Computationally Efficient Signal Reconstruction from Zero-Crossing Sampling by the Three Points DFTs

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***Abstract*—The DFT (Discrete Fourier Transform) is defined for uniformly sampled signals. The inverse DFT (IDFT) may be interpreted as polynomial representation of the discrete signal, that is uniquely represented by the roots of this polynomial and the scaling factor. It is easy to ensure, by reversible transform, that all this roots are real (not complex) and thus can be observed (measured) in the signal. After measuring the times of signal zero-crossings the DFT is computed by expanding polynomial from its roots, and finally uniformly sampled signal is obtained by IDFT. In the paper basic theory of implicit sampling is reviewed and a new algorithm for efficient signal reconstruction from its zero-crossings, based on three points DFTs, is proposed.**

***Keywords*—Discrete Fourier Transform; DFT; sampling; implicit sampling; non-uniform sampling; zero-crossings**

I. INTRODUCTION

Implicit sampling refers to the case when the function is represented by the time instants in which it has a predetermined value, for example its zero-crossings [1-3]. If we assume that the function is a polynomial, then it may be represented by its roots and a scaling factor. This is a non-uniform sampling in the time grid, as opposed to common explicit sampling, when the samples are taken with the constant sampling period.

Hardware realization of implicit sampling analog to digital converter (ADC) requires only time measurements of zero-crossings, the amplitude is not measured, and thus sample and hold and quantization circuits are replaced by precise time measurement circuit.

Implicit sampling has not received much attention in the market yet, because signal reconstruction is more tedious than in the case of explicit sampling when the signal is directly measured. However, the situation may change as more powerful DSP (Digital Signal Processing) tools become available. In [3] implicit sampling based spectrum analyzer is described. The N points DFT is computed from signal zero-crossings with $N^2/2$ computational complexity, and it is claimed that improved dynamic range and speed may be obtained by implicit sampling hardware as compared to conventional ADC. Time signal may further be reconstructed by computing IDFT.

In this paper we propose an algorithm for reconstructing uniformly sampled discrete signal from the zero-crossings of its analog counterpart with $5(N-1)$ computational complexity. Such a low complexity is achieved by applying sequential three points DFTs. In the following basic theory on implicit sampling is reviewed. It is explained how to observe zeros of an arbitrary polynomial in the time domain, and finally the new algorithm for efficient signal reconstruction from its zero-crossings is proposed.

II. DFT AND SIGNAL ZERO-CROSSINGS

The samples x_n of uniformly sampled continuous signal $x(t)$ are obtained as

$$x_n = x(nT_s), \tag{1}$$

where n is the index (the number of the sample), T_s is the sampling period in seconds, and the inverse of T_s is the sampling frequency $F_s=1/T_s$ in hertz. According to *Sampling Theorem* [2] the signal $x(t)$ must be band-limited, and for the low-pass signal, i.e. $X(F)=0$ for $|F|>F_{max}$, where $X(F)$ is the Fourier Transform of $x(t)$ and F_{max} is the highest non-zero frequency component in $X(F)$, sampling frequency F_s must be two times higher than F_{max}, i.e. $F_s>2F_{max}$.

For a uniformly sampled signal x_n having N samples, the DFT and the IDFT (inverse DFT) are periodic in n (time index) and k (frequency index) with a period N [4, 5]. For odd N, $N=2M+1$, the DFT and the IDFT are defined as

$$X_k = \sum_{n=-M}^{M} x_n e^{-j\omega_N kn},\ k=-M,\ldots,-1,0,1,\ldots,M, \tag{2}$$

$$x_n = \frac{1}{N}\sum_{k=-M}^{M} X_k e^{j\omega_N kn},\ n=-M,\ldots,-1,0,1,\ldots,M, \tag{3}$$

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where $\omega_N = \dfrac{2\pi}{N}$ is the DFT frequency step in radians.

