Journal of
Non-Newtonian
Fluid
Mechanics

ELSEVIER J. Non-Newtonian Fluid Mech., 59 (1995) 1-10

Simple elongational flows which are not simple in reality

Stefan Zahorski

Polish Academy of Sciences, Institute of Fundamental Technological Research, ul. Świętokrzyska 21, 00-049 Warsaw, Poland

Received 13 June 1994; in revised form 14 October 1994

Abstract

In many simple elongational flows realized in existing rheometers under the assumption of constant extension rates, the sample radii or heights vary along the axis. This is mainly due to inertial, surface-tension and drag forces acting on a sample subjected to real experimental conditions. Then, the velocity fields are non-uniform in the sample cross-section and the presence of additional shear stresses is necessary. On the basis of the concept of flows with dominating extension (FDEs) we try to discuss the real shapes of samples, the relations between the sample shapes and the velocities, forces and elongational velocities, and the sensitivity of the flow considered to external disturbances. It is shown, among other things, that under realistic conditions the effect of inertial, surface-tension and drag forces is rather small, if their values are moderate.

Keywords: Drag effects; Inertial effects; Sensitivity to external disturbances; Simple elongational flows; Steady extensional flows

1. Introduction

Uniaxial extensional flows or, in other words, simple elongational flows are of great importance in theoretical rheology as well as in polymer processing. In many situations a knowledge of the so-called elongational or extensional viscosity is essential and cannot be replaced by a knowledge of other material functions. To this end, various extensional geometries and various types of rheometers were proposed and applied by the researchers involved in experimental investigations (see, e.g. Refs. [1] and [2]).

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SSDI 0377-0257(95)01332-2

One of the most important and successful devices used for the elongational viscosity determination is the rotary clamp extensometer invented by Meissner [3,4] more than twenty years ago. In this extensometer, in which a sample of polymer melt is extended between two pairs of rollers rotating in opposite directions, a steady extensional flow with constant rate of deformation is realized, which is necessary for proper determination of the elongational viscosity function. Recently, Meissner and Hostettler [5] developed a new rheometer that makes use of metal conveyor belts instead of rotary clamps, and is free from the majority of drawbacks characteristic of previously constructed devices. It should be stressed, however, that apart from many technical improvements concerned with the size of samples, the method of sample support, the range of temperatures etc., the new extensometer realizes exactly the same geometry of elongational flows with constant rate of extension.

Bearing in mind the geometry of flows used in the Meissner type rheometers, we note that in reality inertia effects, surface-tension effects as well as various drag forces, being a result of the surrounding liquid or gaseous medium, may seriously alter the entire picture of simple elongational flow. Under these circumstances the radius (or the height) of a sample is not constant along the axis, the velocity profiles become non-uniform and the presence of shear stresses is necessary to balance forces acting on the free surface. The above facts clearly explain why simple elongational flows realized in the type of rheometer considered are not simple under real experimental conditions.

Thus, in the present consideration, we shall try to answer the following questions.

(1) What is the real shape of a sample subjected, approximately, to steady elongational flows with a constant rate of extension?

(2) How such a shape and non-Newtonian properties of the melt considered may influence velocity fields, stresses and elongational viscosities calcalated on the basis of measurements performed under the assumption of steady elongational flows?

(3) When may the process considered be unstable and what is its sensitivity to small external disturbances imposed on velocities, loads, surface and drag forces?

In what follows we begin with the discussion of momentum (or force) balance along the sample axis, leading to the determination of the approximate sample shape. Under the assumption that the flows considered can be treated as flows with dominating extension (FDEs), introduced and discussed elsewhere [6,7], the corresponding increments in velocities, stresses and viscosities are estimated and discussed in greater detail. At the end, we briefly consider the simplified problem of sensitivity to disturbances in velocities, loads etc., leaving the more exact and complex problem of instability caused by superposed vibrations to the next contribution published separately [8].

