# Business 4720 - Class 14

## Time Series

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# This Class

## What You Will Learn:

- Time Series Models
  - Time series basics
  - Smoothing methods
  - ARIMA models
  - Seasonal models
  - GARCH Models

# Based On

Robert H. Shumway and David S. Stoffer (2017) *Time Series Analysis and Its Applications*, 4th Edition. Springer.
https://www.stat.pitt.edu/stoffer/tsa4/

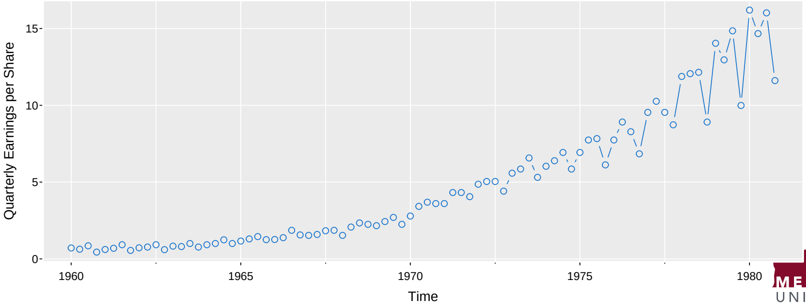
Rob J. Hyndman and George Athanasopoulos (2018) *Forecasting: Principles and Practice*, 2nd edition. OTexts.
https://otexts.com/fpp2/

## Useful Tutorials

- https://github.com/nickpoison/tsa4
- https://a-little-book-of-r-for-time-series.readthedocs.io/en/latest/src/timeseries.html
- https://rc2e.com/timeseriesanalysis
- https://atsa-es.github.io/atsa-labs/chap-tslab.html

# Time Series Application Examples

- Finance: Stock market predictions
- Economics: Unemployment rates
- Social science: School enrollment
- Natural sciences: Global temperature trends
- Ecology: Fish population forecasting
- Epidemiology: COVID incidence rates



# Characteristics of Time Series

## Time-Domain Approach

- Focuses on lagged relationships
- *Example*: How does yesterday's stock performance affect today's stock performance?

## Frequency-Domain Approach

- Focuses on cycles
- *Example*: What is the economic cycle through periods of expansion and recession?

# Time Series Statistical Models

- Moving Averages
- Autoregressive Model
- Random walk with drift
- Signal in noise

# Moving Averages

Example:

$$v_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1})$$

```
# Random numbers
w = rnorm(500,0,1)
# Moving average
v = filter(w, sides=2, filter=rep(1/3,3))
# Plot timeseries
par(mfrow=c(2,1))
tsplot(w,
    main="white noise", col=3, gg=T)
tsplot(v, ylim=c(-3,3),
    main="moving average", col=4, gg=T)
```

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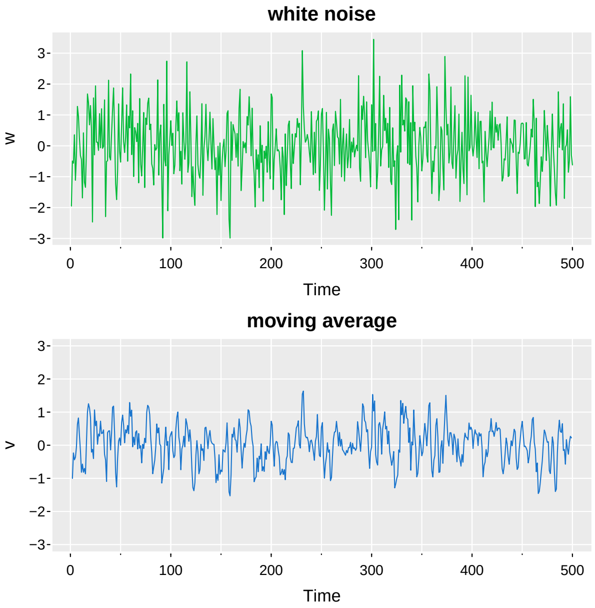
# Moving Averages [cont'd]

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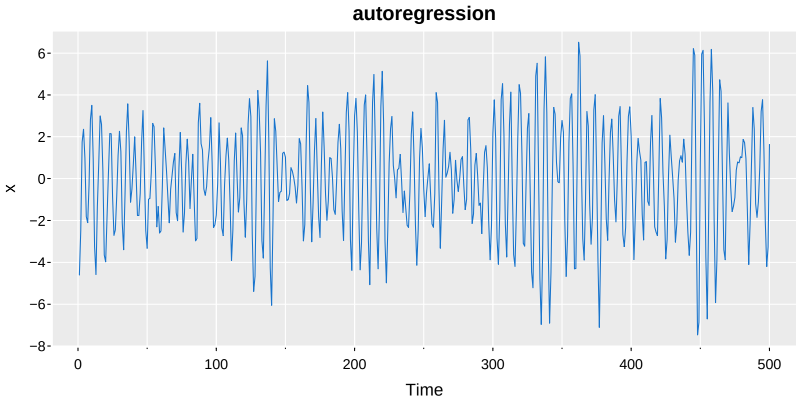
# Autoregressions

Example:

$$x_t = x_{t-1} - 0.9x_{t-2} + w_t$$

```
# Random numbers (errors)
w = rnorm(550,0,1)
# remove first 50 values for startup
x = filter(w, filter=c(1,-.9),
              method="recursive")[-(1:50)]
tsplot(x, main="autoregression", col=4, gg=T)
```

# Autoregressions [cont'd]



# Random Walk with Drift

Example:

$$
\begin{aligned}
x_t &= \delta + x_{t-1} + w_t \\
&= \delta t + \sum_{j=1}^{t} w_j
\end{aligned}
$$

```
w = rnorm(200)
x = cumsum(w)
drift = .2
w.drift = w + drift;
x.drift = cumsum(w.drift)
tsplot(x.drift, ylim=c(-10,55),
       main="random walk", ylab='', col=3, gg=T)
abline(a=0, b=drift, lty=2, col=3)
lines(x, col=4)
abline(h=0, col=4, lty=2)
```

