Statistics Review

EC 320, Set 02

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Spring 2023

Prologue

Housekeeping

The first lab assignment is due Friday at 5p

Due dates

- Problem sets: Tuesday nights at 11:59p
- Lab assignments: Fridays at 5p

The first analytical problem set is due Tuesday. I will post it later tonight. Very short. Very simple

Any issues with R?

- I have office hours Tomorrow from 4p-5p,
- Colleen has office hours Friday, 8a-9a

Motivation

The focus of our course is **regression analysis**–part of the fundamental toolkit for learning from data.

The **underlying theory** is critical to grasp the mechanics and pitfalls

- Make us better practitioners and savvier consumers of science.

Today: Review the essential concepts from Math 243

Warning.

The following review is a lot packed in very briefly though you *should* have learned much of it before. But that being said, it will be overwhelming for most.

Notation

Notation

Data on a variable X are a sequence of n observations, indexed by i:

$$\{x_i : 1, \ldots, n\}.$$

Ex. $n = 5$

i	x_i
1	8
2	9
3	4
4	7
5	2

- i indicates the row number.
- n is the number of rows.
- x_i is the value of X for row i.

Summation

The **summation operator** adds a sequence of numbers over an index:

$$\sum_{i=1}^{n} x_i \equiv x_1 + x_2 + \cdots + x_n .$$

> The sum of x_i from 1 to n.

i	x_i
1	7
2	4
3	10
4	3

$$\sum_{i=1}^{4} x_i = 7 + 4 + 10 + 3 = 23$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i \rightarrow \frac{1}{4}\sum_{i=1}^{4} x_i = 6$$

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i	x_i
1	7
2	4
3	10
4	3

$$\sum_{i=1}^{4} x_i = 7 + 4 + 10 + 3 = 23$$

$$\text{sample average}\left\{ \frac{1}{n}\sum_{i=1}^{n} x_i \rightarrow \frac{1}{4}\sum_{i=1}^{4} x_i = 6 \right.$$

Summation: Rule 01

For any constant c,

$$\sum_{i=1}^{n} c = nc.$$

i	c
1	2
2	2
3	2
4	2

$$\sum_{i=1}^{4} 2 = 4 \times 2$$

$$= 8$$

Summation: Rule 02

For any constant c,

$$\sum_{i=1}^{n} c x_i = c \sum_{i=1}^{n} x_i .$$

i	c	x_i
1	2	7
2	2	4
3	2	10

$$\sum_{i=1}^{3} 2x_i = 2 \times 7 + 2 \times 4 + 2 \times 10$$

$$= 14 + 8 + 20 = 42$$

$$2 \sum_{i=1}^{3} x_i = 2(7 + 4 + 10) = 42$$

Summation: Rule 03

If $\{(x_i, y_i) : 1, \ldots, n\}$ is a set of n pairs, and a and b are constants, then

$$\sum_{i=1}^{n}(ax_i + by_i) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i.$$

i	a	x_i	b	y_i
1	2	7	1	4
2	2	4	1	2

$$\sum_{i=1}^{2}(2x_i + y_i) = 18 + 10 = 28$$

$$2\sum_{i=1}^{2} x_i + \sum_{i=1}^{2} y_i = 2 \times 11 + 6 = 28$$

Summation: Caution 01

The **sum of the ratios** is not the **ratio of the sums:**

$$\sum_{i=1}^{n} x_i/y_i \neq \left(\sum_{i=1}^{n} x_i\right) \Bigg/ \left(\sum_{i=1}^{n} y_i\right)$$

Ex.

If $n = 2$, then $\frac{x_1}{y_1} + \frac{x_2}{y_2} \neq \frac{x_1+x_2}{y_1+y_2}$.

Summation: Caution 02

The **sum of squares** is not the **square of the sums:**

$$\sum_{i=1}^{n} x_i^2 \neq \left(\sum_{i=1}^{n} x_i \right)^2 .$$

Ex.

If $n = 2$, then $x_1^2 + x_2^2 \neq (x_1 + x_2)^2 = x_1^2 + 2x_1 x_2 + x_2^2$.

Cartesian coordinate system

Cartesian plane: 2-D plane defined by two perpendicular number lines:

- x-axis (*horizontal*)
- y-axis (*vertical*)

Using these axes, any point in the plane is described using an ordered pair of numbers (x, y)

Cartesian coordinate system

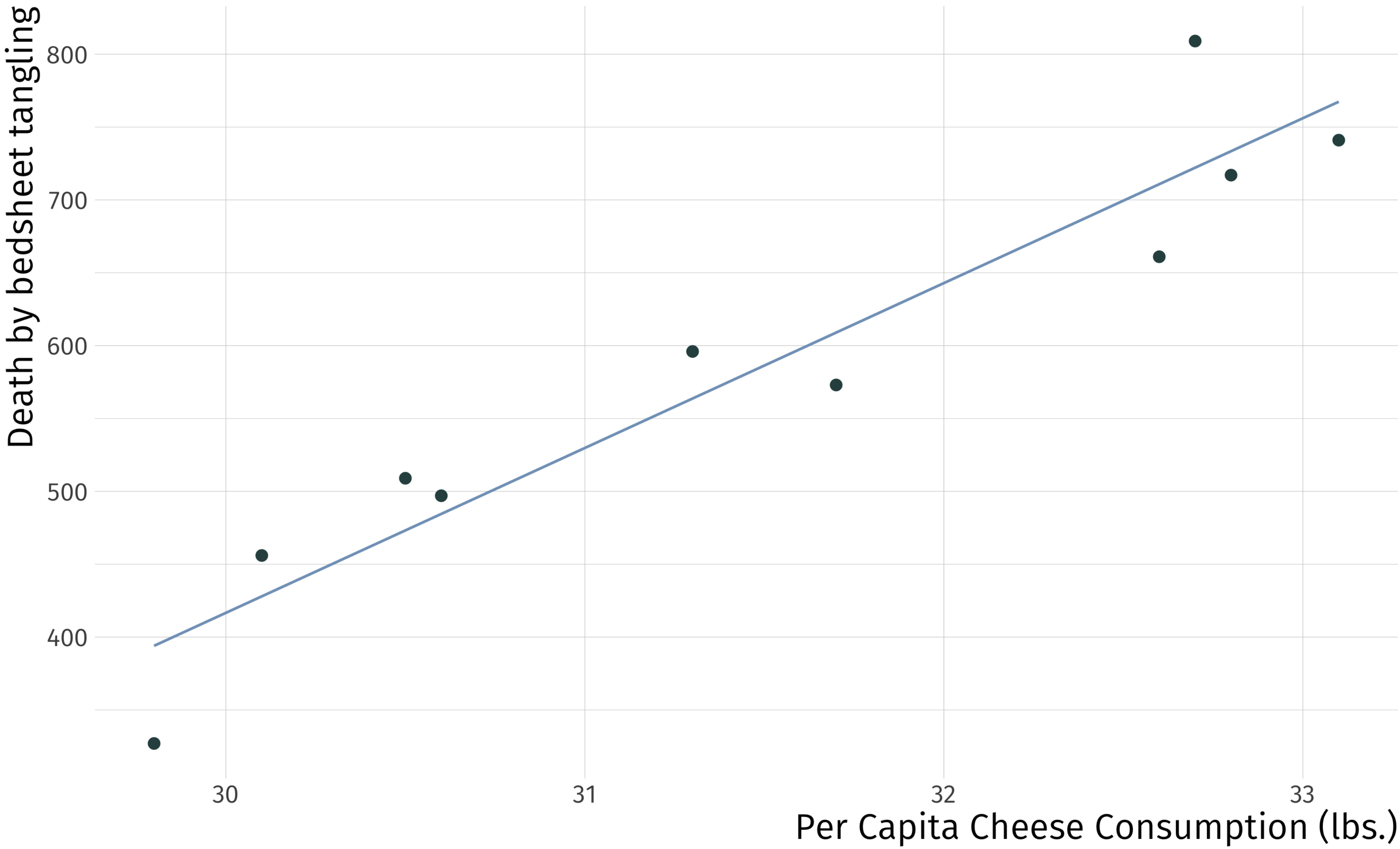
A particular line on this plane takes the form

$$y = a + bx$$

where a is known as the intercept and b is the slope.

Any incremental unit increase in x results in y increasing by b.

Ex.



Death by bedsheet tangling
800
700
600
500
400
30
31
32
33
Per Capita Cheese Consumption (lbs.)

Basic probability

Essential definitions

Experiment:

> Any procedure that is *infinitely repeatable* and has a *well-defined set of outcomes*.

Ex. Flip a coin 10 times and record the number of heads.

Random Variable:

> A variable with *numerical values determined by an experiment or a random phenomenon*.

- Describes the sample space of an experiment.

Essential definitions

Sample Space:

> The set of potential outcomes an experiment could generate

Ex. The sum of two dice is an integer from 2 to 12.

Event:

> A subset of the sample space or a combination of outcomes.

Ex. Rolling a two or a four.

Random variables

Notation: Capital letters for random variables (*e.g.*, X, Y, or Z) and lowercase letters for particular outcomes (*e.g.*, x, y, or z).

Experiment

Flipping a coin.

Events:

Heads or tails.

Random Variable: (X)

Receive $1 if heads, $x_i = 1$, pay $1 if tails, $x_i = -1$

Sample Space:

$\{-1, 1\}$

Discrete random variables

> A random variable that takes a countable set of values.

Bernoulli (*binary*) random variable

> Random variable that takes values of either 1 or 0.

- Characterized by $P(X = 1)$, "the probability of success."
- Probabilities sum to 1: $P(X = 1) + P(X = 0) = 1$
- More generally, if $P(X = 1) = \theta$ for some $\theta \in [0, 1]$, then $P(X = 0) = 1 - \theta$.

Discrete Random Variables: Probabilities

We describe a discrete random variable by listing its possible values with associated probabilities.

If X takes on k possible values $\{x_1, \ldots, x_k\}$, then the probabilities $p_1, p_2, \ldots, p_k$ are defined by

$$p_j = P(X = x_j), \quad j = 1, 2, \ldots, k,$$

where

$$p_j \in [0, 1]$$

and

$$p_1 + p_2 + \cdots + p_k = 1.$$

Discrete Random Variables

Probability density function (pdf)

> The *pdf* of X summarizes possible outcomes and associated probabilities:

$$f(x_j) = p_j, \quad j = 1, 2, \ldots, k.$$

Ex. 2020 Presidential election: 538 electoral votes at stake.

- $\{X : 0, 1, \ldots, 538\}$ is the number of votes won.
- Unlikely that one will win 0 or 538 votes: $f(0) \approx 0$ and $f(538) \approx 0$.
- Nonzero probability of winning an exact majority: $f(270) > 0$.

Discrete random variables *Ex.*

Basketball player goes to the foul line to shoot two free throws.

- X is the number of shots made (either 0, 1, or 2).
- The pdf of X is $f(0) = 0.3$, $f(1) = 0.4$, $f(2) = 0.3$.[1]

Use the pdf to calculate the probability of the **event** that the player makes *at least one shot*, *i.e.*, $P(X \geq 1)$.

$$P(X \geq 1) = P(X = 1) + P(X = 2) = 0.4 + 0.3 = 0.7$$

1. *Note:* the probabilities sum to 1.

Continuous random variables

> A random variable that takes any real value with *zero* probability.

Wait, what?! The variable takes so many values that we can't count all possibilities, so the probability of any one particular value is zero.

Measurement is discrete (*e.g.*, dollars and cents), but variables with many possible values are best treated as continuous.

- *e.g.*, electoral votes, height, wages, temperature, *etc.*

Continuous random variables

Probability density functions also describe continuous random variables.

Difference between continuous and discrete **PDFs**

- Interested in the probability of events within a *range* of values.
- *e.g.* What is the probability of more than 1 inch of rain tomorrow?

Distributions

Distributions

> Function that represents all outcomes of a random variable and the corresponding probabilities.

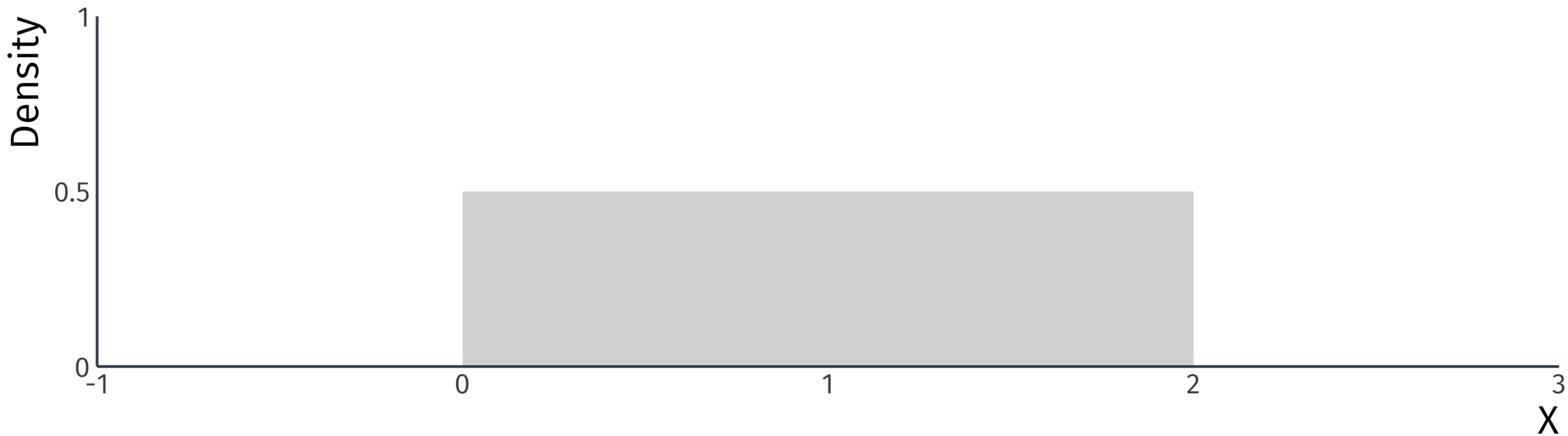
- Summary that describes the spread of data points in a set
- Essential for making inferences and assumptions from data

Key Takeaway: The shape of a distribution provides valuable information

Uniform distribution

The probability density function of a variable uniformly distributed between 0 and 2 is

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

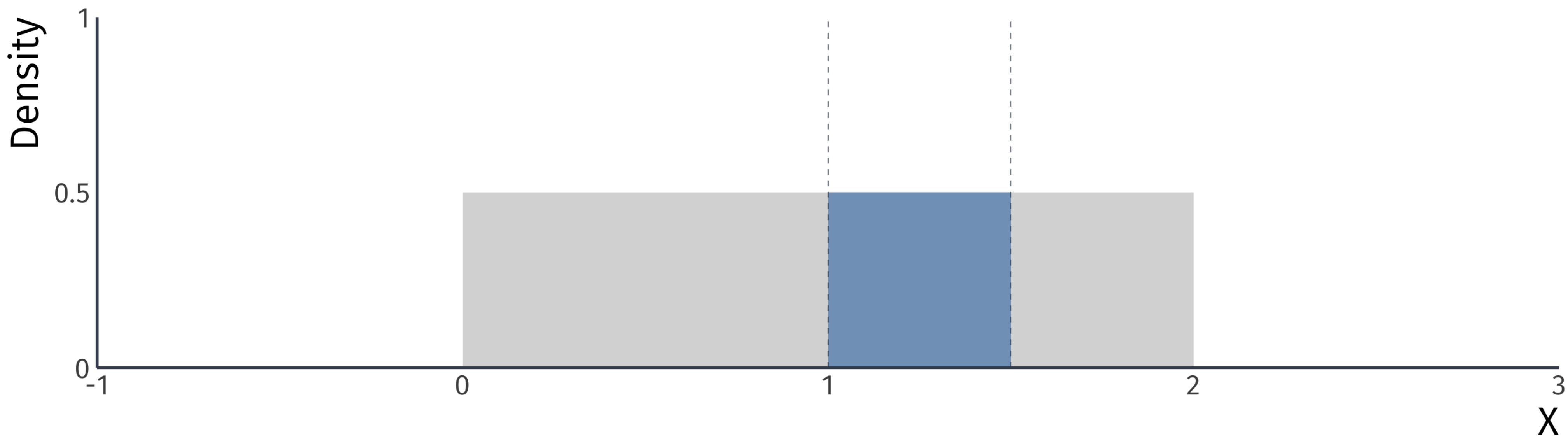


Uniform distribution

By definition, the area under $f(x)$ is equal to 1.

The **shaded area** illustrates the probability of the event $1 \leq X \leq 1.5$.

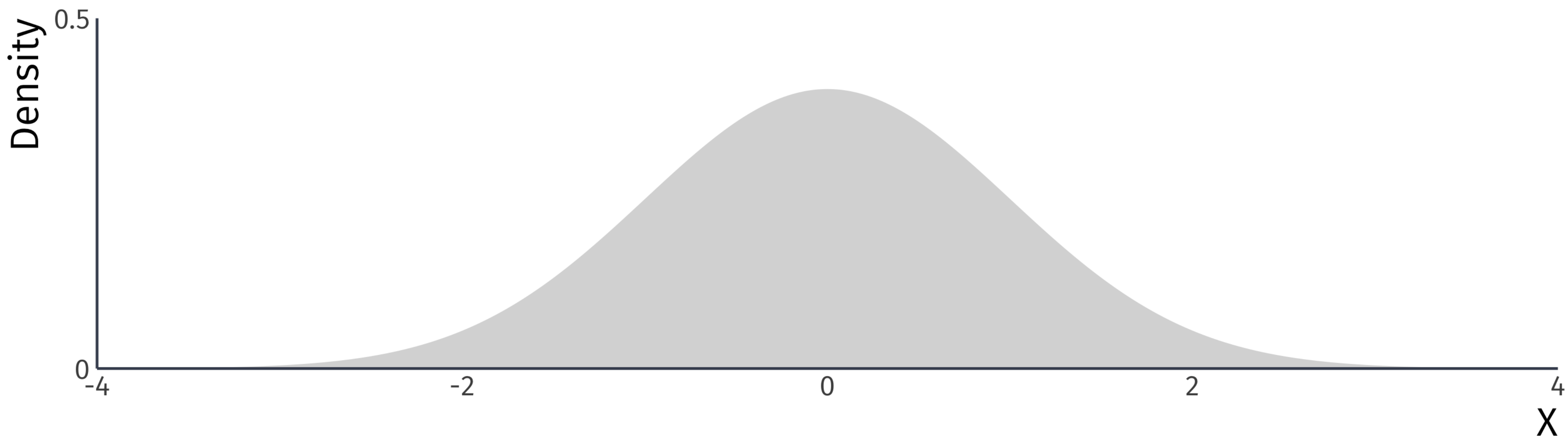
$$P(1 \leq X \leq 1.5) = (1.5 - 1) \times 0.5 = 0.25$$



Normal Distribution

The ***"bell curve"***

- Symmetric: mean and median occur at the same point (*i.e.*, no skew).
- Low-probability events in tails; high-probability events near center.

