MATH2014 Ch. 2 Multiple Integrals (1)

Just don't forget your basic integration skills (e.g. chain rule) $\div \pi$

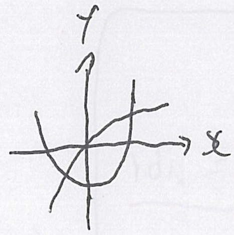
$$\text{volume} = \int_{lb_x}^{ub_x}\int_{ub_y}^{ub_y} f(x,y)\,dy\,dx$$

order can change, to calculate convenient.

Area: $\iint dA$

Avg: $\dfrac{\iint f\,dA}{\iint dA}$

① draw boundary



1. draw
2. intersect
3. push

② choose which way to integrate (order of x/y)

integrating process: like partial, treat irrelevant as const.

$\iint dy\,dx$: 1. shoot arrow $// y$, y bound in terms of x

2. move arrow from x (const. bound)

(remember sorting sound effect?)

tricks! if function separable (no need const. boundary?) $\rightarrow \int_b^a \int_d^c f(x)g(y)\,dy\,dx$

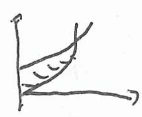
(treat as const.) $= \int_b^a f(x) \int_d^c g(y)\,dy\,dx$

region between = difference $(-)$ (T6Q1)

$\iint c\, f(x,y)\,dA = c\iint f(x,y)\,dA$, $\iint f(x,y) \pm g(x,y)\,dA = \iint f(x,y)\,dA \pm \iint g(x,y)\,dA$

Others / misc: odd / even function, period. irregular integral (∞, whole 1st q.)

if needed, split two parts (diff. intersection)

area union (T7Q1)	$0 < y \le 8$ $0 \le x \le \sqrt{\frac{y}{3}}$ $3x^2 \le y$	$\cup$ $8 \le y \le 12$ $\sqrt{y-8} \le x \le \sqrt{\frac{y}{3}}$ $3x^2 \le y \le x^2+8$	$\rightarrow$  $\int_0^{12}\int_{3x^2}^{x^2+8} dx\,dy$

Change of variables:

$f(x,y) = a$
$f(x,y) = b$
$g(x,y) = c$
$g(x,y) = d$
$\rightarrow$ $u = f(x,y)$, $v = f(x,y)$

☆ use when too difficult.

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$

$\rightarrow x = a r\cos\theta$, $y = b r \sin\theta$, $|J| = abr$

1. define u, v $\quad x = ?,\ y = ? \quad \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$
2. $|J| = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$ $\quad u = ?,\ v = ? \quad \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$
3. lines in terms of u, v
4. rewrite (draw) boundary (including axis!)
5. $\iint ?\ |J|\ du\, dv$

☆☆ $\frac{1}{J}$ if $u = ?$, $v = ?$

☆ J if $x = ?$, $y = ?$

$\int \frac{2}{(1+r^2)^2} \cdot r\, dr = -\frac{1}{2(1+r^2)}$

$\int r\sqrt{a^2+r^2}\, dr = \frac{1}{3}(a^2+r^2)^{3/2}$

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$|J| = r$

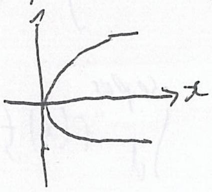
$\underline{x^2+y^2=r^2}$

$\iint ?\ \boxed{r}\, dr\, d\theta$ ☆☆

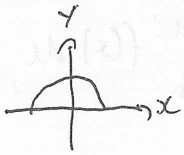
☆ θ from $+$ axis!
Re: P.49 Q6.

1. find r boundary r is +ve!
(usually $[0,$ function$]$)
2. turn θ (split into parts if r is $-$ve / not in domain)
$[0, 2\pi]$
3. rewrite integral (with $|J| = r$!)

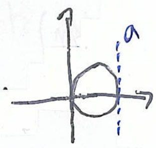
$x = a + by^2$
parabola (rotated)



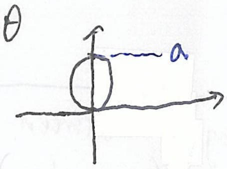
$\sqrt{1-x^2}$
semicircle



$r = a\cos\theta$



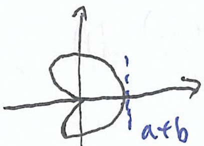
$r = a\sin\theta$



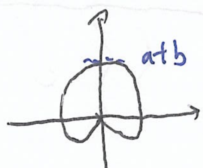
polar common graphs!

