Utility Function Transformations and Money Illusion: Reply and Further Results

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In our 1974 paper, Peter Kalman and I studied the theory of consumer choice with money, in which a generalized real balance effect enters the utility function. It was shown that the class of utility functions which yields illusion-free commodity demand functions (*IFDF*) contains utility functions which are nonhomogeneous. Peter Howitt and Don Patinkin assert that the generalization of the class beyond the "standard" (i.e., homogeneous of degree zero, *HD0*) utility function is spurious, in that the nonhomogeneous functions in the *IFDF* class do not differ *ordinally* from the standard utility function.

Provided below is a complete mathematical characterization of the entire *IFDF* utility class (in the form of an "if and only if" proof). It is shown that the *IFDF* class contains nonhomogeneous utility functions which differ in an ordinal way from the standard utility function.

Consider the "regular" utility function

$$u=f(x,m,p) \tag{1}$$

where x represents the $n-1$ commodities, $x_1,\dots,x_{n-1}$, m is holdings of nominal money balances, and p represents the $n-1$ commodity prices, $p_1,\dots,p_{n-1}$.

The budget constraint is

$$y+L=\sum_{i=1}^{n-1} p_i x_i+m \tag{2}$$

where y is income and L is initial money balances. (Note that the price of m is taken to be unity.) The maximization of (1) over x, m subject to (2) yields the first-order conditions

$$MRS_{ij}(x,m,p)=\frac{\dfrac{\partial f(x,m,p)}{\partial x_i}}{\dfrac{\partial f(x,m,p)}{\partial x_j}}=\frac{p_i}{p_j} \tag{3}$$

$$i=1,\dots,j-1,\,j+1,\dots,n-1$$

$$MRS_{mj}(x,m,p)=\frac{\dfrac{\partial f(x,m,p)}{\partial m}}{\dfrac{\partial f(x,m,p)}{\partial x_j}}=\frac{1}{p_j}$$

$$y+L=\sum_{i=1}^{n-1} p_i x_i+m$$

which, under the usual conditions, can be solved for the demand functions

$$x_i=h^i(p,L,y) \qquad i=1,\dots,n-1 \tag{4}$$

$$m=h^m(p,L,y)$$

These demand functions, which exist in the neighborhood of the solution, are unique. In general these demand functions exhibit money illusion.

Now, the central concern. What is the necessary and sufficient condition for the commodity demand functions in (4) to be free from money illusion? The following proposition is put forth and proved.

PROPOSITION: *The necessary and sufficient condition under which the commodity demand functions are HD0 in (p, L, y) is*

$$\sum_{j,k=1}^{n-1}\frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial x_k \partial p_j}\frac{D_{ki}}{D}p_j+\sum_{j=1}^{n-1}\frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial m\partial p_j}\frac{D_{ni}}{D}p_j+\lambda\frac{D_{ni}}{D}+m\frac{D_{n+1i}}{D}=0 \tag{5}$$

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PROOF:

By Euler's theorem on homogeneous functions we know

$$h^i(p, L, y) = h^i(\alpha p, \alpha L, \alpha y) \tag{6}$$

if and only if

$$\sum_{j=1}^{n-1} \frac{\partial h^i}{\partial p_j} p_j + \frac{\partial h^i}{\partial L} L + \frac{\partial h^i}{\partial y} y = 0 \qquad i = 1, \ldots, n-1 \tag{7}$$

However, from a total differentiation of the first-order conditions we know that[1]

$$\frac{\partial h^i}{\partial p_j} = \lambda \frac{D_{ji}}{D} - \sum_{k=1}^{n-1} \frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial x_k \partial p_j} \frac{D_{ki}}{D} - \frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial m \partial p_j} \frac{D_{ni}}{D} + x_j \frac{D_{n+1i}}{D}$$

$$\frac{\partial h^i}{\partial y} = \frac{\partial h^i}{\partial L} = -\frac{D_{n+1i}}{D}$$

where D is the determinant of the Jacobian of the first-order conditions, D_{ji} is the cofactor of the element in row j column i of the Jacobian, and λ is the Lagrange multiplier. Hence, we have (7) if and only if

$$\sum_{j=1}^{n-1} \left[\lambda \frac{D_{ji}}{D} - \sum_{k=1}^{n-1} \frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial x_k \partial p_j} \frac{D_{ki}}{D} - \frac{\partial^2 (\cdot,\cdot,\cdot)}{\partial m \partial p_j} \frac{D_{ni}}{D} + x_j \frac{D_{n+1i}}{D} \right] p_j - [L + y] \frac{D_{n+1i}}{D} = 0 \tag{8}$$

