STAT2602 Ch.1-2 Fundamentals

Discrete/continuous, independent: $F(x,y) = F_X(x) F_Y(y)$ for all $x, y \in S$ (prop. 1-1)

pdf transformation: $Y = aX + b$, $f_Y(x) = \frac{1}{a} f_X\left(\frac{x-b}{a}\right)$

Empirical distribution: Collect random sample X_i, $F_n(x) \xrightarrow{d} F(x)$ as $n \to \infty$

$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I(X_k \le x)$ # sample $\le x$.

↳ approx $Pr(X \le x) = F(x)$

$f(x) \approx f_n(x) = \frac{1}{n} \sum_{k=1}^{n} I(X_k = x)$ discrete only

e.g. collect data → Relative freq. histogram (class intervals)

↓

approx / obtain $f(x)$, see which interval x lie in.

depend on class interval

$\int_{c_0}^{c_n} h(x)dx = 1$

Expectation $E(X)$:

- $E(u(X)) = \sum_{x \in S} u(x) f(x) < \infty$ (may not exist!)
- uncorrelated: $Cov(X, Y) = E(XY) - E(X)E(Y) = 0$
- $E(ag(X) + bh(X)) = aE(g(X)) + bE(h(X))$
- $E(aX + b) = aE(X) + b$
- g convex, then $g(E(X)) \le E(g(X))$
- $E(X) = \int_0^\infty (1 - F(x))\, dx = \int_0^\infty P(X > x)\, dx$
- $Var(X+Y) = Var(X) + Var(Y) + 2\, Cov(X, Y)$
- $Cov(aX + b, cY + d) = ac\, Cov(X, Y)$
- independent $\Rightarrow$ $E(h(X)\, g(Y)) = E(h(X))\, E(g(Y))$
- if $E(X^2)E(Y^2) < \infty$, $E(XY) \le \sqrt{E(X^2)E(Y^2)}$

$\rho_{XY} = Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}}$

mgf: $E(e^{tX}) = M_X(t) = \sum_{r=0}^{\infty} \mu_r \left(\frac{t^r}{r!}\right)$, $\mu_r = M_X^{(r)}(0)$ for $r = 1, 2, \ldots$

$\int e^{tx} f(x)\,dx$

$\boxed{M_{aX+b}(t) = e^{bt} M_X(at)}$ $\boxed{M_{aX+bY+c}(t) = e^{ct} M_X(at) M_Y(bt)}$

$M_X(t) = M_X(0) + M_X^{(1)}(0)\frac{t}{1!} + M_X^{(2)}(0)\left(\frac{t^2}{2!}\right) + \ldots$

If X_i are independent, $M_{X_i}(t)$ exists, let $Y = X_1 + X_2 + \cdots + X_n$,

$\boxed{M_Y(t) = \prod_{i=1}^{n} M_{X_i} t}$

Re: Distribution of $\bar{X}$

$\boxed{Y = \sum_{i=1}^{n} X_i^2 \sim \chi^2_n \quad X \sim N(0,1)}$

CLT $\boxed{\bar{X} \sim N(\mu, \frac{\sigma^2}{n})}$

Convergence:

in prob.: $\Pr(|Z_n - Z| > \varepsilon) \to 0$ as $n \to \infty$

statistics (functions of random sample): - also random variable itself!

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ $\boxed{\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{(n-1)}}$

$\boxed{X \sim N(\mu, \sigma^2) \quad \sum \frac{(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{(n-1)} \quad \sum \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2_{(n)}}$

e.g. 2.10 $E(X^2) = E(Z_i^2) = 1$

$Var(X^2) = var(Z_i^2) = 2$

z transform: $Z = \frac{X - \mu}{\sigma}$

Resampling (bootstrap): From empirical distribution $F_n(x)$, draw samples again → estimate $f(x)$.

Estimating $E(u(X)) = \int u(x)\,dF(x) \approx \int u(x)\,dF_n(x) = E[u(X^*)]$, $X^* \sim F_n(\cdot)$

1. Generate X_i^* from $F_n(\cdot)$
2. Estimate $E(u(X^*))$ from sample mean of $\{u(X_i^*)\}$

Weak law of large numbers: $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} E(X)$ as $n \to \infty$

If $X_n \xrightarrow{p} \mu$, $Y_n \xrightarrow{p} v$, then $X_n + Y_n \xrightarrow{p} \mu + v$

$X_n \cdot Y_n \to_p \mu v$

$X_n / Y_n \to X_n / Y_n$

Convergence in distribution: $G_n(x) \xrightarrow{p} G(x)$.

