Tutotial Dynamic Epistemic Logic

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Overview

Part I Epistemic Logic

Part II Public Announcement Logic

Part III Action Models

Part IV Expressivity

Part V Probability

Part I

Epistemic Logic

Epistemic logic

What is epistemic logic about?

- I know that p.
- He does not know that p
- He knows whether p
- He knows that I know that she does not know that p

Information

We regard information as something that is relative to a subject who has a certain perspective on the world, called an *agent*, and the kind of information we have in mind is meaningful as a whole, not just loose bits and pieces.

History

- von Wright 1951: An Essay in Modal Logic
- Hintikka 1962: Knowledge and Belief
- Aumann 1976: Agreeing to Disagree
- Fagin, Halpern, Moses and Vardi 1995: Reasoning about Knowledge
- Meyer and van der Hoek 1995: Epistemic Logic for AI and Computer Science

Etymology

- knowledge: ἐπιστήμη
- belief: δόξα

Language

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi$$

Example

- I know that p.

Example

- I know that p.
- $K_a p$

Example

- I know that p.
- $K_a p$
- He does not know that p

Example

- I know that p.
- $K_a p$
- He does not know that p
- $\neg K_b p$

Example

- I know that p.
- $K_a p$
- He does not know that p
- $\neg K_b p$
- He knows whether p

Example

- I know that p.
- $K_a p$
- He does not know that p
- $\neg K_b p$
- He knows whether p
- $K_b p \vee K_b \neg p$

Example

- I know that p.
- $K_a p$
- He does not know that p
- $\neg K_b p$
- He knows whether p
- $K_b p \vee K_b \neg p$
- He knows that I know that she does not know that p

Example

- I know that p.
- $K_a p$
- He does not know that p
- $\neg K_b p$
- He knows whether p
- $K_b p \lor K_b \neg p$
- He knows that I know that she does not know that p
- $K_b K_a \neg K_c p$

Models

A *Kripke model* is a structure $M = \langle S, R, V \rangle$, where

- S is a nonempty set of states.
- R yields an accessibility relation $R_a \subseteq S \times S$ for every $a \in A$
- $V : P \rightarrow \wp(S)$.

Models

A *Kripke model* is a structure $M = \langle S, R, V \rangle$, where

- S is a nonempty set of states.
- R yields an accessibility relation $R_a \subseteq S \times S$ for every $a \in A$
- $V : P \to \wp(S)$.

If all the relations R_a in M are equivalence relations, we call M an *epistemic model*. In that case, we write $\sim_a$ rather than R_a, and we represent the model as $M = \langle S, \sim, V \rangle$.

Example: cards

Example: cards

rwb rbw

wrb wbr

brw bwr

Example: cards

rwb rbw

wrb wbr

brw bwr

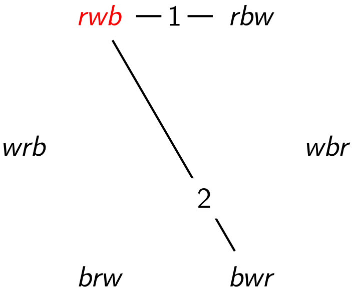
Example: cards

$\color{red}{rwb}$ — 1 — rbw

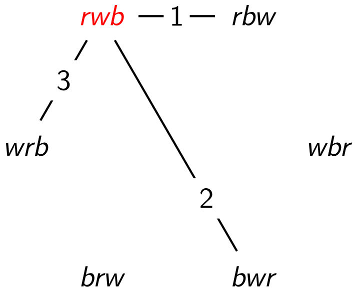
wrb wbr

brw bwr

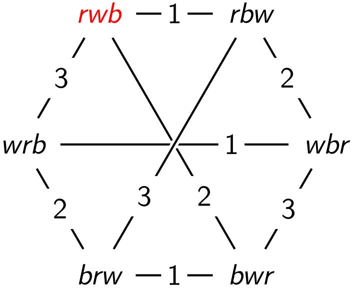
Example: cards



Example: cards



Example: cards



Epistemic modeling

- Given is a description of a situation
- The modeler tries to determine:
  - The set of relevant propositions
  - The set of relevant agents
  - The set of states
  - An indistinguishability relation over these states for each agent

Truth

$$
\begin{array}{lll}
M, s \models p & \text{iff} & s \in V(p) \\
M, s \models (\varphi \wedge \psi) & \text{iff} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models \neg \varphi & \text{iff} & \text{not } M, s \models \varphi \\
M, s \models K_a \varphi & \text{iff} & \text{for all } t \text{ such that } s \sim_a t \\
 & & \text{it holds that } M, t \models \varphi
\end{array}
$$

Axiomatisation

all instantiations of propositional tautologies
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
$K_a\varphi \to \varphi$
$K_a\varphi \to K_aK_a\varphi$
$\neg K_a\varphi \to K_a\neg K_a\varphi$
From φ and $\varphi \to \psi$, infer ψ
From φ, infer $K_a\varphi$

The Byzantine Generals

Imagine two allied generals, a and b, standing on two mountain summits, with their enemy in the valley between them. Generals a and b together can easily defeat the enemy, but if only one of them attacks, he will certainly lose the battle. The generals can communicate using messengers. How can they coordinate their attack?

General knowledge

$$E_B\varphi =_{def} \bigwedge_{b\in B} K_b\varphi$$

Common knowledge

$$C_B\varphi =_{def} \bigwedge_{n\in\mathbb{N}} E_B^n\varphi$$

History

- Lewis 1969: Convention
- Friedell 1969: On the structure of shared awareness
- Aumann 1976: Agreeing to disagree
- Barwise 1988: Three views of common knowledge

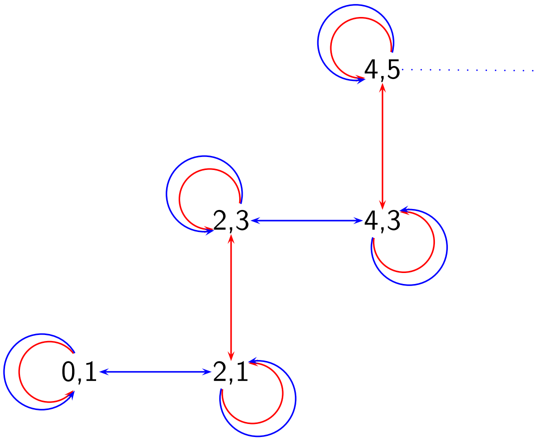
Semantics

$$M, s \models C_B\varphi \text{ iff } M, t \models \varphi \text{ for all } t \text{ such that } s \sim_B^* t$$

Example: consecutive numbers

Two agents, a and b are facing each other. They see a number on each other's head, and those number are consecutive numbers n and $n+1$ for a certain $n \in \mathbb{N}$. This is *common knowledge* among them. Let us assume that a has 2 and b has 3.

A model for consecutive numbers



Axiomatisation

$C_B(\varphi \to \psi) \to (C_B\varphi \to C_B\psi)$
$C_B\varphi \to (\varphi \wedge E_B C_B\varphi)$
$C_B(\varphi \to E_B\varphi) \to (\varphi \to C_B\varphi)$
From φ, infer $C_B\varphi$

Part II

Public Announcement Logic

Overview

What is public announcement logic about?