Using (2) we have (8) if and only if

$$\frac{\lambda}{D} \sum_{j=1}^{n-1} D_{ji} p_j - \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial x_k \partial p_j} \frac{D_{ki}}{D} p_j - \sum_{j=1}^{n-1} \frac{\partial^2 f(\cdot,\cdot,\cdot)}{\partial m \partial p_j} \frac{D_{ni}}{D} p_j - m \frac{D_{n+1i}}{D} = 0 \tag{9}$$

[1]For a detailed derivation, see my paper with Kalman (1972, pp. 343-45).

However, by the theorem on expansion by alien cofactors, we know that

$$\sum_{j=1}^{n-1} D_{ji} p_j + D_{ni} = 0, \qquad i = 1, \ldots, n-1$$

Hence, (9) if and only if (5).

The necessary and sufficient condition in (5) constitutes the most general condition under which the commodity demand functions are *HD0*, and completely characterizes the class of utility functions which yields illusion-free commodity demand functions (i.e., the *IFDF* class). It tells us that the economic responses to the equiproportionate change in p, L, y (i.e., the money illusion experiment), as it impacts through the marginal utility functions and through the budget constraint, must of course be completely offsetting in equilibrium, so that commodity demands remain unchanged. Much more important, the condition in (5) has useful implications. Note that (5) imposes no homogeneity requirement of any kind on the utility function, on the marginal utility functions, or on the marginal rate of substitution functions. The last is particularly important. It is possible for the indifference surfaces to shift in response to the money illusion experiment, and for the marginal rates of substitution in (3) to vary, and still for demand to remain unaltered. We are reminded that the *MRS*s are *functions*, having x, m, and p as arguments, which (for the illusion-free demand outcome) need to exhibit unchanged commodity demand values at the solution point (i.e., in equilibrium), but need not themselves be invariant. The complete characterization of the *IFDF* class thus admits utility functions whose commodity *MRS* functions are nonhomogeneous.[2]

[2]It may be the case that standard utility functions generate all illusion-free demand functions. Just as it may be true that nonhomogeneous utility functions also generate all illusion-free demand functions. The direct characterization of demand, however, is not the issue. The issue is characterization of utility, i.e., the class of utility functions which yield illusion-free demand functions.

Now consider the class of utility functions which are derivable from a standard (i.e., *HD0*) utility function by order-preserving transformations. Let $u(x, m, p)$ be *HD0* in m and p, and ϕ be any positive monotonic transformation of $u(x, m, p)$. The resulting class is given by

(10)

$$\tilde{u}(x, m, p) = \phi[u(x, m, p)], \phi'[u] > 0$$

It is easy to show, since $u(x, m, p)$ is *HD0* in m and p, that *all* utility functions in this class exhibit *homogeneous MRS* functions. The class obviously does not contain utility functions which exhibit *non*homogeneous *MRS* functions. However, we know from the complete characterization of the *IFDF* class derived above that there exist some of the latter (i.e., some nonhomogeneous utility functions having nonhomogeneous *MRS* functions) which yield illusion-free demand functions. Hence, since these nonhomogeneous utility functions are not contained in (10) it must be the case that they differ ordinally from the standard utility function.

A by-product of the complete characterization of the *IFDF* class is a release from the use of homogeneity restrictions as a means for attaining illusion-free demand functions. The results free monetary economists from fixation on homogeneity (in the utility function, in the marginal utility functions, or in the *MRS* functions) and on the invariance of the *MRS* functions. A homogeneity assumption is no longer required in order to have a homogeneity result. That these nonhomogeneous utility functions exist is important in completely characterizing utility theory.[3]

[3]The results reported herein constitute a further, and final, development of the analysis of money illusion first initiated in my 1974 paper with Kalman, and should be taken as representing our final position. A more complete exposition appears in my 1977 paper with Kalman.

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