G_n: cdf of Z_n, G: cdf of Z

$\boxed{g(X_n) \xrightarrow{p} g(\mu) \text{ for continuous } g.}$

$\boxed{E(S^2) = \frac{n-1}{n}\sigma^2 \quad Var(S^2) = \frac{2(\sigma^2)^2(n-1)}{n^2}}$

$\boxed{\text{CLT}: \frac{\bar{X} - E(\bar{X})}{\sqrt{Var(\bar{X})}} = \frac{(\bar{X} - \mu)}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)}$

$E((2+X)^2) = 4 + 4E(X) + E(X^2)$

$Var(X) = E(X^2) - E(X)^2$

$E(X)^2 = Var(X) + E(X)^2$

STAT2602 Ch3 Point Estimation

Aim: Find unknown parameter θ for a distribution

Maximum Likelihood Estimator: Find likelihood function, then maximum point!
(MLE) $\hat{\theta}$

1. Likehood function $L(\theta|x_1, x_2 \dots x_n) = f(x_1|\theta) \dots f(x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$
2. log-likelihood: $\ell(\theta) = \sum_{i=1}^{n} \ln f(x_i|\theta)$
3. differentiate: $\dfrac{\partial \ell(\theta)}{\partial p_i}$, p_i: parameter.
4. Find maximum $(\text{set} = 0)$ $\left(\frac{\partial^2 \ell}{\partial \theta^2} < 0\right)$ (two parameters: solve!)

$F_n(x) = F(x)^n$
$f_n(x) = n F(x)^{n-1} f(x)$

Method of Moments Estimator: Solve parameters as a function of moments
(MME) $\tilde{\theta}$

e.g. $E(X) = \mu_1 = \mu$, $\mu_2 = E(X^2) = Var(X) + E(X)^2 = \sigma^2 + \mu^2$
$\sigma^2 = \mu_2 - \mu_1^2$

not unique

construct $\theta_1 = h(m_1, m_2 \dots m_n)$

Comparing Estimators:

① $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$ unbiased

biased: $E(\hat{\theta}) \neq \theta$

$\lim_{n\to\infty} Bias(\hat{\theta}) = \lim_{n\to\infty} [E(\hat{\theta}) - \theta] = 0$ asymptotically unbiased

Show (asymptotically) unbiased: calc $E(\theta)$ (can have n)

$\int_0^\infty 1 - F(x)\, dx$

② Efficiency: $Eff(\hat{\theta}, \tilde{\theta}) = \dfrac{Var(\tilde{\theta})}{Var(\hat{\theta})}$

$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias(\hat{\theta})^2$ Var ↓, efficiency ↑

③ Consistent $\hat{\theta} \to_p \theta$

Aim: Show $Var(\hat{\theta}) = 0$ as $n \to \infty$

Re: Convergence $(\to_p)$ (WLLN)

expand $\bar{X}^2$ into $\frac{1}{n}\sum_{i=1}^{n} X_i^2$,
then $\to_p E(X^2)$
$g(T(X)) \to_p \theta$

$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu$

$E(S^2) = \frac{1}{n}\sum_{i=1}^{n} E[(X_i - \bar{X})^2]$

$Var(X_1 - \bar{X})$
$= Var\left(X_1 - \frac{X_i}{n}\right)$
$= Var\left(\frac{(n-1)X_1}{n} - \sum_{i=2}^{n} \frac{X_i}{n}\right)$
$= \frac{n-1}{n} \sigma^2$.

$E(X^2) = Var(X) + E(X)^2$
$Var(X) = E(X^2) - E(X)^2$

Re: Transformation

$P(g(X) \le x) = P(X \le g^{-1}(x))$

$E(X^2) = \int_0^\infty x^2\, dF(x)$

$Var(X) = E(X^2) - E(X)^2$

$\boxed{Var(X+Y) = Var(X) + Var(Y)}$

$Var(S^2) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$

$(\sigma^2)^2 \; Var\left(\frac{1}{\sigma^2} \cdot S^2\right)$

$(\sigma^2)^2 \; \frac{1}{n} \sum_{i=1}^{n} \frac{(X_i - \bar{X})}{\sigma^2}$

$= \frac{(\sigma^2)^2 \cdot 2(n-1)}{n}$

$\dfrac{\hat{\theta} - \theta}{\sqrt{1/I_n(\theta)}} \to_d N(0,1)$

$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias(\hat{\theta})^2$

T: Sufficient Statistic:

From the data sample, what information you need to get θ?

