STAT2601 Ch. 4 Discrete Distributions (1)

$\underline{X}$: random variable e.g. no. of aces

$X \overset{a.s}{=} Y \Leftrightarrow Pr(X=Y)=1$

$X(w)=Y(w), \forall w \in \Omega$

$X \overset{d}{=} Y \Leftrightarrow F_X(t)=F_Y(t)\ \forall t$

coin $\times 5$, $X=\#H$, $Y=\#T$, $X \neq Y$

$X(\Omega) = \{0, 1, 2, 3\}$ ← support (domain)

$X(AAA) = 3$

$X(ANN) = X(NAN) = X(NNA) = 1$

$Pr(X=?) = Pr(\{?\}) = [0,1]$

"probability mass function"

$\underline{p_X}(?) = Pr(X=?)$, $\underline{? \in X(\Omega)}$ write the support! ⚠

↓

$p_X(x) = F_X(x) - F_X(x-1)$

properties: ① $p(x) \geq 0$ for all $x \in X(\Omega)$ ☆

② $\sum_{x \in X(\Omega)} p(x) = 1$ ☆

③ $Pr(X \in A) = \sum_{x \in A} p(x)$, $A \subset X(\Omega)$

graphically:

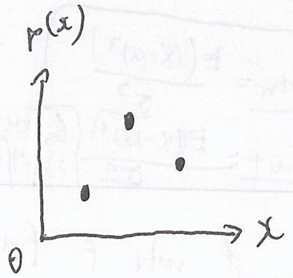


table:

u	1	2	3	4
$p(u)$	k	k	k^2	k^2

function $p_X(x) = \begin{cases} \frac{x}{6} & \text{for } x \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$

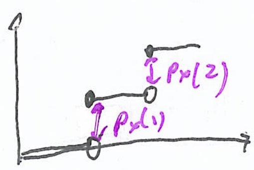
"cumulative distribution function" $\underline{F_X}(?) = Pr(X \leq ?) = \sum_{t \leq ?} p(t)$, $-\infty < ? < \infty$

property: ① non decreasing ③ right continuous

② $F(-\infty)=0$ ④ step size $= p_X(x)$

$F(\infty)=1$

graphically:



function: $F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{6} & \text{for } 1 \leq x < 2 \\ \frac{1}{2} & \text{for } x \geq 3 \end{cases}$

dot (=) pos!

Expected value: $E(X) = \sum_{x \in X(\Omega)} x\, p(x) = \mu$

change input, prob. unchange

$$E(g(X)) = \sum_{x \in X(\Omega)} g(x)\, p(x) \neq g(E(X)) !!$$

$E(X^2) = \sum x^2 p(x)$

Re: Ch.5 SA2 Q5

$$E(aX+c) = aE(X) + c$$

variance: $Var(X) = E((X-\mu)^2) = \sum (x-\mu)^2 p(x)$

expand $(X-\mu)^2$, $E(X)=\mu$

$$\boxed{= E(X^2) - E(X)^2} \geq 0$$

or $E(X^2) - \mu^2$

$$Var(aX+c) = a^2 Var(X)$$

"uniquely identify distribution"

moment generating function mgf $M_X(t)$

$E(X^r)$: r^{th} moment of X

$E((X-b)^r)$: r^{th} moment of X about b

r^{th} central moment when $b=\mu$

$= E(e^{tX}) = \sum e^{tx} p(x)$

$\boxed{M_X^{(r)}(0) = E(X^r)}$ $M_X(0)=1$ $(\sum p(x) = 1)$

Skew $= \frac{E((X-\mu)^3)}{\sigma^3}$

Kurt $= \frac{E((X-\mu)^4)}{\sigma^4}$

$G_X(e^t) = E(t^X)$ $R_X(t) = \ln M_X(t)$, $R_X^{(r)}(0) = E((X-\mu)^r)$, $R_X'(0) = E(X)$

Example: W: random variable of value of fair die. (14-15 S1 Assignment 2 Q1)

support: $\{1,2,3,4,5,6\}$

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

pmf: $p_W(x) = \begin{cases} \frac{1}{6} & \text{for } x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$

cmf: $F_W(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x}{6} & \text{if } x = 1,2,3,4,5,6 \\ 1 & \text{if } x > 6 \end{cases}$

variance $= E(W^2) - [E(W)]^2$

$\sum_{w=1}^{6} w^2 p(w) - \left(\frac{7}{2}\right)^2$

$= \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} - \left(\frac{7}{2}\right)^2$

expectation: $\sum_{w=1}^{6} w\, p(w) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$.