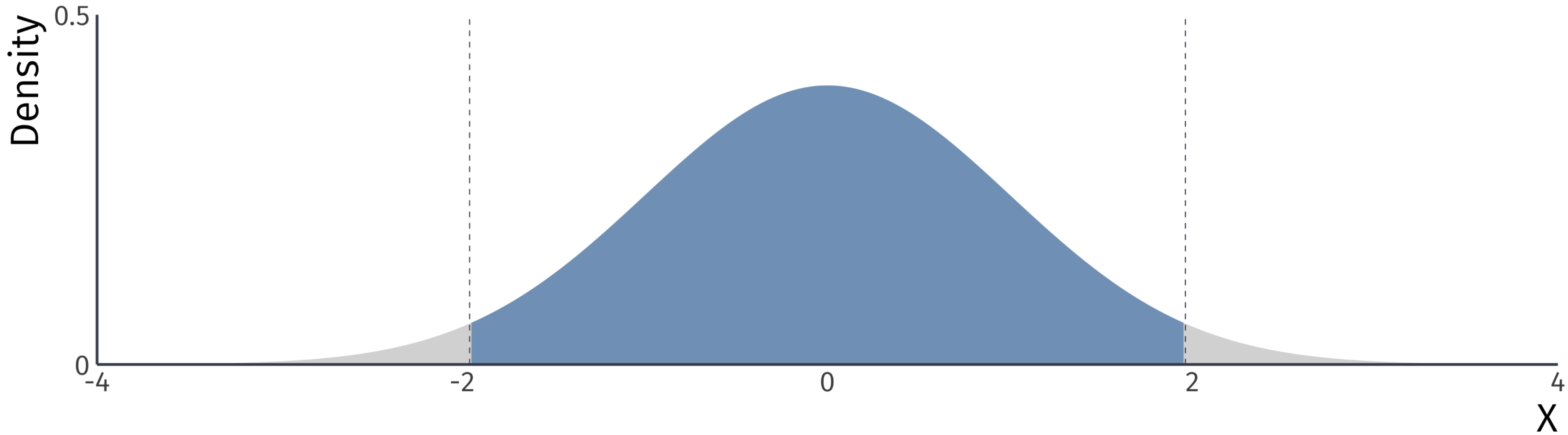


Normal Distribution

The **shaded area** illustrates the probability of the event $-2 \leq X \leq 2$.

- "Find area under curve" = use integral calculus (or, in practice, `R`).

$$P(-2 \leq X \leq 2) \approx 0.95$$



Normal Distribution

Continuous distribution where x_i takes the value of any real number ($\mathbb{R}$)

- Domain spans the entire real line
- Centered on the distribution mean μ

Rule 1: The probability that the random variable takes a value x_i is 0 for any $x_i \in \mathbb{R}$

Rule 2: The probability that the random variable falls between $[x_i, x_j]$ range, where $x_i \neq x_j$, is the area under $p(x)$ between those two values

The area above represents $p(x) = 0.95$. The values $\{-1.96, 1.96\}$ represent the 95% confidence interval for μ.

Moments

Moments

> Quantitative measures used to describe the shape and characteristics of a probability distribution[1]

Summarize and understand the important features of a distribution

First moment: **Mean**

Second moment: **Variance**

Third moment: Skewness

Fourth moment: Kurtosis

$\vdots$

1. See this video for a more in depth description of moments (click here)

Expected Value

> Describes the *central tendency* of distribution in a single number.[1]

Density functions describe the entire distribution, but sometimes we just want a summary.

Other summary statistics we may be interested in include

- Median
- Standard deviation
- 25th percentile
- 75th percentile

1. *Central tendency* = typical value to expect upon drawing from the distribution.

Expected Value (discrete)

> The expected value of a discrete random variable X is the weighted average of its k values $\{x_1, \ldots, x_k\}$ and their associated probabilities:

$$
\begin{aligned}
E(X) &= x_1 P(x_1) + x_2 P(x_2) + \cdots + x_k P(x_k) \\
&= \sum_{j=1}^{k} x_j P(x_j).
\end{aligned}
$$

AKA: **Population mean**

Expected Value *Ex.*

Rolling a six-sided die once can take values $\{1, 2, 3, 4, 5, 6\}$, each with equal probability. *What is the expected value of a roll?*

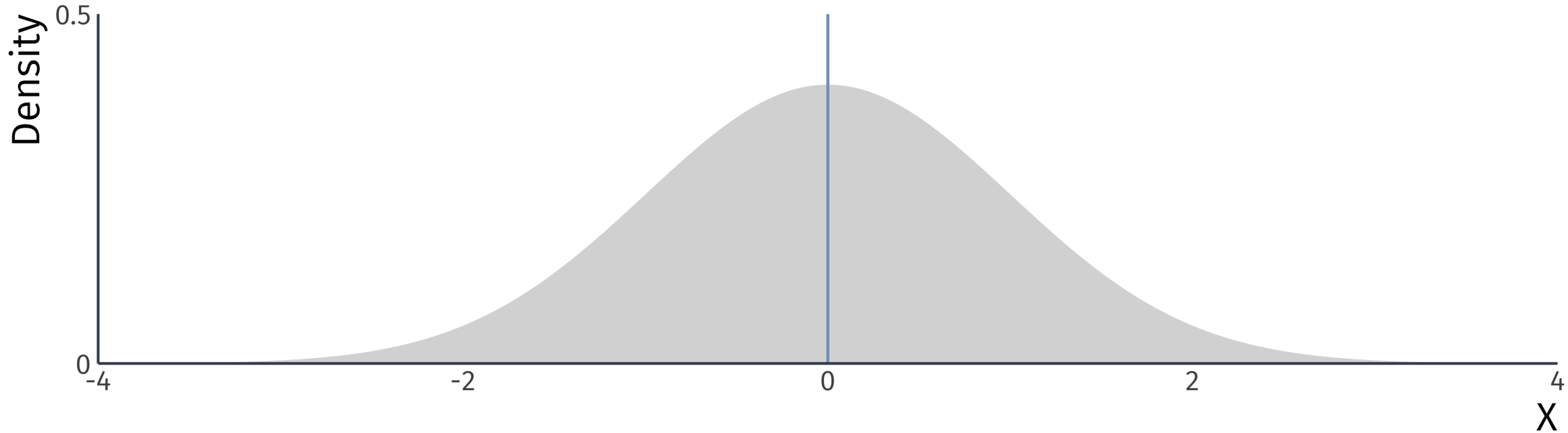
$$
\begin{aligned}
\text{E}(\text{Roll}) = 1 \times \frac{1}{6} &+ 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} \\
&+ 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5
\end{aligned}
$$

Note: The **EV** can be a number that isn't a possible outcome of X.

Expected value (continuous)

If X is a continuous random variable and $f(x)$ is its probability density function, then the expected value of X[1] is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$



1. *Note:* x represents the particular values of X.

Expected value: Rule 01

For any constant c, $E(c) = c$. *Ex.*

- $E(5) = 5$.
- $E(1) = 1$.
- $E(4700) = 4700$.

Expected value: Rule 02

For any constants a and b, $E(aX + b) = aE(X) + b$.

Ex. Suppose X is the high temperature in degrees Celsius in Eugene during August. The long-run average is $E(X) = 28$. If Y is the temperature in degrees Fahrenheit, then $Y = 32 + \frac{9}{5}X$. What is $E(Y)$?

$$E(Y) = 32 + \frac{9}{5}E(X) = 32 + \frac{9}{5} \times 28 = 82.4$$

Expected value: Rule 03

If $\{a_1, a_2, \ldots, a_n\}$ are constants and $\{X_1, X_2, \ldots, X_n\}$ are random variables, then

$$E(a_1 X_1 + a_2 X_2 + \cdots + a_n X_n) = a_1 E(X_1) + a_2 E(X_2) + \cdots + a_n E(X_n)$$

In English, **the expected value of the sum** = **the sum of expected values**.

Expected value: Rule 03

The expected value of the sum = **the sum of expected values**.

Ex. Suppose that a coffee shop sells X_1 small, X_2 medium, and X_3 large caffeinated beverages in a day. The quantities sold are random with expected values $E(X_1) = 43$, $E(X_2) = 56$, and $E(X_3) = 21$. The prices of small, medium, and large beverages are 1.75, 2.50, and 3.25 dollars. What is expected revenue?

$$
\begin{aligned}
E(1.75X_1 + 2.50X_2 + 3.35X_3) &= 1.75E(X_1) + 2.50E(X_2) + 3.25E(X_3) \\
&= 1.75(43) + 2.50(56) + 3.25(21) \\
&= 283.5
\end{aligned}
$$

Expected value: Caution

Previously, we found that the expected value of rolling a six-sided die is $\mathrm{E}\left(\mathrm{Roll}\right) = 3.5$.

- If we square this number, we get $[\mathrm{E}(\mathrm{Roll})]^2 = 12.25$.

Is $\left[\mathrm{E}\left(\mathrm{Roll}\right)\right]^2$ *the same as* $\mathrm{E}\left(\mathrm{Roll}^2\right)$*?*

$$
\begin{aligned}
\mathrm{E}\left(\mathrm{Roll}^2\right) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} \\
+ 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} \\
\approx 15.167 \neq 12.25.
\end{aligned}
$$

No!

Expected value: Caution

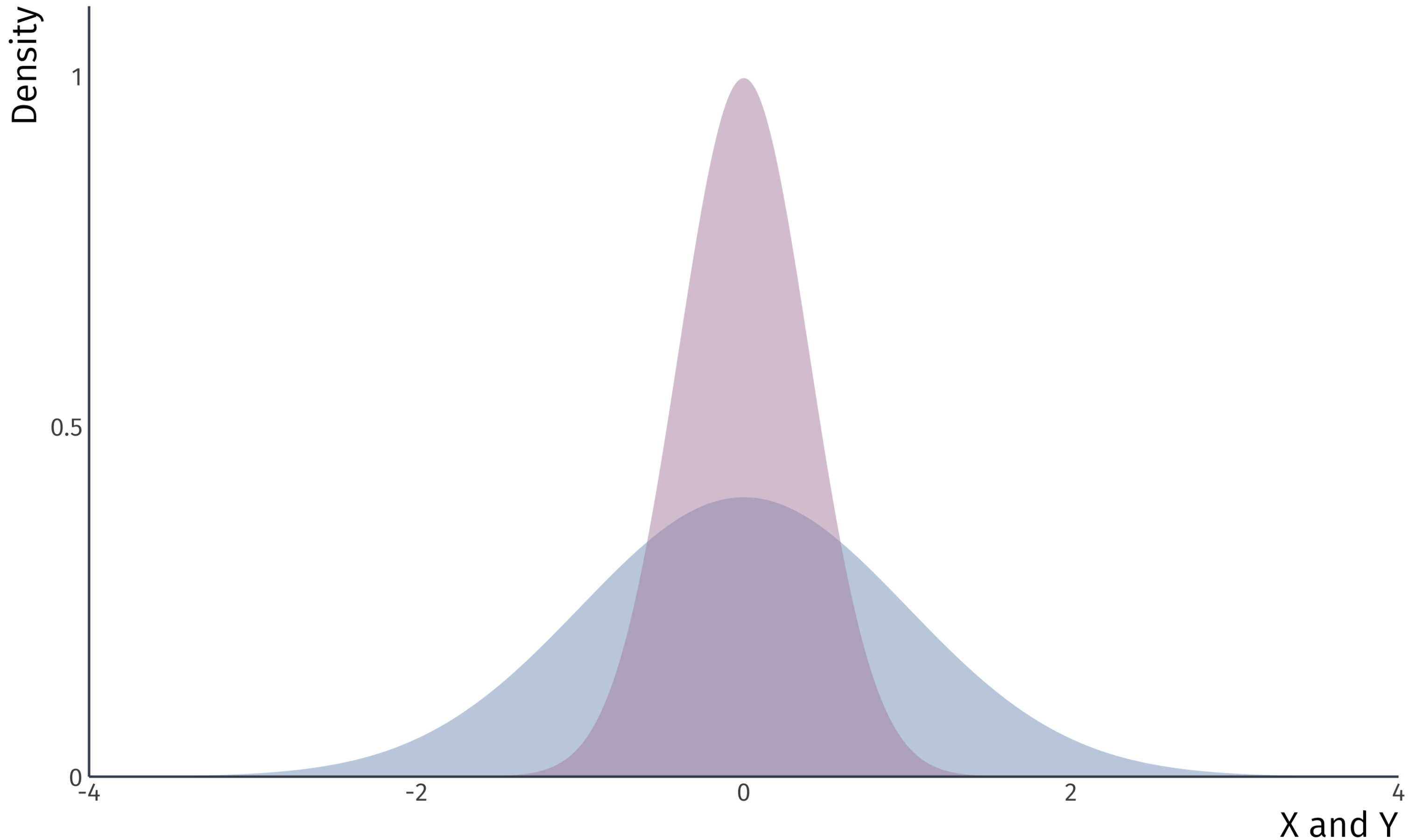
Except in special cases, **the transformation of an expected value** is note **the expected value of a transformed random variable**.

For some function $\mathrm{g}(\cdot)$, it is typically the case that

$$\mathrm{g}\left(\mathrm{E}(X)\right) \neq \mathrm{E}\left(\mathrm{g}(X)\right).$$

Variance

Random variables X and Y share the same population mean, but are distributed differently.



Variance (σ^2)

> Tells us how far X deviates from μ, *on average*:

$$\text{Var}(X) \equiv \mathbf{P}\left((X-\mu)^2\right) = \sigma^2$$

Where: $\mu = E(X)$.

How tightly is a random variable distributed about its mean?

Describe the distance of X from its population mean μ as the squared difference: $(X-\mu)^2$.

- Distributing the terms above yields $\sigma^2 = E(X^2 - 2X\mu + \mu^2) = E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$.

Variance: Rule 01

$\text{Var}(X) = 0 \iff X$ is a constant.

- A random variable that never deviates from its mean has zero variance.

Wait what? How can a random variable be a constant?? Because a constant fits the technical definition of a random variable[1]. It's just not-so-random

1. See this link for a more technical explanation

Variance: Rule 02

For any constants a and b, $\text{Var}(aX + b) = a^2 \, \text{Var}(X)$.

Ex. Suppose X is the high temperature in degrees Celsius in Eugene during August. If Y is the temperature in degrees Fahrenheit, then $Y = 32 + \frac{9}{5}X$. What is $\text{Var}(Y)$?

$$\text{Var}(Y) = \left(\frac{9}{5}\right)^2 \text{Var}(X) = \frac{81}{25} \, \text{Var}(X)$$

Standard Deviation (σ)

> The positive square root of the variance:

$$\text{sd}(X) = +\sqrt{\text{Var}(X)} = \sigma$$

Rule 01: For any constant c, $\text{sd}(c) = 0$.

Rule 02: For any constants a and b, $\text{sd}(aX + b) = |a|\ \text{sd}(X)$.

Note: The same as variance, almost

Standardizing a random variable

When we're working with a random variable X with an unfamiliar scale, it is useful to **standardize** it by defining a new variable Z:

$$Z \equiv \frac{X - \mu}{\sigma}.$$

Z has mean 0 and standard deviation 1. How?

- First, some simple trickery: $Z = aX + b$, where $a \equiv \frac{1}{\sigma}$ and $b \equiv -\frac{\mu}{\sigma}$.
- $E(Z) = aE(X) + b = \mu\frac{1}{\sigma} - \frac{\mu}{\sigma} = 0.$
- $\text{Var}(Z) = a^2\text{Var}(X) = \frac{1}{\sigma^2}\sigma^2 = 1.$

Covariance

> For two random variables X and Y, the covariance is defined as the expected value (or mean) of the product of their deviations from their individual expected values:

$$\text{Cov}(X, Y) \equiv E\left[(X - \mu_X)(Y - \mu_Y)\right] = \sigma_{xy}$$

Idea: Characterize the relationship between random variables X and Y.

- **Positive correlation:** When $\sigma_{xy} > 0$, then X is above its mean when Y is above its mean, *on average*.
- **Negative correlation:** When $\sigma_{xy} < 0$, then X is below its mean when Y is above its mean, *on average*.

Covariance: Rule 01

Statistical independence:

> If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$.

- If X and Y are independent, then $\text{Cov}(X, Y) = 0$.

Caution: $\text{Cov}(X, Y) = 0$ **does not imply** that X and Y are independent.

- $\text{Cov}(X, Y) = 0$ means that X and Y are *uncorrelated*.

Covariance: Rule 02

For any constants a, b, c, and d,

$$\text{Cov}(aX + b, cY + d) = ac\,\text{Cov}(X, Y)$$

Correlation Coefficient

A problem with covariance is that it is sensitive to units of measurement.