2. The real shape of extended samples

For simple elongational flows, an axial velocity gradient, expressed in cylindrical coordinates (r,θ,z) can be written as

$$[\nabla V] = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} V', \qquad V' = \text{const.}, \tag{2.1}$$

where $V = V'z$ is the extensional velocity at the ends and the prime denotes differentiation with respect to z.

The constitutive equations of an incompressible simple fluid (see Ref. [9]), i.e.

$$\boldsymbol{T} = -p\mathbf{1} + \beta_1 \boldsymbol{A}_1 + \beta_2 \boldsymbol{A}_1^2, \qquad \text{tr}\, \boldsymbol{A}_1 = 0, \tag{2.2}$$

where $\boldsymbol{T}$ is the stress tensor and p the hydrostatic pressure, $\boldsymbol{A}_1$ denotes the first Rivlin–Ericksen kinematic tensor, and $\beta_i(V')$ $(i = 1,2)$ are the material functions, lead to the following expressions:

$$T^{33} - T^{11} = 3\beta V', \tag{2.3}$$

where $\beta(V')$ defined through

$$\eta_E \equiv 3\beta = \bar{\beta}_1 + \bar{\beta}_2 V', \tag{2.4}$$

is simply related to the elongational (longitudinal) viscosity function η_E.

In reality, the sample subjected to the action of various inertial, surface-tension and drag forces does not preserve its cylindrical shape in the sense that its radius varies along the axis. We may assume, however, that under the quasi-elongational approximation (see Ref. [10]), the velocity gradient (2.1) is valid for the so-called fundamental flow, necessary in the concept of flows with dominating extension (FDEs) [6,7].

The axial momentum balance, equivalent to the balance of axial forces (compare Refs. [2] and [10]), can be written in the form:

$$F_{ext} \equiv F_0 = 3\beta V' \pi R^2 + F, \tag{2.5}$$

where $R(z)$ is the variable radius of a sample, and $F_0 = 3\beta V' \pi R_0^2$ denotes the extensional force for $z = 0$, i.e. in the middle of a sample (Fig. 1). force F may be composed of inertia, surface-tension and drag forces, respectively, i.e.

$$F = \rho V^2 \pi R^2 + F_{st} + F_d. \tag{2.6}$$

It is obvious that, under the assumption of steady elongation, the radius R as well as the radius R_0 at $z = 0$ both depend on time in the following way:

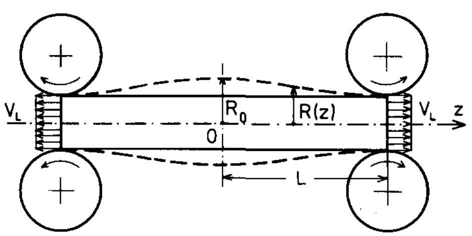

Fig. 1. The sample in simple elongational flows. The broken line shows the real (exaggerated) shape.

$$R = R(t=0)\exp\left(-\frac{V't}{2}\right), \qquad R_0 = R_0(t=0)\exp\left(-\frac{V't}{2}\right). \tag{2.7}$$

and the forces F and F_0 depend on time in the way resulting from the balance (2.5).

On the basis of Eq. (2.5) we directly obtain

$$R = R_0\left(1-\frac{F}{F_0}\right)^{1/2}, \qquad R' = -\frac{R_0}{2F_0}\frac{F'}{[1-(F/F_0)]^{1/2}}, \tag{2.8}$$

where the ratio F/F_0 must be independent of time.

Since always $F < F_0$, the radii $R_L = R(L)$ at both ends of a sample are less than the radius R_0. An inflection point on the sample profile and the corresponding change of curvature are determined by

$$R'' \geq 0, \tag{2.9}$$

where

$$R'' = -\frac{R_0}{2F_0[1-(F/F_0)]}\left\{F''(1-F/F_0)^{1/2} + \frac{F'^2}{2F_0}\frac{1}{[1-(F/F_0)]^{1/2}}\right\}. \tag{2.10}$$

Thus, the condition (2.9) leads to

$$(F_0 - F)F'' \leq -\tfrac{1}{2}F'^2, \qquad \text{if } F \neq F_0. \tag{2.11}$$

The above equality treated as a differential equation for F, gives the following solution:

$$\frac{F}{F_0} = 1 - \{[\sqrt{1-(F_1/F_0)} - 1]\xi + 1\}^2, \tag{2.12}$$

where $\xi = z/L$ denotes the axial dimensionless coordinate, and $F_1 = F(1)$.