- After it is announced that p, everyone knows that p
- After it is announced that φ, it is common knowledge that φ
- After it is announced that none of the children know that they are muddy, all the muddy children know that they are muddy.

History

- Plaza 1989: Logics of Public Communications
- Gerbrandy & Groeneveld 1997: Reasoning about Information Change
- Baltag, Moss & Solecki 1998: The Logic of Common Knowledge, Public Announcements, and Private Suspicions

Language

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid C_B\varphi \mid [\varphi]\varphi$$

Examples

- After it is announced that p, everyone knows that p

Examples

- After it is announced that p, everyone knows that p
- $[p]Ep$

Examples

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Examples

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- $[p]Ep$
- After it is announced that φ, it is common knowledge that φ
- $[\varphi]C\varphi$
- After it is announced that none of the children know that they are muddy, all the muddy children know that they are muddy.

Examples

- After it is announced that p, everyone knows that p
- $[p]Ep$
- After it is announced that φ, it is common knowledge that φ
- $[\varphi]C\varphi$
- After it is announced that none of the children know that they are muddy, all the muddy children know that they are muddy.
- $[\bigwedge_{a\in A} \neg K_a m_a] \bigwedge_{a\in B} K_a m_a$

Semantics

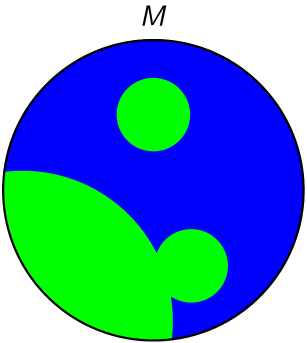
$$M, s \models [\varphi]\psi \quad \text{iff} \quad M, s \models \varphi \text{ implies } M|\varphi, s \models \psi$$

Model restriction

$M|\varphi = \langle S', \sim', V' \rangle$ with:

$$\begin{array}{lll} S' & =_{def} & [\![\varphi]\!]_M \\ \sim'_a & =_{def} & \sim_a \cap ([\![\varphi]\!]_M \times [\![\varphi]\!]_M) \\ V'_p & =_{def} & V_p \cap [\![\varphi]\!]_M \end{array}$$

As a picture



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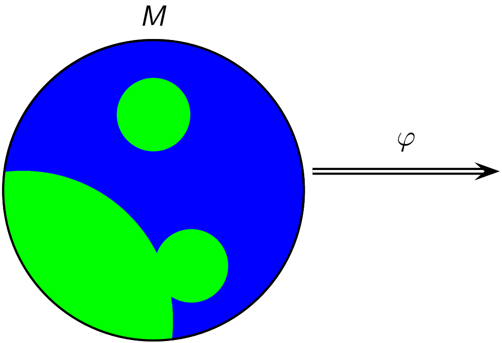

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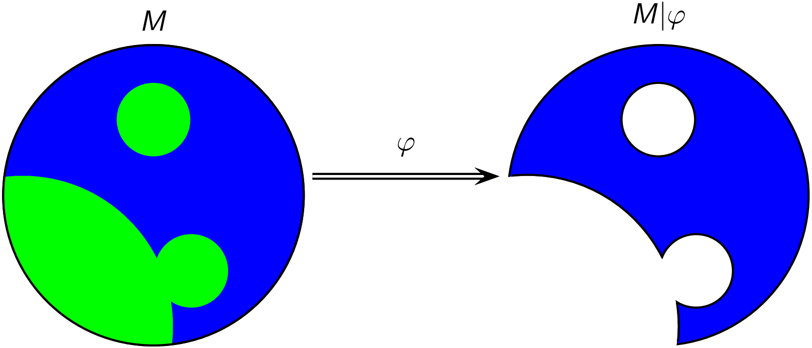
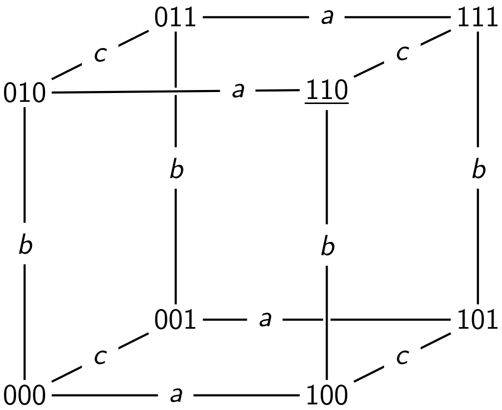


Example: the muddy children

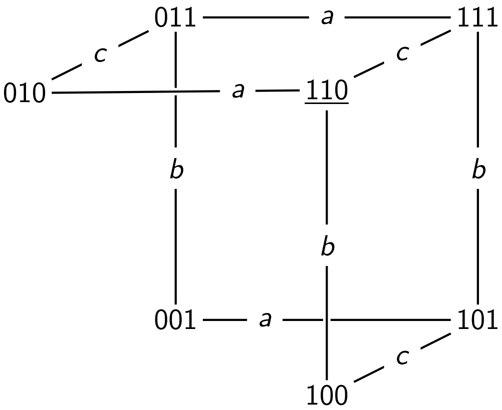
A group of children has been playing outside and are called back into the house by their father. The children gather round him. As one may imagine, some of them have become dirty from the play and in particular: they may have mud on their forehead. Children can only see whether other children are muddy, and not if there is any mud on their own forehead. All this is commonly known, and the children are, obviously, perfect logicians. Father now says: "At least one of you has mud on his or her forehead." And then: "Will those who know whether they are muddy please step forward." If nobody steps forward, father keeps repeating the request. What happens?

As a picture



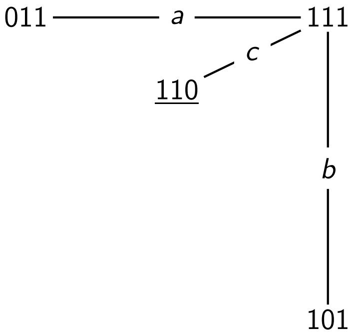
The children can see each other

As a picture



At least one of you has mud on his or her forehead.

As a picture



Will those who know whether they are muddy please step forward?

As a picture

<u>110</u>

Will those who know whether they are muddy please step forward?

Axiomatization

$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
$[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$
From φ, infer $[\psi]\varphi$
From $\chi \rightarrow [\varphi]\psi$ and $\chi \wedge \varphi \rightarrow E_B\chi$, infer $\chi \rightarrow [\varphi]C_B\psi$

Theorem (Plaza, Gerbrandy)

Every formula in the language of public announcement logic without common knowledge is equivalent to a formula in the language of epistemic logic.

Part III

Action Models

Overview

Epistemic modeling

- Given is a description of a situation
- The modeler tries to determine:
  - The set of relevant propositions
  - The set of relevant agents
  - The set of states
  - An indistinguishability relation over these worlds for each agent

Dynamic modeling

- Given is a description of a situation and an event that takes place in that situation.
- The modeler first models the epistemic situation, and then tries to determine:
  - The set of possible events
  - The preconditions for the events
  - An indistinguishability relation over these events for each agent

History

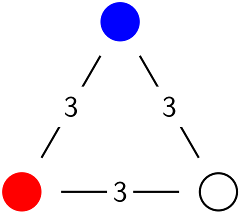
- Baltag, Moss & Solecki 1998: The Logic of Common Knowledge, Public Announcements, and Private Suspicions

Action models

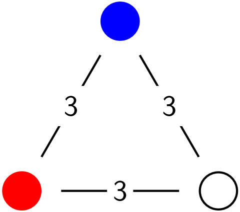
An action model M is a structure $\langle \mathsf{S}, \sim, \mathsf{pre} \rangle$

- S is a *finite* domain of action points or events
- $\sim_a$ is an equivalence relation on S
- $\mathsf{pre} : \mathsf{S} \rightarrow \mathcal{L}$ is a preconditions function that assigns a precondition to each $\mathsf{s} \in \mathsf{S}$.