$r(T)$, T are sufficient.

exponential family: $f(x;\theta) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{s} p_i(\theta)\, t_i(x)\right)$ (take out / group exp)

complete sufficient $\Rightarrow$ $T(X) = \left(\sum_{j=1}^{n} t_1(X_j), \sum_{j=1}^{n} t_2(X_j), \dots\right)$

e.g. $f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2+\mu^2-2\mu x}{2\sigma^2}\right)$

$= \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)\right] \exp\left(\frac{n\mu}{\sigma^2}\frac{x}{n} - \frac{n}{2\sigma^2}\frac{x^2}{n}\right)$

factorization:

$L(\theta) = g(T;\theta)\, h(x)$ ← samples

uniform: $f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{\beta-\alpha} I(\alpha \le x_i \le \beta)$ → must have indicator for uniform dis.

$= \left(\frac{1}{\beta-\alpha}\right)^n I(\alpha \le x_i \le \beta)$

$= \left(\frac{1}{\beta-\alpha}\right)^n I(\alpha \le \min X_i)\, I(\beta \ge \max X_i)$

UMVUE: Best unbiased estimator

unique

↳ $E(\hat{\theta}) = \theta$ (unbiased)

complete statistic: $E(z(T)) = 0$ for all θ $\Rightarrow$ $P(z(T)=0)$ for all θ.

$E(\phi(T)) = \theta$ (by chance?)

↳ ratio → $\phi^{-1}(\theta)$

CRLB method: 1) Find l.b. of $Var(\hat{\theta})$ 2) Find $\hat{\theta}$ that satisfy l.b.

$$I_n(\theta) = n I(\theta) = n \cdot E\left[\left(\frac{\partial \log f(X;\theta)}{\partial\theta}\right)^2\right]$$

$$= n\left(-E\left(\frac{\partial^2 \log f(X;\theta)}{\partial\theta^2}\right)\right)$$ ← 1 observation

i.e. $E(X^2) = Var(X) + E(X)^2$

$$Var(\hat{\theta}) \ge \frac{1}{I_n(\theta)} = \frac{1}{n \cdot I(\theta)} \rightarrow \text{CRLB}$$

Prove UMVUE: show $Var(\hat{\theta}) = \frac{1}{nI(\theta)}$

$$\frac{\hat{\theta}-\theta}{\sqrt{1/I_n(\theta)}} \to_d N(0,1)$$

or

STAT2602 Chapter 5 Hypothesis Testing

Hypothesis: Simple: completely specify , $p = x$
Composite: does not completely specify! $p \neq x$, $p > x$

$\pi(\theta_0) \leftarrow$ "size"

"kill a good guy" Null Hypothesis: H_0 : Reject wrongly $\rightarrow$ Type I error, α

"let go a bad man" Alt. Hypothesis: H_1 : Reject wrongly $\rightarrow$ Type II error. β

$\boxed{\text{Power function } \pi(\theta):}$ prob. reject H_0 when true value is θ.

$\alpha(\theta)$: $\Pr(X \in RR \mid \theta_0)$

$$\pi(\theta) = \begin{cases} \alpha(\theta) & \text{for } \theta \in \Omega_0 \\ 1-\beta(\theta) & \text{for } \theta \in \Omega_1 \end{cases}$$

$\boxed{\text{power} = \Pr(\text{reject } H_1 \mid H_0 \text{ true})}$

☆ Rejection region / critical region: set of values s.t. H_0 rejected re: test statistic $W(X)$

$\downarrow$

$\boxed{W(X) \geq K,\ |W(X)| \geq K,\ W(X) \leq K,\ \cup W(X) \geq K_2,}$ K, K_1, K_2 depend on α