It is seen from Fig. 2 that the existence of an inflection point along the $F(\xi)$ plot, ($F'' = 0$) at some point before that at which the profile curvature is changed (ξ_i) is a necessary condition for Eq. (2.9) to be satisfied. Otherwise, the curvature of the sample profile remains negative as for the concave plot $F(\xi)$ shown also in Fig. 2.

It is worth mentioning that in the case of square or rectangular cross-sections of a sample, and this is the case recently investigated by Meissner and Hostettler [5], we obtain

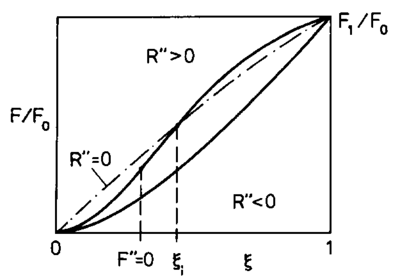

Fig. 2. The existence of an inflection point on the plot of the force $F(\xi)$.

$$A = A_0(1 - F/F_0), \qquad A_0 = A_0(t = 0)\exp(-V't) \tag{2.13}$$

where $A(z)$ denotes the variable area of sample cross-section and $A = h \cdot b$, where h is the height and b the width. From the relations:

$$A' = -F'/3\beta V', \qquad A'' = -F''/3\beta V', \tag{2.14}$$

it is seen that the conditions: $h'' > 0$ and simultaneously $b \approx$ const., lead to

$$F'' \leq 0. \tag{2.15}$$

Thus, in the case of plane deformations, an inflection point on the sample profile exactly corresponds to the inflection point on the $F(\xi)$ plot.

In an interesting particular case when all effects but inertia can be neglected, we obtain

$$R = \frac{R_0}{(1 + Re_z)^{1/2}}, \tag{2.16}$$

and

$$\begin{aligned} R' &= -\frac{R_0}{3}\frac{\rho V'z}{\beta}\frac{1}{(1 + Re_z)^{3/2}} \\ R'' &= -\frac{R_0 \rho V'}{3\beta}\frac{1}{(1 + Re_z)^{5/2}}(1 - 2Re_z), \end{aligned} \tag{2.17}$$

where

$$Re_z = \frac{\rho V^2}{3\beta V'} = \frac{\rho V'z^2}{3\beta}. \tag{2.18}$$

denotes the z-dependent Reynolds number.

The condition (2.9) can be satisfied, if

$$\xi^2 = z^2/L^2 \geq \frac{1}{2Re}, \tag{2.19}$$

where

$$Re = \frac{\rho V_{\mathrm{L}}^2}{3\beta V'} = \frac{\rho V'L^2}{3\beta}. \tag{2.20}$$

Therefore, in the case considered, an inflection point along the sample profile may occur, if $Re > 0.5$.

3. Velocities, stresses and viscosities in a real sample

As we mentioned earlier, the real samples have variable cross-sections along their axes; they are thicker in the middle and thinner at both ends at the rotary clamps. Thus, the presence of additional velocity distributions and shear stresses is necessary to satisfy the corersponding boundary conditions at the free surface.

