infty} T_n(x) \iff R_n(x) = 0$. But for some functions, it may depend on the value of x

Theorem 3.4. Radius of Convergence Consider the power series $\sum_{n\to\infty} c_n(x-a)^n, \exists R$ so either
$R = 0$: the series converges only at $x = a$

- $R = +\infty$: the converges for all x
- $0 < R < +\emptyset$: the series converges absolutely for $|x-a| < R$, but diverges for $|x-a| > R$

Example 3.2. 1. $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ converges to e^x for all x

2. $\sum_{n=1} \frac{(-1)^{n+1}}{n} x^n$ converges to $\ln(1+x)$ for $-1 \le x < 1$
3. $\sum_{n=1} n!x^n$ converges only when $x = 0$

3.3.4 Properties of Taylor's Series

In general, exchanging two limits can be very dangerous. However, Taylor's series have a good property

Theorem 3.5 (Term-by-Term Differentiation and Integration)**.** If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$, then we can do term-by-term differentiation and integration within the radius of convergence. i.e.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \int_0^x f(t)dt \quad = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

Example 3.3.

1. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1 \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$
2. $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, |x| < 1 \Rightarrow \operatorname{atan} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$