Utility Function Transformations and Money Illusion: Comment

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In a recent exchange in this *Review*,[1] Richard Dusansky and Peter Kalman (hereafter, D-K) have claimed that the standard condition that utility functions be homogeneous of degree zero (*HD0*) in money and prices is not a necessary one for demand functions to be free of money illusion. They have proposed a "more general" set of conditions that imply illusion-free demand functions. Clower and Riley (hereafter, C-R) have claimed that, on the contrary, the standard condition serves completely to characterize the class of illusion-free demand functions, and that the conditions of D-K achieve only a spurious generality. However, D-K have apparently rebutted this claim by pointing out what is actually a minor error in the paper by C-R.[2]

The purpose of the present note is to confirm C-R's claim by demonstrating the nature of D-K's spurious generality in a simple way that goes back to fundamental utility theory. In particular, we shall show that D-K's conditions are more general only in the sense that they allow a consumer whose ordinal utility is *HD0* to choose an index of cardinal utility that depends upon the vector of money prices. The spuriousness of this generality arises from the well-known fact that (under conditions of certainty) no operational significance can be attached to the choice of a cardinal utility index. Correspondingly, D-K's conditions are operationally equivalent to the standard *HD0* condition, in the sense that they generate the same demand functions.

More specifically, consider one consumer with the utility function $u(x)$ and a second one with the utility function $f(x)$ defined by

$$f(x) \equiv \phi(u(x)) \tag{1}$$

where $x=(x_1,\ldots,x_n)$ is the consumption vector, and ϕ is a strictly increasing function, which (following Paul Samuelson) we will call the consumer's index of cardinal utility. It is a commonplace of consumer theory[3] that the marginal rates of substitution of two such consumers are identical, for

$$\frac{\dfrac{\partial f}{\partial x_i}(\ )}{\dfrac{\partial f}{\partial x_j}(\ )} \equiv \frac{\dfrac{d\phi}{du}(\ )\cdot\dfrac{\partial u}{\partial x_i}(\ )}{\dfrac{d\phi}{du}(\ )\cdot\dfrac{\partial u}{\partial x_j}(\ )} \equiv \frac{\dfrac{\partial u}{\partial x_i}(\ )}{\dfrac{\partial u}{\partial x_j}(\ )} \tag{2}$$

Correspondingly, each consumer will exhibit exactly the same demand functions. Thus the choice of the cardinal index has no operational significance.

Assume now that the second consumer chooses his cardinal index in accordance with the phases of the moon; that is, when the phase of the moon is λ he chooses the cardinal index $\phi^\lambda(\ )$. In this case the consumer's utility function can be defined as

$$f(x,\lambda) \equiv \phi^\lambda(u(x)) \tag{3}$$

One might get the impression that a consumer with such a utility function could be

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[1] See Dusansky and Kalman (1974, 1976) and Robert Clower and John Riley.

[2] The error was C-R's omission (equation (3), p. 184) of the inconsequential $h(p)$, as in our equation (8) below.

[3] See Vilfredo Pareto, pp. 539–45, John Hicks, pp. 306–07, and Paul Samuelson, pp. 98–99.

called a lunatic in the primal sense of the term, because his utility is affected by lunar phases. However, it would be misleading to claim that we had in this way generalized the standard theory of consumer's behavior to incorporate the case of lunacy. For in the present case the only role played by λ is to choose the inconsequential cardinal index which cancels out as in (2) above. In other words, the utility function defined by (3) deals only with lunatics who say they feel better (or worse) when the full moon comes out, but do not behave any differently. Thus it allows only a spurious generality.