STAT2601 Ch. 4 Discrete Distributions (2) parameter, pmf, $E(X)$, $Var(X)$, mgf given!

$$\boxed{X \sim Ber(p)}$$
X: r.v ; $\sim$: follows ; Ber: distribution family ; p: parameter

Bernoulli Trial (binary): Same success probability p, independent (~~memoryless?~~)

Binomial - $B(n,p)$ "x success in n Bernoulli trials"

$X_i \sim Ber(p)$
$\underbrace{\Sigma X_i \sim B(n,p)}_{\text{fix } n}$


↑ then ↓

e.g. # of "6" in 9 die tosses. $X \sim B(9, \frac{1}{6})$

mode: $\lfloor (n+1)p \rfloor$ (floor function)

$$\frac{p(k+1)}{p(k)} = \frac{(n-k)p}{(k+1)(1-p)}$$

$Pr(X \le 2)$, $X \sim Bin$
$= Pr(X=0) + Pr(X=1) + Pr(X=2)$

e.g. 4.13 "linear transform (ECON MC)":
$Pr(X=15) = \binom{50}{15}(0.2)^{15}(0.8)^{35}$
$Y = 2X - 0.5(50-X) = 2.5X - 25$
$E(Y) = 2.5E(X) - 25$

☆ B vs NB

$Pr(B(n,p) \ge r) = Pr(NB(r,p) \le n)$
$Pr(B(n,p) < r) = Pr(NB(r,p) > n)$
"$<r$ in n $= >n$ to have r"

Geometric - $Geo(p)$ "x trial needed for 1^{st} success"

$$F(x) = Pr(X \le x) = \sum_{i=1}^{x} p(i) = \boxed{1-(1-p)^x = F(x)}$$

e.g. 4.16 pokemon collection: let X_i: # of trials after $(i-1)^{th}$ new card for i^{th} new
$X_i \sim Geo\left(\frac{N-(i-1)}{N}\right)$ $E(\Sigma X_i) = N\sum_{i=1}^{N}\frac{1}{i}$

memoryless: $Pr(X > a+b) = Pr(X > a)\,Pr(X > b)$
("hot hand") $Pr(X > b \mid X > a) = Pr(X > b-a)$ Re: conditional prob.

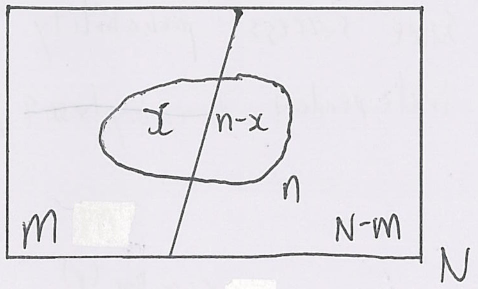
Negative Binomial - $NB(r,p)$ "x trial needed for exactly r success"

e.g. 4.18 F & P coin
H (F): m, $r = N-m$
T (P): n, $s = N-n$
F need r more head before P have s fail
i.e. rH in $< r+s$ trials
$$Pr(X < r+s) = \sum_{i=r}^{r+s-1} \binom{i-1}{r-1} p^r (1-p)^{i-r}$$

$X_i \sim Geo(p)$
$\Sigma X_i \sim NB(n,p)$
fix r

Hypergeometric - $Hyp(N, m, n)$

"have x of 'special' out of drawing n from pool of m 'special' out of m"



pmf: $\dfrac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$

m choose x special — all cases — remaining choose remain

don't throw, understand.

