STAT 2601 Ch.2 Counting (1)

2.1: Product rule: break down task into multiple steps → $n_1 \cdot n_2$ ways.

☆ mind if there is any permutation

e.g. choosing ppl one by one → $16 \times 14 \times 12 \times 10 \div 4!$

2.2: Sum rule: (combining the pool of choices) (cases)

2-3: Subtraction rule! (inclusion-exclusion principle)

$$Pr(E_1 \cup E_2 \cup E_3) = Pr(E_1) + Pr(E_2) + Pr(E_3) - \left(Pr(E_1 \cap E_2) + Pr(E_2 \cap E_3) + Pr(E_1 \cap E_3)\right) + Pr(E_1 \cap E_2 \cap E_3)$$

2-4: Division rule (divide by n when repeat n times)

→ circular table vs in a line

D A B C (circle) : $\frac{4!}{4}$ (fix one, relative) ABCD : $4!$

bracelet: flip + same

→ choosing people: $\div (n!)$ → choose who first no effect on result. ref. nCr.

watch circle order (adjacent) (Assignment 1 Q3c)

2-5: Permutation : LETTERS: $\frac{7!}{1!\,1!\,1!\,2!\,2!}$

nPr ← start from n, r numbers.

n ppl, put r in a line.

$\boxed{\text{☆ divide by groups of same item!}}$

multiple groups: multiply!

multiple ways: add! × → +

2-6: Combination

nCr $\binom{n}{r}$

n ppl, put r in a group. ☆ divide by $(!)$

Divide into 2 teams:

$\frac{10C5}{2}$ → order is irrelevant between 2 teams.

Common skills in combination:

① Glueing. → permutation within glue.

(B₁ B₂) (G₁) (G₂) (G₃)

permutation = $4! \times 2!$

② Arrows. (with fixed baskets)

e.g. routing problem (DP!?)

U U U

↑ ↑

R R R R

or: permutation of U×3, R×4.

$\frac{7!}{3!\,4!}$

${}_nH_r$

↓

n baskets

throw r balls into them

③ Inclusion-Exclusion

e.g. no. of paths not pass thr. center:

= total − (start to center) × (center to end)

④ split case of overlap. → (question choosing)

⑤ Birthday problem! (none same) $= \frac{365\,P\,n}{365^n}$

none same with you: $\left(\frac{364}{365}\right)^{n-1}$

⑥ Throwing / sitting step by step

☆ no. of distinct → ${}_nH_r$ / possibilities each step.

vs P(distinct) ∝ divide by n! if choose. (die, committee)

∝ full house → case step by step ${}_nC_r$.

STAT 2601 Ch.2 Counting (2)

(replace: can choose same box ×2?)

Box: Distinct

		With replacement	Without replacement
Balls:	Ordered (distinguishable) (distinct)	n^r	$_nP_r$
	Unordered (indistinguishable) (same)	$_nH_r = \binom{n+r-1}{r}$	$_nC_r$ $\binom{n}{r}$

or generating function (opt!)

$_nH_r$:

n baskets,
r balls.

① $x_1+x_2+x_3=12$ $x_i \geq 0$ $_3H_{12}$.

② $x_1+x_2+x_3=12$ $x_i > 0$. (≥ 1) $_3H_{12-3}$ let $y_i = x_i - 1$

③ $x_1+x_2+x_3 < 12 \to$ ⓪ $x_i \geq 0 \to$ ④ $_4H_{12-1}$

$\leq 12 \to$ ④ $>0 \to$ ⓪

④ $1 \leq x_1 < x_2 < x_3 \leq 12$, $\binom{12}{3}$

④.① $1 \leq x_1 \leq x_2 \leq x_3 \leq 12$ $_{12}H_3$

$\{1, 2, \cdots 12\}$

Allocation:

- n distinct objects, r distinct groups (size)

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} \quad n_1+\cdots+n_r=n$$

- n same object, r distinct groups $_rH_n$
- n distinct objects, r distinct groups (no size) r^n
- n distinct object, r same group (size) $\frac{n!}{(5!)^r \cdot r!}$ same

o o o o o o o o o

arrange → swap within group.

☆☆☆ ⑤ $x_1, x_2, x_3 \leq 12$, $x_{i+1} - x_i > 1$ (no consecutive)

$x_i > 0$.

⑥ $x_1+x_2+x_3=10$ re: Ch.2 online

$x_i < 20 \to y_i = 20 - x_i$

$y_1+y_2+y_3 = 60-10 = 50$

$a_2 > a_1 + 1$, $a_1 < a_2 - 1$

$a_3 > a_2 + 1$, $a_2 - 1 < a_3 - 2$

$a_3 \leq 12$ $a_3 - 2 \leq 12 - 2$

$0 < a_1 < a_2 - 1 < a_3 - 2 \leq 12 - 2$

④: $\binom{12-2}{3}$

In general, $\geq$: put x balls.

$>$: change to $\geq$.

When adding new box, see $<$ or $\leq$.
(+1) (+0)

box upper limit
↓
Inclusion-exclusion.