According to the concept of FDEs [6,7], we assume the following full velocity gradient:

$$[\nabla V^*] = [\nabla V] + \begin{bmatrix} \dfrac{\partial u}{\partial r} & 0 & \dfrac{\partial u}{\partial z} \\ 0 & \dfrac{u}{r} & 0 \\ \dfrac{\partial w}{\partial r} & 0 & \dfrac{\partial w}{\partial z} \end{bmatrix}, \tag{3.1}$$

where the first term is determined by Eq. (2.1), and u and w denote additional radial and axial velocity components, respectively. We assume, moreover, that in the flow considered the ratio of sample radius to its length is a small quantity $\epsilon = R_0/L \ll 1$ and that the perturbed form of the constitutive equation (2.2) amounts to

$$\begin{aligned} \boldsymbol{T}^* = & -p\boldsymbol{1} + \beta_1 \boldsymbol{A}_1 + \beta_2 \boldsymbol{A}_1^2 + \beta_1 \boldsymbol{A}_1^+ + \beta_2 (\boldsymbol{A}_1^2)^+ \\ & + \frac{\mathrm{d}\beta_1}{\mathrm{d}V'} V'\boldsymbol{A} + \frac{\mathrm{d}\beta_2}{\mathrm{d}V'} V'\boldsymbol{A}_1^2 + \dots, \end{aligned} \tag{3.2}$$

where the crosses denote incremental terms.

The above equations introduced into the corresponding equations of equilibrium, after retaining terms of the highest order of magnitude with respect to ϵ (ϵ^0), lead to the following governing equation (compare Refs. [6,7]):

$$\frac{1}{\mathrm{r}} \frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right) + \frac{1}{2\beta}\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}V'} + \frac{\mathrm{d}\beta_2}{\mathrm{d}V'} V'\right) \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2 = \frac{C(z)}{\beta}, \tag{3.3}$$

where $C(z)$ depends on z only.

The boundary conditions are assumed in the following form:

$$\int_0^R w \, 2\pi r \mathrm{d}r = 0 \tag{3.4}$$

and

$$R'(T^{*33} - T^{*11}) = T^{*13} \tag{3.5}$$

for small R'. The first condition (3.4) expresses the fact that at both ends of a sample the mean velocities are constant and equal to $V_{\mathrm{L}} = V'L$, while the second one (3.5) results from the balance of forces acting on the free surface as described by Kase [11]. Here, for simplicity, surface-tension effects have been omitted (see Ref. [10]).

The non-linear equation (3.3) can be solved by the perturbation method applied to slightly non-Newtonian fluids, if the following quantity is small enough (see Ref. [7]):

$$k = \frac{1}{\beta}\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}V'} + \frac{\mathrm{d}\beta_2}{\mathrm{d}V'} V'\right) V' \tag{3.6}$$

Under the Newtonian approximation ($k \equiv 0$) the solution is of the form:

$$w_0 = \tfrac{3}{2}\frac{R'}{R}V'\left(r^2 - \frac{R^2}{2}\right) \tag{3.7}$$

and

$$w = w_0 + \tfrac{9}{4}kV'R\left(\frac{R'}{R}\right)^2 V'\left(r^2 - \frac{R^2}{2}\right) + \tfrac{9}{16}kV'R^2\frac{\partial}{\partial z}\left(\frac{R'}{R}\right)^2\left(r^2 - \frac{R^2}{2}\right)$$
$$- \tfrac{9}{32}kV'\frac{\partial}{\partial z}\left(\frac{R'}{R}\right)^2\left(r^4 - \frac{R^4}{3}\right), \tag{3.8}$$

for more general non-Newtonian cases.

Taking into account Eqs. (2.8), we arrive at

$$V^*|_{\substack{r=R,0\\ z=L}} = V_L\left(1 \mp \frac{F'}{8\beta_0\pi V_L}\right) \tag{3.9}$$

in Newtonian cases, and at

$$V^*|_{\substack{r=R,0\\ z=L}} = V_L\left[1 \mp \frac{F'}{8\beta_0\pi V_L} \mp \frac{k}{32}\frac{F'^2}{\beta^2 V_L V'\pi R_0^2(1 - F/F_0)^{1/2}}\right] \tag{3.10}$$

in more general cases. The quantity β_0 denotes the Newtonian value of β, independent of V'.