By putting a new variable $z_n = e^{j\omega_N n}$ into (3) we get

$$\begin{aligned} x_n &= \frac{1}{N}\sum_{k=-M}^{M} X_k z_n^k \\ &= \frac{1}{N}(X_{-M} z_n^{-M} + \ldots + X_0 z_n^0 + \ldots + X_M z_n^M) \\ &= a\prod_{i=1}^{2M}(z_n - z_i) \end{aligned} \tag{4}$$

where

$$z_i = e^{j\omega_N n_i} \tag{5}$$

are the roots of the polynomial of variable z_n and n_i are the time instances of signal $x(t)$ zero-crossings, $t_i = n_i T_s$. In general n_i are real numbers (i.e. not necessarily integers).

In (4) x_n is represented as a polynomial of z_n with degree $2M$ that has $2M$ roots z_i (a is a scaling factor). A real $x(t)$ has either real or complex conjugate roots. If we want all $2M$ roots to be real then the polynomial must change the sign $2M$ times in the considered interval. To ensure this, it is enough to subtract a sinusoid that changes sign $2M$ times and has an amplitude that is higher than the maximum absolute value of $x(t)$. This is known as the *Implicit Sampling Theorem* proved in [1]. The sampling sinusoid is defined as

$$s(t) = A_{\sin}\cos(2\pi F_{\sin} t), \tag{6}$$

where $A_{sin} > \max|x(t)|$, and $F_{sin} > F_{max}$.

The real signal $y(t)$ having all real zeros is

$$y(t) = x(t) - s(t). \tag{7}$$

The spectrum of $y(t)$ is band-limited but wider than the spectrum of $x(t)$. After uniform sampling the sampling sinusoid is

$$s_n = A_s\cos(M\omega_N n). \tag{8}$$

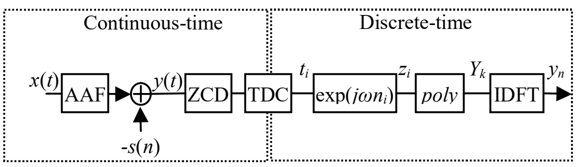

Fig. 1. Implicit non-uniform ADC: AAF – anti-aliasing filter, ZCD – Zero Crossing Detector, TDC - Time to Digital Converter, *poly* – expansion of the polynomial from its roots.

In the $2M+1$ points DFT of uniformly sampled $y(t)$ (7) the sampling sinusoid s_n is present in bins Y_{-M} and Y_M, and the analyzed signal x_n lays in bins from Y_{-M+1} to Y_{M-1}.

The block diagram of implicit non-uniform zero-crossing ADC is depicted in Fig. 1. Analog signal $x(t)$ must be bounded and band-limited. The cutoff frequency of low-pass anti-aliasing filter AAF is smaller than the frequency F_{sin} of sampling sinusoid $s(t)$ (6). Analog signal $y(t)$ with only real zeros is obtained by subtracting sampling sinusoid $s(t)$ from $x(t)$ (7). According to the *Implicit Sampling Theorem* [1] the amplitude of this sinusoid must be higher than the maximum absolute value of $x(t)$, and the frequency F_{sin} of this sinusoid must be higher than the maximum frequency F_{max} present in the spectrum of $x(t)$. Exemplary analog signals $x(t)$ and $s(t)$ are depicted in Fig. 2(a). The zeros in $y(t)$, shown in Fig. 2(b), are detected by zero crossing detector (ZCD), and the time instances t_i of zeros in signal $y(t)$ are measured by Time to Digital Converter (TDC). So far the processing is analog, once having vector t_i we switch to the digital domain. The complex zeros of (4) are computed using (5), where n_i are the real indices of $y(t)$ zero-crossings

$$n_i = \frac{t_i}{T_{\sin}}\frac{2N}{N-1}, \quad T_{\sin} = \frac{1}{F_{\sin}}, \tag{9}$$

where T_{sin} is the period of sampling sinusoid in seconds and F_{sin} is its frequency in hertz. Because the signal $x(t)$ is real all zeros z_i are complex conjugate. Next, the polynomial (4) is expanded by function *poly* in Fig. 1 (*poly* is the name of Matlab function used in simulations) to obtain the DFT Y_k of uniformly sampled signal y_n. Scaling factor a is obtained from known amplitude A_{sin} of sampling sinusoid $s(t)$ (6). For frequency analysis we may end at this point, but if the discrete uniformly sampled signal y_n is needed then it is obtained by IDFT. The signal x_n is also reconstructed by IDFT from DFT Y_k with bins Y_{-M} and Y_M equaled to zero.