Example: showing a card

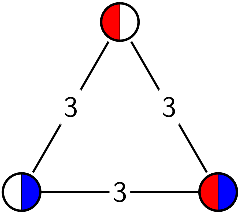


Example: showing a card

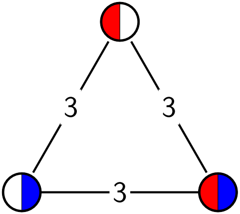


- $\mathsf{S} = \{\mathsf{r}, \mathsf{w}, \mathsf{b}\}$
- $\sim_1 = \{(\mathsf{s}, \mathsf{s}) \mid \mathsf{s} \in \mathsf{S}\}$
- $\sim_2 = \{(\mathsf{s}, \mathsf{s}) \mid \mathsf{s} \in \mathsf{S}\}$
- $\sim_3 = \mathsf{S} \times \mathsf{S}$
- $\mathsf{pre}(\mathsf{r}) = r_1$
- $\mathsf{pre}(\mathsf{w}) = w_1$
- $\mathsf{pre}(\mathsf{b}) = b_1$

Example: whispering



Example: whispering



- $S = \{r, w, b\}$
- $\sim_1 = \{(s, s) \mid s \in S$
- $\sim_2 = \{(s, s) \mid s \in S$
- $\sim_3 = S \times S$
- $\text{pre}(r) = \neg r_1$
- $\text{pre}(w) = \neg w_1$
- $\text{pre}(b) = \neg b_1$

What do you learn from an action?

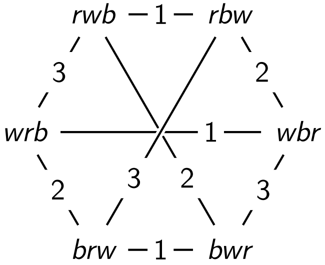
- Firstly, if you can distinguish two actions, then you can also distinguish the states that result from executing the action.
- Secondly, you do not forget anything due to an action. States that you could distinguish before an action are still distinguishable.

Product update

Given are an epistemic state (M, s) with $M = \langle S, \sim, V \rangle$ and an action model (M, s) with $\mathsf{M} = \langle \mathsf{S}, \sim, \mathsf{pre} \rangle$. The result of executing (M, s) in (M, s) is $M', (s, \mathsf{s})$ where $M' = \langle S', \sim', V' \rangle$

- $S' = \{(s, \mathsf{s}) \mid s \in S, \ \in \mathsf{S}, \ \text{and} \ \ M, s \models \mathsf{pre}(\mathsf{s})\}$
- $(s, \mathsf{s}) \sim'_a (t, \mathsf{t})$ iff $s \sim_a t$ and $\mathsf{s} \sim_a \mathsf{t}$
- $(s, \mathsf{s}) \in V'_p$ iff $s \in V_p$

In a picture: showing a card



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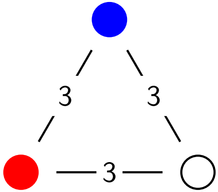
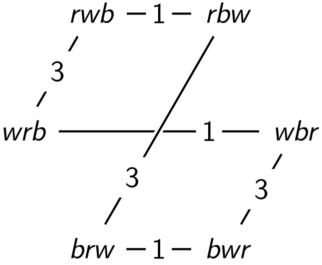
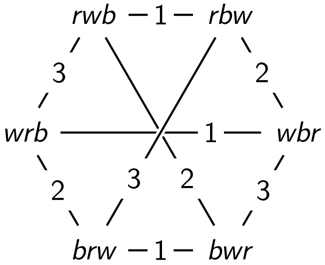




In a picture: whispering



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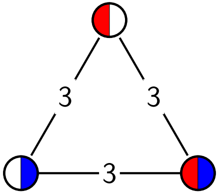

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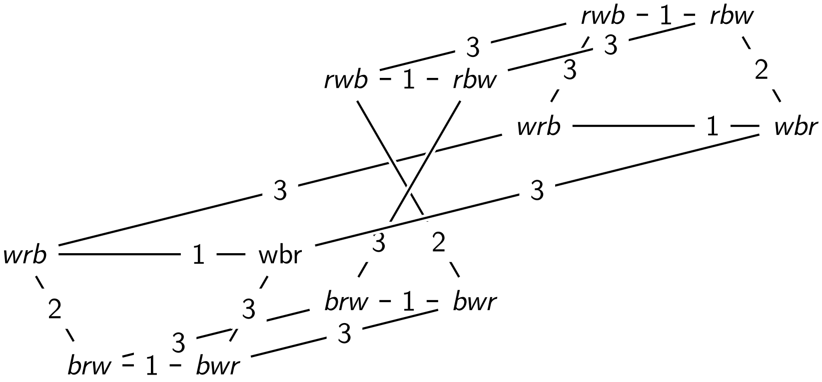


Language

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid C_B\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

Semantics

$$
\begin{array}{lll}
M, s \models p & \text{iff} & s \in V_p \\
M, s \models \neg\varphi & \text{iff} & M, s \not\models \varphi \\
M, s \models \varphi \wedge \psi & \text{iff} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models K_a\varphi & \text{iff} & \text{for all } s' \in S : s \sim_a s' \text{ implies } M, s' \models \varphi \\
M, s \models C_B\varphi & \text{iff} & \text{for all } s' \in S : s \sim_B^* s' \text{ implies } M, s' \models \varphi \\
M, s \models [\mathsf{M}, \mathsf{s}]\varphi & \text{iff} & \text{if } M, s \models \mathsf{pre}(\mathsf{s}), \text{ then } M \otimes \mathsf{M}, (s, \mathsf{s}) \models \varphi
\end{array}
$$

Syntax and semantics

- Are syntax and semantics clearly separated?

Syntax and semantics

- Are syntax and semantics clearly separated?
- Yes!