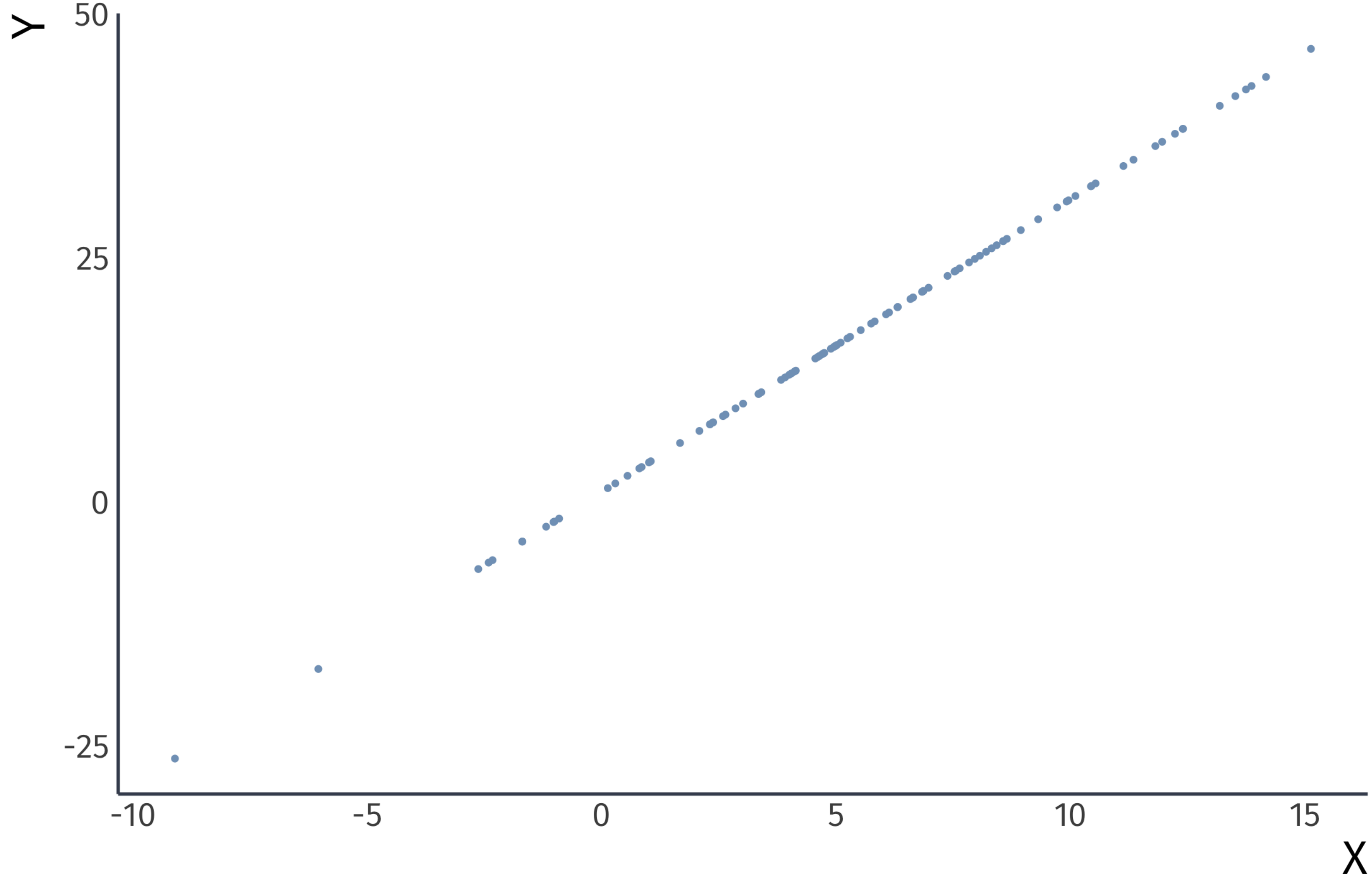
The **correlation coefficient** solves this problem by rescaling the covariance:

$$\text{Corr}(X, Y) \equiv \frac{\text{Cov}(X, Y)}{\text{sd}(X) \times \text{sd}(Y)} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

- Also denoted as ρ_{XY}.
- $-1 \leq \text{Corr}(X, Y) \leq 1$
- Invariant to scale: if I double Y, $\text{Corr}(X, Y)$ will not change.

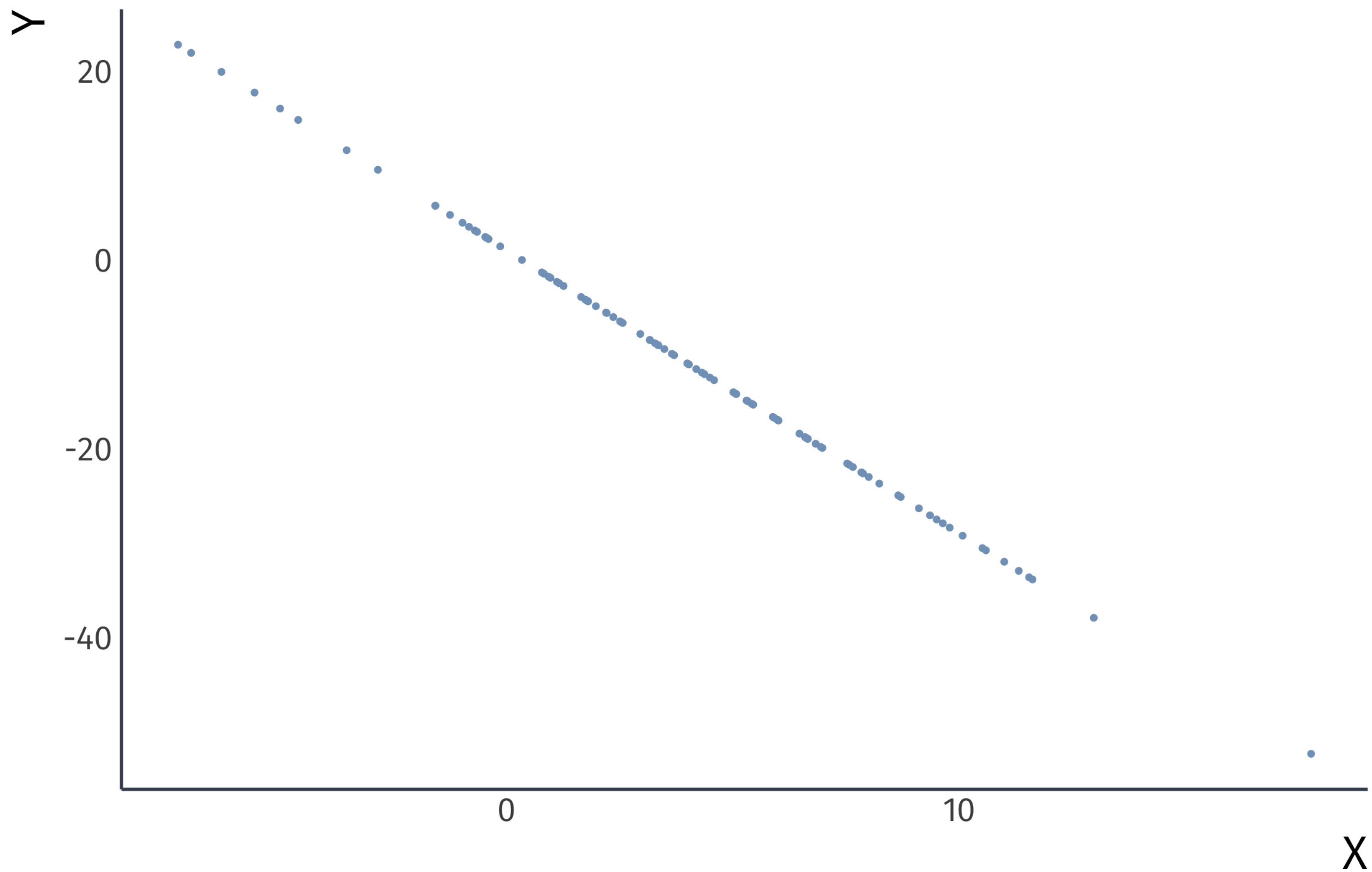
Correlation Coefficient

Perfect positive correlation: $\text{Corr}(X, Y) = 1$.



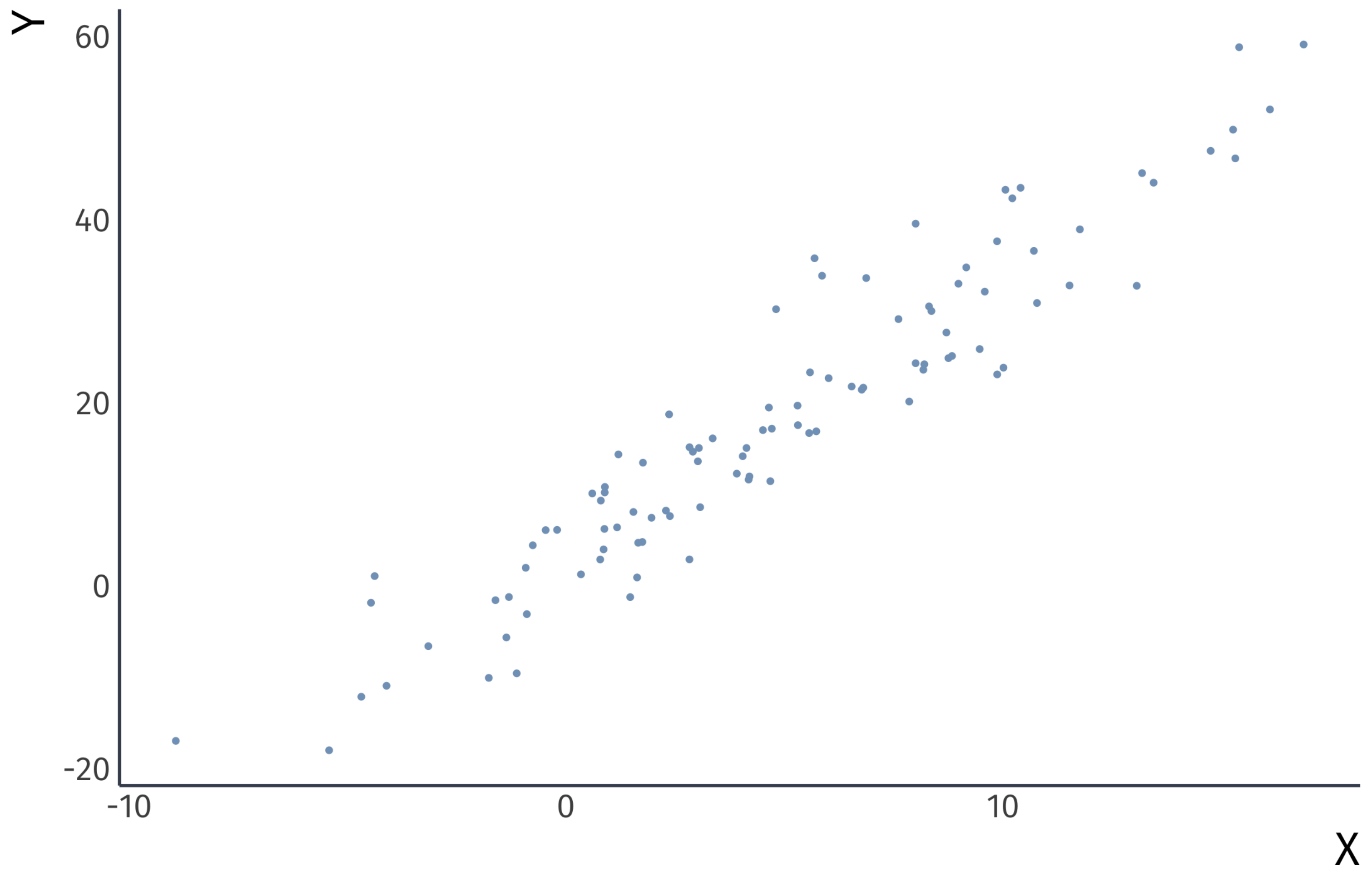
Correlation Coefficient

Perfect negative correlation: $\text{Corr}(X, Y) = -1$.



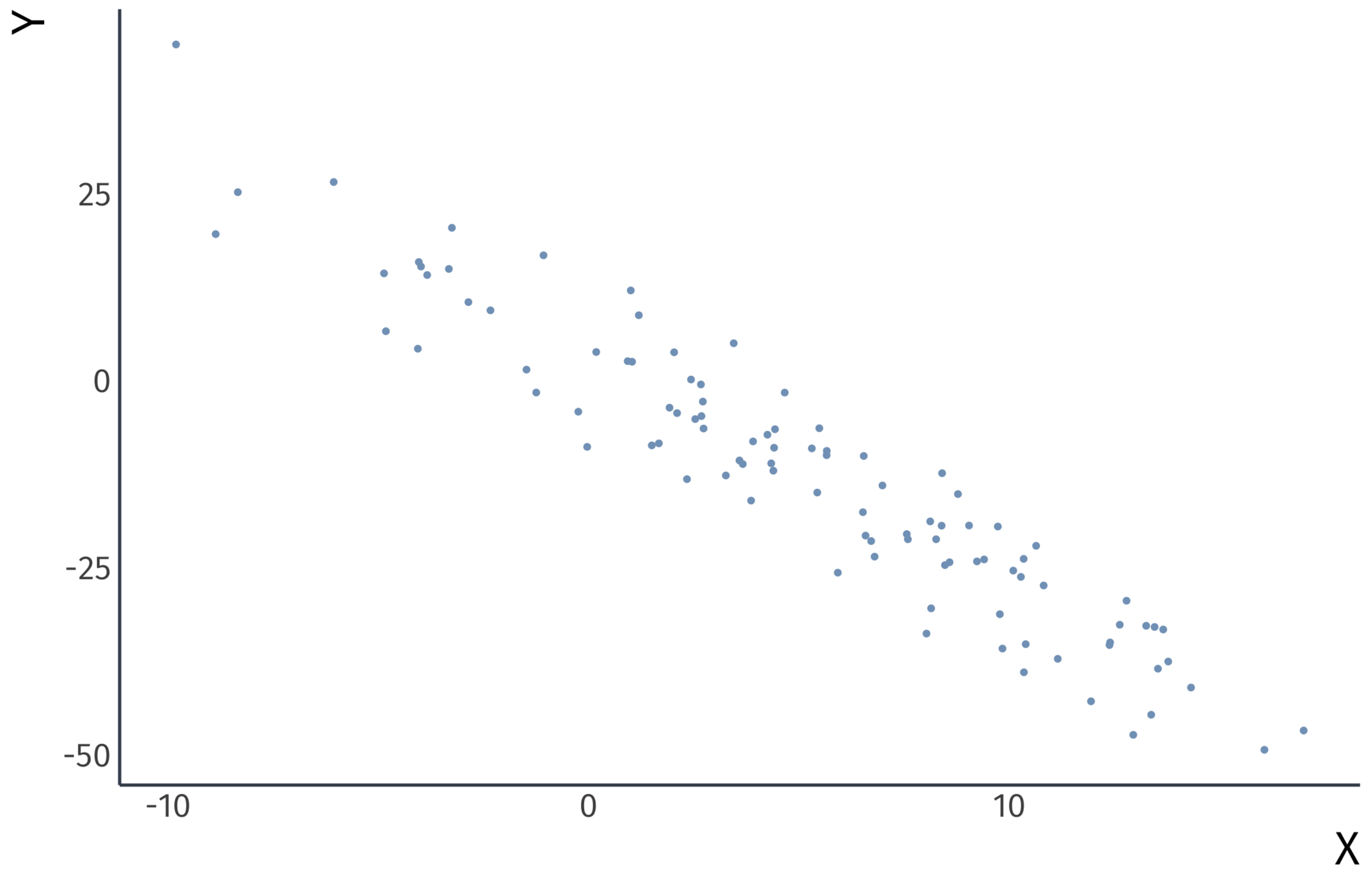
Correlation Coefficient

Positive correlation: $\text{Corr}(X, Y) > 0$.



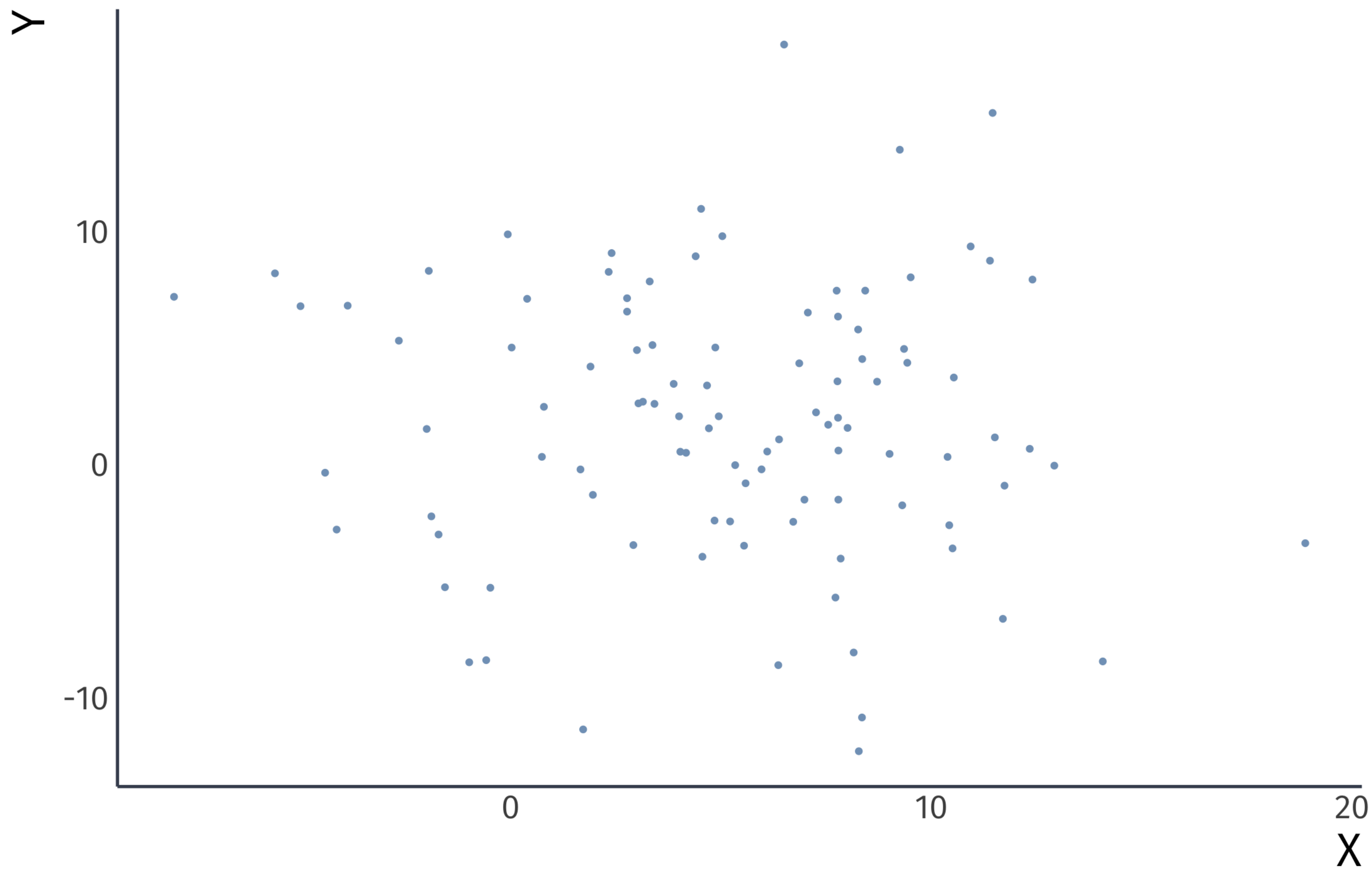
Correlation Coefficient

Negative correlation: $\text{Corr}(X, Y) < 0$.



Correlation Coefficient

No correlation: $\text{Corr}(X, Y) = 0$.



Variance: Rule 03

For constants a and b,

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y).$$

- If X and Y are uncorrelated, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- If X and Y are uncorrelated, then $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Estimators

Estimators

Why do we estimate things?

Suppose we want to know the average height of the population in the US

- We have a sample 1 million Americans

How can we use these data to estimate the height of the population?

Estimators

Estimand:

> Quantity that is to be estimated in a statistical analysis

Estimator:

> A rule (or formula) for estimating an unknown population parameter given a sample of data.

Estimate:

> A specific numerical value that we obtain from the sample data by applying the estimator.