# Random Walk with Drift [cont'd]

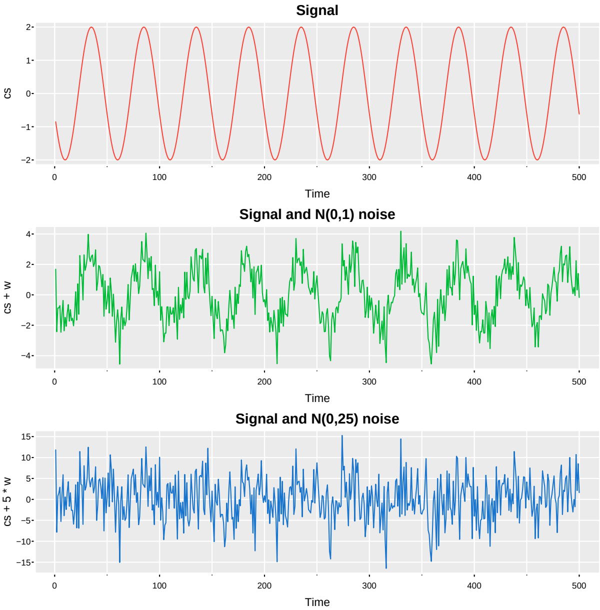
# Signal in Noise

Example:

$$
\begin{aligned}
x_t &= A\cos(2\pi\omega t + \phi) \\
A &= 2 && \text{amplitude} \\
\omega &= 1/50 && \text{frequency} \\
\phi &= .6\pi && \text{phase shift}
\end{aligned}
$$

```r
# Create signal
cs = 2*cos(2*pi*1:500/50 + .6*pi)
w = rnorm(500,0,1)
# Overlay with gaussian noise and plot
par(mfrow=c(3,1), mar=c(3,2,2,1), cex.main=1.5)
tsplot(cs,
    main='Signal', col=2, gg=T)
tsplot(cs+w,
    main='Signal and N(0,1) noise', col=3, gg=T)
tsplot(cs+5*w,
    main='Signal and N(0,25) noise', col=4, gg=T)
```

# Signal in Noise [cont'd]



# Basic Time Series Operations in R

Building a time series:

```
# Creating a time series object with monthly data
ts_data <- ts(1:24, frequency = 12, start = c(2020, 1))
```

Dealing with missing values:

```
library(zoo)
# Introduce NA values
ts_data[c(5, 10, 15)] <- NA
# Remove leading/trailing NA values
trimmed_ts <- na.trim(ts_data)
# Last Observation Carried Forward
locf_ts <- na.locf(ts_data)
# Interpolate NA values
approx_ts <- na.approx(ts_data)
```

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# Basic Time Series Operations in R

First and last observations:

```
head(ts_data)
tail(ts_data)
```

Merging time series:

```
# Creating another time series
ts_data2 <- ts(c(1:24), frequency = 12, start = c(2020, 7))

# Find intersection of two time series
intersect_ts <- ts.intersect(ts_data, ts_data2)

# Find union of two time series
union_ts <- ts.union(ts_data, ts_data2)
```

# Basic Time Series Operations in R

Lagging a series (shifting forward or backward in time):

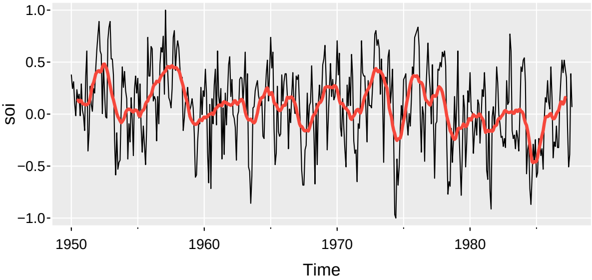
```r
# Positive k shifts backwards
lag(ts_data, 6)
# Negative k shifts forwards
lag(ts_data, -3)
```

# Smoothing a Time Series

- Moving average
- Kernel smoothing
- KNN regression / Lowess
- Smoothing splines

$$m_t = \sum_{j=-k}^{k} a_j x_{t-j} \quad \text{where} \quad \sum_{j=-k}^{k} a_j = 1$$

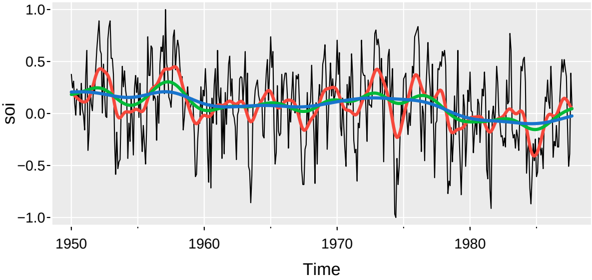
```
f = c(.5, rep(1, 11), .5)/12
filter(soi, sides=2, filter=f)
```



# Kernel Smoothing

$$a_i(t) = K\left(\frac{t-i}{b}\right) / \sum_{j=1}^{n} K\left(\frac{t-j}{b}\right) \text{ and } K(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right)$$

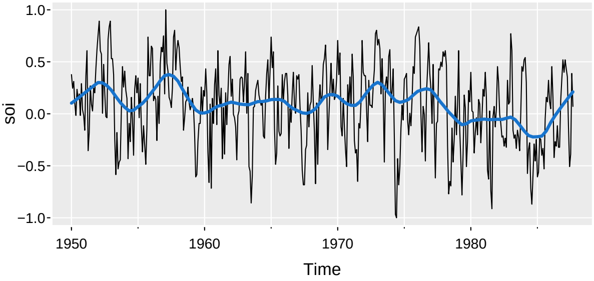
```
ksmooth(time(soi), soi, 'normal', bandwidth=1)
```



# Locally Weighted Regression

- Identify proportion *f* of closest points
- Use weighted least squares regression to predict values

```
lowess(soi, f=0.1)
```

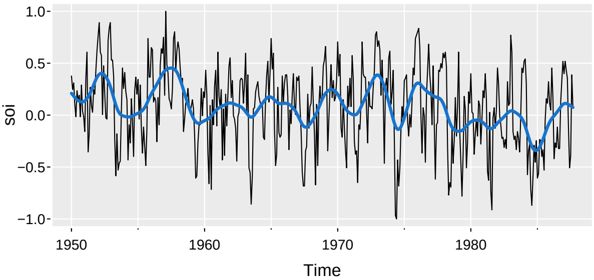


# Smoothing Splines

- Penalized polynomial regression
- Fit $m_t = \beta_0 + \beta_1 t + \beta_2 t^2$
- Minimize loss function:

$$\sum_{t=1}^{n}(x_t - m_t)^2 + \lambda \int \left(\frac{\mathrm{d}^2 m}{\mathrm{d}t^2}\right)^2 dt$$

```
smooth.spline(time(soi), soi, spar=0.5)
```