Along the same lines, we might suppose that instead of looking at phases of the moon, the second consumer looks at the vector of prices, $p=(p_1,\ldots,p_n)$ when deciding which cardinal utility index to use. Thus the utility function would be defined as

$$(4) \qquad f(x,p)\equiv\phi^p(u(x))$$

where for each p, ϕ^p is monotonically increasing. Just as before, one would hardly want to claim that this involves a generalization of the standard theory to incorporate Veblenesque snob effects by virtue of the inclusion of the price vector in the utility function: for once again the marginal rates of substitution of the two consumers will be identical, in view of the fact that

$$(5) \qquad \frac{\frac{\partial f}{\partial x_i}(\;)}{\frac{\partial f}{\partial x_j}(\;)}\equiv\frac{\frac{d\phi^p}{du}(\;)\cdot\frac{\partial u}{\partial x_i}(\;)}{\frac{d\phi^p}{du}(\;)\cdot\frac{\partial u}{\partial x_j}(\;)}\equiv\frac{\frac{\partial u}{\partial x_i}(\;)}{\frac{\partial u}{\partial x_j}(\;)}$$

In the standard theory of consumer choice with money, the consumer is assumed to possess a utility function $u(x,m,p)$ which is *HD0* in m and p, where m is the nominal demand for money. The consumer maximizes this utility function subject to the budget constraint

$$(6) \qquad \sum_{i=1}^{n} p_i x_i + m = W$$

where W denotes the money value of the consumer's initial endowment. The budget constraint (6), together with the first-order conditions:

$$(7) \qquad \frac{\partial u}{\partial x_i}(\;)\Big/\frac{\partial u}{\partial m}(\;)=p_i \quad i=1,\ldots,n$$

define the demand functions $\bar{x}(p,W)$, $\bar{m}(p,W)$. These functions are easily seen to be free of money illusion; i.e., $\bar{x}(\;)$ is *HD0* and $\bar{m}(\;)$ is homogeneous of degree one (*HD1*) in p and W.[4]

Just as before, the choice of the particular cardinal utility index does not affect the marginal rates of substitution in (7) and hence has no operational significance. In particular, the consumer's demand functions would be just the same if his utility function were defined by $u(x,m,p)$ or by $\phi^\lambda[u(x,m,p)]$ or by $\phi^p[u(x,m,p)]$. In this last case, the utility function generally would not be *HD0* in m and p. For if m and p were doubled, the value of $u(\;)$ would be unaffected; but since the change in p might dictate the choice of a different cardinal utility index, the value of $\phi^p[u(\;)]$ might be affected. However, it is obvious that such a utility function would not imply any operationally meaningful form of money illusion.

Now, this is really what is involved in D-K's so-called "generalization" of the standard theory. For despite the technical complexity of their contribution, the class of utility functions defined by their proposed conditions is simply

$$(8) \quad f(x,m,p)\equiv(p_1)^k u(x,m,p)+h(p)$$

where $u(\;)$ is differentiable and *HD0* in m and p, and $h(p)$ is any function. Their conditions (1974, p. 118) are that $\partial f(\;)/\partial x_i$ be homogeneous of degree k for all i, and that $\partial f(\;)/\partial m$ be homogeneous of degree $k-1$, in m and p.[5] Define $u(x,$

[4]To see this note that if $u(\;)$ is *HD0*, then the marginal rates of substitution in (7) are *HD1* in m and p (see R. G. D. Allen, p. 324). Thus if m is doubled but x unchanged when p and W are doubled, (6) and (7) will continue to hold. See also Patinkin, pp. 289–90.

[5]Actually D-K would have seen immediately that something was wrong with these conditions if they had noticed that the conditions apply directly to marginal utility functions, which in ordinal utility theory have no operational significance except indirectly through their ratios, the marginal rates of substitution.