Similarly, we obtain for the stress difference

$$T^{*33} - T^{*11} = 3\beta_0 V'\left[1 + \tfrac{3}{8}k\frac{\beta}{\beta_0}\frac{F'r^2}{(F_0 - F)^2}\right], \tag{3.11}$$

and for the elongational viscosity averaged over the cross-section,

$$\langle\eta_E\rangle = 3\beta_0\left[1 + \tfrac{3}{16}\frac{\beta}{\beta_0}R_0\frac{F'r^2}{(F_0 - F)F_0}\right], \tag{3.12}$$

where

$$\langle\ \rangle = \frac{1}{\pi R^2}\int_0^R (\ )2\pi r\,\mathrm{d}r. \tag{3.13}$$

It can easily be verified that the additional velocities in Eqs. (3.9) and (3.10) as well as the additional elongational viscosity in Eq. (3.12) are, in general, small quantities. For instance, if the force F depends on z almost linearly (see Ref. [10]), i.e. $F \approx F'L$, $F' \approx$ const., we see that the second terms on the right-hand sides of Eqs. (3.9) and (3.12) can be presented as

$$\mp\tfrac{3}{8}\frac{R_0^2}{L^2}\frac{F}{F_0} \quad \text{and} \quad \tfrac{3}{16}k\frac{\beta}{\beta_0}\frac{R_0^2}{L^2}\frac{F^2/F_0^2}{(1 - F/F_0)}, \tag{3.14}$$

respectively. Since the viscoelastic parameter k and the ratio R_0/L are small by definition (for slightly non-Newtonian fluids, and for the concept of FDEs), the final effect of the terms (3.14) on the corresponding velocities and elongational viscosities may be meaningful only for relatively large ratios F/F_0, i.e. for high inertia and/or drag forces F as compared with extensional forces F_0.

It is noteworthy that for purely inertial flows (with drag effects neglected), we obtain

$$V^*|_{\substack{r=R,0\\ z=L}} = V_{\mathrm{L}}\left[1 \mp \tfrac{3}{4}\frac{Re}{(1+Re)^2}\frac{R_0^2}{L^2} \pm \tfrac{9}{8}k\frac{\beta}{\beta_0}\frac{Re^2}{(1+Re)^{7/2}}\frac{R_0^3}{L^3} \left\{\begin{matrix} +\tfrac{3}{8} \\ -\tfrac{3}{16}\end{matrix}\right. k\frac{\beta}{\beta_0}\frac{Re^2(1-Re)}{(1+Re)^5}\frac{R_0^4}{L^4}\right], \tag{3.15}$$

$$T^{*33} - T^{*11}|_{\substack{r=R\\ z=L}} = 3\beta_0 V'\left[1 + \tfrac{3}{2}k\frac{\beta}{\beta_0}\frac{Re^2}{(1+Re)^3}\frac{R_0^2}{L^2}\right] \tag{3.16}$$

and

$$\langle \eta_{\mathrm{E}} \rangle = 3\beta_0\left[1 + \tfrac{3}{4}k\frac{\beta}{\beta_0}\frac{Re^2}{(1+Re)^3}\frac{R_0^2}{L^2}\right], \tag{3.17}$$

where the Reynolds number Re has been defined by Eq. (2.20). More detailed inspection of maxima of the expressions composed of Re in Eqs. (3.14), (3.16) and (3.17) leads to the following numerical coefficients: 0.19, 0.10, 0.01 for the velocity, 0.22 for the stress, and 0.11 for the elongational viscosity. Thus, the effect of inertial forces is rather small and may be observable only for relatively high values of the parameter k and the ratio R_0/L.