$Hyp(N, m, n) \stackrel{d}{=} Hyp(N, n, m)$

(SA4 Q3)

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e.g. 4.20 — catch and recapture (fish)

Catch m, tag m, recatch n, count $X = i$ "max. likelihood estimation"

$$\frac{P_i(N)}{P_i(N-1)} = \frac{(N-m)(N-n)}{N(N-m-n+i)} \geq 1 \text{ when } N \leq \frac{mn}{i}, \; N = \left\lfloor \frac{mn}{i} \right\rfloor$$

Poisson - $Po(\lambda)$

"have x in continuous interval given rate λ"

Aim: solve for λ (mean)

$$p(k+1) = \frac{\lambda}{k+1} p(k) \qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Binomial approx: $p(x) = \binom{n}{x} p^x (1-p)^{n-x} \hat{=} \dfrac{e^{-np}(np)^x}{x!}$ $\quad n \geq 100, \; np \leq 10$

Binomial: n discrete events, each with success chance p

Poisson: Infinite attempts, each with infinitesimal chance of success, "model events that happen large # times, but very rarely"

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e.g. 4.21 — customers come to store

$\lambda = 30/\text{hr.}$ $\quad N(\frac{1}{6})$

$Pr(\geq 6 \text{ in } 10 \text{ min})$ $\quad X \sim Po(30 \div 6)$

$$= 1 - \sum_{x=0}^{6} \frac{e^{-5} 5^x}{x!}$$

calculate new # based on rate and new interval!

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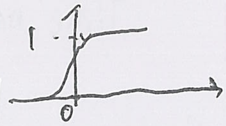
cdf $\; R_X(t) = \ln M_X(t), \; R_X^{(r)}(0) = E((X-\mu)^r), \; R_X'(0) = E(X)$, $R_X''(0) = \sigma^2$

pgf $\; G_X(z) = E(z^X) \quad G_X'(1) = E(X(1)^{X-1}) = E(X) \quad G_X''(1) = E(X(X-1))$

differentiate, put $z=0$ for probabilities!

mgf $\; M_X(t) = E(e^{tX}), \; M_X^{(r)}(t) = E(X^r)$



$F(x) = \int_{-\infty}^{x} f(t)\, dt$, $-\infty < x < \infty$. $\int_{-\infty}^{\infty} f(x)\, dx = F(\infty) = 1$

increasing, ~~continuous~~

$\boxed{f(x) = F'(x)}$ → probability density function . $f(x) \geq 0$ can ≥ 1

↓ prove pdf.

$Pr(a \leq X \leq b) = F(b) - F(a) = \boxed{\int_a^b f(x)\, dx}$

$Pr(X = a) = 0$

↓

piecewise integration!

$$F(x) = \begin{cases} 0 & \text{for } x<1 \\ \frac{x-1}{3} & \text{for } 1 \leq x < 2 \\ \frac{2x-3}{3} & \text{for } 2 \leq x < 3 \\ 1 & \text{for } x > 3 \end{cases}$$

diff. ← integral bound (2 pieces)

Expectation: Same rules as discrete, but use integration here. watch integral bounds/domain!

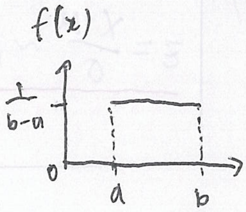
$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$, $\sigma = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = E(X^2) - \mu^2$

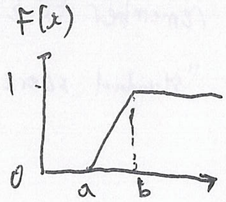
$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$ $R_X(t) = \ln M_X(t)$

Common continuous distributions: No special meaning, given by question (recognize pdf)

Uniform - $U(a,b)$

Constant pdf.





$$F(x) = \begin{cases} 0 & \text{for } x \leq a \\ \frac{x-a}{b-a} & \text{for } a < x < b \\ 1 & \text{for } x \geq b \end{cases}$$

Standard uniform: $a=0, b=1$ $X \sim U(0,1)$

$Y = cX + d$ $Y \sim U(d, c+d)$ $c>0$

Re: linear transform $Y \sim U(c+d, d)$ $c<0$

Exponential - $Exp(\lambda)$

" amount of time needed between (or first) success in rate λ " time unit!

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$Pr(a \leq X \leq b) = e^{-\lambda a} - e^{-\lambda b}$

$Pr(X > a) = 1 - F(a)$ (Re: Poisson Process)

$Pr(X > a+b) = Pr(a)\ Pr(b)$

$Pr(X > a+b \mid X > a) = Pr(X > b)$ } memoryless

e.g. 5.5 machine failure	X be failure time; $X \sim Exp(2)$ $\lambda = 2$ failures/month $E(X) = \frac{1}{\lambda}$ ↓ avg fail time $Pr(X>4) = 1 - F(4) = 1 - (1 - e^{-2(4)}) = e^{-8}$