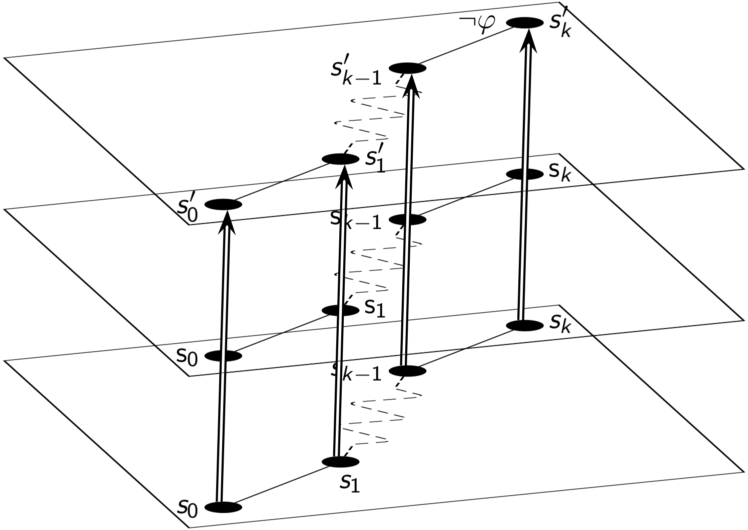
Axiomatization

$[\mathsf{M}, \mathsf{s}]p \leftrightarrow (\mathsf{pre}(\mathsf{s}) \rightarrow p)$
$[\mathsf{M}, \mathsf{s}]\neg\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \rightarrow \neg[\mathsf{M}, \mathsf{s}]\varphi)$
$[\mathsf{M}, \mathsf{s}](\varphi \land \psi) \leftrightarrow ([\mathsf{M}, \mathsf{s}]\varphi \land [\mathsf{M}, \mathsf{s}]\psi)$
$[\mathsf{M}, \mathsf{s}]K_a\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \rightarrow \bigwedge_{\mathsf{s}\sim_a\mathsf{t}} K_a[\mathsf{M}, \mathsf{t}]\varphi)$
$[\mathsf{M}, \mathsf{s}][\mathsf{M}', \mathsf{s}']\varphi \leftrightarrow [(\mathsf{M}, \mathsf{s}); (\mathsf{M}', \mathsf{s}')]\varphi$
From φ, infer $[\mathsf{M}, \mathsf{s}]\varphi$
Let (M, s) be an action model and let a set of formulas χ_t for every t such that $\mathsf{s} \sim_B \mathsf{t}$ be given. From $\chi_\mathsf{t} \rightarrow [\mathsf{M}, \mathsf{t}]\varphi$ and $(\chi_\mathsf{t} \land \mathsf{pre}(\mathsf{t})) \rightarrow K_a\chi_\mathsf{u}$ for every $\mathsf{t} \in \mathsf{S}$ such that $\mathsf{s} \sim_B \mathsf{t}$, $a \in B$ and $\mathsf{t} \sim_a \mathsf{u}$, infer $\chi_\mathsf{s} \rightarrow [\mathsf{M}, \mathsf{s}]C_B\varphi$.

Axiomatization

$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
$[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$
From φ, infer $[\psi]\varphi$
From $\chi \rightarrow [\varphi]\psi$ and $\chi \wedge \varphi \rightarrow E_B\chi$, infer $\chi \rightarrow [\varphi]C_B\psi$

A picture



Theorem (Baltag, Moss & Solecki)

Every formula in the language of action model logic without common knowledge is equivalent to a formula in the language of epistemic logic.

Bisimulation

$(S, R, V) \leftrightarrows (S', R', V')$
$\exists B \subseteq S \times S'$ such that if sBs', then for all $a \in A$

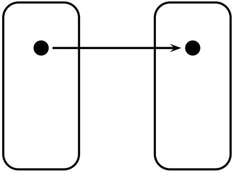
Atoms $s \in V_p$ iff $s' \in V'_p$ for all $p \in P$

Forth if $sR_a t$, then there is a $t' \in S'$ such that $s' R'_a t'$ and $t \mathfrak{R} t'$

Back if $s' R'_a t'$, then there is a $t \in S$ such that $sR_a t$ and $t \mathfrak{R} t'$

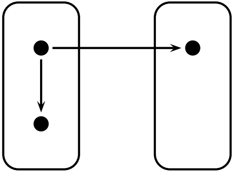
In a picture: bisimulation

Forth



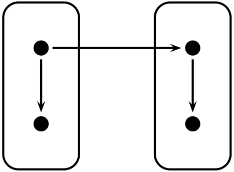
In a picture: bisimulation

Forth



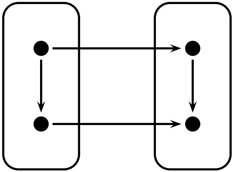
In a picture: bisimulation

Forth



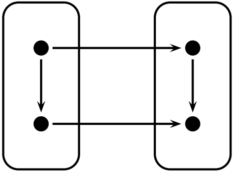
In a picture: bisimulation

Forth

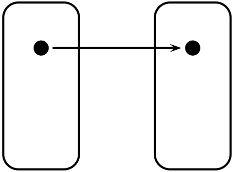


In a picture: bisimulation

Forth

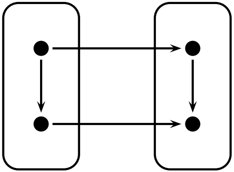


Back

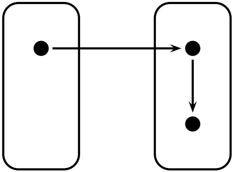


In a picture: bisimulation

Forth

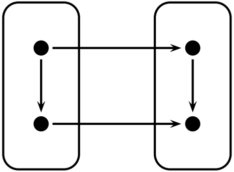


Back

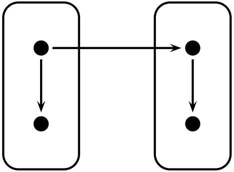


In a picture: bisimulation

Forth

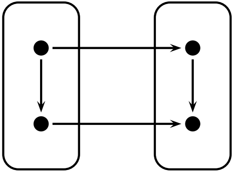


Back



In a picture: bisimulation

Forth



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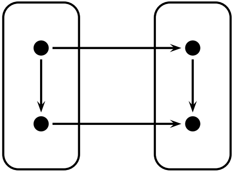

Bisimulation: theorem

If $M, s \leftrightarroweq M' s'$, then $M, s \models \varphi$ iff $M', s' \models \varphi$ for all φ.

Bisimulation: theorem: proof

By induction on φ

Base case For all $p \in P$: if $M, s \leftrightarroweq M' s'$, then $M, s \models p$ iff $M', s' \models p$.

Bisimulation: theorem: proof

By induction on φ

Base case: For all $p \in P$: if $M, s \leftrightarroweq M's'$, then $M, s \models p$ iff $M', s' \models p$.

Induction hypothesis: If $M, s \leftrightarroweq M's'$, then $M, s \models \varphi$ iff $M', s' \models \varphi$.
If $M, s \leftrightarroweq M's'$, then $M, s \models \psi$ iff $M', s' \models \psi$.

Bisimulation: theorem: proof

By induction on φ

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If $M, s \leftrightarroweq M's'$, then $M, s \models \psi$ iff $M', s' \models \psi$.

Induction step

- $\neg\varphi$
- $\varphi \wedge \psi$
- $K_a\varphi$
- $C_B\varphi$
- $[\mathrm{M}, \mathrm{s}]\varphi$

Bisimulation: theorem: proof

By induction on φ

Base case: For all $p \in P$: if $M, s \leftrightarroweq M' s'$, then $M, s \models p$ iff $M', s' \models p$.

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Induction step

- $\neg\varphi$
- $\varphi \wedge \psi$
- $K_a\varphi$
- $C_B\varphi$
- $[\mathsf{M}, \mathsf{s}]\varphi$

But why would $M \otimes \mathsf{M}, (s, \mathsf{s})$ be bisimilar to $M' \otimes \mathsf{M}, (s', \mathsf{s})$.