$r = a + b\cos\theta$

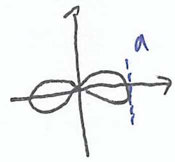
+ ! $\frac{a}{b} \geq 1$



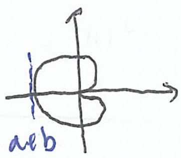
$r = a + b\sin\theta$



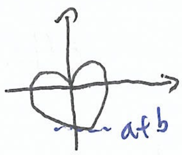
$r^2 = a^2\cos 2\theta$



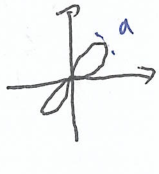
$r = a - b\cos\theta$



$r = a - b\sin\theta$



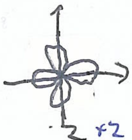
$r^2 = a^2\sin 2\theta$

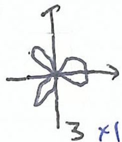


cos: $\leftrightarrow$

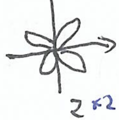
sin: $\updownarrow$

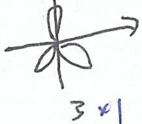
$r = a\cos n\theta$





$r = a\sin n\theta$





MATH 2014 Ch. 2 Multiple Integrals (2)

Triple integrals: $\iiint f(x,y,z)\,dz\,dy\,dx$

volume $= \iiint dV$

avg $= \dfrac{\iiint f(x,y,z)\,dV}{\text{volume}}$

mass $= \iiint \delta\,dV$

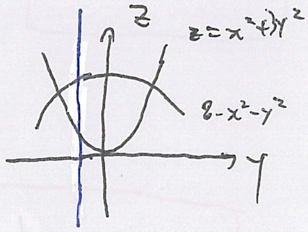
δ: density

center of mass (centroid): $\bar{x} = \dfrac{\iiint x\,dV}{\iiint dV}$

imagine: x, y on paper
z out of paper

① let x or $y = 0$, find z intersect point. $z-x$ / $z-y$ graph

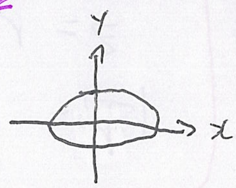
eg. $z = x^2 + 3y^2$
$z = 8 - x^2 - y^2$



② find projection (shadow) on x-y plane, normal double in. x-y graph

$x^2 + 3y^2 = 8 - x^2 - y^2$ set z both sides equal.

$x^2 + 2y^2 = 4$



③ write integral

$$\int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-y^2-x^2} dz\,dy\,dx$$

☆ eqt of plane: $ax + by + cz + d = 0$ → $c \neq 0$, divide by z.
(e.g. 20-21) then sub points, find eqt of plane.

change of variables: $|J| = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}$

$x = ?$
$y = ?$
$z = ?$ $\quad |J| = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Cylindrical Coordinates: (cylinder)

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ |J| &= r \end{aligned}}$$

1. z limits, (usually given?)
2. r limits
3. θ limits

$$\int_0^{2\pi}\int_0^2\int_0^{r^2} r \; dz\,dr\,d\theta$$

not $\frac{r^2}{2}$!

$$= \int_0^{2\pi} d\theta \int_0^2 rz\Big|_0^{r^2} dr$$

$$= 2\pi \cdot \int_0^2 r^3\,dr = 8\pi$$

vol sphere $= \frac{4}{3}\pi r^3$

Spherical Coordinates

(sphere: $\rho = 7$, cone: $z = r$)

$$\boxed{\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \\ |J| &= r^2\sin\phi \end{aligned}}$$

1. r limit
2. ϕ limit
3. θ limit (usually $[0, 2\pi]$)

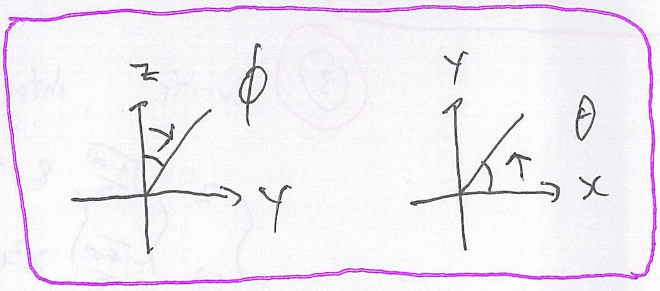
$$\iiint dr\,d\phi\,d\theta$$

" $x-y$: $\sin\phi$ " $\quad x^2+y^2 = r^2\sin^2\phi$

memorize J

$x = au$
$y = bv$
$z = cw$

ϕ is from z axis!



Special: 2-sphere (A2 Q7) (T8 Q2)

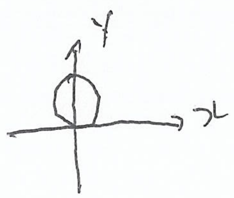
$$x^2+y^2+(z-R)^2 \le R^2$$

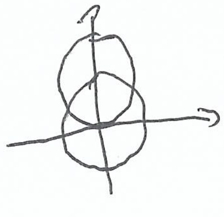
$$r^2\sin^2\phi + (r\cos\phi - R)^2 \le R^2$$

$$r^2 - 2rR\cos\phi \le 0$$

$$r \le 2R\cos\theta$$

$$x^2 + (y-r)^2 = r^2$$





$$x^2+y^2+z^2 \le R^2$$

$$r \le R$$

$$\{r \le R\} \cap \{r \le 2R\cos\theta\}$$

$$2R\cos\theta \le R$$

$$\theta \ge \frac{\pi}{3} \rightarrow \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^R J\,dV + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} J\,dV$$

MATH 2014 Ch.3 Linear Algebra "Matrix time is calculator time!"