# Hands-On Exercises

1. Generate 100 observations from the autoregression model $x_t = -.9x_{t-2} + w_t$ with $\sigma_w^2 = 1$
   1.1 Smooth the time series using a moving average filter $v_t = (x_t + x_{t-1} + x_{t-2} + x_{t-3})/4$
   1.2 Plot $x_t$ as a line and superimpose $v_t$
   1.3 Comment on the behaviour of $x_t$ and how applying the moving average filter changes that behavior
2. Repeat the smoothing but with $x_t = \cos(2\pi t/4)$
3. Repeat the smoothing but with added $N(0, 1)$ noise, that is smooth $x_t = \cos(2\pi t/4) + w_t$
4. Compare and contrast the three exercises: How does the moving average change each series

---

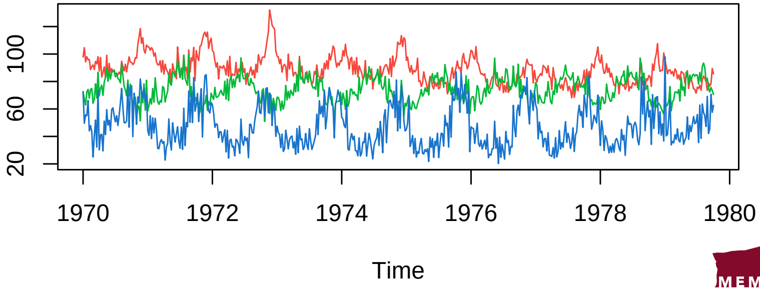
- Use `x = filter(w, filter=c(a, b), method="recursive")`
- Use `v = filter(x, rep(1/4, 4), sides=1)` as above.
- Use `c = cos(2*pi*1:500/4)` as above.
- Use `lines(...,lty=2)` to add lines to a plot

# Time Series Regression Example

Epidemiology Example:

- Cardiovascular mortality `cmort`
- Temperate `tempr`
- Particulate pollution `part`

```r
# IMPORTANT: Use ts.plot, NOT tsplot:
ts.plot(cmort, tempr, part, col=2:4)
```



# Time Series Regression Example [cont'd]

Fit different linear regression models (at the same time points):

```
# Center Temperature
temp = tempr - mean(tempr)

# Square Temperature
temp.2 = temp^2

# Fit different linear models and provide summaries
summary(lm(cmort ~ time(cmort)))
summary(lm(cmort ~ time(cmort) + temp))
summary(lm(cmort ~ time(cmort) + temp + temp.2))
summary(lm(cmort ~ time(cmort) + temp + temp.2 + part))
```

```
# Lag the temperature
temp.l.2 = lag(temp, 2)
temp.l.4 = lag(temp, 4)

# Intersect all time series to
# omit leading/trailing NA
temp.df <- ts.intersect(cmort, time(cmort), part,
                        temp, temp.2, temp.l.2,
                        temp.l.4,
                        dframe=TRUE)
# Fit the linear model including lagged temperature
summary(lm(cmort ~ time.cmort. + temp + temp.2 +
                   temp.l.2 + temp.l.4 + part,
           data=temp.df))
```

# Strict Stationarity

A **strictly stationary** time series is one for which the probabilistic behaviour of every collection of values

$$\{x_{t1}, x_{t2}, \ldots, x_{tk}\}$$

is identical to that of the shifted set

$$\{x_{t1+h}, x_{t2+h}, \ldots, x_{tk+h}\}.$$

That is,

$$\Pr\{x_{t1} \leq c_1, \ldots, x_{tk} \leq c_k\} = \Pr\{x_{t1+h} \leq c_1, \ldots, x_{tk+h} \leq c_k\}$$

# Weak Stationarity

A **weakly stationary** time series is a finite variance process such that

1. the mean is constant and does not depend on time: $\mu_t = \mu$
2. the autocovariance $\gamma$ depends on $s$ and $t$ only through their difference $h = |s - t|$.

Let $s = t + h$, then:

$$\gamma(s, t) = \gamma(t + h, t) = cov(x_{t+h}, x_t) = cov(x_h, x_0) = \gamma(h)$$

and

$$\rho(h) = \gamma(h)/\gamma(0)$$

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# Measures of Dependence

## Autocovariance

- Covariance between two points $t, t+h$ on time series $x$

$$\gamma(h) = cov(x, x_{t+h}) = E[(x_t - \mu)(x_{t+h} - \mu)]$$

- Large autocovariance $\rightarrow$ smooth time series
- Small autocovariance $\rightarrow$ choppy time series

## Sample Autocovariance for Lag $h$

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_t - \bar{x})(x_{t+h} - \bar{x})$$

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# Measures of Dependence [cont'd]

## Autocorrelation Function (ACF)

$$\rho_x(h) = \frac{\gamma(t+h,t)}{\sqrt{\gamma(t+h,t+h)\gamma(t,t)}} = \frac{\gamma(h)}{\gamma(0)} \quad \text{(weak stationarity)}$$

## Sample Autocorrelation for Lag *h*

$$\hat{\rho}_x(h) = \frac{\hat{\gamma}(h)}{\sqrt{\hat{\gamma}(h)\hat{\gamma}(0)}} = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \quad \text{(weak stationarity)}$$

# Measures of Dependence [cont'd]

The large-sample distribution of $\hat{\rho}_x(h)$ is normal with mean 0 and standard deviation

$$\sigma_{\hat{\rho}_x} = 1/\sqrt{n}$$

*if the processes is independent white noise*.