Estimators *Ex.*

Suppose we want to know the average height of the population in the US

- We have a sample 1 million Americans

Estimand: The population mean (μ)

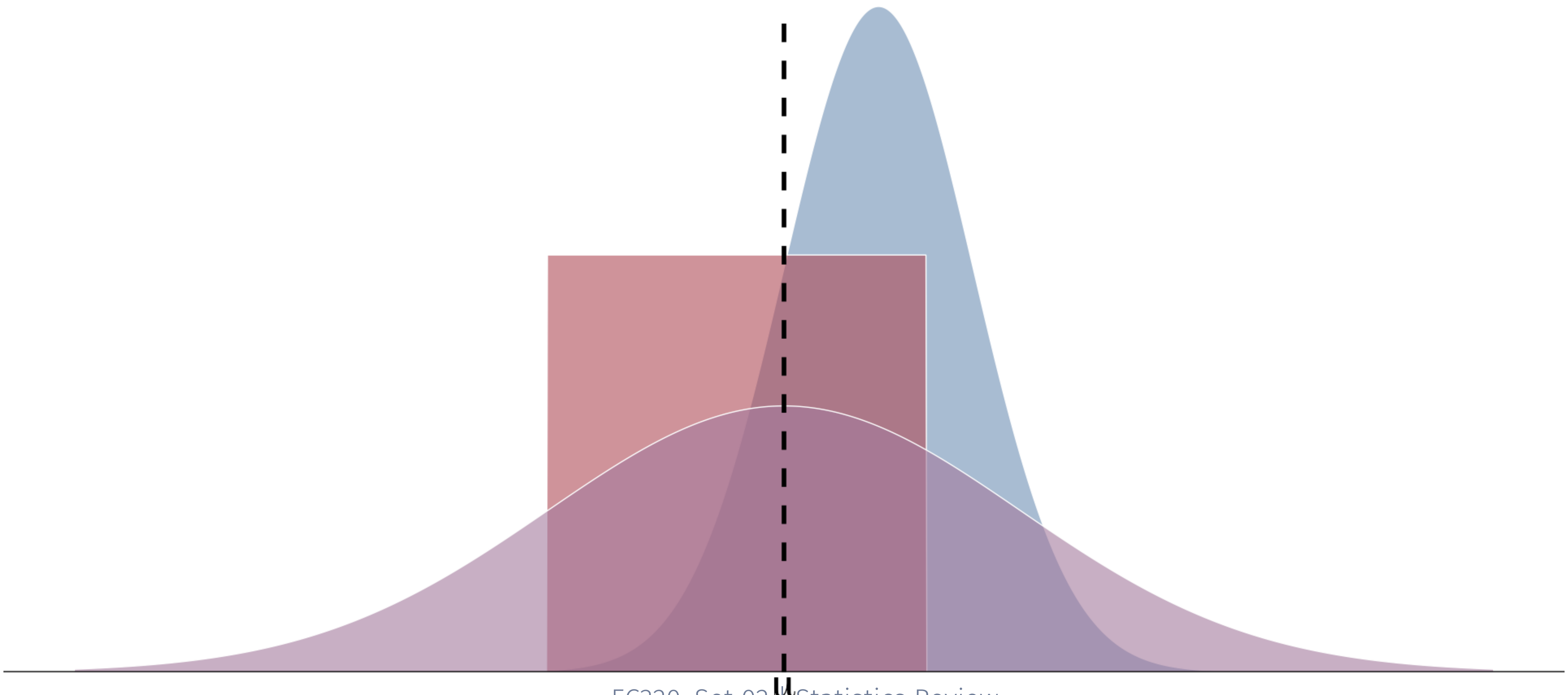
Estimator: The sample mean ($\bar{X}$)

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Estimate: The sample mean ($\hat{\mu} = 5'6''$)

Properties of estimators

Imagine that we want to estimate an unknown parameter μ, and we know the distributions of three competing estimators. *Which one should we use?*



Properties of estimators

Question What properties make an estimator reliable?

Answer (1): **Unbiasedness**

On average, does the estimator tend toward the correct value?

More formally: Does the mean of estimator's distribution equal the parameter it estimates?

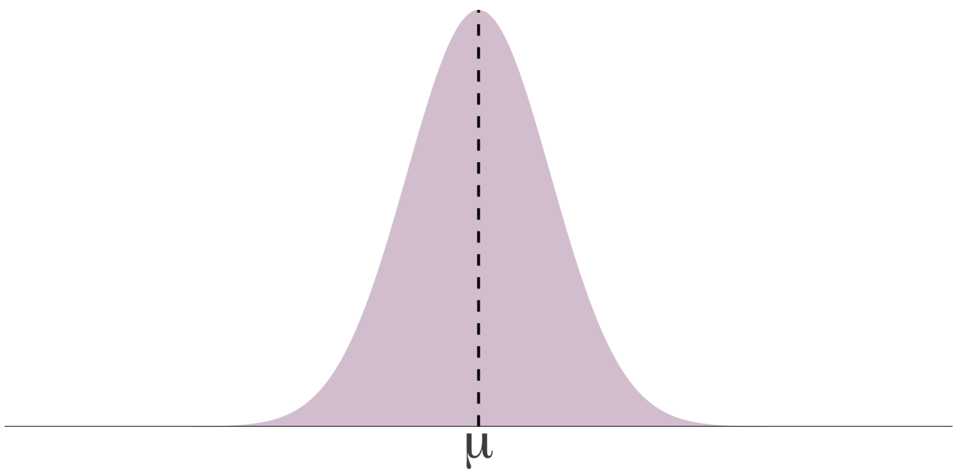
$$\text{Bias}_\mu(\hat{\mu}) = \mathbb{E}[\hat{\mu}] - \mu$$

Properties of estimators

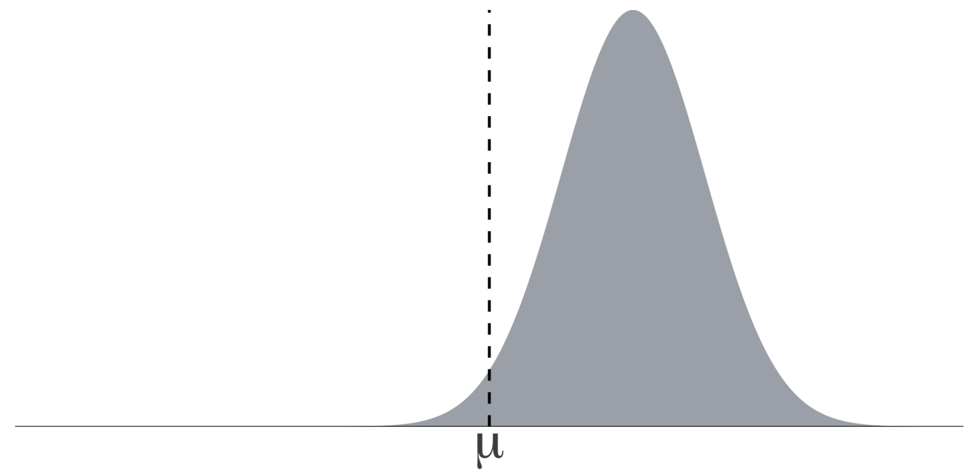
Question What properties make an estimator reliable?

A01: **Unbiasedness**

Unbiased estimator: $\mathbb{E}[\hat{\mu}] = \mu$



Biased estimator $\mathbb{E}[\hat{\mu}] \neq \mu$



Unbiasedness example

Is the sample mean $\frac{1}{n}\sum_{i=1}^{n} x_i = \hat{\mu}$ an unbiased estimator of the population mean $E(x_i) = \mu$?

$$
\begin{aligned}
\mathbb{E}[\hat{\mu}] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[x_i] \qquad \} \quad \text{rule 3} \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu \qquad\qquad \} \quad \text{by definition} \\
&= \mu
\end{aligned}
$$

Properties of estimators

Question What properties make an estimator reliable?

A02: **Efficiency** *(low variance)*

The central tendencies (means) of competing distributions are not the only things that matter. We also care about the *variance* of an estimator.

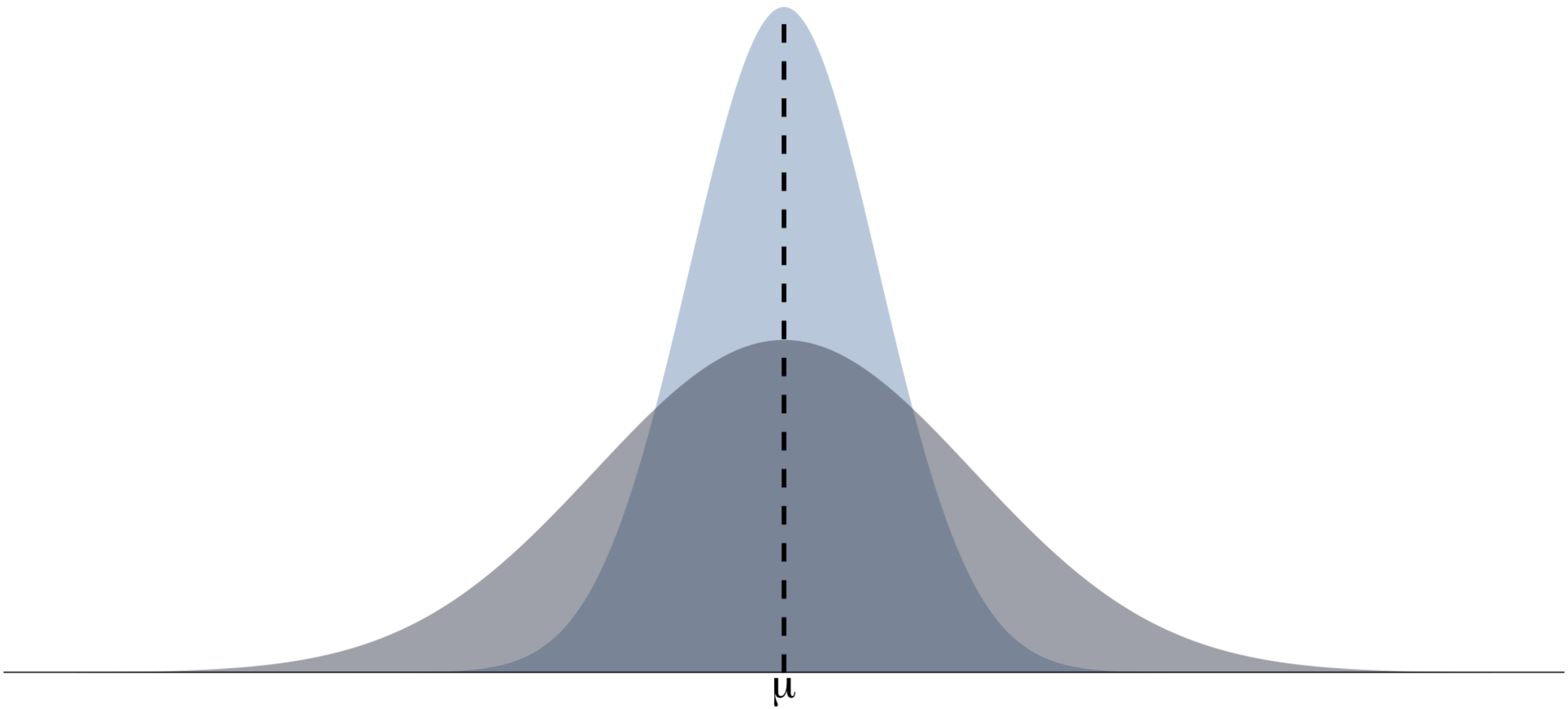
$$\mathrm{Var}(\hat{\mu}) = \mathbb{E}\left[(\hat{\mu} - \mathbb{E}[\hat{\mu}])^2\right]$$

Lower variance estimators produce estimates closer to the mean in each sample.

Properties of estimators

Question What properties make an estimator reliable?

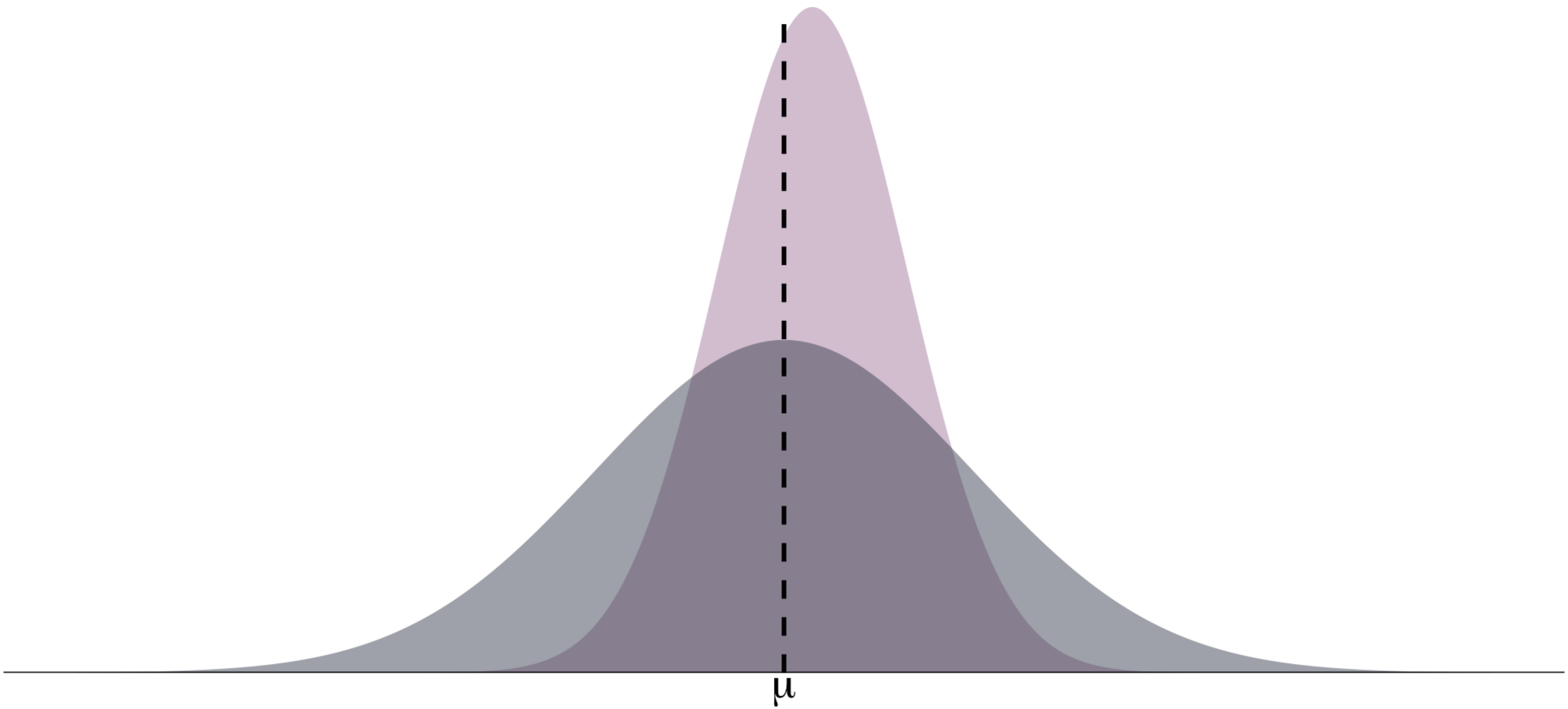
A02: **Efficiency** *(low variance)*



The bias-variance tradeoff

Should we be willing to take a bit of bias to reduce the variance

In econ, in causal inference we emphasize unbiasedness



Unbiased estimators

In addition to the sample mean, there are several other unbiased estimators we will use often.

- **Sample variance** estimates variance σ^2.
- **Sample covariance** estimates covariance σ_{XY}.
- **Sample correlation** estimates the pop. correlation coefficient ρ_{XY}.

Unbiased estimators

Sample variance, S_X^2, is an unbiased estimator of the pop. variance σ^2

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - X)^2.$$

Sample covariance, S_{XY}, is an unbiased estimator of the pop. covariance, σ_{XY}

$$S_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - X)(Y_i - Y).$$

Unbiased estimators

Sample correlation r_{XY} is an unbiased estimator of the pop. correlation coefficient ρ_{XY}

$$r_{XY} = \frac{S_{XY}}{\sqrt{S_X^2}\sqrt{S_Y^2}}.$$

Sampling

Sampling

Population:

> A group of items or events we would like to know about.

Ex. Americans, games of chess, cats in Eugene, etc.

Parameter:[1]

> a value that describes that population

Ex. Mean height of American, average length of a chess game, median weight of the kitties

1. Parameter of interest is the parameter that the researcher seeks to learn about

Sampling

Sample:

> A survey of a subset of the population.

Ex. Respondents to a survey, random sample of econ students at the UO

Often we aim to draw observations *randomly* from the population

- Advantageous as it becomes a **representative sample** of the population...

Sampling distributions

Focus: Populations vs Samples

- How can we make inferences about a **population** based on a small **sample** of the population?
- How do we learn about an unknown population *parameter of interest*?

Challenge: Usually missing data of the entire population.

Solution: Sample from the population and estimate the parameter.

- Draw n observations from the population, then use an *estimator*.

Sampling distributions

There are myriad ways to produce a sample,[1] but we will restrict our attention **to simple random sampling**, where

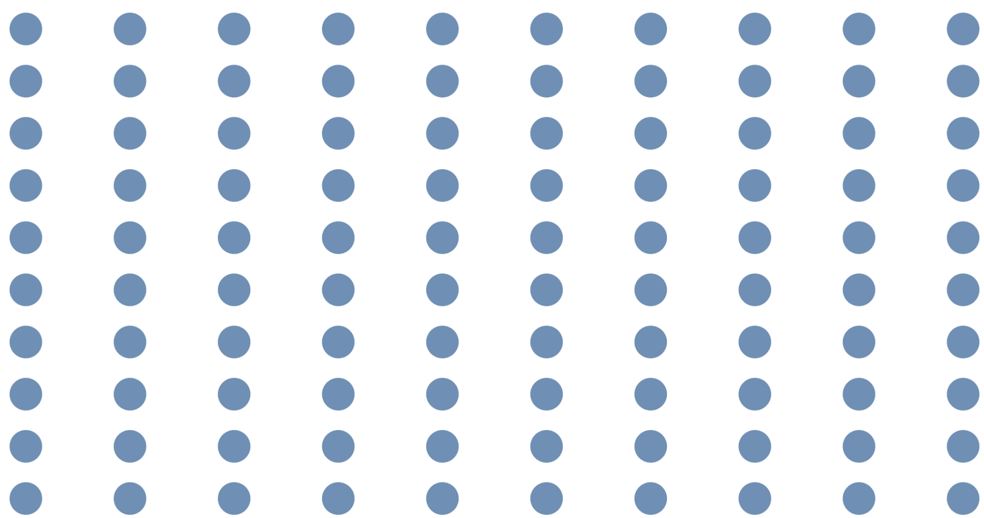
1. Each observation is a random variable.
2. The n random variables are independent.

Life becomes much simpler for the econometrician.