Preservation of bisimulation: theorem

If $M, s \leftrightarroweq M', s'$, then $M \otimes \mathsf{M}, (s, \mathsf{s}) \leftrightarroweq M' \otimes \mathsf{M}, (s', \mathsf{s})$.

Preservation of bisimulation: theorem: proof

- Let B be a bisimulation for $M, s \leftrightarroweq M', s'$.

Preservation of bisimulation: theorem: proof

- Let B be a bisimulation for $M, s \leftrightarroweq M', s'$.
- Let $(t, \mathrm{t})B'(t', \mathrm{t})$ iff tBt'.

Preservation of bisimulation: theorem: proof

- Let B be a bisimulation for $M, s \underline{\leftrightarrow} M', s'$.
- Let $(t, \mathrm{t})B'(t', \mathrm{t})$ iff tBt'.
- Atoms
- Back
- Forth

Preservation of bisimulation: theorem: proof

- Let B be a bisimulation for $M, s \leftrightarroweq M', s'$.
- Let $(t, \mathrm{t})B'(t', \mathrm{t})$ iff tBt'.
- Atoms
- Back
- Forth

But why would t and t' both satisfy pre(t)?

Part IV

Expressivity

Overview

What is expressive power?

$\mathcal{L}_1 \preceq \mathcal{L}_2$ iff for every formula $\varphi_1 \in \mathcal{L}_1$ there is a formula $\varphi_2 \in \mathcal{L}_2$ such that $\varphi_1 \equiv \varphi_2$.

What is expressive power?

$$
\begin{array}{lll}
\mathcal{L}_1 \preceq \mathcal{L}_2 & \text{iff} & \text{for every formula } \varphi_1 \in \mathcal{L}_1 \text{ there} \\
& & \text{is a formula } \varphi_2 \in \mathcal{L}_2 \text{ such that} \\
& & \varphi_1 \equiv \varphi_2. \\
\mathcal{L}_1 \equiv \mathcal{L}_2 & \text{iff} & \mathcal{L}_1 \preceq \mathcal{L}_2 \text{ and } \mathcal{L}_2 \preceq \mathcal{L}_1
\end{array}
$$

What is expressive power?

$\mathcal{L}_1 \preceq \mathcal{L}_2$	iff	for every formula $\varphi_1 \in \mathcal{L}_1$ there is a formula $\varphi_2 \in \mathcal{L}_2$ such that $\varphi_1 \equiv \varphi_2$.
$\mathcal{L}_1 \equiv \mathcal{L}_2$	iff	$\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \preceq \mathcal{L}_1$
$\mathcal{L}_1 \prec \mathcal{L}_2$	iff	$\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_1 \not\equiv \mathcal{L}_2$

Example: propositional logic

Let us take two propositional languages:

$$\begin{array}{lcl} \varphi & ::= & \bot \mid p \mid \varphi \to \varphi \\ \varphi & ::= & p \mid \varphi \dagger \varphi \end{array}$$

where

φ	ψ	$\varphi \dagger \psi$
1	1	0
1	0	0
0	1	0
0	0	1

Equally expressive

$$
\begin{array}{lcl}
t_1(p) & = & p \\
t_1(\varphi \dagger \psi) & = & ((t_1(\varphi) \to \bot) \to t_1(\psi)) \to \bot
\end{array}
$$

$$
\begin{array}{lcl}
t_2(p) & = & p \\
t_2(\bot) & = & ((p \dagger p) \dagger p) \\
t_2(\varphi \to \psi) & = & ((t_2(\varphi) \dagger t_2(\varphi)) \dagger t_2(\psi)) \dagger ((t_2(\varphi) \dagger t_2(\varphi)) \dagger t_2(\psi))
\end{array}
$$

Equally expressive

$$
\begin{array}{lcl}
t_1(p) & = & p \\
t_1(\varphi \dagger \psi) & = & ((t_1(\varphi) \to \bot) \to t_1(\psi)) \to \bot \\
\varphi \dagger \psi & \equiv & ((\varphi \to \bot) \to \psi) \to \bot
\end{array}
$$

$$
\begin{array}{lcl}
t_2(p) & = & p \\
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t_2(p) & = & p \\
t_2(\bot) & = & ((p \dagger p) \dagger p) \\
t_2(\varphi \to \psi) & = & ((t_2(\varphi) \dagger t_2(\varphi)) \dagger t_2(\psi)) \dagger ((t_2(\varphi) \dagger t_2(\varphi)) \dagger t_2(\psi)) \\
\varphi \to \psi & \equiv & ((\varphi \dagger \varphi) \dagger \psi) \dagger ((\varphi \dagger \varphi) \dagger \psi)
\end{array}$$

Another language

$$\varphi ::= p \mid (\varphi \bigtriangledown \psi) \mid (\varphi_1 \leftrightarrow \varphi_2)$$

φ	ψ	$(\varphi \bigtriangledown \psi)$
0	0	0
0	1	1
1	0	1
1	1	0

Public announcement logic

$$
\begin{array}{lcl}
t(p) & = & p \\
t(\neg\varphi) & = & \neg t(\varphi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t(K_a\varphi) & = & K_a t(\varphi) \\
t([\varphi]p) & = & t(\varphi) \rightarrow p \\
t([\varphi]\neg\varphi) & = & t(\varphi) \rightarrow \neg[t(\varphi)]t(\psi) \\
t([\varphi]K_a\psi) & = & t(\varphi) \rightarrow K_a[t(\varphi)]t(\psi) \\
t([\varphi][\psi]\chi) & = & t([\varphi \wedge [\varphi]\psi]\chi)
\end{array}
$$

Model comparison games

- Two models: $M = (S, R, V)$ and $M' = (S', R', V')$.

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Model comparison games

- Two models: $M = (S, R, V)$ and $M' = (S', R', V')$.
- Two players: spoiler and duplicator.
- Two starting states: $s \in S$ and $s' \in S'$.
- The number of rounds: n.

Model comparison games

In each round spoiler initiates one of the following scenario's:

forth-move Spoiler chooses an agent a and a state t such that $(s, t) \in R_a$. Duplicator responds by choosing a world t' such that $(s', t') \in R'_a$. The output of this move is (t, t').

back-move Spoiler chooses an agent a and a world t' such that $(s', t') \in R'_a$. Duplicator responds by choosing a world t such that $(s, t) \in R_a$. The output of this move is (t, t').

Model comparison games

If either player cannot perform a prescribed action, that player loses. If the output worlds differ in their atomic properties for P, spoiler wins the game. If spoiler has not won after all n rounds, duplicator wins the game.

Example

Example

$$\Box(\Diamond p \wedge \Diamond \neg p)$$

Theorems

If for all $n \in \mathbb{N}$ duplicator has a winning strategy for the n-round game on M, s and M', s', then $M, s \equiv M', s'$.

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If for all $n \in \mathbb{N}$ duplicator has a winning strategy for the n-round game on M, s and M', s', then $M, s \equiv M', s'$.

Let P be finite.

If $M, s \equiv M', s'$, then for all $n \in \mathbb{N}$ duplicator has a winning strategy for the n-round game on M, s and M', s'.