Gamma - $\Gamma(\alpha, \lambda)$

"Time needed to achieve α arrivals"

$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\Gamma(n) = (n-1)!$

Gamma $\Leftrightarrow$ Poisson: Calculating probabilities

$$F(t) = Pr(T_n \le t) = 1 - Pr(T_n > t)$$
$$= 1 - Pr(N(t) < n)$$
$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$\{T_n > t\} = \{N_t < n\}$

need $> t$ time for n ; $< n$ in t time

$Y = bX$, $Y \sim \Gamma(\alpha, \frac{\lambda}{b})$

χ^2: $\alpha = \frac{r}{2}$, $\lambda = \frac{1}{2}$

$X \sim \chi^2(r)$ deg of freedom r

e.g. 5.7/5.8 phone calls

rate: 20 call/hr. $T_8 \sim \Gamma(8, 20)$ in hours. Poisson sum

$E(T_8) = \frac{8}{20} = 2/5$.

$Pr(T_8 > 0.8) = 1 - Pr(T_8 \le 0.8)$

new λ: $16 = 20 * 0.8$

$= 1 - \sum_{k=0}^{7} \frac{16^k e^{-16}}{k!}$ only 7 in T_8

$Y = 40T_8 \sim \Gamma(8, 1/2) \equiv \chi^2(16)$ ☆ χ^2 (appendix)

$Pr(T_8 > 0.8) = 1 - Pr(T_8 \le 0.8)$
$= 1 - Pr(Y \le 32) = 1 - 0.99 = 0.01$

Normal - $N(\mu, \sigma^2)$

μ: location parameter σ^2: scale parameter

Standard normal: $N(0,1)$

$\Phi(x) = \int_{-\infty}^{x} \phi(x)dx$

$\Phi(x) + \Phi(-x) = 1$

$\phi(x) = \phi(-x)$

Z transformation

probability from table!

remember to & symmetry!

"standard score"

e.g. 5.9 $Pr(25 \le X \le 40)$

$X \sim N(30.5, 4.5^2)$

$= Pr\left(\frac{25-\mu}{\sigma} \le Z \le \frac{40-\mu}{\sigma}\right)$

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$= Pr(-1.22 \le Z \le 2.11)$
$= 0.3888 + 0.4826$
$= 0.8714$

if $X \sim N(0,1)$, $X^2 \sim \chi^2(1)$

Normal approximation to Binomial $\rightarrow Y \sim (np, np(1-p))$

$np \ge 5$, $n(1-p) \ge 5$

$Pr(X = a) \approx Pr(a - 0.5 < Y < a + 0.5)$

e.g. 5.15 (4.22) $X \sim B(8000, \frac{1}{1000})$

$\mu = 8$, $\sigma^2 = 8000 \times \frac{1}{1000} \times \frac{999}{1000} = 7.992$

$Y \sim N(8, 7.992)$

$Pr(X < 8) = Pr(X \le 7) \approx Pr(Y \le 7.5)$

$= Pr\left(Z \le \frac{7.5-8}{\sqrt{7.992}}\right) \approx 0.4298$

Beta - $Beta(\alpha, \beta)$

$\alpha = \beta = 1$: $U(0,1)$

Beta integral! $B = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

inverted! $= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Gamma integral! $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx$

inverted! $\rightarrow = \frac{\Gamma(\alpha)}{\lambda^\alpha}$

Re: A4 Q3b $E(x^4)$

STAT2601 Ch.6 Other properties of distributions

$$F(t) = 1 - e^{-\int_0^t \lambda(s)ds}$$

$$f(x) = -S'(x)$$

Survival function

Hazard rate function

$$S(t) = Pr(X > t) = \int_t^\infty f(x)\,dx = 1 - F(t)$$

$$\lambda(t) = \frac{f(t)}{S(t)}$$ → - conditional probability intensity that a t-unit old item die instantly at time t

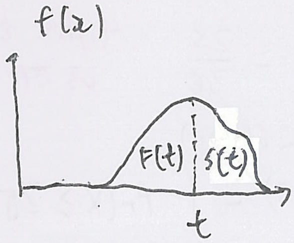
$\propto$ memory-less

$$= \frac{F'(t)}{1-F(t)}$$

$$= -\frac{d}{dt}\ln S(t)$$ - uniquely determine distribution

$$S(t) = e^{-\int_0^t \lambda(s)\,ds}$$

$$f(x) = \lambda(t)\, e^{-\int_0^t \lambda(s)\,ds}$$



Indicator function

$$\mathbb{1}_{\{x \in (0,1)\}} = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$ → simply function compact way

