

|   |   |   |
|---|---|---|
| 1 | Calculus                                    | 2 |
| 2 | Linear Algebra                              | 4 |
| 3 | Optimization                                | 4 |
| 4 | Discrete Math and Probability Distributions | 4 |
| 5 | Statistical Estimation                      | 4 |
| 6 | Scientific Computing and Numerical Methods  | 4 |

# Calculus

## 1.1 Partial Differentiation (MATH2014)

### Prove continuity via polar coordinates

Q-type: Let  $f(x, y) = \begin{cases} ? & \text{if } (x, y) \neq (0, 0) \\ \text{const.} & \text{if } (x, y) = (0, 0) \end{cases}$ . Is  $f$  continuous?

To prove a function is **continuous**, a strategy is:

1. Change to polar:  $\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases}$
2. Prove  $\sin\theta$  and  $\cos\theta$  is bounded, e.g. via sandwich thm.
  - $\lim_{r \rightarrow 0} |f(\theta)| \leq |c|$

To prove a function is **discontinuous**, some strategies are:

- (A) Change to polar, set  $\lim_{r \rightarrow 0}$ , then choose two  $\theta$  where value of  $f(x, y)$  is different under limit.
- (B) Fix  $x = 0$  or  $y = 0$ , then set  $\lim_{(x, y) \rightarrow (0, 0)}$  "along the half line of  $y = 0, x > 0$  (or some other half-axis)", and compare this with true value of  $f(0, 0)$  without limit.

### Partial derivatives

To partial d. on  $x$ , perform d. as normal on  $x$  while treating others ( $y, z$ ) as constants.

Notation:  $f_x = \frac{\partial f}{\partial x}$

Mixed partial derivatives:  $f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$  (**order of fraction is inverted!**)

If  $f_{xy}$  is continuous in a neighbourhood of a point, then  $f_{yx} = f_{xy}$ .

### Chain rule

$$\frac{\partial z}{\partial ?} = \underbrace{\frac{\partial z}{\partial u} \frac{\partial u}{\partial ?} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial ?}}_{\text{For each input } (u, v), \text{ one by one}}$$

Recommend: Calculate each  $\frac{\partial z}{\partial u}$  individually, then combine and sub.

Special:  $(a^x)' = a^x - \ln(a)$

### Total differential

Linear approximation for  $\Delta w$  when  $(x, y)$  changes from  $(x_0, y_0)$  to  $(x_0 + \Delta x, y_0 + \Delta y)$  ]

- Useful for approximating functions where the input value deviates little from another value where computation is easier (e.g. 3.95 vs 4).

$$\Delta w \approx dw = w_x(x_0, y_0)\Delta x + w_y(x_0, y_0)\Delta y$$

Q-type: Given maximum relative errors of each input value, find the maximum relative error of the function output value:

A: Still write the above equation, but divide whole equation by  $w$ , then rearrange to plug in inequalities. i.e.  $\frac{\Delta w}{w} \approx \frac{w_x \Delta x}{w} + \frac{w_y \Delta y}{w}$

### Taylor's formula (Second order)

Quadratic approximation

$$f(x, y) \approx f(x_0, y_0) + [\Delta x f_x(x_0, y_0) + \Delta y f_y(x_0, y_0)] + \frac{1}{2!} [\Delta x]^2 f_{xx}(x_0, y_0) + 2\Delta x \Delta y f_{xy}(x_0, y_0) + [\Delta y]^2 f_{yy}(x_0, y_0)]$$

Recommend: Calculate partials separately first!

### Newton-Raphson Method (finding solutions via iterations)

- (A) Root estimation: Start from some  $x_0$ , then  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- (B) Simul. eqt:  $\begin{cases} f(x, y) = \dots = 0 \\ g(x, y) = \dots = 0 \end{cases}, \begin{cases} f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y = -f(x_0, y_0) \\ g_x(x_0, y_0)\Delta x + g_y(x_0, y_0)\Delta y = -g(x_0, y_0) \end{cases}$

### Relative extrema

1. Solve  $\begin{cases} f_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) = 0 \end{cases}$
2. Let  $A = f_{xx}, B = f_{xy}, C = f_{yy}, H = AC - B^2$  (draw a table!)
 

|                                    |                          |
|------------------------------------|--------------------------|
| • $H > 0, A > 0$ : Relative minima | • $H < 0$ : Saddle point |
| • $H > 0, A < 0$ : Relative maxima | • $H = 0$ : Inconclusive |
3. **Find extrema on the "boundary"**: x-axis, y-axis,  $(0, 0)$ ,  $(\infty, \infty)$

**Beware of missing roots!** e.g.  $\div 0, \pm$

- When dividing by a variable, split into two cases:  $\begin{cases} \text{Case 1: Normal} \\ \text{Case 2: } \div 0 \end{cases}$

**Lagrange multiplier (Optimization with constraints)**

Optimize  $f(x, y, z)$  with constraints  $g(x, y, z) = 0$  and  $h(x, y, z) = 0$ . For optimization within a region, find all relative extrema in range, then use Lagrange on the boundary.

$$1. \text{ Solve } \begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g = 0 \\ h = 0 \end{cases}$$

- Use two equations at a time!
  - Divide by variable! (consider  $= 0$  and  $\neq 0$ )
- Determine minimum/maximum for each root
  - Determine if global minimum/maximum exists
    - Think: Can you make the function infinitely small/large?
    - E.g. Let  $x = a$ ,  $\lim_{a \rightarrow \infty}$

Tip: For optimization of distance, optimize without  $\sqrt{\dots}$  first, add it back at the end.

**1.2 Multiple Integrals (MATH2014)****Volume of a region**

For area, treat  $f(x, y) \equiv 1$

$$\int_{lb_x}^{ub_x} \underbrace{\int_{lb_y}^{ub_y} f(x, y) dy}_{\text{Order can be changed for more convenient calculation}} dx$$

- Draw boundary
- Choose which way to integrate (order of  $x/y$ ) (for  $y \rightarrow x$ , write  $dy dx$ )
- $y$  bounds (in terms of  $x$ ): Shoot an arrow  $\parallel y$  to the graph, write intersects
- $x$  bounds (constant): Move arrow from left to right, just covering whole graph
  - Remember the sorting sound effect?

Integrating process: Like partial differentiation, treat irrelevant variables as const.

**Misc. tricks in multiple integration**

- Separable function
- Odd/even function, period

**Change of variables****Polar coordinates****Common polar graphs**

## 2 Linear Algebra

## 3 Optimization

## 4 Discrete Math and Probability Distributions

## 5 Statistical Estimation

## 6 Scientific Computing and Numerical Methods