∝ infinite

Binomial theorem / Trinomial / Negative binomial → Generating function

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$(x+y+z)^n = \sum_{\substack{(a,b,c):\\ a+b+c=n}} \binom{n}{a,b,c} x^a y^b z^c \qquad \binom{n}{a,b,c} = \frac{n!}{a!\,b!\,c!}$$

↓

n ppl

groups of a, b, c

↓

$(x+y+z)^n$

coeff $(x^a y^b z^c)$

ref. tut 2 Q16

$$(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

sum of GS: $\dfrac{a(1-r^n)}{1-r}$ n = no. of terms

Generating function: solve $x_1 + x_2 + x_3 + \dots = k$ with constraints

① Write down individual

$A(x) = 1 + x + x^2 + x^3 + \dots$

$B(x) = 1 + x + x^2$

$C(x) = x + x^2 + x^3 + x^4$

power can be -ve (re. 2.26)

② Transform into sum of GS, multiply.

$$G(x) = A(x)\,B(x)\,C(x)$$

$$= \frac{1}{1-x} \cdot \frac{1-x^3}{1-x} \cdot \frac{x(1-x^4)}{1-x}$$

-ve binomial thm

$$= x(1-x^3)(1-x^4) \cdot \sum_{r=0}^{\infty} \binom{3+r-1}{r} x^r$$

③ expand, find $\boxed{\text{coeff}}$ of x^r to find no. of sol for $x_1 + x_2 + x_3 = r$.

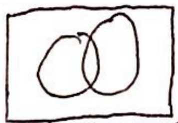
→ For each term in (front polynomial), find corresponding $n+r$ term at the back

STAT2601 Ch.3 Probability (1)

Set properties:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



"If all else fails venn diagram."

$Pr(\bar{A} \cap B) = Pr(B) - Pr(A \cap B)$

Re: venn diagram

$Pr(A \cup \bar{B}) = Pr(A) + Pr(\bar{B}) - Pr(A \cap \bar{B})$

$$A \cup B = A \cup (B \cap \bar{A}) = B \cup (A \cap \bar{B})$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

→ Translate $\cap$ into $\cup$, then $1 - Pr$

(ordering in probability?)

add them up!

(throwing a die)

$Pr(A \cap B) = 0$

disjoint / Mutually exclusive: $Pr\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} Pr(E_i)$

independent: $Pr(A \cap B) = Pr(A)\,Pr(B)$

$Pr(A) = Pr(A|B)$

Both AB occur

Inclusion-exclusion (re. JL Ch.2): $\boxed{Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)}$

Boole's inequality: $Pr\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} Pr(E_i)$ any event

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Derangement: None of the objects go to their correct positions

let $E_i = Pr(\text{correct envelope})$

$P(\text{derangement}) = 1 - Pr\left(\bigcup_{i=1}^{4} E_i\right)$

$$Pr(E_1) = \frac{3!}{4!} = \frac{1}{4} \qquad Pr(E_1 \cap E_2) = \frac{2!}{4!} = \frac{1}{12}$$

$$Pr(E_1 \cap E_2 \cap E_3) = \frac{1}{4!} = \frac{1}{24} \qquad Pr(E_1 \cap E_2 \cap E_3 \cap E_4) = \frac{1}{4!} = \frac{1}{24}$$

Inclusion exclusion principle: $Pr\left(\bigcup_{i=1}^{4} E_i\right) = \sum_{i=1}^{4} \frac{1}{4} - \sum_{i<j} \frac{1}{12} + \sum_{i<j<k} \frac{1}{24} - \frac{1}{24}$

$$= \frac{5}{8}$$

$$P(\text{derangement}) = 1 - \frac{5}{8} = \frac{3}{8}$$

$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}$

$\lim_{n \to \infty} = e^{-1}$

→ If cannot calculate $\cup$, try $\bar{A}$, $\cap$, inclusion-exclusion → $\cap$

Advanced inclusion - exclusion:

$$P(\text{derangement}) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots (-1)^n \frac{1}{n!}$$

① Derangement:

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r (n-r)!$$

(all) (some correct)

$$= n! - \left[\binom{n}{1}(n-1)! - \cdots + (-1)^{n+1}\binom{n}{n}(n-n)!\right]$$

choose 1 correct, shuffle others.

② no empty box : same balls, distinct box:

$${}_nH_r - \binom{n}{1}{}_{n-1}H_r + \binom{n}{2}{}_{n-2}H_r - \binom{n}{3}{}_{n-2}H_r$$

(naive?)

choose 1 empty, A others.

③ one empty box: distinct balls, distinct box

$$\binom{n}{1}\left((n-1)^r - \binom{n-1}{1} \times (n-2)^r + \binom{n-1}{2} \times (n-3)^r - \cdots\right)$$

1 empty 1+0 empty 1+1 empty 1+2 empty

rule: 1. naive

2. $- + - +$ empty/fit

STAT 2601 Ch.3 Probability (2)

Conditional probability $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \rightarrow Pr(A\cap B) = Pr(A|B)\,Pr(B)$

☆ asking questions → answer: how many cases does "ans" have?
"what is behind what I tell you?"

if all else fails, list out cases.