$m, p) \equiv f(x, m/p_1, p/p_1)$ and $\hat{h}(x, m, p) \equiv f(x, m, p) - (p_1)^k u(x, m, p)$. Then (8) holds if $\hat{h}(x, m, p)$ can be expressed as a function only of p. This will be the case if $\partial \hat{h}(x, m, p)/\partial x_i \equiv \partial \hat{h}(x, m, p)/\partial m \equiv 0$ for all i; and this condition follows immediately from D-K's homogeneity conditions. That (8) implies D-K's conditions follows immediately from the fact that, because $u(\;)$ is *HD0*, therefore $\partial u(\;)/\partial x_i$ is *HD0* for all i, and $\partial u/\partial m$ is homogeneous of degree minus one, in m and p (see Allen, p. 324).

Clearly every utility function in this class is of the form $\phi^p[u(x, m, p)]$, where, for any given p, $\phi^p[\;]$ is an increasing function. Thus the only sort of added generality allowed for by D-K's conditions is the spurious generality of permitting the consumer to vary the cardinal utility index when prices are changed. As C-R put it, the only sense in which the standard *HD0* condition is, as D-K claim, overly restrictive, is that it ignores the case in which the consumer says he feels better (or worse) when money holdings and all money prices are doubled, but does not behave any differently.

To set the record straight, we show that a necessary and sufficient condition for demand functions to be free from money illusion is that they be derivable from a utility function that is *HD0* in m and p. Sufficiency is proven in footnote 4. To prove necessity, suppose that the illusion-free demand functions $\bar{x}(\;)$, $\bar{m}(\;)$ are derivable from the utility function: $f(x, m, p)$, which is not *HD0*. Then it suffices to show that they are also derivable from the utility function: $u(x, m, p) \equiv f(x, m/p_1, p/p_1)$, which is *HD0*. Take any values (p^0, W^0) of prices and wealth, and define $x^0 \equiv \bar{x}(p^0, W^0)$, $m^0 \equiv \bar{m}(p^0, W^0)$. We want to show that x^0 and m^0 are also the quantities that will be chosen by the consumer with utility function $u(\;)$ when faced with (p^0, W^0). This will be the case if the first-order conditions

$$(9) \qquad \frac{\dfrac{\partial u}{\partial x_i}(x^0, m^0, p^0)}{\dfrac{\partial u}{\partial m}(x^0, m^0, p^0)} = p_i^0 \qquad i = 1, \ldots, n$$

are satisfied. To show that they are indeed satisfied, note that, since $\bar{x}(\;)$ and $\bar{m}(\;)$ are free of money illusion, therefore $x^0 = \bar{x}(p', W')$ and $m' = \bar{m}(p', W')$; where $(p', W', m') \equiv (1/p_1^0)(p^0, W^0, m^0)$. Therefore, since $\bar{x}(\;)$ and $\bar{m}(\;)$ are derivable from $f(\;)$, the first-order conditions:

$$(10) \qquad \frac{\dfrac{\partial f}{\partial x_i}(x^0, m', p')}{\dfrac{\partial f}{\partial m}(x^0, m', p')} = p_i' \qquad i = 1, \ldots, n$$

must hold. Equation (9) can be derived directly from (10) using the chain rule of calculus.

Thus D-K have merely shown that illusion-free demand functions may also be derivable from operationally equivalent cardinal utility functions that are not *HD0*. Indeed D-K could have gone further and noted that they may be derivable from any utility function that has the same marginal rates of substitution as an *HD0* function whenever (7) is satisfied;[6] but once again this added generality has no operational significance. From the operational viewpoint, then, the traditional theory of money illusion is reaffirmed.

[6]When (7) is not satisfied the utility functions need not satisfy any homogeneity conditions, because these values of (x, m, p) can never correspond to a point on the demand function. See Robert Pollak, pp. 70–71. This set of utility functions is not of the form $\phi^p[u(x, m, p)]$ where $u(\;)$ is *HD0*. Nor, as Menahem Yaari has pointed out to us, is the transformation employed in our proof that any illusion-free demand function is derivable from an *HD0* utility function. This means that the class of utility functions that are operationally equivalent to *HD0* functions is broader than the class of monotonic transformations, which in turn is broader than the class defined by D-K's conditions.

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