Linearity of Expectation:

$$E(X_1 + X_2 + \cdots X_n) = E(X_1) + E(X_2) + \cdots E(X_n)$$

even if $X_1, X_2 \ldots X_n$ dependent

$$E(X_i) = p(x \in A)$$

↓ calculate X_i individually!

e.g. 6-4 lift stop, m ppl, n floor.

let X_i be indicator that elevator stop at floor i

$$Pr(X_i = 1) = 1 - Pr(\text{nobody get off at } i) = 1 - \left(\frac{n-1}{n}\right)^m$$

$$E(X) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$= n\left(1 - \left(\frac{n-1}{n}\right)^m\right)$$

e.g. 6-3: X_i = Book start at i

e.g. 6-5: new run start at i (coin) ($i > 1$)

$$E(X) = \sum_{x=0}^\infty x\,p(x) = \sum_{x=0}^\infty [1 - F(x)]$$

$x \geq 0$

$$= \int_0^\infty x f(x)\,dx = \int_0^\infty 1 - F(x)\,dx$$

Proof: rotate summation / integral order.

$$\int_0^\infty x f(x)\,dx = \int_0^\infty \int_0^x f(x)\,dy\,dx$$

$$= \int_0^\infty \int_y^\infty f(x)\,dx\,dy$$

$$= \int_0^\infty Pr(X > y)\,dy$$

$$= \int_0^\infty Pr(X > x)\,dx$$

Res: $g(x) = \frac{S(x)}{\mu}$

reverse integral order.

e.g. (slides) throwing 4 dice

sum = $E(X_1 + X_2 + X_3 + X_4) = 4 \times 3.5$

min: let $M = \min(X_1, X_2, X_3, X_4)$

for $j = 1 \ldots 6$ $Pr(M \geq j)$

$$= Pr(X_1 \geq j)\,Pr(X_2 \geq j) \cdots Pr(X_4 \geq j)$$

$$= \left(\frac{6-(j-1)}{6}\right)^4. \quad E(M) = \sum_{j=1}^{6} Pr(M \geq j)$$

sum (largest 3) = E(sum all − min)

Inequalities

① Markov's : $\boxed{Pr(X \geq c) \leq \frac{E(X)}{c}}$. $x \geq 0$, $c > 0$

② Chebyshev's : $\boxed{Pr(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}}$ → $Pr(|X-\mu| \leq k\sigma) \geq 1-\frac{1}{k^2}$

"deviation greater than"

e.g. (slides/6-6): $\mu = 50$, $\sigma = 5$, X is discrete

$Pr(X > 75) = Pr(X \geq 76) \leq \frac{E(X)}{76} = \frac{25}{38}$ tight bound vs $\leq Pr(X \geq 75)$

$Pr(40 < X < 60) = Pr(40-50 < X-50 < 60-50)$
$= Pr(-10 < X-\mu < 10) = Pr(|X-\mu| < 2\sigma)$
$= 1 - Pr(|X-\mu| \geq 2\sigma)$
$\geq 1 - \frac{1}{2^2}$

lower/upper bound ∝ "tight"

$Pr(40 < X < 70) = Pr(40-50 < X-50 < 70-50)$
$= Pr(-10 < X-\mu < 20)$ shrink range
$\geq Pr(-10 < X-\mu < 10)$

≥ if followed by ≥, vice versa

Distributions of a mixed type

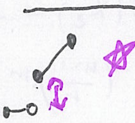
∝ truncated / censored

e.g. 6-7 bulb testing

$X = \begin{cases} T & \text{if } T < 2 \\ 2 & \text{if } T \geq 2 \end{cases}$ $T \sim Exp(1)$ ☆☆☆ split jump point!