Hence, the approximate 95% confidence interval on the ACF is

$$-\frac{1}{n} \pm \frac{2}{\sqrt{n}}$$

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# Measures of Dependence [cont'd]

## Partial Autocorrelation Function (PACF)

The PACF is the correlation between $x_{t+h}$ and $x_t$ with the linear dependence of $\{x_{t+1}, \ldots, x_{t+h-1}\}$ on each, removed:

$$\phi_{hh} = \begin{cases} \rho(1) & h = 1 \\ \operatorname{corr}(x_{t+h} - \hat{x}_{t+h}, x_t - \hat{x}_t) & h \geq 2 \end{cases}$$

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# Autocorrelation Function (ACF) Example 1

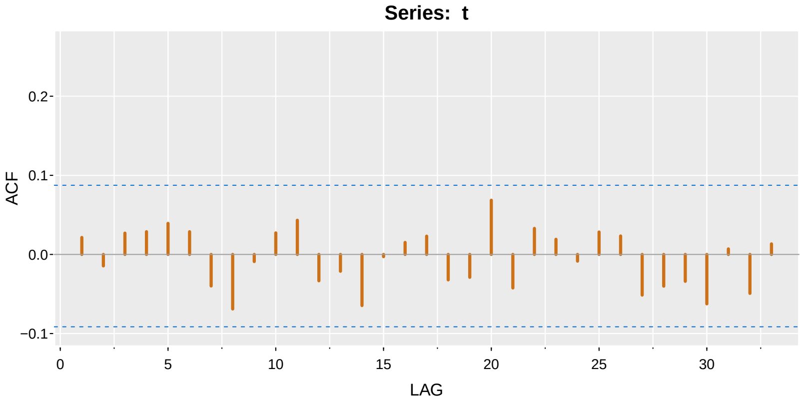
Gaussian white noise example:

```
library(astsa)
t <- ts(rnorm(500))

# The hard way:
cor(ts.intersect(t, lag(t,1), dframe=T))
cor(ts.intersect(t, lag(t,2), dframe=T))
# etc.

# The easy way:
# Without plot
acf <- acf1(t, plot=FALSE)
# With plot
acf1(t, gg=T, col=7, lwd=3)
```

# Autocorrelation Function (ACF) Example 1 [cont'd]



# Autocorrelation Function (ACF) Example 2

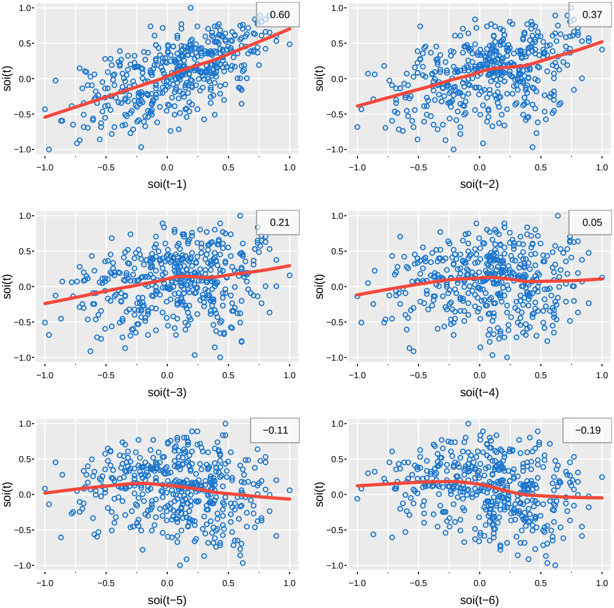
Example using the `soi` data set (sea level air pressure index) from the `astsa` library:

```
library(astsa)
?soi

# Compute and plot the ACF for different lags
acf1(soi, gg=T, co=3, lwd=2)

# Scatterplot of original versus
# or lags up to 6, with ACF values
lag1.plot(soi, max.lag = 6,
    gg=T, col=4, lwl=3)
```

# Autocorrelation Function (ACF) Example 2 [cont'd]



# Dealing with Non-Stationarity

## Log Transformation

$$y_t = \log x_t$$

## Box-Cox power transformation

$$y_t = \begin{cases} (x_t^\lambda - 1)/\lambda & \lambda \neq 0 \\ \log x_t & \lambda = 0 \end{cases}$$

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# Dealing with Non-Stationarity

Assume that $x_t = \mu_t + y_t$ where $\mu_t$ is the trend and $y_t$ a stationary process.

## Detrending

- Estimate trend, e.g. with an LM such as $\mu_t = \beta_0 + \beta_1 t$
- Work with residuals, e.g. $\hat{y}_t = x_t - \hat{\mu}_t = x_t - \hat{\beta}_0 - \hat{\beta}_1 t$

```
# Simulate a time series with a linear trend
t <- ts(1:100 + rnorm(100) * 10)
# Fit a linear model to the time series
trend_model <- lm(t ~ time(t))
# Calculate detrended series
detrended <- residuals(trend_model)
# Plot original and detrended
par(mfrow=c(2,1))
tsplot(t, type="l", main="original",col=3,gg=T)
tsplot(detrended, type="l", main="detrend",col=2,gg=T)
```

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# Detrending Example

# Dealing with Non-Stationarity

Assume that $x_t = \mu_t + y_t$ where $\mu_t$ is the trend and $y_t$ a stationary process.

## Differencing

- Model the trend stochastically as a random walk with drift: $\mu_t = \delta + \mu_{t-1} + w_t$ where $w_t$ is white noise
- Differencing then yields $x_t - x_{t-1} = (\mu_t + y_t) - (\mu_{t-1} + y_{t-1}) = \delta + w_t + y_t - y_{t-1}$ which is stationary

- First difference eliminates linear trend
- Second difference eliminates quadratic trend
- ...