S5 Spines

$$\bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \underset{p}{\bullet}$$

S5 Spines

$$
\begin{tikzcd}
\bullet \arrow[r, dash, "a"] & \bullet \arrow[r, dash, "b"] & \bullet \arrow[r, dash, "a"] & \bullet \arrow[r, dash, "b"] & \bullet \arrow[r, dash, "a"] & \bullet \arrow[r, dash, "b"] & \bullet \arrow[r, dash, "a"] & \underset{p}{\bullet}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, dash, "a"] & \bullet \arrow[r, dash, "b"] & \bullet \arrow[r, dash, "a"] & \bullet \arrow[r, dash, "b"] & \bullet \arrow[r, dash, "a"] & \underset{p}{\bullet}
\end{tikzcd}
$$

S5 Spines

$$\bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \underset{p}{\bullet}$$

$$\bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \underset{p}{\bullet}$$

$$\bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}} \bullet \overset{b}{\rule{2em}{0.4pt}} \bullet \overset{a}{\rule{2em}{0.4pt}}$$

S5 Spines

$$\bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \underset{p}{\bullet}$$

$$\bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \underset{p}{\bullet}$$

$$\bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}} \bullet \overset{b}{\text{———}} \bullet \overset{a}{\text{———}}$$

$C_{ab}\neg p$

Theorem

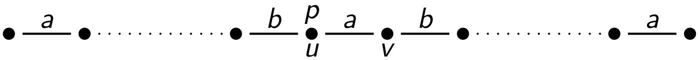
Epistemic logic with common knowledge is more expressive than epistemic logic without common knowledge.

Moves for common knowledge

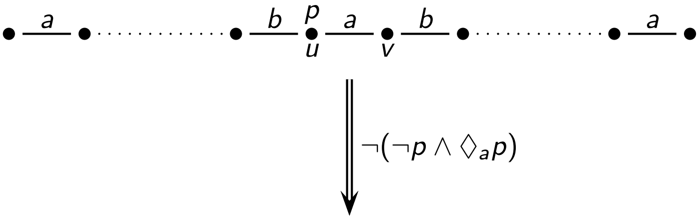
C-forth-move Spoiler chooses a group B and a state t such that sR_B^*t. Duplicator responds by choosing a state t' such that $s'R_B^*t'$. The output of this move is (t, t').

C-back-move Spoiler chooses a group B and a state t' such that $s'R_B^*t'$. Duplicator responds by choosing a state t such that sR_B^*t. The output of this move is (t, t').

Left or right?



Left or right?



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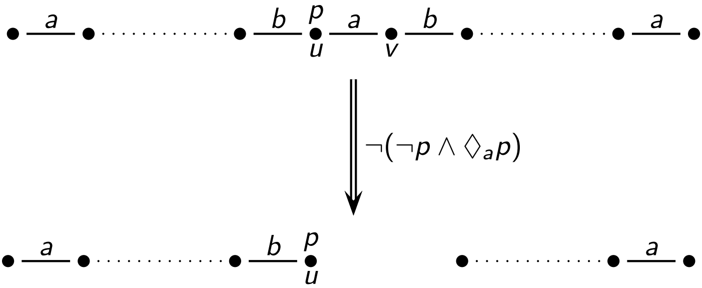

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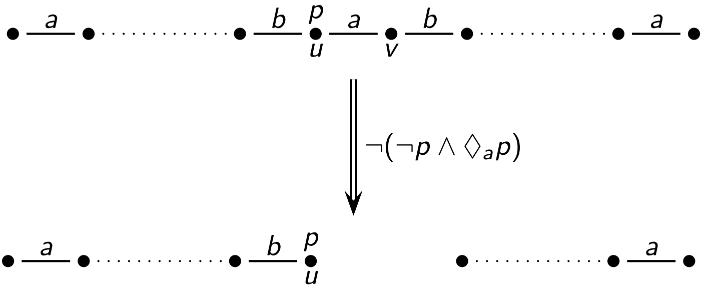

$$[\neg(\neg p \wedge \Diamond_a p)]C_{ab}p$$

Theorem

Public announcement logic with common knowledge is more expressive than epistemic logic with common knowledge.

Expressive power

$$\begin{tikzcd} \mathrm{S5C} \arrow[r] & \mathrm{PAC} \\ \mathrm{S5} \arrow[u] \arrow[r, leftrightarrow] & \mathrm{PA} \arrow[u] \end{tikzcd}$$

Theorem

Spoiler chooses a number $r < n$, and sets $W_1 \subseteq S_1$ and $W_2 \subseteq S_2$, with the current $s_1 \in W_1$ and likewise $s_2 \in W_2$.

Stage 1 Duplicator chooses states w in $W_1 \cup W_2$, $\overline{w}$ in $\overline{W_1} \cup \overline{W_2}$. Then spoiler and duplicator play the r-round game for these worlds. If duplicator wins this subgame, she wins the n-round game.

Stage 2 : Otherwise, the game continues in the relativized models $M|W_1, s_1$ and $M_2|W_2, s_2$ over $n - r$ rounds.

Substitution: semantics

A substitution assigns a (complex) formula to a propositional variable.

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Notation: $p := K_a q$

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Notation: $p := K_a q$

$$(M, w) \models [p := \varphi]\psi \quad \text{iff} \quad (M^{p:=\varphi}, w) \models \psi$$

where $M^\sigma = (W, R, V^{p:=\varphi})$:

$$\begin{array}{l} V_p^{p:=\varphi} = [\![\varphi]\!]_M \\ V_q^{p:=\varphi} = V_q \end{array}$$

Exercitives

- You're disqualified.
- I choose George.
- You're fired.
- I sentence you to death.
- I pronounce you husband and wife.

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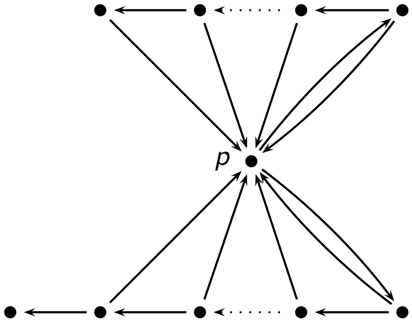
$p := \top$

The past

$$[p := \psi][\varphi][p]\chi$$

Hourglasses



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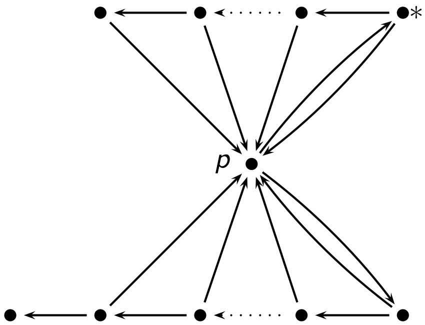

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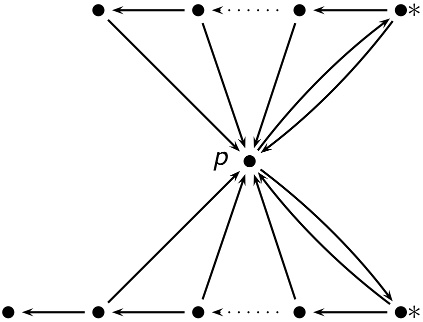