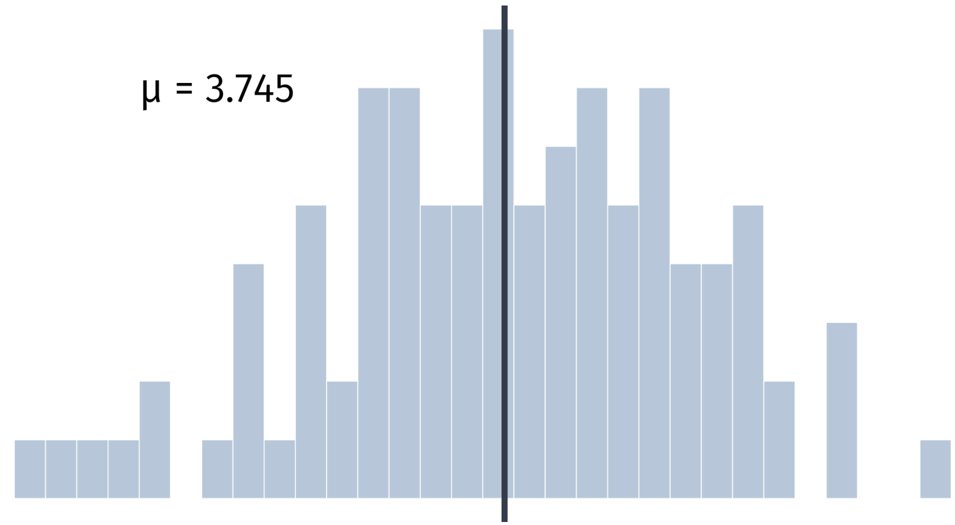
1. Only a subset of these can help produce reliable statistics.

Population *vs.* sample

Question: *Why do we care about population vs. sample?*



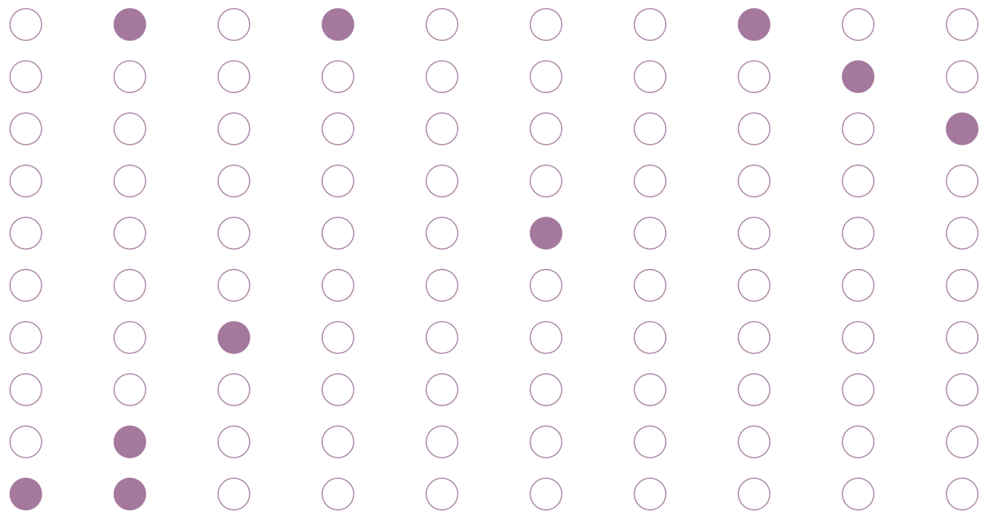
Population



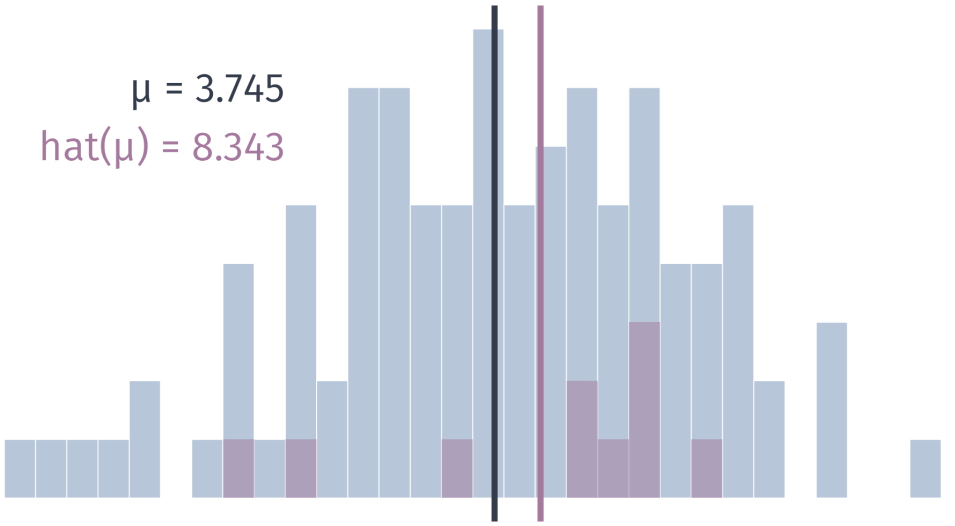
Population relationship

Population *vs* sample

Question: Why do we care about population vs. sample?



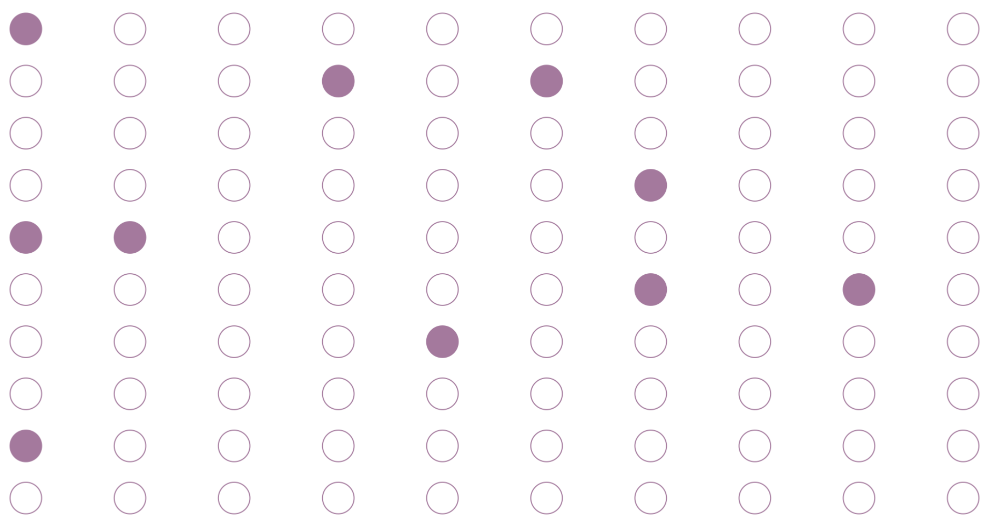
10 random individuals



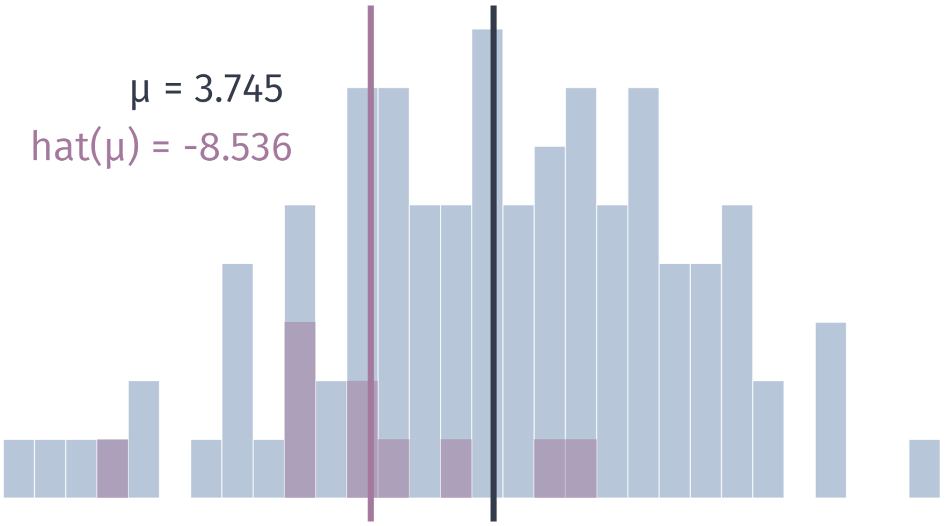
Population relationship

Population *vs* sample

Question: *Why do we care about population vs. sample?*



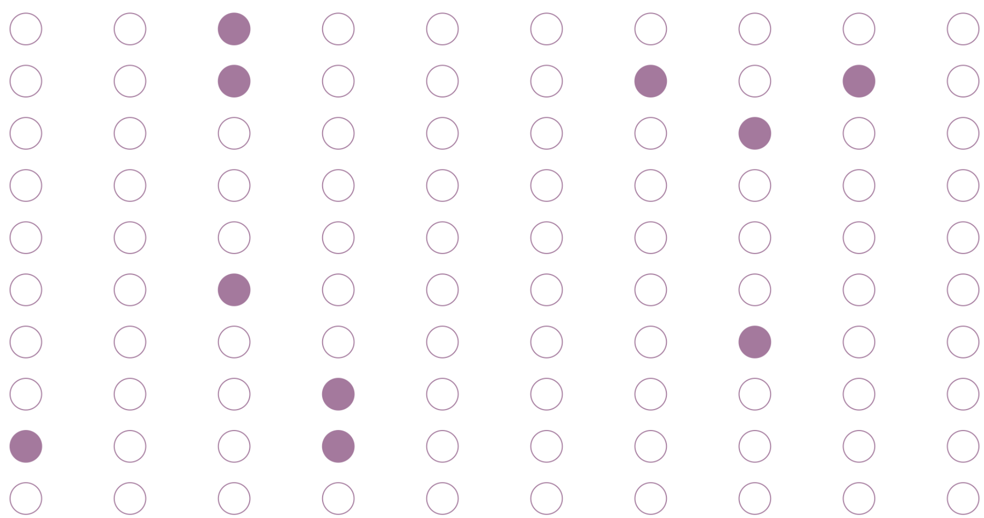
10 random individuals



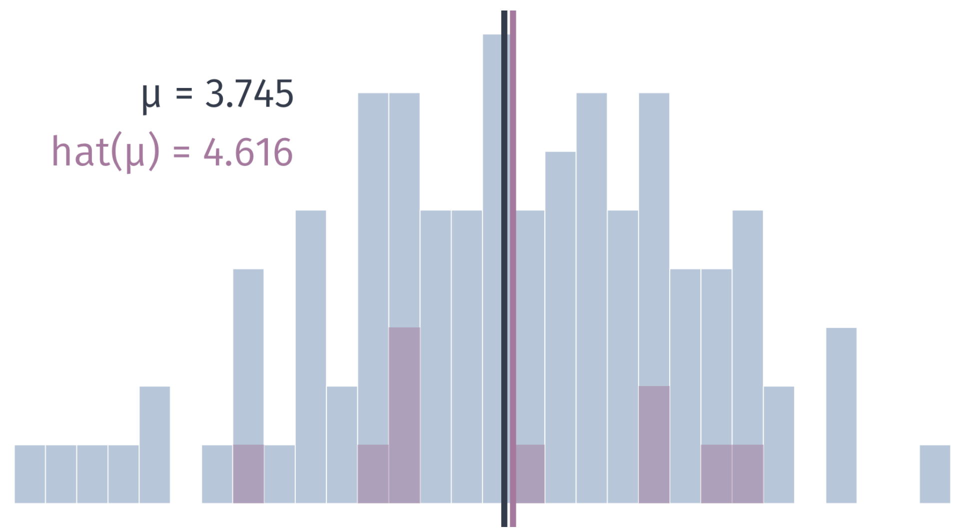
Population relationship

Population *vs* sample

Question: *Why do we care about population vs. sample?*



10 random individuals



Population relationship

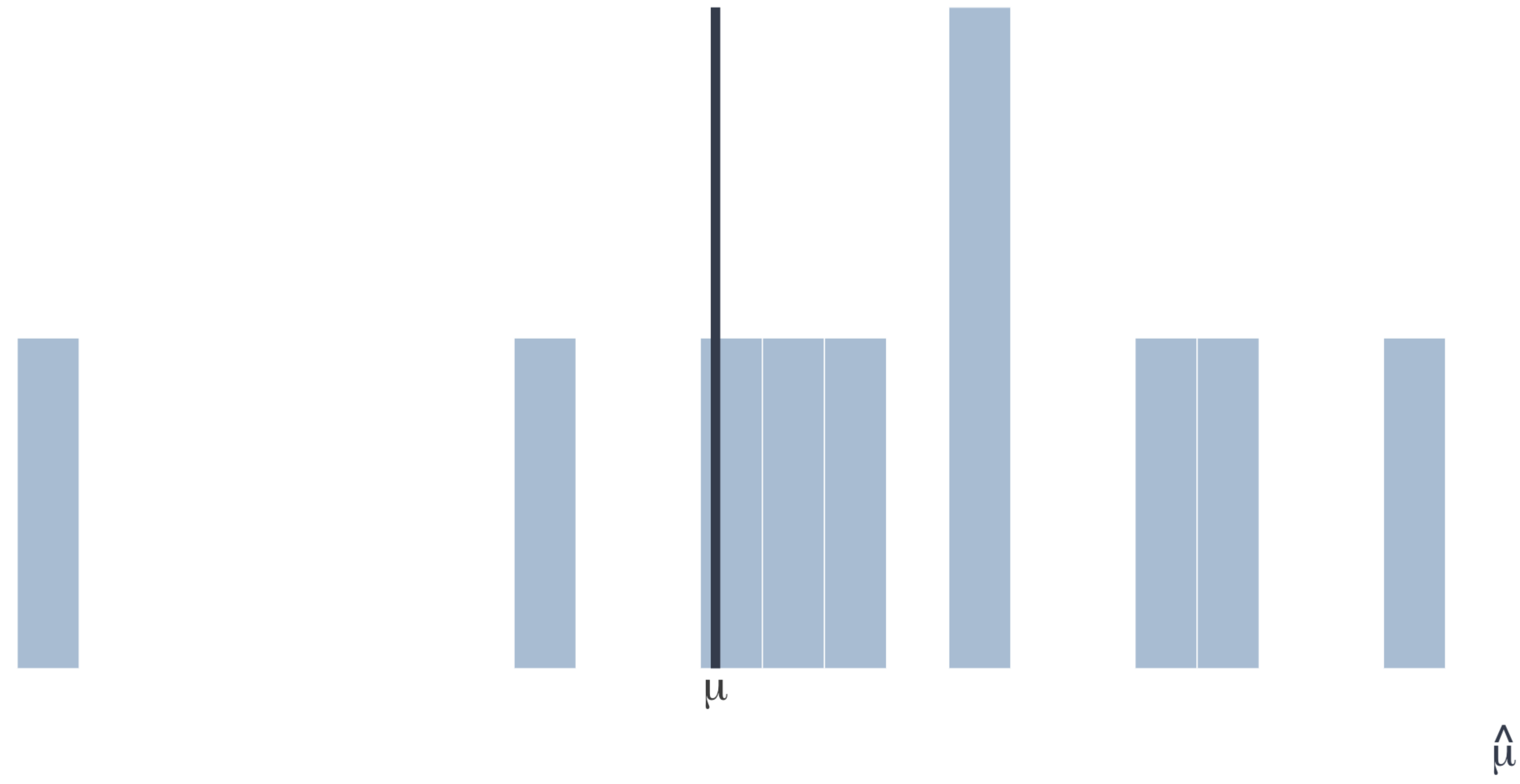
Let's repeat this **10,000 times** and then plot the estimates.

(This exercise is called a Monte Carlo simulation.)

How in the world do I do that

▶ Show the code

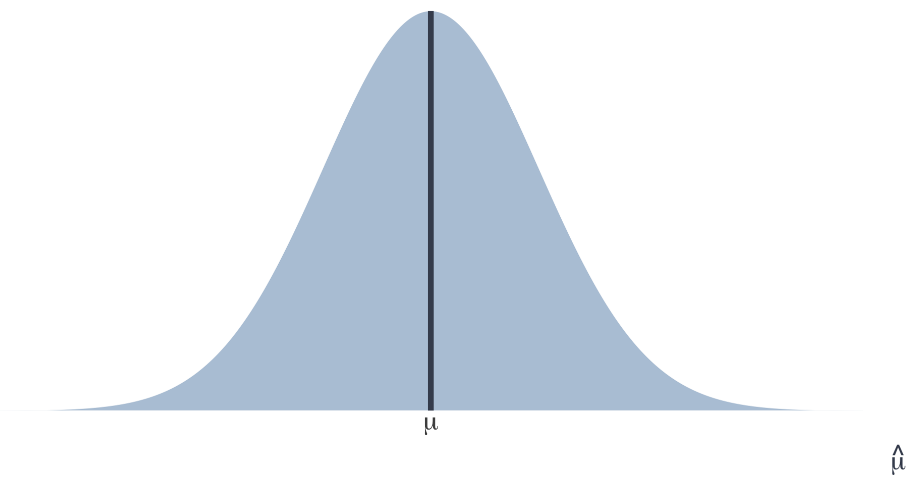
10 100 1,000 10,000 100,000



Regular resampling means of 10 obs at a time

Population *vs.* sample

Question: *Why do we care about population vs. sample?*



As the number of samples approach infinity

On average, the mean of the samples are close to the population mean

- Some individual samples can miss the mark.
- The difference between individual samples and the population creates **uncertainty**

Population *vs.* sample

Question: Why do we care about population vs. sample?

Answer: Uncertainty matters.

- $\hat{\mu}$ is a random variable that depends on the sample.
- We don't know if our sample is representative of the population.
- Individual sample means can be biased
- We have to keep track of this uncertainty.

Population distributions

Consider the following argument *(this slide scrolls down)*

Suppose we have some estimator $\hat{\theta}$ for a parameter θ:

- θ is unobserved, but assume $\hat{\theta}$ follows a probability distribution $p(\hat{\theta})$
- We hypothesize some value, say $\theta = 2.5$
- We use our estimator $\hat{\theta}$ to calculate an estimate. $\hat{\theta} = 45$
- If we make an ***assumption*** of the distribution of $\hat{\theta}$, we can calculate the probability of getting $\hat{\theta} = 45$ when $\theta = 2.5$ is true.
- For sake of argument, let's say that the probability that $\theta = 2.5$ if we observe $\theta = 45$ is less than 0.001

We can say

> *if θ really was 2.5, then the probability of getting $\hat{\theta} = 45$ is super super low. Thus the probability that θ is actually 2.5 is super super low".*

- We can make statements about the true value of θ just by knowing the distribution of our preferred estimator $\hat{\theta}$

But what distribution should we be assuming?