System of linear eqt!

$x_1 - x_2 + 2x_3 = 1$
$2x_1 + x_2 + x_3 = 8$
$x_1 + x_2 = 5$

Augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right)$

(000: homogeneous: trivial / non-trivial sol)

Gaussian elimination program??

Row echlon form! (upper triangular)
1. All nonzero rows above all-zero rows
2. Each leading entry of a row to the right of LE of prev.
3. All entries below a LE are zero

Reduced row echlon
4. LE in each row is 1
5. Each LE is only nonzero in column.

Rank! # of non-zero rows in reduced row form (full rank → unique solution)

free variable: $x = \begin{pmatrix} 3-t \\ 0 \\ t \end{pmatrix}$ where $t \in \mathbb{R}$ (Infinitely many sol)

inconsistent / no sol! $0 = a$, $a \neq 0$

Matrix! A_{mn} : m row, n column   multiplication! $A_{mk} B_{kn} \to C_{mn}$

$AI = IA = A$, $AB \neq BA$, $AC = AB \not\Leftrightarrow B = C$, $AB = 0 \not\Leftrightarrow A = 0, B = 0$, $(A+B)^T = A^T + B^T$, $(AB)^T = B^T A^T$

Submatrix! $A(2|3)$: remove row 2 and col 3

Determinant! column/row expansion (any is ok)
recursively det (A) for @ item in col/row, multiply item

$\begin{pmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \end{pmatrix}$

$\det(AB) = \det A \det B$, $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$

program again!

$\begin{cases} - - - = 1\ 0\ 0 \\ - - - = 0\ 1\ 0 \\ - - - = 0\ 0\ 1 \end{cases}$

det of upper triangular matrix! multiply diagonal items

row swapping: $\det A = -\det B$; multiply row/col! $\det A = k \det B$
minus row/col: unchanged.

$(A^{-1})^T = (A^T)^{-1}$

Singular: $\det(A) = 0 \to$ no unique sol, non invertible

$(AB)^{-1} = A^{-1}B^{-1}$
$(A^n)^{-1} = (A^{-1})^n$

Inverse: $A^{-1}A = I$ $(A | I) \sim (I | A^{-1})$ Gaussian elimination

☆☆☆
☆☆☆

Eigenvalue (λ), Eigenvector ($\vec{x}$) $A\vec{x} = \lambda\vec{x}$, $(A-\lambda I)\vec{x} = 0$

(1) solve λ: $\det(A - \lambda I) = 0$

put $\lambda = E3$, quadratic/cubic eqt

2×2: double root, 1 eigenvector, cannot diagonalize.

3×3: ??? → diagonalizable: n linearly independent EV.

(2) Find $\vec{x}$: put $\lambda = ?$

Put x_1, x_2 free 3×3: Gaussian elimination $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

☆ 2 free variables

$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x_1 = t \\ x_2 = u \end{array} \to \begin{pmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} x_2$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

represent x_n in terms of x_1

1st non zero entry is '1'

(3) Power: $A^k = PD^kP^{-1} = (v_1 v_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(v_1 v_2)^{-1}$

verification/ans checking: $AP = PD$

square root of matrix: $(PD^{1/2}P^{-1})^2 = PDP^{-1} = A$

$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$

$\lambda_1 = 3$, $\lambda_2 = -1$

$\lambda = 5$ $\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$8x_1 - 4x_2 = 0$

$x_2 = 2x_1$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{v_1} x_1$

$\lambda = -1$ $\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$4x_1 + 0x_2 = 0$

$x_1 = 0$

$\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{v_2} x_2$

$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}$

Vectors: one column, n rows : $\mathbb{R}^n$, arrow of movement

- Linear combination: $a_1\vec{x_1} + a_2\vec{x_2} = \vec{b}$; Span: set of all linear combination
- Linearly independent: $x_1\vec{v_1} + x_2\vec{v_2} + \dots + x_n\vec{v_n} = 0$ only trivial solution ($\det \neq 0$)

$[a\,b\,c]^T\,[d\,e\,f]^T\,[g\,h\,i]^T \to \left[\begin{array}{cc|c} a & b & c \\ d & e & f \\ g & h & i \end{array}\right] \to$ gaussian elimination → check case for no sol → linear combination (2015 T3 Q3)

exam skill!

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix} = \begin{pmatrix} \lambda_1^n a & \lambda_2^n b & \lambda_3^n c \\ \lambda_1^n d & \lambda_2^n e & \lambda_3^n f \\ \lambda_1^n g & \lambda_2^n h & \lambda_3^n i \end{pmatrix}$, multiply: dot dot dot both hands.