# Differencing [cont'd]

## Difference Operator

$$\nabla x_t = x_t - x_{t-1}$$

## Backshift / Lag Operator

$$\begin{aligned}
\mathrm{B}\, x_t &= x_{t-1} \\
\mathrm{B}^k\, x_t &= x_{t-k} \\
\nabla x_t &= (1 - \mathrm{B}) x_t \\
\nabla^2 x_t &= (1 - \mathrm{B})^2 x_t = (1 - 2\,\mathrm{B} + \mathrm{B}^2) x_t = x_t - 2x_{t-1} + x_{t-2} \\
\nabla^d &= (1 - \mathrm{B})^d
\end{aligned}$$

# Differencing in R

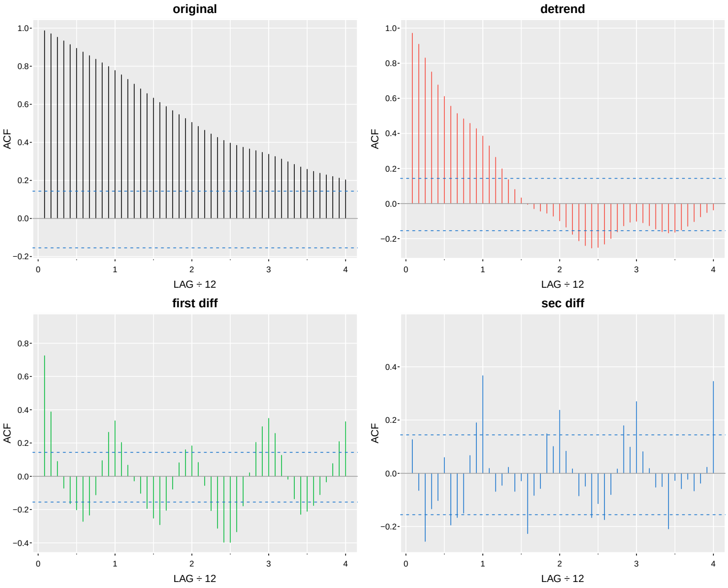
```
set.seed(42)
# Simulating a time series with trend
t <- ts(cumsum(rnorm(100)))

tsplot(diff(t, differences = 1), type="l",
    main="first difference", col=4,gg=T)
tsplot(diff(t, differences = 2), type="l",
    main="second difference", col=5,gg=T)
```

# Detrending and Differencing in R

Compare the ACF of the original, detrended, and differenced series:

```r
acf1(chicken, max.lag=48,
     main="original", col=1, gg=T)
acf1(resid(fit), max.lag=48,
     main="detrend", col=2, gg=T)
acf1(diff(chicken), max.lag=48,
     main="first diff", col=3, gg=T)
acf1(diff(chicken, differences=2), max.lag=48,
     main="sec diff", col=4, gg=T)
```

# Detrending and Differencing in R



# Hands-On Exercises

1. Extend the mortality, temperature and pollution/particulate model by adding another component to the regression that accounts to the particulate four weeks prior; that is, add $P_{t-4}$ to the regression.
2. Draw a scatterplot matrix of $M_t$, $T_t$, $P_t$ and $P_{t-4}$, then calculate the pairwise correlations between them. Compare the relationship between $M_t$ and $P_t$ versus $M_t$ and $P_{t-4}$
3. Detrend the `soi` time series data by fitting a regression of $S_t$ on time $t$. Is there a significant trend in the surface pressure?
4. Use two different smoothing techniques to estimate the trend in the global temperature series `gtemp_both` in the `astsa` library.

Consider the two weekly time series `oil` and `gas` in the `astsa` library. The oil series is in dollars per barrel, while the gas series in in cents per gallon.

1. Plot the data on the same graph. Do you believe the series are stationary?
2. Apply the transformation $y_t = \nabla \log x_t$ to the data for both series
3. Plot the transformed series on the same graph, and calculate the ACFs for both series
4. Plot the CCF of the transformed series and comment.

# ARIMA Models

- **AR**: Autoregressive models
- **MA**: Moving average models
- **ARMA**: Autoregressive moving-average models
- **ARIMA**: Autoregressive integrated moving-average models (for non-stationary models with trend)

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# AR(p) Models

An **autoregressive model** of order $p$ is of the form:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$$

where $w_t$ is white noise and $\phi_i$ are model parameters.

In contrast to an ”ordinary” regression model, the $x_i$ are random effects, not fixed, because each $x_i$ has an associated error term $w_t$.

The **autoregressive operator** is defined using the backshift operator as

$$\begin{aligned}\phi(B) &= 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \\ &= \left(1 - \sum_{j=1}^{p} \phi_j B^j\right)\end{aligned}$$

so that the AR(p) model becomes:

$$\phi(B) x_t = w_t$$

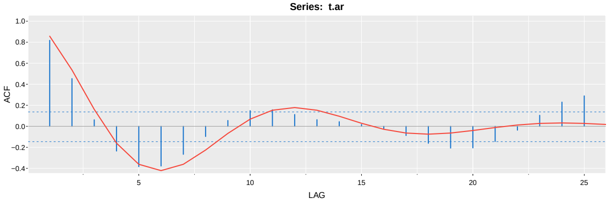
# AR(p) Models – Theoretical ACF and Simulations

## Theoretical ACF of an AR(2) model

```
ARMAacf(ar=c(1.5, -.75), lag.max=10)
```

## Simulate an ARIMA(2,0,0) model with those AR coefficients

```
t.ar = arima.sim(list(ar=c(1.5, -.75)), n=200)
# Compute and plot the ACF of the simulated series
acf1(t.ar, max.lag=25, gg=T, lwd=2, col=4)
# Add the theoretical values for comparison
lines(ARMAacf(ar=c(1.5, -.75), lag.max=26)[-1],
      lwd=2, col=2)
```



# MA(p) Models

A **moving average model** of order $q$ is defined as:

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

where $w_t$ are Gaussian errors and $\theta_i$ are model parameters.

The **moving average operator** is defined using the backshift operator as

$$\begin{aligned}\theta(B) &= 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q \\ &= \left(1 + \sum_{j=1}^{q} \theta_i B^i\right)\end{aligned}$$

so that the MA(q) model becomes:

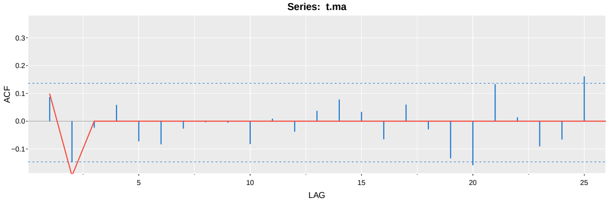
$$x_t = \theta(B) w_t$$

Theoretical ACF of an MA(2) model

```
ARMAacf(ma=c(1.5, -.75), lag.max=10)
```

Simulate an ARIMA(0,0,2) model with those MA coefficients

```
t.ma = arima.sim(list(ma=c(1.5, -.75)), n=200)
# Compute and plot the ACF of the simulated series
acf1(t.ma, gg=T, lwd=2, col=4)
# Add the theoretical values for comparison
lines(ARMAacf(ma=c(1.5, -.75), lag.max=26)[-1],
      lwd=2, col=2)
```



# ARMA(p, q) Models

A time series is **ARMA(p,q)** if it is stationary and

$$x_t = \alpha + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

where $\phi_p \neq 0, \theta_q \neq 0$, $\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$ and $w_t$ is Gaussian.