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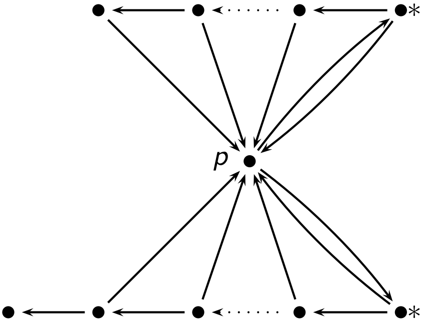
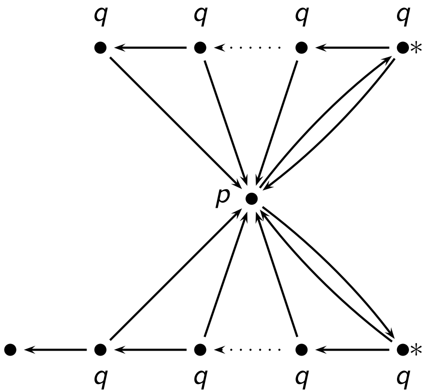


$$[q := \Diamond p][\neg p]Cq$$

Hourglasses



$$[q := \Diamond p][\neg p]Cq$$

Hourglasses

$$\begin{tikzcd} \overset{q}{\bullet} & \overset{q}{\bullet} \arrow[l] & \overset{q}{\bullet} \arrow[l, dotted] & \overset{q}{\bullet}{*} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \bullet & \underset{q}{\bullet} \arrow[l] & \underset{q}{\bullet} \arrow[l] & \underset{q}{\bullet} \arrow[l, dotted] & \underset{q}{\bullet}{*} \arrow[l] \end{tikzcd}$$

$$[q := \Diamond p][\neg p]Cq$$

Theorem

Public announcement logic with common knowledge and substitutions is more expressive than public announcement logic with common knowledge interpreted over the class of K models.

Notes

- Model comparison games are due to Ehrenfeucht and Fraïssé for first-order modal logic.
- The adaptation for modal logic seems to be folklore.
- The adaptation for common knowledge is due to Baltag Moss and Solecki.
- The adaptation for public announcements is due to van Benthem, van Eijck and Kooi.
- The relative expressive power result for S5 and S5C is folklore.
- The relative expressive power result for S5C and PAC is due to Baltag, Moss and Solecki.
- The relative expressive power result for PAC and PACS is due to Kooi.

Part V

Probability

Overview

Introduction

Probabilistic Epistemic Logic

Public announcements

Probabilistic Dynamic Epistemic Logic

How are logic and probability related?

- Uncertain reasoning
  - fuzzy logic
  - inductive logic
- Reasoning about uncertainty
  - probability logic

A card game

There are two players: a and b. Player a randomly draws a card from an ordinary deck of 52 cards. Player b wins 10 euro if she guesses correctly which card a drew. Before player b guesses she can ask one yes/no question, which is truthfully answered by player a. Which question is best?

It does not matter!

$$\begin{array}{cccc} \frac{26}{52} \times \frac{1}{26} & = & \frac{1}{52} & \\ \frac{26}{52} \times \frac{1}{26} & = & \frac{1}{52} & + \\ \hline & & \frac{2}{52} & \end{array} \qquad \begin{array}{cccc} \frac{1}{52} \times \frac{1}{1} & = & \frac{1}{52} & \\ \frac{51}{52} \times \frac{1}{51} & = & \frac{1}{52} & + \\ \hline & & \frac{2}{52} & \end{array}$$

It does not matter!

$$
\begin{array}{lcll}
\frac{26}{52} \times \frac{1}{26} & = & \frac{1}{52} & \\
\frac{26}{52} \times \frac{1}{26} & = & \frac{1}{52} & + \\
\hline
 & & \frac{2}{52} &
\end{array}
\qquad
\begin{array}{lcll}
\frac{1}{52} \times \frac{1}{1} & = & \frac{1}{52} & \\
\frac{51}{52} \times \frac{1}{51} & = & \frac{1}{52} & + \\
\hline
 & & \frac{2}{52} &
\end{array}
$$

It does not matter!

$$
\begin{array}{rcll}
\frac{x}{52} \times \frac{1}{x} & = & \frac{1}{52} & \\
\frac{1-x}{52} \times \frac{1}{1-x} & = & \frac{1}{52} & + \\
\hline
& & \frac{2}{52} &
\end{array}
$$

The Monty Hall Dilemma

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

Some responses

- "I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error . . ."
  – Robert Sachs, Ph.D., George Mason University

Some responses

- “I’m very concerned with the general public’s lack of mathematical skills. Please help by confessing your error ...” – Robert Sachs, Ph.D., George Mason University
- “You blew it, and you blew it big! ... You seem to have difficulty grasping the basic principle at work here ... There is enough mathematical illiteracy in this country, and we don’t need the world’s highest IQ propagating more. Shame!” – Scott Smith, Ph.D., University of Florida

Responses to the reply

- "I am in shock that after being corrected by at least three mathematicians, you still don't see your mistake." – Kent Ford, Dickinson State University

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- "...Albert Einstein earned a dearer place in the hearts of the people after he admitted his errors." – Frank Rose, Ph.D., University of Michigan
- "...If all those Ph.D.s were wrong, the country would be in serious trouble." – Everett Harmann, Ph.D., U.S. Army Research Institute

Responses to the reply

- "I am in shock that after being corrected by at least three mathematicians, you still don't see your mistake." – Kent Ford, Dickinson State University
- "...Albert Einstein earned a dearer place in the hearts of the people after he admitted his errors." – Frank Rose, Ph.D., University of Michigan
- "...If all those Ph.D.s were wrong, the country would be in serious trouble." – Everett Harmann, Ph.D., U.S. Army Research Institute
- "Maybe women look at math problems differently than men." Don Edwards, Sunriver, Oregon.

Language

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid q_1\mathbf{P}_a(\varphi_1) + \cdots + q_n\mathbf{P}_a(\varphi_n) \geq q$$

Language

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$$\begin{array}{lcl}
\mathbf{P}_a(\varphi) \leq q & = & -\mathbf{P}_a(\varphi) \geq -q \\
\mathbf{P}_a(\varphi) < q & = & \neg(\mathbf{P}_a(\varphi) \geq q) \\
\mathbf{P}_a(\varphi) > q & = & \neg(\mathbf{P}_a(\varphi) \leq q) \\
\mathbf{P}_a(\varphi) = q & = & \mathbf{P}_a(\varphi) \geq q \wedge \mathbf{P}_a(\varphi) \leq q \\
\mathbf{P}_a(\varphi) \geq \mathbf{P}_a(\psi) & = & \mathbf{P}_a(\varphi) - \mathbf{P}_a(\psi) \geq 0
\end{array}$$

Language

$$K_a \mathbf{P}_a(\varphi) = q$$

$$\mathbf{P}_a(\mathbf{P}_b(\varphi) = q)q'$$

Models

$M = \langle S, R, V, P \rangle$ such that:

