1 Calculus

1.1 Partial Differentiation (MATH2014)

Prove continuity via polar coordinates

Q-type: Let $f(x,y) = \begin{cases} ? & \text{if } (x,y) \neq (0,0) \\ const. & \text{if } (x,y) = (0,0) \end{cases}$. Is f continuous?

To prove a function is **continuous**, a strategy is:

1. Change to polar: $\begin{cases} x = r \cdot cos\theta \\ y = r \cdot sin\theta \end{cases}$
2. Prove $sin\theta$ and $cos\theta$ is bounded, e.g. via sandwich thm.
   - $\lim_{r\to 0} |f(\theta)| \leq |c|$

To prove a function is **discontinuous**, some strategies are:

(A) Change to polar, set $\lim_{r\to 0}$, then choose two θ where value of $f(x,y)$ is different under limit.

(B) Fix $x = 0$ or $y = 0$, then set $\lim_{(x,y)\to(0,0)}$ "along the half line of $y = 0, x > 0$ (or some other half-axis)", and compare this with true value of $f(0,0)$ without limit.

Partial derivatives

To partial d. on x, perform d. as normal on x while treating others (y, z) as constants.

Notation: $f_x = \frac{\partial f}{\partial x}$

Mixed partial derivatives: $f_x y = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$ **(order of fraction is inverted!)**

If f_{xy} is continuous in a neighbourhood of a point, then $f_{yx} = f_{xy}$.

Chain rule

$$\frac{\partial z}{\partial ?} = \underbrace{\frac{\partial z}{\partial u}\frac{\partial u}{\partial ?} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial ?}}_{\text{For each input}(u,v)\text{,one by one}}$$

Recommend: Calculate each $\frac{\partial z}{\partial u}$ individually, then combine and sub.

Special: $(a^x)' = a^x - ln(a)$

Total differential

Linear approximation for Δw when (x,y) changes from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$]

- Useful for approximating functions where the input value deviates little from another value where computation is easier (e.g. 3.95 vs 4).

$$\Delta w \approx dw = w_x(x_0,y_0)\Delta x + w_y(x_0,y_0)\Delta y$$

Q-type: Given maximum relative errors of each input value, find the maximum relative error of the function output value:

A: Still write the above equation, but divide whole equation by w, then rearrange to plug in inequalities. i.e. $\frac{\Delta w}{w} \approx \frac{w_x \Delta x}{w} + \frac{w_y \Delta y}{w}$

Taylor's formula (Second order)

Quadratic approximation

$$f(x,y) \approx f(x_0,y_0) + [\Delta x f_x(x_0,y_0) + \Delta y f_y(x_0,y_0)] + \frac{1}{2!}\left[[\Delta x]^2 f_{xx}(x_0,y_0) + 2\Delta x \Delta y f_{xy}(x_0,y_0) + [\Delta y]^2 f_{yy}(x_0,y_0)\right]$$

Recommend: Calculate partials separately first!

Newton-Raphson Method (finding solutions via iterations)

(A) Root estimation: Start from some x_0, then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(B) Simul. eqt: $\begin{cases} f(x,y) & = ... = 0 \\ g(x,y) & = ... = 0 \end{cases}$, $\begin{cases} f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y = -f(x_0,y_0) \\ g_x(x_0,y_0)\Delta x + g_y(x_0,y_0)\Delta y = -g(x_0,y_0) \end{cases}$

Relative extrema

1. Solve $\begin{cases} f_x(x_0,y_0) = 0 \\ f_y(x_0,y_0) = 0 \end{cases}$
2. Let $A = f_{xx}, B = f_{xy}, C = f_{yy}, H = AC - B^2$ (draw a table!)
   - $H > 0, A > 0$: Relative minima
   - $H > 0, A < 0$: Relative maxima
   - $H < 0$: Saddle point
   - $H = 0$: Inconclusive
3. **Find extrema on the "boundary"**: x-axis, y-axis, $(0,0)$, (∞,∞)

Beware of missing roots! e.g. $\div 0$, $\pm$

- When dividing by a variable, split into two cases: $\begin{cases} \text{Case 1:} & \text{Normal} \\ \text{Case 2:} & \div 0 \end{cases}$

Lagrange multiplier (Optimization with constraints)

Optimize $f(x,y,z)$ with constraints $g(x,y,z)=0$ and $h(x,y,z)=0$. For optimization within a region, find all relative extrema in range, then use Lagrange on the boundary.

1. Solve $\begin{cases} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g &= 0 \\ h &= 0 \end{cases}$
   - Use two equations at a time!
   - Divide by variable! (consider $=0$ and $\neq 0$)
2. Determine minimum/maximum for each root
3. Determine if global minimum/maximum exits
   - Think: Can you make the function infinitely small/large?
   - E.g. Let $x=a$, $\lim\limits_{a\to\infty}$

Tip: For optimization of distance, optimize without $\sqrt{\dots}$ first, add it back at the end.

1.2 Multiple Integrals (MATH2014)

Volume of a region

For area, treat $f(x,y)\equiv 1$

$$\underbrace{\int_{lb_x}^{ub_x} \underbrace{\int_{lb_y}^{ub_y} f(x,y)\,dy}_{}\;dx}_{\text{Order can be changed for more convenient calculation}}$$

1. Draw boundary
2. Choose which way to integrate (order of x/y) (for $y\to x$, write $dy\,dx$)
3. y bounds (in terms of x): Shoot an arrow $\parallel y$ to the graph, write intersects
4. x bounds (constant): Move arrow from left to right, just covering whole graph
   - Remember the sorting sound effect?

Integrating process: Like partial differentiation, treat irrelevant variables as const.

Misc. tricks in multiple integration

(A) Separable function: The function can be represented as $f(x)\cdot g(y)$
Seems no need constant boundary?

$$\int_b^a \int_d^c f(x)\cdot g(y)\,dy\,dx = \int_b^a f(x) \int_d^c g(y)\,dy\,dx$$

(B) $\int\int c\,f(x,y)\,dA = c\int\int f(x,y)\,dA$,
$\int\int f(x,y)\pm g(x,y)\,dA = \int\int f(x,y)\,dA \pm \int\int g(x,y)\,dA$

(C) Odd/even function, period

Change of variables

1. Define u, v
2.
3.

Polar coordinates

Common polar graphs

2 Linear Algebra

3 Optimization

4 Discrete Math and Probability Distributions

5 Statistical Estimation

6 Scientific Computing and Numerical Methods