This can be written as

$$\phi(B) x_t = \theta(B) w_t$$

# Equivalent Models

## ARMA to MA

Every ARMA model has an equivalent (infinite) MA model

```
ARMAtoMA(ar = c(-.5), ma = c(-.9), lag.max=10)
```

## ARMA to AR

An invertible ARMA model has an equivalent (infinite) AR model with coefficients $pi_j$ and $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$.

```
library(astsa)
ARMAtoAR(ar = c(-.5), ma = c(-.9), lag.max=10)
```

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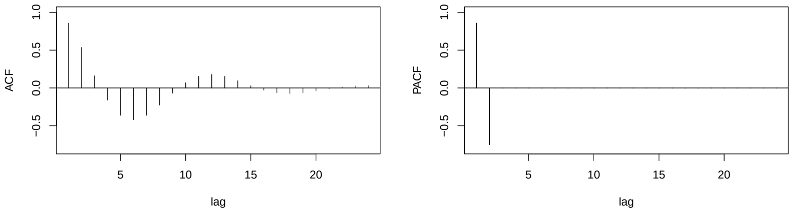
# Partial Autocorrelation Function (PACF)

```
ACF = ARMAacf(ar=c(1.5,-.75), ma=0, 24)[-1]
PACF = ARMAacf(ar=c(1.5,-.75), ma=0, 24, pacf=TRUE)

plot(ACF, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
plot(PACF, type="h", xlab="lag", ylim=c(-.8,1))
abline(h=0)
```



# ARMA(p, q) Models – Model Selection

| | AR(p) | MA(q) | ARMA (p, q) |
|---|---|---|---|
| ACF | Tails off | Cuts off after laq $q$ | Tails off |
| PACF | Cuts off after lag $p$ | Tails off | Tails off |

Source: Shumway&Stoffer, Table 3.1

# ARIMA(p, d, q) Models

If the AR operator can be factorized by $(1 - B)$, then:

$$\phi(B) = \left(1 - \sum_{j=1}^{p'} \phi_j B^j\right) = \left(1 - \sum_{j=1}^{p'-d} \phi_j B^j\right)(1 - B)^d$$

With $p = p' - d$, the **ARIMA(p,d,q)** model is then:

$$\left(1 - \sum_{j=1}^{p} \phi_j B^j\right)(1 - B)^d x_t = \left(1 + \sum_{j=1}^{q} \theta_j B^j\right) w_t$$

This can be generalized to:

$$\left(1 - \sum_{j=1}^{p} \phi_j B^j\right)(1 - B)^d x_t = \delta + \left(1 + \sum_{j=1}^{q} \theta_j B^j\right) w_t$$

where $\delta = \mu(1 - \phi_1 - \cdots - \phi_p)$

# Building ARIMA Models

1. Plot the data
2. Possibly transform the data
3. Assess stationarity
4. Possibly difference the data
5. Identify the dependence orders (p, q) of the model
6. Estimate parameters
7. Model diagnostics
8. Model selection

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# Identifying Dependence Order

## Identifying $d$

- Slow decay in sample ACF $\hat{\rho}(h)$ indicates need for differencing
- Over-differencing can introduce dependence where non exists
- Difference once using `diff(x, differences = 1)`, then check ACF again

## Preliminary $p$ and $q$

- Examine sample ACF and PACF of differenced data $\nabla^d x_t$

| | AR(p) | MA(q) | ARMA (p, q) |
|---|---|---|---|
| ACF | Tails off | Cuts off after lag $q$ | Tails off |
| PACF | Cuts off after lag $p$ | Tails off | Tails off |

Source: Shumway&Stoffer, Table 3.1

# Example

Examine data and transform:

```
# Plot data
plot(gnp)
# Plot ACF
acf2(gnp, 50)
# Log transform, and first order differencing
gnpgr = diff(log(gnp))
# Plot transformed and differenced data
plot(gnpgr)
# Plot ACF of transformed and differenced data
acf2(gnpgr, 24)
```

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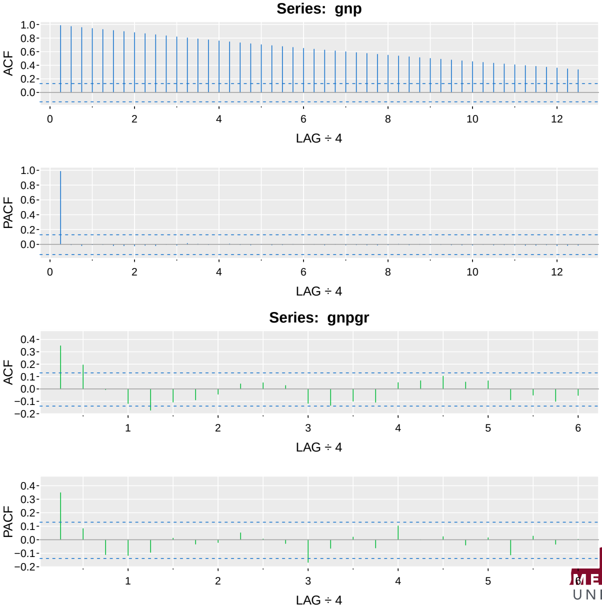
# Example [cont'd]

Original
ACF and
PACF

Transformed
and differ-
enced
ACF and
PACF

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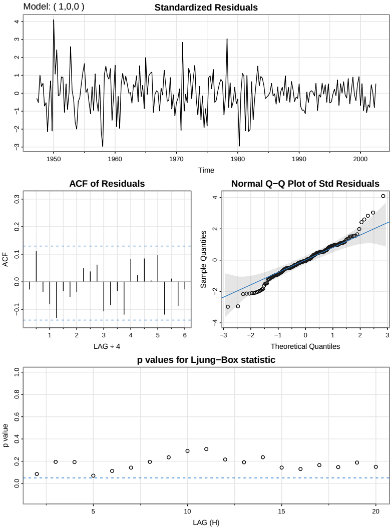
# Example [cont'd]