$\boxed{\text{p-value}}$: H_0 rejected at $\alpha \iff$ p-value $\leq \alpha$

RR: case 1: $\{W(X) \leq K\}$, $p = \max_{\theta \in \Omega_0} P_\theta(W(X) \leq W(x))$

case 2: $\{W(X) \geq K\}$, $p = \max_{\theta \in \Omega_0} P_\theta(W(X) \geq W(x)\}$

case 3: $\{|W(X)| \geq K\}$, $p = \max_{\theta \in \Omega_0} P_\theta(|W(X)| \geq |W(x)|\}$

representation observed

Aim: From $W(X)$, build distribution (eg. Z, χ^2) then lookup table

$\boxed{\text{Goodness of fit tests}}$ H_0: No preference H_1: Preference

$$RR: \left\{-2\ln\Lambda = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \boxed{\sum_{i=1}^{m} \frac{O_i^2}{E_i} - n \geq \chi^2_{\alpha, m-1}}\right\} \quad E_i = \frac{n}{m}$$

Re: SCNC1111

$\boxed{\text{Pearson chi-sq. test of independence}}$ H_0: X, Y independent H_1: Not independent

$$RR: \left\{ -2\ln\Lambda = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{O_{ij}^2}{E_{ij}} - n = \boxed{n\left(\sum_{i=1}^{r}\sum_{j=1}^{c} \frac{O_{ij}^2}{n_i n_j} - 1\right) \geq \chi^2_{\alpha, (r-1)(c-1)}} \right.$$

Most powerful tests: Simple $H_0: \theta = \theta_0$ vs Simple $H_1: \theta = \theta_1$, power at H_1 max

$\boxed{\text{Likelihood ratio test}}$ $\boxed{\dfrac{L(\theta_0)}{L(\theta_1)} \le k}$ → find form of RR

uniformly most powerful: composite H_1, $\pi(\theta) \ge \tilde{\pi}(\theta)$ $\forall \theta \in \Omega_1$

↳ Check uniform: let $\lambda_1 \in \Theta_1$, does k depend on λ_1?

$\boxed{\text{Generalized likelihood ratio test:}}$ $\Lambda = \boxed{\dfrac{L(\Omega_0)}{L(\Omega)} \le k}$, $\Omega_1 = \Omega \setminus \Omega_0$, $0 \le k \le 1$

$\lambda > \lambda_0 \to \lambda = \lambda_1$, $\lambda_1 > \lambda_0$

$L(\Omega_0)$ = max for Ω_0

$L(\Omega)$ = max for Ω, MLE.

Q-Type: Find RR:

① Construct $L(\theta)$

② From inequality, find form of RR

③ From sufficient statistic of $L(\theta)$, find distribution, set α.

①.1 Find max val of $L(\Omega_0)$, $L(\Omega)$ may have to set piecewise

$\boxed{\text{Think: what increase/decrease} \to \text{lower than } k\text{?}}$

→ $\bar{X}$: set $M_{\bar{X}}(t) = \prod_{i=1}^{n} M_X\left(\frac{1}{n}t\right)$

S^2: $X \sim N(\mu, \sigma^2)$, $\frac{nS^2}{\sigma^2} \sim \chi^2_{(n-1)}$

↳ $E(S^2) = \frac{n-1}{n}\sigma^2$

$Var(S^2) = 2\sigma^4 \frac{(n-1)}{n^2}$

Q-Type: Find p-value

e.g. $\max P(W(X) \le w(x))$

$= \max_{\mu \ge \mu_0} P_\mu\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \le w(x)\right)$

$= \max_{\mu \ge \mu_0} P_\mu\left(Z \le w(x) + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$

$= P(Z \le w(x))$

$\frac{\bar{X}-\mu}{S/\sqrt{n-1}} \sim t_{n-1}$ $\quad \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$g(w(x)) \sim F(D)$ ← cdf

$U \sim \chi^2_{r_1}$, $V \sim \chi^2_{r_2}$, $\frac{U/r_1}{V/r_2} \sim F_{r_1, r_2}$

$\alpha = Pr\left(g^{-1}(D_{1-\alpha/2}) \le \theta \le g^{-1}(D_{\alpha/2})\right)$

Ch 4: Interval Estimation - $\alpha \downarrow$, width $\uparrow$

e.g. T7Q4: $f_Y(y) = \frac{1}{\theta} e^{-y/\theta}$ $\quad F_Y(y) = 1 - e^{-\frac{y}{\theta}}$

$\alpha = P(a < Y < b)$,

$F_Y(a) = \alpha/2$,

$F_Y(b) = 1 - \alpha/2$,

$a\,b$ in terms of θ

→ rearrange θ. don't sub f_Y!

$Z \sim N(0,1)$

$V \sim \chi^2_k$

$T = \frac{Z}{\sqrt{V/k}} \sim t_k$