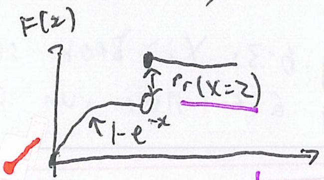
$Pr(X=2) = Pr(T \geq 2) = 1-(1-e^{-2}) = e^{-2}$

$Pr(X \leq x) = Pr(T \leq x) = 1-e^{-x}$, $0 < x < 2$



$1(\frac{1}{4}) + \int_1^2 x(0)\,dx$
$+ 2(\frac{1}{6}) + \int_2^3 x(\frac{1}{2})\,dx$

$F(x) = \begin{cases} 0 & x \leq 0 \\ 1-e^{-x} & 0 < x < 2 \\ 1 & x \geq 2 \end{cases}$

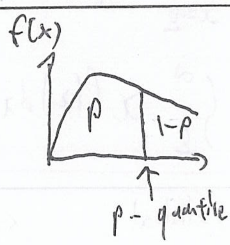


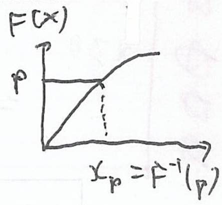
split

$E(X) = \int_0^2 xe^{-x}\,dx + 2e^{-2} = 1-e^{-2}$ ← piecewise integration (or $1-F(x)$)

Quantiles

$x_p : F^{-1}(p)$: p-quantile / 100p percentile





Discrete case: smallest x s.t. $F(x) \geq p$

lower bound

STAT2601 ch.7 Multivariate Distributions + Transformation (6, 7, 9)

Discrete

joint pmf

$$p(x,y) = Pr(X=x, Y=y), \quad x \in X(\Omega), \quad y \in Y(\Omega)$$

$\downarrow \uparrow x$

marginal pmf

$$\boxed{p_X(x) = Pr(X=x) = \sum_{y \in Y(\Omega)} p(x,y)}$$ ↳ tabular form

$$\boxed{p_Y(y) = Pr(Y=y) = \sum_{x \in X(\Omega)} p(x,y)}$$ → sum col / row

ignore other term

Continuous

$$Pr((X,Y) \in C) = \iint_{(x,y) \in C} f(x,y)\,dx\,dy$$ → continuous

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$Pr(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

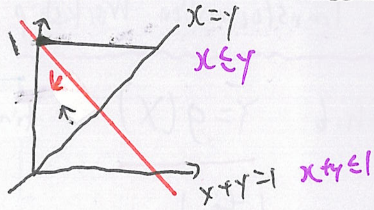
marginal pdf →

$$f_X(x) = \int_{y\text{-lo}}^{y\text{-hi}} f(x,y)\,dy \qquad f_Y(y) = \int_{x\text{-lo}}^{x\text{-hi}} f(x,y)\,dx$$

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x} F_{XY}(x,y)$$

e.g. CW Q1.

$$f(x,y) = \begin{cases} 60x(1-y)^2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



☆ linear programming!

find <> side

$$f_X(x) = \int_x^1 60(1-y)^2\,dy$$ ← // y axis

$$= 60x\left[\frac{(1-y)^3}{3}\right]_1^x = 20x(1-x)^3 \cdot 1_{(0 \le x \le 1)}$$ $X \sim Beta(2,4)$

$$f_Y(y) = \int_0^y 60(1-y)^2\,dx$$ $Y \sim Beta(3,3)$

$$= 60(1-y)^2\left[\frac{x^2}{2}\right]_2^y = 30y^2(1-y)^2 \cdot 1(0 \le y \le 1)$$

$$Pr(X+Y<1) = \int_0^{1/2}\int_x^{1-x} f(x,y)\,dy\,dx$$

Y> ↑

X> →

Independence

$p(x,y) = p_X(x)\,p_Y(y)$ $\quad f(x,y)$ factorized to $g(x)h(y)$, rectangle region ☆

prove dependent: $Pr(V \in A \text{ and } V \in B) \neq Pr(V \in A)\,Pr(V \in B)$

Expectation

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\,p(x,y)$$

no need independent ↕

$$= \boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy}$$

$E(XY) = \iint xy f(x,y)$

$E(X+Y) = E(X) + E(Y)$ $\quad E(g_1(X)\,g_2(Y)) = E(g_1(X))\,E(g_2(Y))$

independent ↕

$Var(X+Y) = Var(X-Y)$

$Var(X) + Var(Y)$

$E(g(X_1))$ found via marginal pmf/pdf of X

$$M_{aX+bY}(t) = M_X(at)\,M_Y(bt)$$

distribution summation → multiply mgf

Re: transformation (2 of 1)