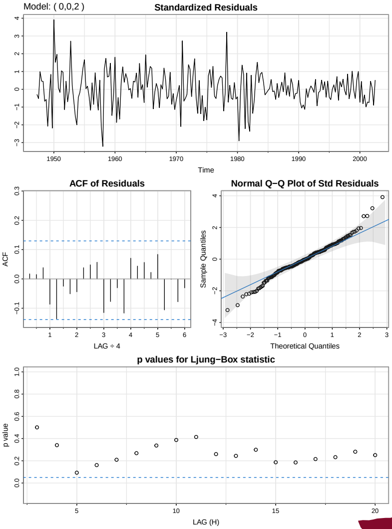
Fit initial models

```
# Fit an AR(1) model
sarima(gnpgr, 1, 0, 0)
# Fit an MA(2) model
sarima(gnpgr, 0, 0, 2)
# Models are roughly equivalent
ARMAtoMA(ar=0.35, ma=0, 10)
```

The models show similar fit and all coefficients are significant.

# Example [cont'd]

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# Example [cont'd]

## Diagnostics

- Standardized residuals are Gaussian ($\mu = 0$, $sd = 1$)
- Residuals are not autocorrelated
- Residual ACF are Gaussian with $\mu = 0$ and $sd = 1/\sqrt{n}$
- Ljung-Box statistic $Q$ of the error ACF $\hat{\rho}_e$ for different maximum lags $H$ is larger than the $1 - \alpha$ quantile of the $\chi^2_{H-p-q}$ distribution (i.e. the test statistic is not statistically signifantly different from 0)

$$Q = n(n+2) \sum_{h=1}^{H} \frac{\hat{\rho}_e^2(h)}{n-h}$$

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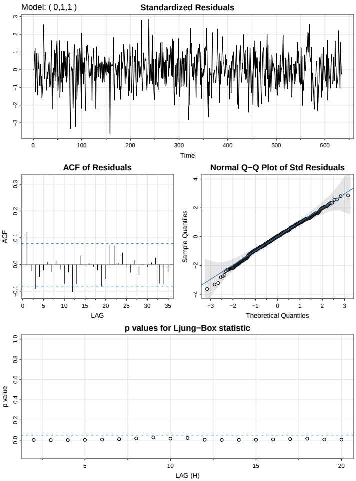
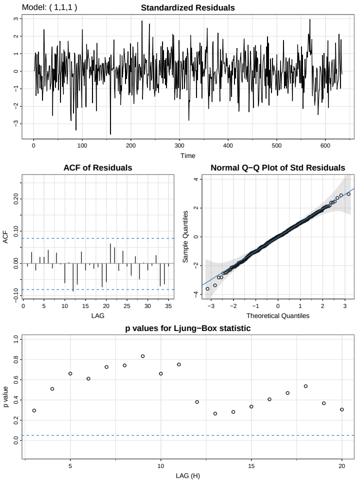
# Mis-Fit Example

```
sarima(log(varve), 0, 1, 1, no.constant=T)
sarima(log(varve), 1, 1, 1, no.constant=T)
```



# Example [cont'd]

## Model Selection

For MLE estimated models, model choice is typically based on information criteria

- Functions of the log-likelihood $L$
- Adjusted for model complexity (number of parameters) $k$
- Adjusted for sample size $n$
- Express *relative* quality of fit: **Smaller is better**

$$\text{AIC} = -2 \log L + 2k$$

Akaike Information Criterion

$$\text{AICc} = AIC + \frac{2k(k+1)}{n-k-1}$$

Akaike Information Criterion, corrected

$$\text{BIC} = -2 \log L + k \log n$$

Bayesian Information Criterion

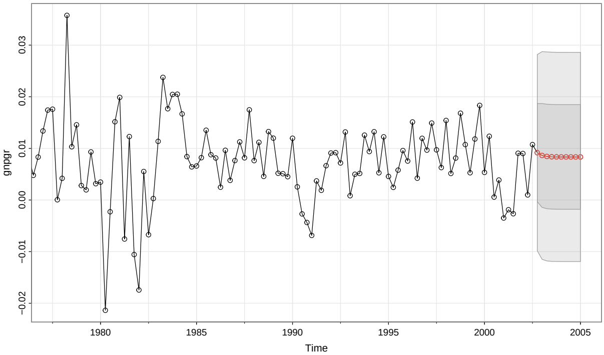
# Example [cont'd]

```
> sarima(gnpgr, 1, 0, 0)

AIC = -6.44694  AICc = -6.446693  BIC = -6.400958
```

```
> sarima(gnpgr, 0, 0, 2)

AIC = -6.450133  AICc = -6.449637  BIC = -6.388823
```

# Example – Forecasting

```
forecasts <- sarima.for(gnpgr, n.ahead=10, p=1,d=0,q=0)
```




ARMA predictions quickly settle to the mean with a constant prediction error

# General Autoregressive Conditional Heteroscedastic (GARCH) Models

ARCH models the variance of a series of returns:

$$r_t = \frac{x_t - x_{t-1}}{x_{t-1}} \qquad \text{("Return")}$$

**Example**: ARCH(1) Model – The variance depends on the prior return.

$$r_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2$$

where $\epsilon_t$ is Gaussian.