The variable radius of a sample has been determined, as a first approximation, on the basis of quasi-elongational fundamental flow through Eqs. (2.8). As a second approximation we can take into account the expression for stresses presented in Eq. (3.11). After introducing into the equation of balance (2.5), we arrive at

$$R = R_0\left(1 - \frac{F}{F_0}\right)\left[\left(1 - \frac{F}{F_0}\right) + \tfrac{3}{16}kR_0^2\frac{F'^2}{F_0^2}\right]^{-1/2}, \tag{3.18}$$

or, alternatively, at

$$R = R_0\left(1 - \frac{F}{F_0}\right)\left[\left(1 - \frac{F}{F_0}\right) + \tfrac{3}{16}k\frac{R_0^2}{L^2}\frac{F^2}{F_0^2}\right]^{-1/2}, \tag{3.19}$$

if the particular linear dependence: $F \approx F'L$, $F' \approx$ const., can be used. The above expressions introduced into Eqs. (3.7) or (3.8) lead to new additional velocity fields for non-Newtonian fluids. For $k \equiv 0$, we immediately return to the previous results.

4. Sensitivity to external disturbances

Leaving the problem of dynamic instability to the next contribution [8], we try, in a simplified way, to answer the question whether constant or periodic disturbances imposed on velocities (or their gradients), loads or drag forces lead to a failure of the flow considered. Such a failure may occur, for instance if $R \to 0$ at some arbitrary place along the axis.

To this end, we assume that in the case of realistic disturbances it may happen that

$$F^* \geq F_0^*, \tag{4.1}$$

where $F^* = F + \Delta F$, $F_0^* = F_0 + \Delta F_0$, and Δ denotes either a positive or negative increment of the corrsponding quantities. Thus, inequality (4.1) directly leads to

$$\frac{\Delta F}{F_0} - \frac{\Delta F_0}{F_0} \geq 1 - \frac{F}{F_0} \geq 0, \tag{4.2}$$

where the ratio $\Delta F_0/F_0$ can be presented as

$$\frac{\Delta F_0}{F_0} = \frac{\Delta V'}{V'} + 2\frac{\Delta R_0}{R_0} + \frac{\Delta \beta}{\beta}, \tag{4.3}$$

if independent disturbances of the velocity gradient V', the radius R_0 and the viscosity β are admissable.

Therefore, a failure of the process considered is possible, if the increments of drag forces are positive and those of extensional forces are negative. The increments may be less for pretty large ratios F/F_0, i.e. for relatively high contributions of inertia and drag forces. For example, if $F/F_0 \approx 0.4$, the inequality (4.2) is satisfied for $\Delta F/F_0 \approx 0.1$ and $\Delta V'/V' = \Delta R_0/R_0 = \Delta\beta/\beta_0 \approx -0.1$.

5. Final conclusions

Trying to answer the questions posed at the very beginning of the paper, we may conclude the following.

(1) Under the action of real inertial, surface-tension and drag forces, the radius or height of a sample is not constant along the axis. A sample is thicker in the middle and thinner at both ends connected with the rotary clamps. An inflection point on the sample profile (changing its curvature) is possible either for Reynolds numbers greater than 0.5 or in the case in which a plot of the drag force $F(z)$ has an inflection point for $z \in (0,L]$. An inspection of the following data: $\eta_E \approx 10^3$–10^4 Pa s, $V' \approx 1\ s^{-1}$, $\rho = 1.1$–$1.2\ g\ cm^{-3}$ and the sample dimensions used by Meissner and Hostettler [5] shows that inertial effects are very small and may lead to an inflection point for samples longer than 70–200 cm.

(2) The velocities, stress (or forces) and elongational viscosities calculated on the basis of measurements weakly depend on the non-Newtonian properties (parameter k), the ratios R_0/L and the ratios F/F_0. The additional terms in the expressions for velocities, stresses and viscosities, resulting from the fact that in reality the flow considered is not a purely elongational one, may be essential (being of order of several percent) only for large non-Newtonian effects, for rather short and thick samples and for higher values of surface-tension and/or drag forces as compared with extensional forces exerted by the rollers. The inertia effects for smaller velocities are of less importance.

(3) The sensitivity of elongational flows to external disturbances imposed on velocities, loads and drag forces requires further more detailed studies based on dynamic unsteady solutions [8]. A simplified picture of failure of the process

considered is possible, in principle, if disturbances are sufficiently high. The probability of failure is greater for positive increments of surface-tension and drag forces and simltaneously negative increments of extensional forces.

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