Core course for *Decision Analytics*, *Quantitative Finance*, *Risk Management* and *Statistics* Majors:

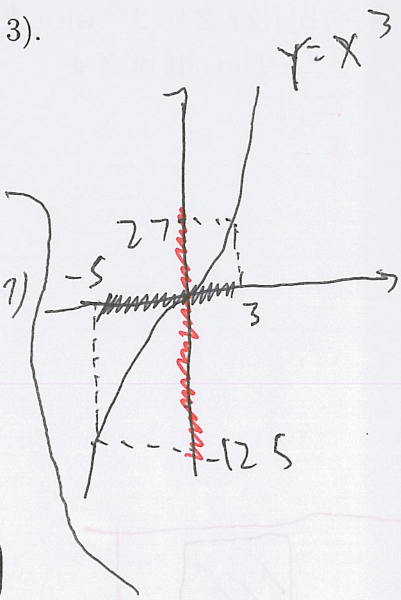
STAT2601B Probability and Statistics I (2017-2018 Second Semester)

Chapter 6 - Examination Question 3 (December 13, 2014)

3. Let X be a random variable having a uniform distribution $\mathrm{U}(-5, 3)$.

   (a) Find the probability density function of $Y = X^3$.

   (b) Find the probability density function of $Z = X^2$.



Solution:

3. (a) For $Y = X^3$ with $X \sim \mathrm{U}(-5, 3)$. The support of Y is $(-125, 27)$
   The cdf of Y is

$$f_X(x) = \begin{cases} 1/8 & \text{if } x \in (5,3) \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 5 \\ \dfrac{x-(-5)}{3-(-5)} & \text{if } -5 < x < 3 \\ 1 & \text{if } x > 3 \end{cases}$$

uniform distribution

$$\begin{aligned}
F_Y(y) &= \Pr(Y \le y) \\
&= \Pr(X^3 \le y) \\
&= \Pr(X \le y^{\frac{1}{3}}) = F_X(y^{1/3}) \\
&= \begin{cases} 0, & \text{for } y < -125; \\ \dfrac{y^{\frac{1}{3}} - (-5)}{3 - (-5)}, & \text{for } -125 \le y < 27; \\ 1, & \text{for } y \ge 27. \end{cases} \\
&= \begin{cases} 0, & \text{for } y < -125; \\ \dfrac{1}{8}(y^{\frac{1}{3}} + 5), & \text{for } -125 \le y < 27; \\ 1, & \text{for } y \ge 27. \end{cases}
\end{aligned}$$

inverse

Therefore, the pdf of Y is

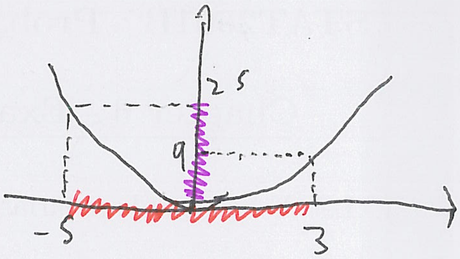
differentiation

$$f_Y(y) = F'_Y(y) = \begin{cases} \dfrac{1}{24y^{\frac{2}{3}}}, & \text{for } -125 < y < 27 \text{ and } \boxed{y \neq 0}; \\ 0, & \text{otherwise.} \end{cases}$$

3. Let X be a random variable having a uniform distribution $\mathrm{U}(-5,3)$.

   (a) Find the probability density function of $Y = X^3$.

   (b) Find the probability density function of $Z = X^2$.

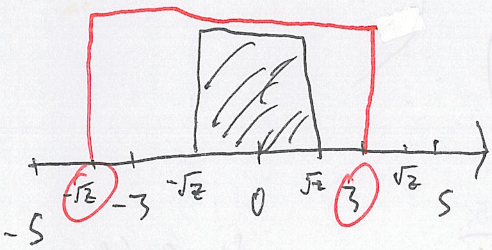


Solution:

3. (b) For $Z = X^2$ with $X \sim \mathrm{U}(-5,3)$. The support of Z is $[0, 25)$
   The cdf of Z is

$$
\begin{aligned}
F_Z(z) &= \Pr(Z \le z) \\
&= \Pr(X^2 \le z) \\
&= \Pr(-\sqrt{z} \le X \le \sqrt{z})
\end{aligned}
$$

inverse



$$
= \begin{cases}
0, & \text{for } z < 0; \\
\displaystyle\int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{3-(-5)}\,dx, & \text{for } 0 \le z < 9; \\
\displaystyle\int_{-\sqrt{z}}^{3} \frac{1}{3-(-5)}\,dx + \int_{3}^{\sqrt{z}} 0\,dx, & \text{for } 9 \le z < 25; \\
1, & \text{for } z \ge 25.
\end{cases}
$$

$$
= \begin{cases}
0, & \text{for } z < 0; \\
\dfrac{2\sqrt{z}}{8} = \dfrac{\sqrt{z}}{4}, & \text{for } 0 \le z < 9; \\
\dfrac{3+\sqrt{z}}{8}, & \text{for } 9 \le z < 25; \\
1, & \text{for } z \ge 25.
\end{cases}
$$

Therefore, the pdf of Z is

$$
f_Z(z) = F_Z'(z) = \begin{cases}
\dfrac{1}{8\sqrt{z}}, & \text{for } 0 < z < 9; \\
\dfrac{1}{16\sqrt{z}}, & \text{for } 9 \le z < 25; \\
0, & \text{otherwise.}
\end{cases}
$$

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