① Boy-girl paradox: $\{BB, BG, GB\}$ vs $\{BB, BG\}$

② Monty hall: how many choices to open? 1 vs 2

AB conditionally independent on C: $Pr(A \cap B|C) = Pr(A|C)\,Pr(B|C)$
$Pr(A|B\cap C) = Pr(A|C)$

Tree diagram: $Pr(A\cap B\cap C) = Pr(A) \times Pr(B|A) \times Pr(C|A\cap B)$

Total probability / Bayes Theorem: $Pr(A) = Pr(A|?)\,Pr(?)$ for all cases of ?

$$\boxed{Pr(B_j|A) = \frac{Pr(A|B_j)\,Pr(B_j)}{\sum Pr(A|B_i)\,Pr(B_i)}}$$

in denominator

→ $\frac{\text{branch}}{\text{all branches}}$

memorize: $\boxed{\frac{\text{"flip, self"}}{\text{"cases"}}}$

Bayesian statistics ($\overset{CW}{\theta_1\theta_2\theta_3}$)

$$\Rightarrow \frac{Pr(\theta_1\theta_2\theta_3|E)\,Pr(E)}{Pr(\theta_1\theta_2\theta_3|E) + Pr(\theta_1\theta_2\theta_3|E^c)}$$

$$\Rightarrow Pr(\theta_4|\theta_1\theta_2\theta_3) = Pr(\theta_4|\theta_1\theta_2\theta_3\cap E) + Pr(\theta_4|\theta_1\theta_2\theta_3\cap E^c)$$

$Pr(\theta_4|E)\,Pr(E|\theta_1\theta_2\theta_3) + Pr(\theta_4|E^c)\,Pr(E^c|\theta_1\theta_2\theta_3)$

Proofs in probability:

WLOG, $Pr(A) \geq Pr(B)$

$\max\{Pr(A), Pr(B)\} = Pr(A)$

$Pr(A\cup B) = Pr(A) + Pr(B) - \underbrace{Pr(A\cap B)}_{\geq 0}$

$\geq \max\{Pr(A), Pr(B)\}$

if all else fails, M.I.

main pt:
minusing subset, ≥ 0,
$A\cap B \leq A, B$
$A\cup B \geq A, B$,
cases: ≥ 1, <1

if $Pr(A) + Pr(B) \geq 1$,
$Pr(A\cup B) \leq \min\{1, Pr(A)+Pr(B)\}$
$= 1$

if $Pr(A) + Pr(B) < 1$
$Pr(A\cup B) = Pr(A) + Pr(B) - Pr(A\cap B)$
$\leq Pr(A) + Pr(B)$

Recurrence relation: 1, 2 then to infinity and beyond!

JL: Tips

$$p_{current} = p_{\text{last state ideal}} \times (\text{"Pr(travel to current ideal state")})$$

"ways to travel to ideal from last step"

$$+ \left(Pr(\text{not ideal state, last one}) \right) \times (\text{"Pr(travel to ideal here)"})$$

usually $(1-p_{k-1})$

often: $p_k = p_{k-1} \times ? + (1-p_{k-1}) \times ??$

$p_k = a + b\,p_{k-1}$

or: $x_n = A\alpha^n + B\beta^n$

$x_n = A\alpha^n + Bn\alpha^n$

$$\boxed{p_k = b^{k-1}\left(p_1 - \frac{a}{1-b}\right) + \frac{a}{1-b}}$$

enter inside program!

Re: SCNC1111 Difference eqt.

at infinity, $p = a + bp$, solve:

multiple roots: try each (e.g. 3.22)

"minus both sides take common factor if $p_{n-1} < ?$ $p_n < ?$"

$$p_n - \frac{1}{3} = \frac{3}{4}\left(p_{n-1} + \frac{1}{3}\right)\left(p_{n-1} - \frac{1}{3}\right)$$

if $p_{n-1} < \frac{1}{3}$, $p_n < \frac{1}{3}$. $p_2 = \frac{1}{4} < \frac{1}{3}$, $p_n < \frac{1}{3}$

$p = \lim_{n \to \infty} p_n = \frac{1}{3}$

or, ignore sequence. direct calculate $\to$ $p = Pr(\text{win condition}) + Pr(\text{lose condition}) \cdot Pr(\text{go back}) \cdot p$

(e.g. 3.21) (game)

T5: eg. 3.22 bacteria

$$p_n = \boxed{\text{die}} + \boxed{\text{self}} \times p_{n-1} + \boxed{\text{split}} \times p_{n-1}^2$$

T5: Gambling "beat the casino" $\to$ state is money, not turn.

e.g. 3.20 coin tossing

Re: Penny's game

$\to$ split cases of HT TH HH TT

then see next step (retossing; restart?)

Tournament: $\frac{n-1}{\binom{n}{2}} = \frac{2}{n}$