Covariance

variability + correlation: $\sigma_{XY} = Cov(X, Y)$

calculation: brute force (usually many 0) (or independent)

$$= E[(X-\mu_X)(Y-\mu_Y)]$$

$$= E(XY) - \mu_X\mu_Y$$

properties:

① $\sigma_{XY} > 0$: $X\uparrow Y\uparrow$ $\quad \sigma_{XY} < 0$: $X\uparrow Y\downarrow$

② $X' = aX+b \quad Y' = cY+d$

$Cov(X', Y') = ac\, Cov(X, Y)$

③ $Cov(X, Y) = Cov(Y, X)$, $Cov(X, c) = 0$

$Cov(X, X) = Var(X)$

④ $Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j X_j\right)$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j\, Cov(X_i, Y_j)$

⑤ $Var(X+Y) =$

$Var(X) + Var(Y) \pm 2Cov(X,Y)$

Correlation

$\rho = \rho_{XY} = Corr(X, Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$

Independent: $Var(X+Y) = Var(X-Y)$

$= Var(X) + Var(Y)$

$X \perp\!\!\!\perp Y \Rightarrow E(X,Y) \overset{\not\Leftarrow}{=} E(X)E(Y)$

$\Longleftrightarrow Cov(X,Y) = 0$

$\Longleftrightarrow X\ Y$ uncorrelated

JL Transformation Workshop (Ch 6, 7, 9): cdf? mgf? pdf/pmf? f/g? expectation? Pr?

Ch. 6: $Y = g(X)$

1 to 1

☆ range of $g(X) \to Y$ support

$Y = X^2$. $X \sim U(-5, 3)$

$\int_{-\sqrt{z}}^{\sqrt{z}} f_X(x)\,dx \cdot \quad 0 \le z \le 9$

$\int_{-\sqrt{z}}^{3} f_X(x)\,dx \quad 9 \le z \le 25$

$\propto$ what / how many $X \to Y$?

$\to$ draw graph $y = g(x)$

from certain y, get what x

$F_Y(y) = Pr(Y \le y)$

$= Pr(g(X) \le y)$ sub

$= Pr(X \le g^{-1}(X))$ make LHS X

$= F_X(g^{-1}(y))$ put F_X

$f_Y(y) = F'_X(g^{-1}(y))$

$\Sigma p_X(x) \cdot \frac{d}{dy} y$ $\quad = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$

$E(Y) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \longrightarrow$ special integral (Beta / Gamma / mgf)

$= \int_{-\infty}^{\infty} x f_Y(x)\,dx$

$Y = X^2$ $\quad F_Y(y) = Pr(Y \le y)$

$= Pr(X^2 \le y)$

$Y = F(X)$, $Y \sim U(0,1)$

$= Pr(-\sqrt{y} \le X \le \sqrt{y})$

$= 2\Phi(\sqrt{y}) - 1$

$f_X(y) = F'_Y(y)$

$= 2\phi(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}$

Discrete: $P_Y(y) = Pr(Y = y)$

sub $= Pr(g(X) = y)$

make LHS X $= Pr(X = g^{-1}(y))$

put P_X $= P_X(g^{-1}(y))$

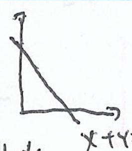
Ch. 7/9: $Z = g(X, Y)$

2 to 1

$E(g(X,Y)) = \iint_{\Omega} g(x,y) f(x,y)\,dx\,dy$

$M_{aX+bY}(t) = M_X(at)\, M_Y(bt)$

$F_Z(u) = Pr(Z \le u)$



sub $= Pr(X + Y \le u)$

MATH 2014 $= \int_0^u \int_0^{u-x} f_{XY}(x,y)\,dy\,dx$

$X \perp\!\!\!\perp Y$ $= \int_0^u f_X(x) \int_0^{u-x} f_Y(y)\,dy\,dx$

$\vdots \quad f_Z(u) = F'_Z(u)$

$M_{X+Y}(t) = M_X(t)\, M_Y(t) \quad X \perp\!\!\!\perp Y$

Discrete: $P_Z(t) = Pr(Z = t)$

$= Pr(X + Y = t)$

dummy variable $= \sum_{r=0}^{t} Pr(X = r, Y = t-r)$

$X \perp\!\!\!\perp Y$ $= \sum_{r=0}^{t} Pr(X = r)\, Pr(Y = t-r)$

$\vdots$