- $S \neq \emptyset$
- $R : A \to \wp(S \times S)$
- $V : Atoms \to \wp(S)$
- $P : (A \times S) \to (S \rightharpoonup [0,1])$; such that

$$\forall a \in A \forall s \in S \sum_{v \in dom(P(a,s))} P(a,s)(v) = 1$$

Truth

$$
\begin{array}{lll}
M, s \models p & \text{iff} & s \in V(p) \\
M, s \models \neg\varphi & \text{iff} & M, s \not\models \varphi \\
M, s \models (\varphi \wedge \psi) & \text{iff} & M, s \models \varphi \text{ and } (M, s \models \psi \\
M, s \models K_a\varphi & \text{iff} & (M, t) \models \varphi \text{ for all } t \\
 & & \text{such that } sR_at \\
M, s \models \sum_{i=1}^{n} q_i \mathbf{P}_a(\varphi_i) \geq q & \text{iff} & \sum_{i=1}^{n} q_i P(a, s)(\varphi_i) \geq q
\end{array}
$$

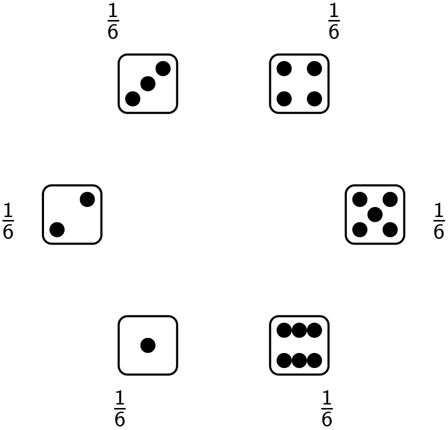
Some additional axioms / restrictions

$$
\begin{aligned}
&\textbf{(PD)} \quad \mathbf{P}_a(\varphi) = 1 \to \neg \mathbf{P}_a(\neg\varphi) = 1 \\
&\textbf{(P4)} \quad \textstyle\sum_{i=1}^{n} q_i \mathbf{P}_a(\varphi_i) \geq q \to \mathbf{P}_a(\sum_{i=1}^{n} q_i \mathbf{P}_a(\varphi_i) \geq q) = 1 \\
&\textbf{(P5)} \quad \neg \textstyle\sum_{i=1}^{n} q_i \mathbf{P}_a(\varphi_i) \geq q \to \mathbf{P}_a(\neg \sum_{i=1}^{n} q_i \mathbf{P}_a(\varphi_i) \geq q) = 1 \\
&\textbf{(CONS)} \quad K_a\varphi \to \mathbf{P}_a(\varphi) = 1
\end{aligned}
$$

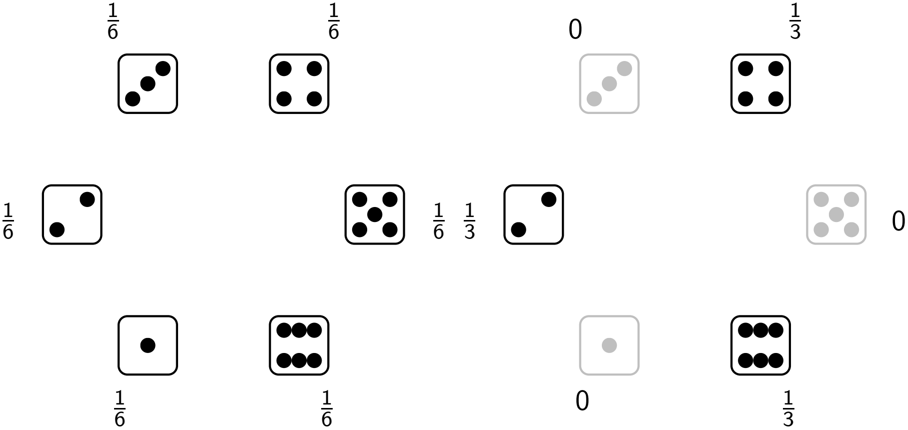
Conditional probability

$$\text{If } \mathbf{P}(B) \neq 0, \text{ then } \mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

Conditional Probability



Conditional Probability



What's the difference

generalized Ramsey Axiom
vs.
conditional certainty

$$
\begin{aligned}
[\psi]K\varphi \quad &\leftrightarrow \quad K(\psi \to [\psi]\varphi) \\
\mathbf{P}(\psi) > 0 \to (\mathit{Cert}(\varphi \mid \psi) \quad &\leftrightarrow \quad \mathit{Cert}(\psi \to \varphi))
\end{aligned}
$$

Observation

Theorem (Halpern)

The logic of single-agent certainty is KD45.

Theorem (Wajsberg)

In KD45 every formula is equivalent to a formula with modal depth 1.

Alternative semantics for public announcement

$$M, s \models [\varphi]\psi \text{ iff } M|\varphi, s \models \psi$$

$$\begin{array}{lcl} S' & = & S \\ R_a & = & \{(s,t) \mid (s,t) \in R_a \text{ and } (M,t) \models \varphi\} \\ V' & = & V \end{array}$$

Alternative semantics for public announcement

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$$\begin{array}{lcl}
S' & = & S \\
R_a & = & \{(s,t) \mid (s,t) \in R_a \text{ and } (M,t) \models \varphi\} \\
V' & = & V \\
dom(P'(a,s)) & = & \begin{cases} dom(P(a,s)) & \text{if } P(a,u)(\varphi) = 0 \\ \{v \in dom(P(a,s)) \mid (M,t) \models \varphi\} & \text{otherwise} \end{cases}
\end{array}$$

Alternative semantics for public announcement

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P'(a,u)(v) & = & \begin{cases} P(a,u)(v) & \text{if } P(a,u)(\varphi) = 0 \\ \dfrac{P(a,u)(v)}{P(a,u)(\varphi)} & \text{otherwise} \end{cases}
\end{array}$$

Example: the mumbling child

You do not know whether Mary, your three year old daughter, took a cookie or not. You assign probability 0.5 to either case. You ask Mary whether she took the cookie: you know that if she did, she will speak the truth with probability .3. (If she did not steal the cookie, she will, of course, not lie about that.)
Mary mumbles something – you did not quite get whether her answer was 'yes' or 'no.' You think that you observed her saying 'yes' with probability 0.8, but there is a 0.2 chance that she said 'no.'

Three sources of probability

- prior probabilities
- occurrence probabilities
- observation probabilities

Update models

$U = (E, R, \Phi, \text{pre}, P)$ where:

- E
- $R : A \rightarrow \wp(E \times E)$
- Φ
- pre assigns to each precondition $\varphi \in \Phi$ a probability distribution over E
- For each $a \in A$, P_a is a probability function over R_a equivalence classes in E such that $P_a(e) > 0$ for each e.

Product Update

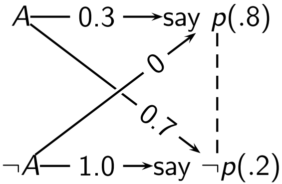
$M \otimes U = (S', R', P', V')$

- $S' = \{(s, e) \mid s \in S, e \in E \text{ and } \mathrm{pre}(s, e) > 0\}$
- $(s, e)R'_a(s', e')$ iff sR_as' and eR_ae'
- $P'_a((s, e), (s', e')) :=$

$$\frac{P_a(s)(s') \cdot \mathrm{pre}(s', e') \cdot P_a(e)(e')}{\displaystyle\sum_{s'' \in S, e'' \in E} P_a(w)(w'') \cdot \mathrm{pre}(s'', e'') \cdot P_a(e)(e'')} \text{ if denominator } > 0$$

  and 0 otherwise
- $V'((s, e)) = V(s)$

The mumbler again



Thank You!