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# Example: Combined AR(1) and ARCH(1) Model

AR(1) Model with ARCH(1) Errors:

$$x_t = \phi_0 + \phi_1 x_{t-1} + \sigma_t \epsilon_t \quad \text{where} \quad \sigma_t = \alpha_0 + \alpha_1 x_{t-1}^2$$

Example: US GNP Data

```
u = sarima(diff(log(gnp)), 1, 0, 0)
acf2(resid(u$fit)^2, 20)
```

Squared residuals may have some dependence.

```
library(fGarch)
summary(garchFit(~arma(1,0)+garch(1,0), diff(log(gnp))))
```

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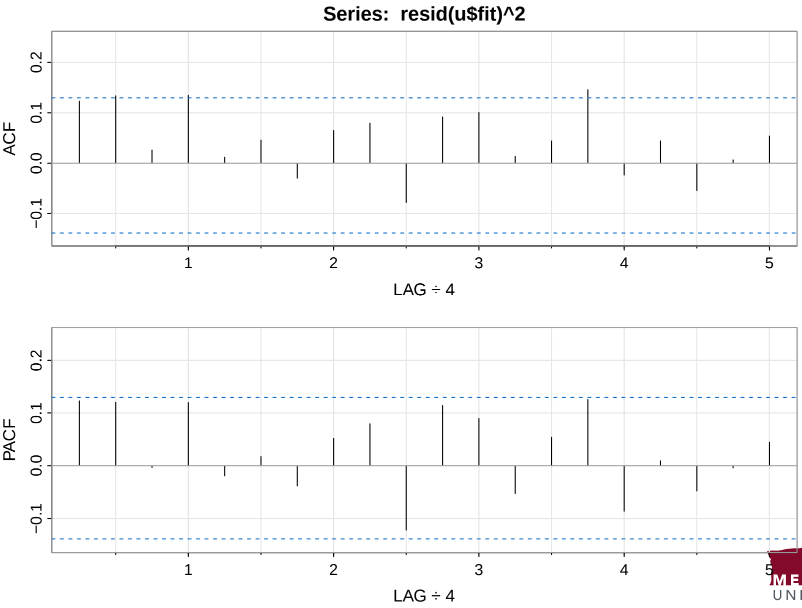
# Example: Combined AR(1) and ARCH(1) Model



# ARCH Extensions

## ARCH(q) Model

Extend the ARCH(1) Model to multiple previous returns:

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \cdots + \alpha_p r_{t-p}^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i r_{t-i}^2$$

## GARCH(p, q) Model

Variance depends also on prior variances:

$$\begin{aligned}\sigma_t^2 &= \omega + \alpha_1 r_{t-1}^2 + \cdots + \alpha_q r_{t-q}^2 \\ &\qquad + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2 \\ &= \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2\end{aligned}$$

# GARCH Example – DJIA Returns

```r
library(fGarch)
# Log transform
djiar = diff(log(djia$Close))[-1]
# Fit an AR(1) + GARCH(1,1) model
djia.g <- garchFit(~arma(1,0)+garch(1,1), data=djiar)
# Show summary information
summary(djia.g)
# Different plots available
plot(djia.g, which=3)
```

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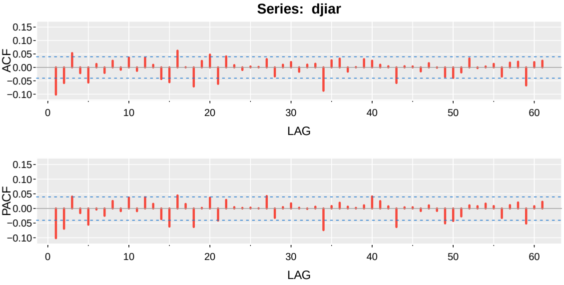
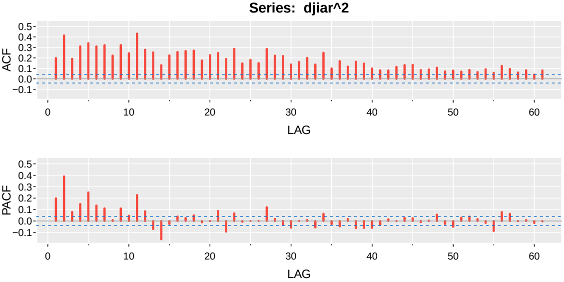


Series: djiar
ACF
PACF
LAG

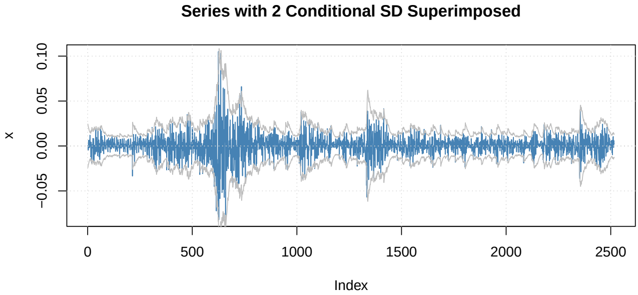



Series: djiar^2
ACF
PACF
LAG

# GARCH Example – DJIA Returns [cont'd]

## Results

```
         Estimate  Std. Error  t value Pr(>|t|)
mu      8.585e-04   1.470e-04    5.842 5.16e-09 ***
ar1    -5.532e-02   2.023e-02   -2.735 0.006238 **
omega   1.610e-06   4.459e-07    3.611 0.000305 ***
alpha1  1.244e-01   1.660e-02    7.496 6.55e-14 ***
beta1   8.700e-01   1.526e-02   57.022  < 2e-16 ***
shape   5.979e+00   7.917e-01    7.551 4.31e-14 ***
---
Log Likelihood:
 8249.619    normalized:  3.27756
```



# Appendix – Basic Time Series Functions in R

| | |
|---|---|
| `filter` | Filters time series, through moving averages or autoregression |
| `lag` | Creates a lagged version of a time series by shifting the time-base back |
| `diff` | Creates lagged differences |
| `plot.ts` | Plot a time series |
| `ts.plot` | Plot multiple time series |
| `lag.plot` | Scatterplot of lagged values |
| `acf` | ACF and plot |
| `ccf` | CCF and plot |

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# Appendix – Basic Time Series Functions in R

| | |
|---|---|
| `time` | Creates the vector or times at which a time series was sampled |
| `cycle` | Gives the positions in the cycle of each observation |
| `frequency` | Number of samples per unit time |
| `ts.intersect` | Bind time series together that have a common frequency. Restrict to time covered by all series |
| `ts.union` | Bind time series together that have a common frequency. Pad with NA if necessary |
| `ar` | Fit an autoregressive model |
| `arima` | Fit an ARIMA model |

# Appendix – Time Series Functions in the `astsa` library

| | |
|---|---|
| `tsplot` | Plot a time series |
| `acf1` | ACF and plot |
| `ccf2` | CCF and plot |
| `sarima` | Fit seasonal ARIMA models (and nice diagnostic plots) |
| `lag1.plot` | Scatterplot of lagged values |

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