The Central Limit Theorem[1]

Theorem

> *Let $x_1, x_2, \ldots, x_n$ be a random sample from a population with mean $\mathbb{E}[X] = \mu$ and variance $\mathrm{Var}(X) = \sigma^2 < \infty$, let X be the sample mean. Then, as $n \longrightarrow \infty$, the function $\frac{\sqrt{n}(X-\mu)}{S_x}$ converges to a Normal Distribution with mean 0 and variance 1.*

- CLT states that when $n \longrightarrow \infty$, the sample mean will be normally distributed.
- The Law of Large Number (LLN) states that as $n \longrightarrow \infty$, the sample converges on the population mean.

1. A more in depth explanation + visualization

Hypothesis testing

Hypothesis Testing

How do we assess an estimate of the population mean?

- How likely is it that we have observed this estimate?
- Is it just a coincidence?
- Is is *statistically distinguishable* from a hypothesized value?
- Should we update out prior beliefs?

We can conduct *statistical tests* to address these questions.

Hypothesis Testing

Null hypothesis $(H_0): \mu = \mu_0$

Alternative hypothesis $(H_1): \mu \neq \mu_0$

There are four possible outcomes of our test:

1. We **fail to reject** the null hypothesis and the null is true.
2. We **reject** the null hypothesis and the null is false.
3. We **reject** the null hypothesis, but the null is actually true (*Type I error*).
4. We **fail to reject** the null hypothesis, but the null is actually false (*Type II error*).

Hypothesis Testing

Four possible outcomes

We **fail to reject** the null hypothesis and the null is true.

- The defendant was acquitted and he didn’t do the crime.

We **reject** the null hypothesis and the null is false.

- The defendant was convicted and he did the crime.

Hypothesis Testing

Four possible outcomes

We **reject** the null hypothesis, but the null is actually true.

- The defendant was convicted, but he didn't do the crime!
- **Type I error** (a.k.a. *false positive*)

We **fail to reject** the null hypothesis, but the null is actually false.

- The defendant was acquitted, but he did the crime!
- **Type II error** (a.k.a. *false negative*)

Hypothesis Testing

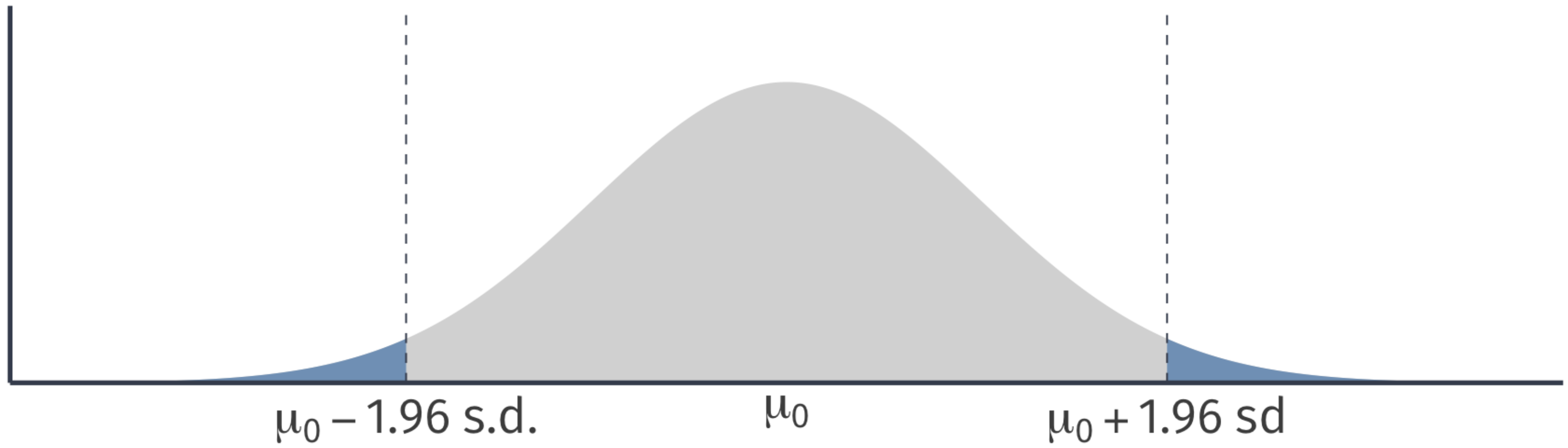
$\hat{\mu}$ is random: it could be anything, even if $\mu = \mu_0$ is true.

- But if $\mu = 0$ is true, then $\hat{\mu}$ is unlikely to take values far from zero.
- As the variance of $\hat{\mu}$ shrinks, we are even less likely to observe "extreme" values of $\hat{\mu}$ (assuming $\mu = \mu_0$).

Our test should take extreme values of $\hat{\mu}$ as evidence against the null hypothesis, but it should also weight them by what we know about the variance of $\hat{\mu}$.

- For now, we'll assume that the variable of interest X is normally distributed with mean μ and standard deviation σ.

Hypothesis Testing



Reject (H_0) if $\hat{\mu}$ lies in the **rejection region**.

Hypothesis Testing

Reject (H_0) if $|z| = \left| \dfrac{\hat{\mu} - \mu_0}{\mathrm{sd}(\hat{\mu})} \right| > 1.96.$

What happens to z as $|\hat{\mu} - \mu_0|$ increases?

What happens to z as $\mathrm{sd}(\hat{\mu})$ increases?

Hypothesis Testing

The formula for the z statistic assumes that we know $\text{sd}(\hat{\mu})$.

- In practice, we don't know $\text{sd}(\hat{\mu})$, so we have to estimate it.

If the variance of X is σ^2, then

$$\sigma^2_{\hat{\mu}} = \frac{\sigma^2}{n}.$$

- We can estimate σ^2 with the sample variance S^2_X.

Hypothesis Testing

The formula for the z statistic assumes that we know $\mathrm{sd}(\hat{\mu})$.

- In practice, we don't know $\mathrm{sd}(\hat{\mu})$, so we have to estimate it.

The sample variance of the sample mean is

$$S_{\hat{\mu}}^2 = \frac{1}{n(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Hypothesis Testing

The **standard error** of $\hat{\mu}$ is the square root of $S^2_{\hat{\mu}}$:

$$\text{SE}(\hat{\mu}) = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (X_i - X)^2} \; .$$

- Standard error = sample standard deviation of an estimator.

When we use $\text{SE}(\hat{\mu})$ in place of $\text{sd}(\hat{\mu})$, the z statistic becomes a t statistic:

$$t = \frac{\hat{\mu} - \mu_0}{\text{SE}(\hat{\mu})} \, .$$

Hypothesis Testing

The **standard error** of $\hat{\mu}$ is the square root of $S^2_{\hat{\mu}}$:

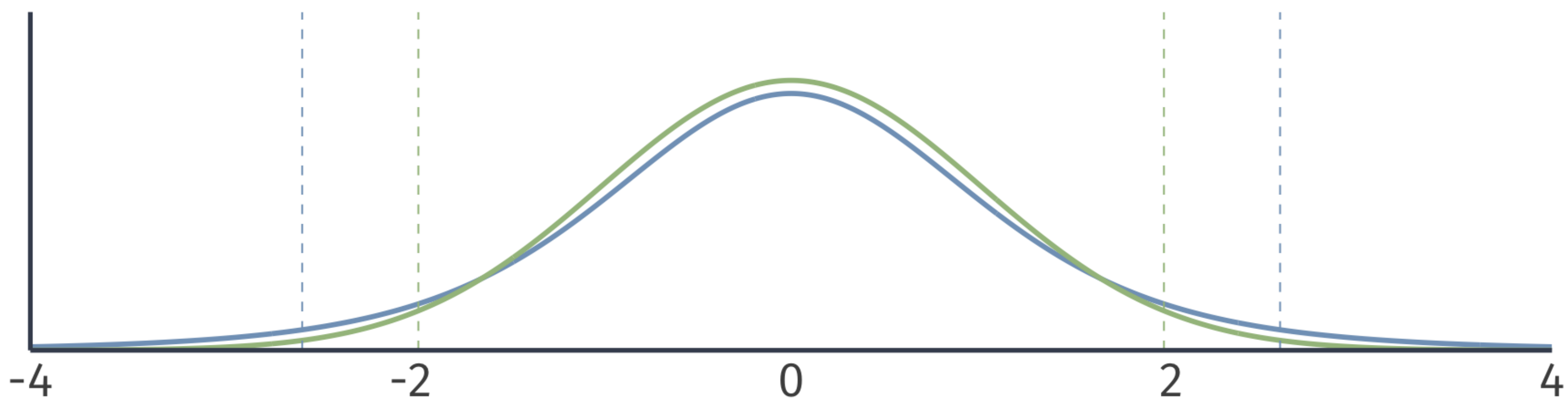
$$\text{SE}(\hat{\mu}) = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (X_i - X)^2} \,.$$

- Standard error = sample standard deviation of an estimator.
- Unlike the $\text{SD}(\hat{\mu})$, $\text{SE}(\hat{\mu})$ varies from sample to sample.
- *Consequence:* t statistics do not necessarily have a normal distribution.

Hypothesis Testing

Normal distribution vs. t distribution

- A normal distribution has the same shape for any sample size.
- The shape of the t distribution depends the **degrees of freedom**.

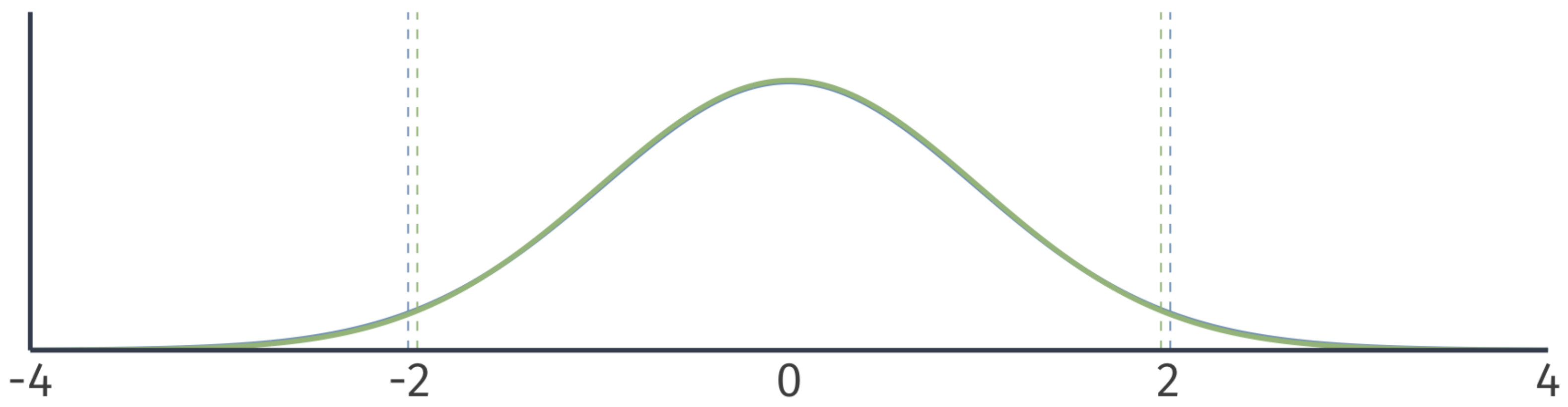


- Degrees of freedom = 5.

Hypothesis Testing

Normal distribution vs. t distribution

- A normal distribution has the same shape for any sample size.
- The shape of the t distribution depends the **degrees of freedom**.

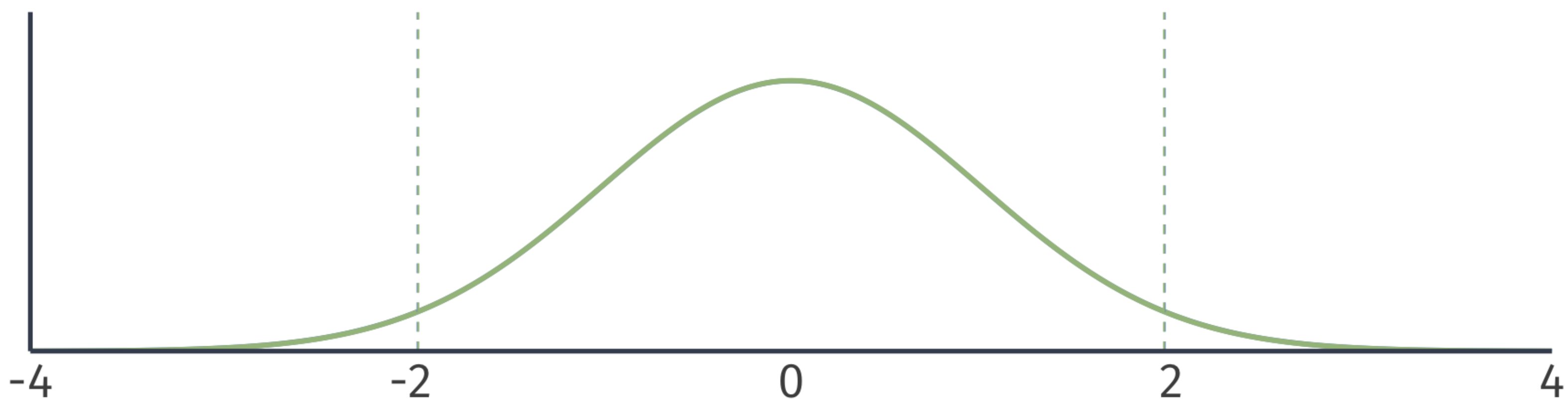


- Degrees of freedom = 50.

Hypothesis Testing

Normal distribution vs. t distribution

- A normal distribution has the same shape for any sample size.
- The shape of the t distribution depends the **degrees of freedom**.



- Degrees of freedom = 500.

Hypothesis Testing

Two sided t Tests

To conduct a t test, compare the t statistic to the appropriate **critical value** of the t distribution.

- To find the critical value in a t table, we need the degrees of freedom and the significance level α.

Reject (H_0) at the $\alpha \cdot 100$-percent level if

$$|t| = \left| \frac{\hat{\mu} - \mu_0}{SE(\hat{\mu})} \right| > t_{crit}.$$

On Your Own

As the term progresses, we will encounter additional flavors of hypothesis testing and other related concepts.

You may find it helpful to review the following topics from Math 243:

- Confidence intervals
- One-sided t tests
- p values

Hypothesis testing is an essential tool. Yet the traditional way of teaching hypothesis testing can be unintuitive.

It took me several tries (classes) to fully understand the concept

If you can program, you have direct access to the fundamental ideas in statistics

To demonstrate, consider *hypothesis testing*

In order to do that, we need a problem...

Does drinking beer make you more attractive to mosquitos?

Hypothesis testing

Though it sounds silly, this research question is important

- Malaria is transmitted via mosquito
- Most model for malaria transmission historically assume equal risk of mosquito bites across individuals
- Though, good evidence of *heterogenous* propensity bites exist
- Understanding which people might have higher propensity for bites may allow for interventions that reduce the impact of malaria

OPEN ACCESS Freely available online

PLoS one

Beer Consumption [redacted] Human Attractiveness to Malaria Mosquitoes

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Abstract

Background: Malaria and alcohol consumption both represent major public health problems. Alcohol consumption is rising in developing countries and, as efforts to manage malaria are expanded, understanding the links between malaria and alcohol consumption becomes crucial. Our aim was to ascertain the effect of beer consumption on human attractiveness to malaria mosquitoes in semi field conditions in Burkina Faso.

Here is the data. Treatment group in **blue**.

Treatment group:

20	25	28	21	27
29	18	20	17	31
23	24	28	19	24
31	24	21	20	19
27	20	21	26	27

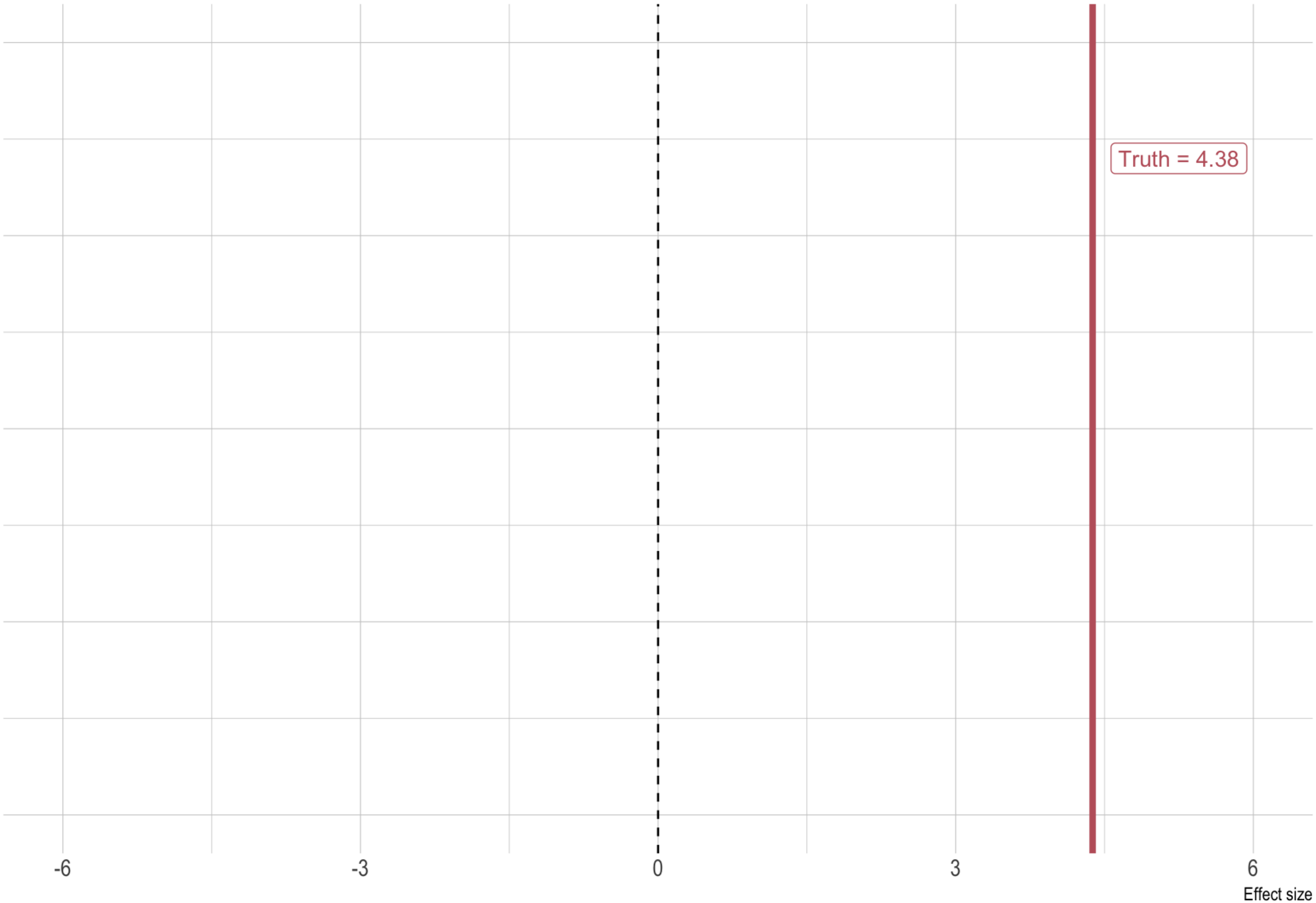
Control group:

24	18	20
13	22	20
24	19	23
19	15	22
12	21	16
21	22	15

Treatment mean: 23.6 **Control mean:** 19.22

Difference in means: **4.38**

Plot the true difference



Suppose the **difference** is coincidental. Then the labels don't matter

20	25	28	21	27
29	18	20	17	31
23	24	28	19	24
31	24	21	20	19
27	20	21	26	27

24	18	20
13	22	20
24	19	23
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Treatment mean: 23.6 **Control mean:** 19.22

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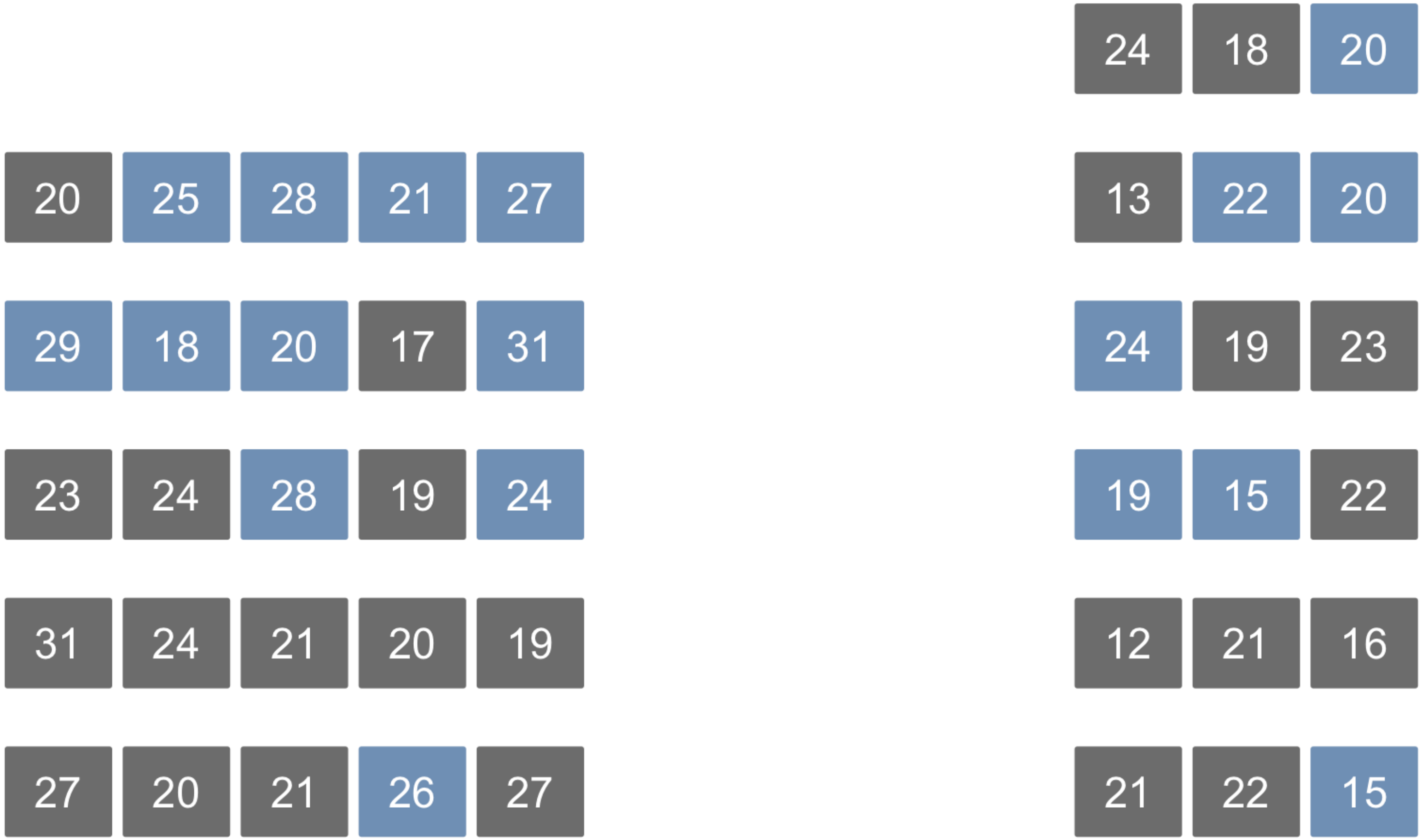
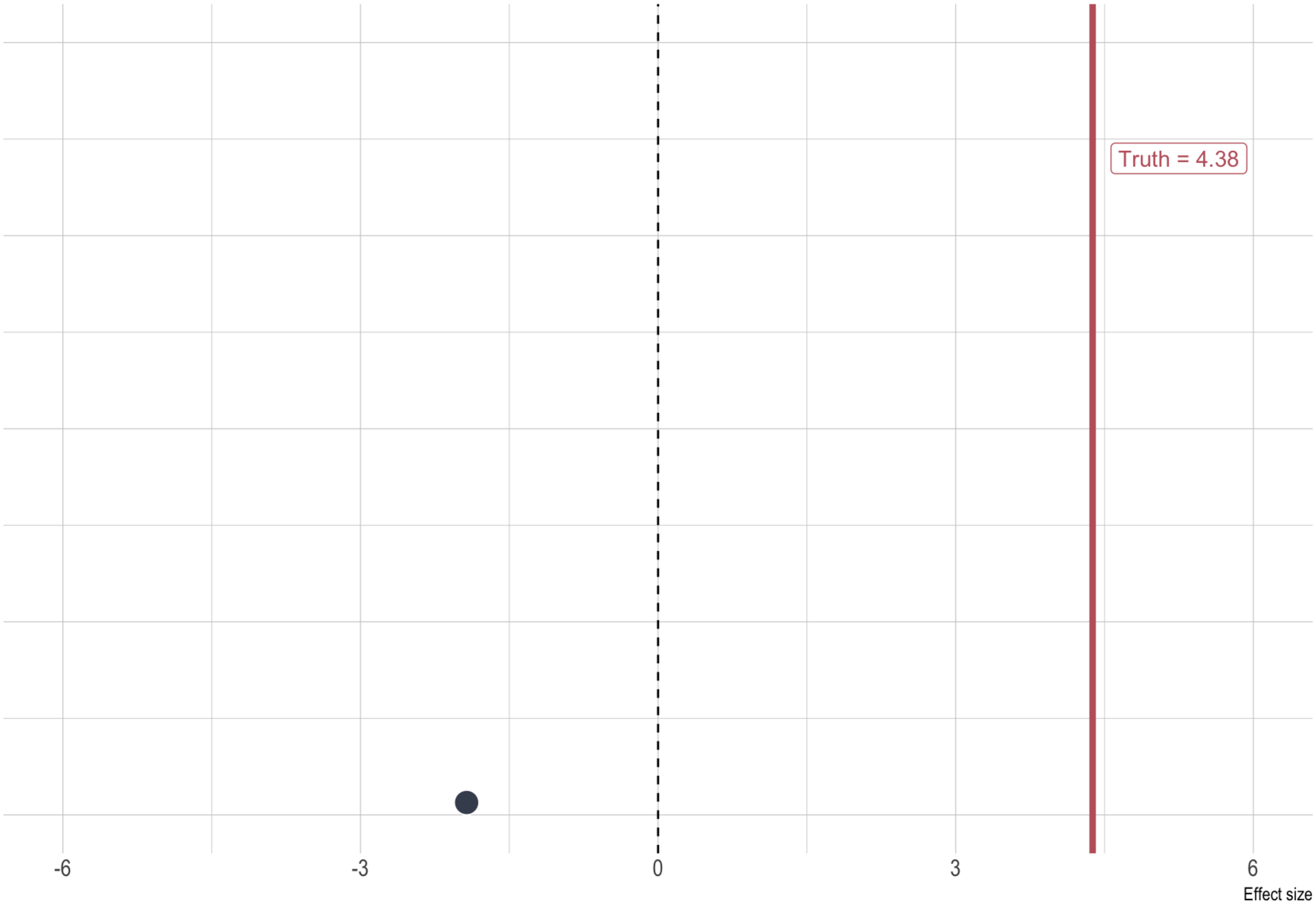


Treatment mean: 20.96 **Control mean:** 22.89

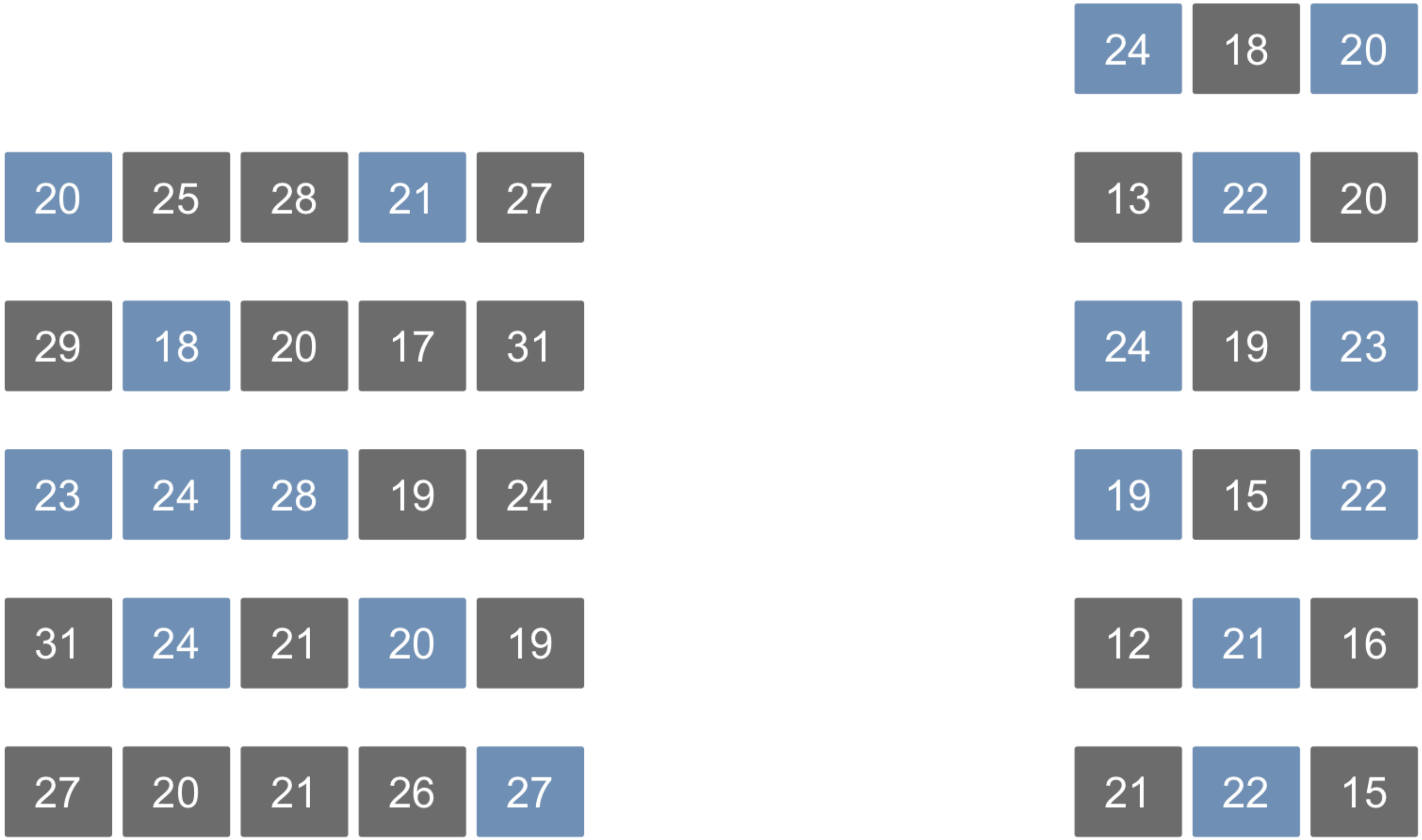
False difference in means : **-1.93**

Plot the *"fake"* difference



And do it again

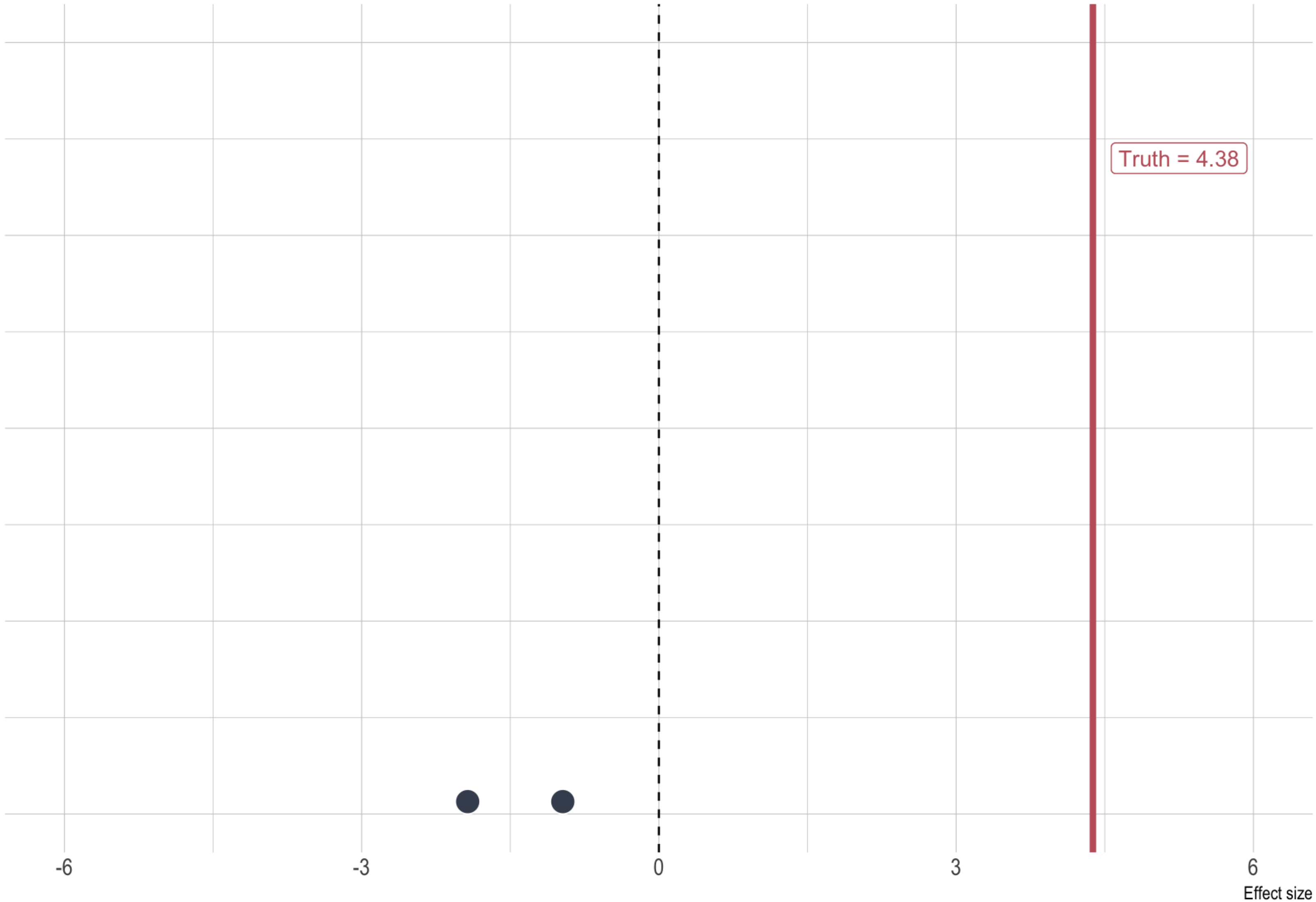
Labels don't matter. Assign treatment randomly. Find the difference.



