Graph Planarity

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Graph planarity: a graph is planar if and only if it can be drawn on a plane without its edges intersecting or crossing each other.

Kuratowski's Theorem

A graph is planar if and only if it does not contain a subgraph that can be transformed to $K_{3,3}$ or K_5 by inserting or erasing vertices of degree 2.

K_5: a complete graph on five vertices (this is not planar)

$K_{3,3}$: a complete bipartite graph with three vertices in each subset (this is not planar)

Euler's Formula

If G is a connected, planar graph, then any plane graph for G has r = m - n + 2 regions (Proof by induction). (r = region, m = edges, n = states).

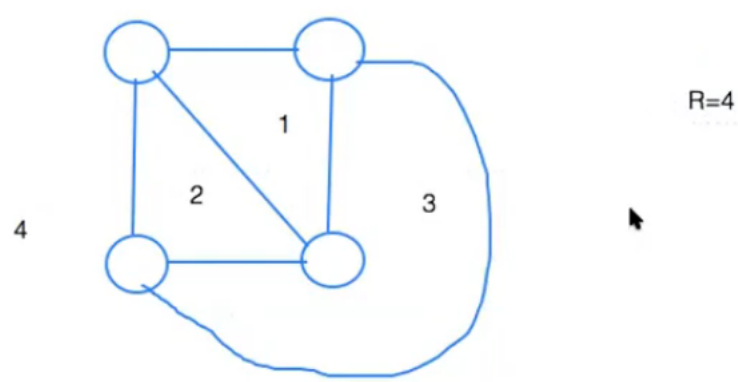

Figure 1: Example of what a region is. The graph above has 4 regions

Euler's Formula: Corollary

Corollary: If G is a connected planar graph with $m > 1$, then $m \leq 3n - 6$

$3r \geq 2m$ (each region is bordered by at least 3 edges, edges get double counted)

- Ex. From the graph in figure 1. Region 1 and region 2 share an edge, and that edge gets double counted to ensure that each region is bordered by 3 edges.

To construct a formula that relates m and n, we need to use the inequality equations to get rid of r.

$R \geq 2m/3$

$M - N + 2 \geq 2M/3$

$M/3 + 2 \geq N$

$M/3 > N - 2$

$M > 3N - 6$, corollary is true

Consequences of Planarity

- The graph can be drawn "nicely" and visualized more easily
- $m = \Theta(n)$. So, Bellman-Ford, which takes $\Theta(nm)$ in general, takes $\Theta(n^2)$ for a planar graph
- Some graph problems are NPC in general graphs, but in P for planar graphs (Ex. max-clique, graph coloring)

Geometric (Planar Graphs)

List of geometric (planar graph problem) we will see in this class

- Euclidean minimum spanning trees
- Euclidean TSP
- Rectilinear/Euclidean Steiner trees
- Voronoi Diagrams
- Delaunay Triangulation
- Relative neighborhood graph
- Gabriel graph

Geometric Dual G* of a Planar Graph G

To create a geometric dual:

Place vertex in each region of G

If two regions have an edge x in common, join the corresponding vertices by and edge x* crossing only x

Always gives a planar pseudo-graph

Applications of Duality

Map coloring $\equiv$ Planar graph Coloring

Don't want neighboring states sharing the same color (this is planar graph coloring problem)

VLSI floor planning (rectangular dual)

Generally, any dissection of a plan into regions can be represented by a planar graph

Algorithmic Problems Related to Planarity

Planarity testing: given a graph, is it planar? This can be done in O(n + m) time. Two algorithms:

1. Hopcroft-Tarjan Algorithm
2. Lempel-Even-Cedarbaum and Booth-Leuker

Planar embedding: If graph is planar, draw it in a plane so that the edges don't cross

Straight-line embedding: each edge is a straight line segment