Treatment mean: 21.36 **Control mean:** 22.33

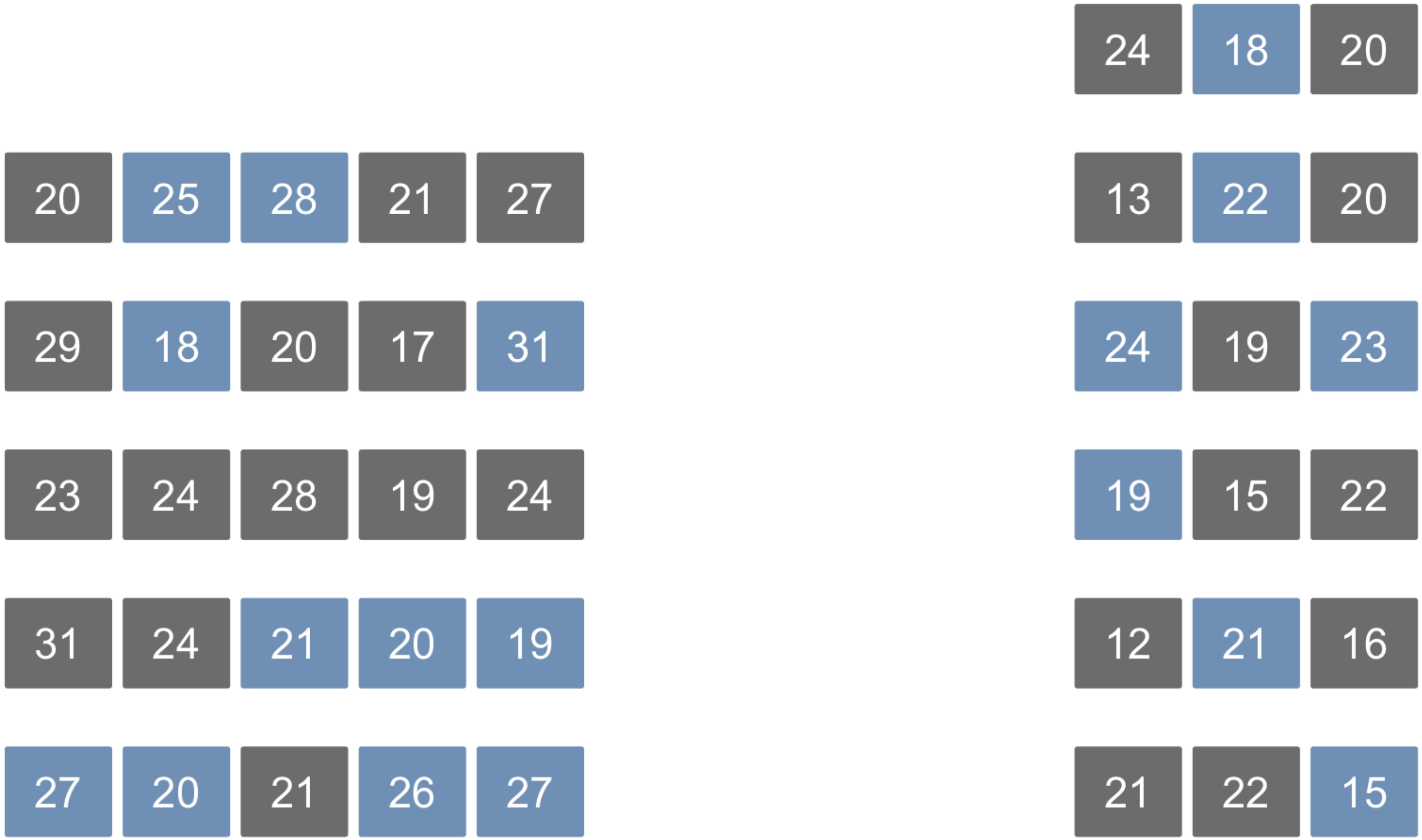
False difference in means : **-0.97**

Plot the difference



And do it again (3 times)

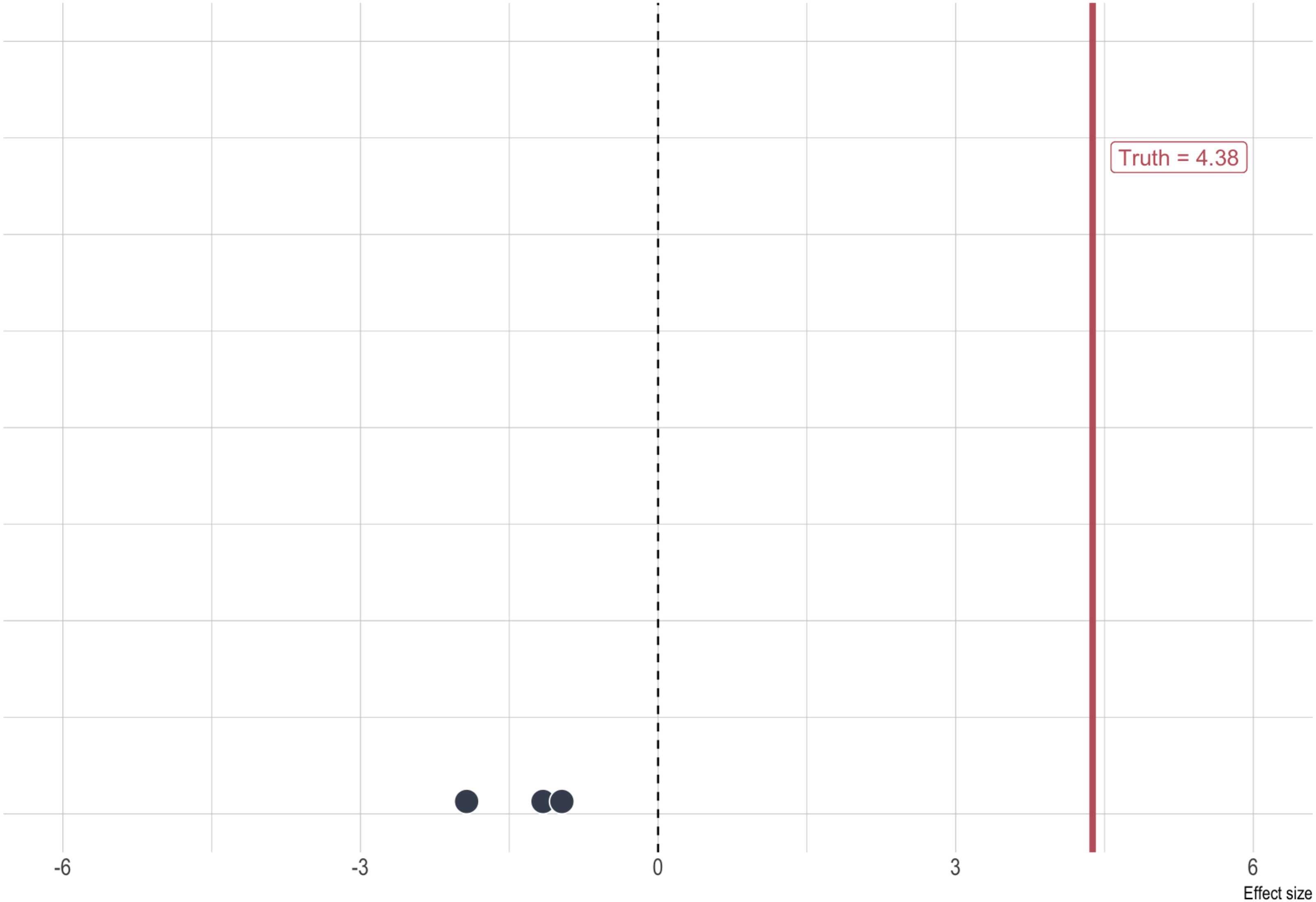
Labels don't matter. Assign treatment randomly. Find the difference.



Treatment mean: 21.28 **Control mean:** 22.44

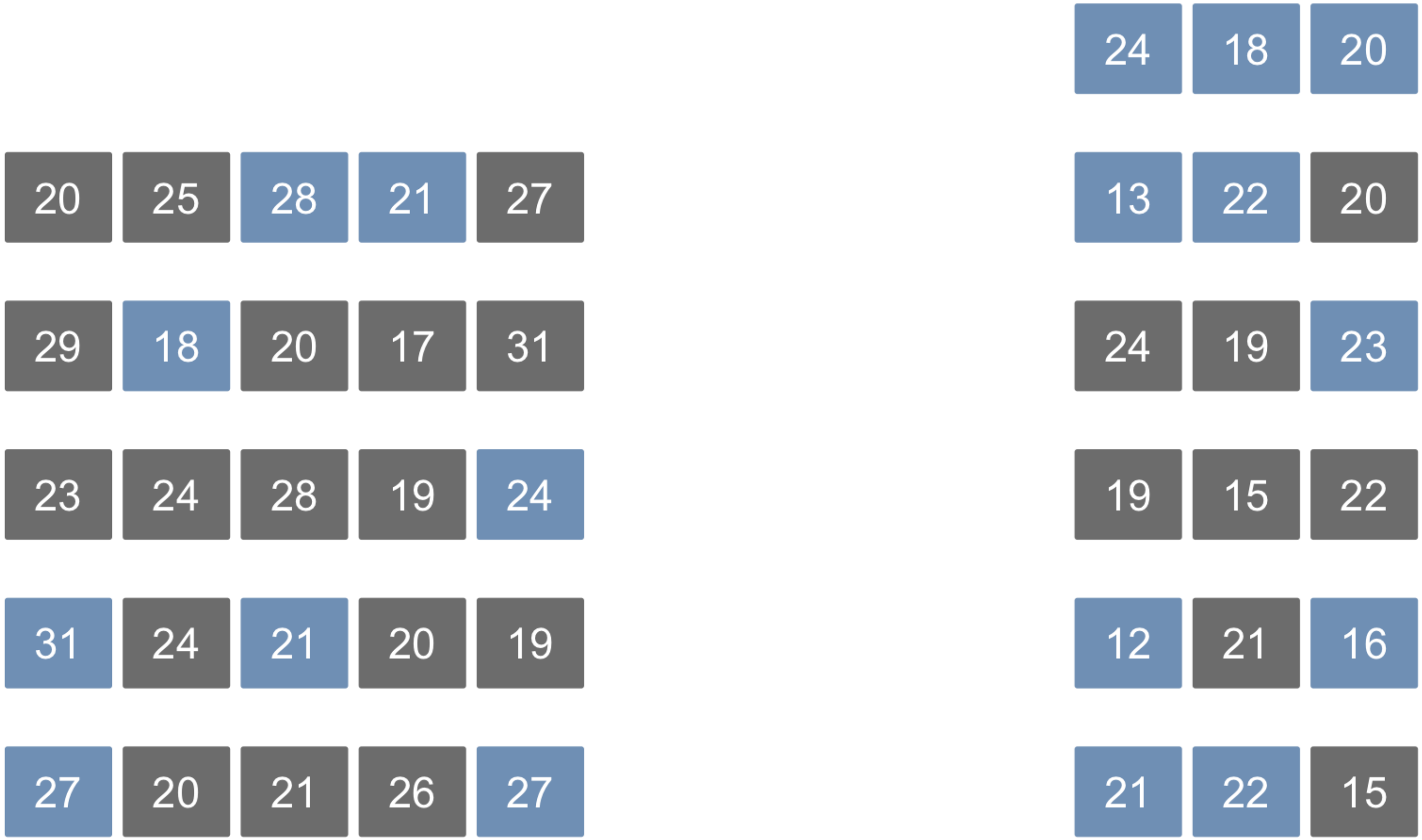
False difference in means : **-1.16**

Plot the differences



And do it again (4 times)

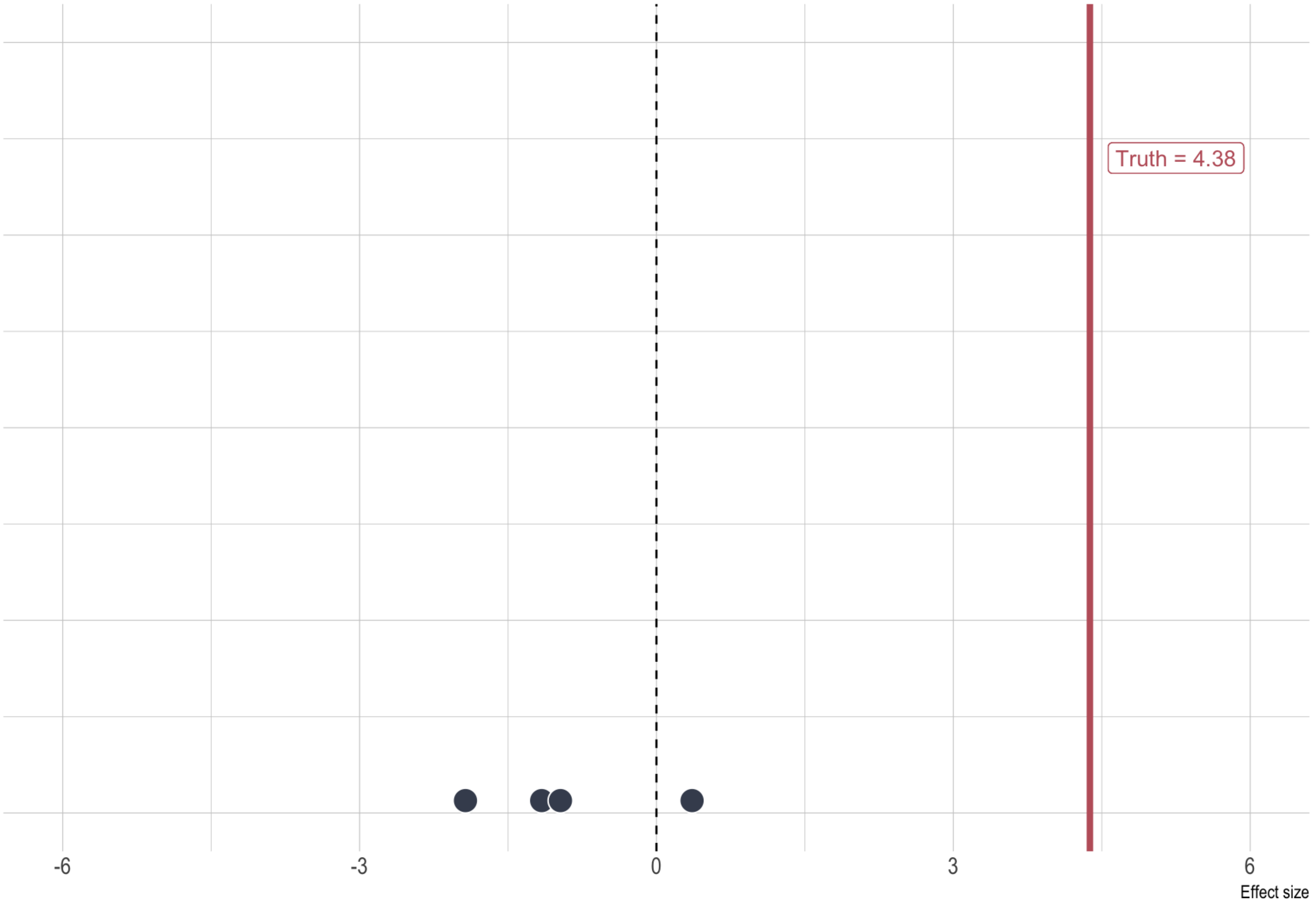
Labels don't matter. Assign treatment randomly. Find the difference.



Treatment mean: 21.92 **Control mean:** 21.56

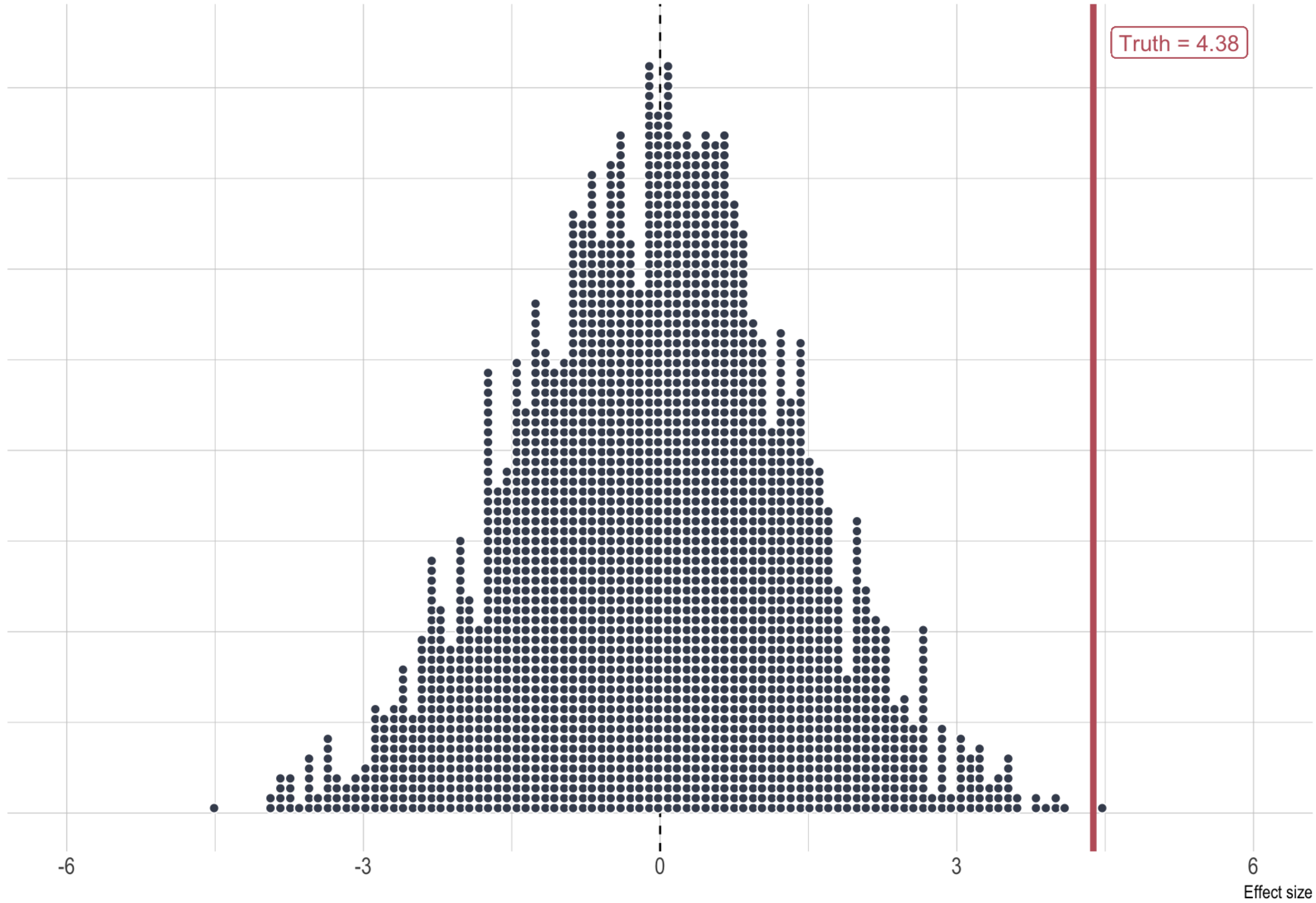
False difference in means : **0.36**

Plot the differences



And do it again (2,500 times)

Plot all 2,500 differences.



Data types

Data

There are **two** broad types of data

1. **Experimental data**

> Data generated in controlled, laboratory settings[1]

Ideal for **causal identification**, but difficult to obtain

- Logistically intractable
- Expensive
- Morally repugnant

1. *Note:* Experiments can often occur outside the lab (eg randomized control trials and A/B testing)

Data

There are **two** broad types of data

1. **Experimental data**
2. **Observational data**

> Data generated in non-experimental settings

Types of observational data:

- Surveys
- Census
- Administrative data
- Environmental data
- Transaction data
- Text and image data

Commonly used though poses challenges to **causal identification**

Data types: Cross sectional

> Sample of individuals from a population at a point in time

Ideally collected using **random sampling**

- **random sampling** + **sufficient sample size** = **representative sample**
- Non-random sampling is more common and difficult to work with

Note: Used extensively in applied microeconomics[1] and is the main focus of this course

1. Applied microeconomics = Labor, health, education, public finance, development, industrial organization, and urban economics

Data types: Time series

> Observations of variables over time

Ex.

- Quarterly GDP
- Annual infant mortality rates
- Daily stock prices

Complication: Observations are not independent draws

- eg GDP this quarter is highly correlated to GDP last quarter

More advanced methods needed[1]

1. See EC 421 and EC 422

Data types: Pooled cross sectional

> Cross sections from different points in time

Useful for studying relationship that change over time.

Again, requires more advanced methods[1]

1. See EC 421 and many of the 400-level applied classes

Data types: Panel data

> Time series for each cross sectional unit

Ex. Daily attendance across my class

Can control for *unobserved* characteristics

Again, requires more advanced methods[1]

1. See EC 421 and many of the 400-level applied classes

Data types: Messy data

Analysis ready dataset are rare. Most data are *messy*

Data wrangling is a non-trivial part of an economist or data scientist/analyst's job

R has a suite of packages[1] that facilitate data wrangling:

- The `tidyverse`: `readr`, `tidyr`, `dplyr`, `ggplot2` + others

1. More than one suite, but for this class we use the `tidyverse`