It is observed in Fig. 2(a, b) that the information contained in the signal $x(t)$ is represented in $y(t)$ as zero-crossings, which are displacements of zero-crossings of sampling sinusoid $s(t)$. It is also evident that the sampling speed in uniform and non-uniform sampling is similar because the sampling frequency F_s in uniform sampling and the frequency of sampling sinusoid F_{sin} in non-uniform sampling must both be higher than the highest frequency F_{max} of non-zero component in the spectrum of $x(t)$. In Fig. 2(d) similar number of samples is observed for uniform and non-uniform sampling in the observation interval.

A. Example

Consider the discrete signal

$$x_n = 0.5 + 2\cos(\omega_N n) + \cos(3\omega_N n + \pi/2) \tag{10}$$

with M=5 (thus N=2M+1=11), n=-5, ...,5, $\omega_N = 2\pi/N$ rad. Note that the discrete signal (10) is periodic. The continuous-time counterpart signal $x(t)$ is depicted in Fig. 2(a) and uniformly sampled discrete-time signal x_n is shown, with star markers in Fig. 2(d).

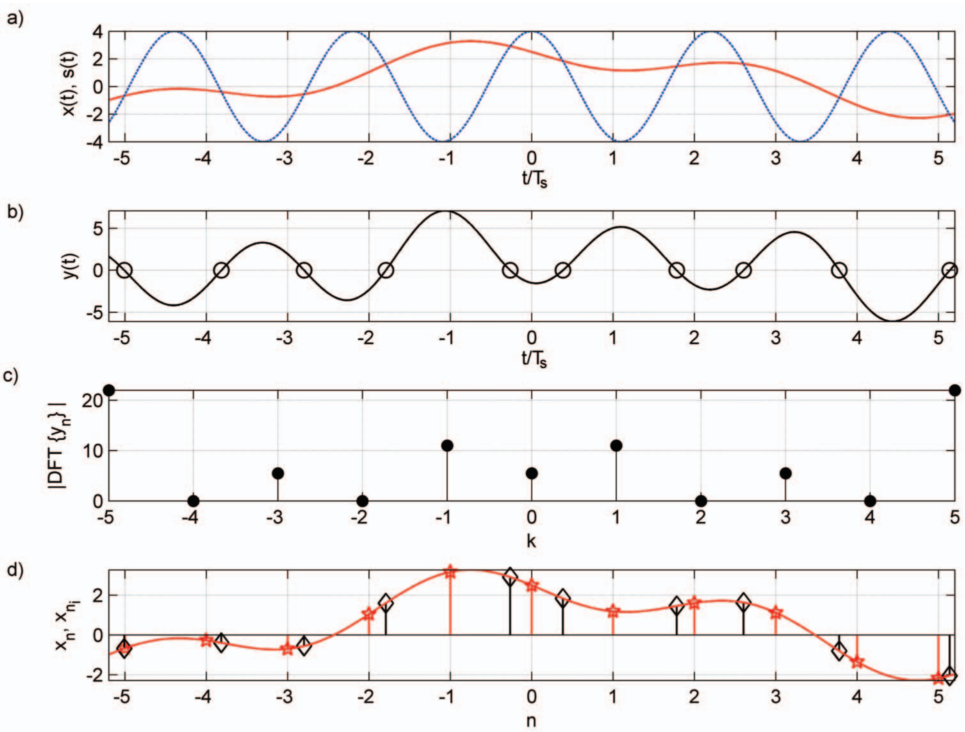

Fig. 2. An example of implicit zero-crossing sampling and signal reconstruction by the N points DFT: a) analyzed signal $x(t)$ (red continuous line) and sampling sinusoid $s(t)$ (blue dotted line), b) signal $y(t)$ (black continuous line) with all real zeros (black circles), c) DFT of y_n calculated from z_i (5), d) analyzed signal: uniform sampling (red stars), non-uniform sampling (black diamonds).

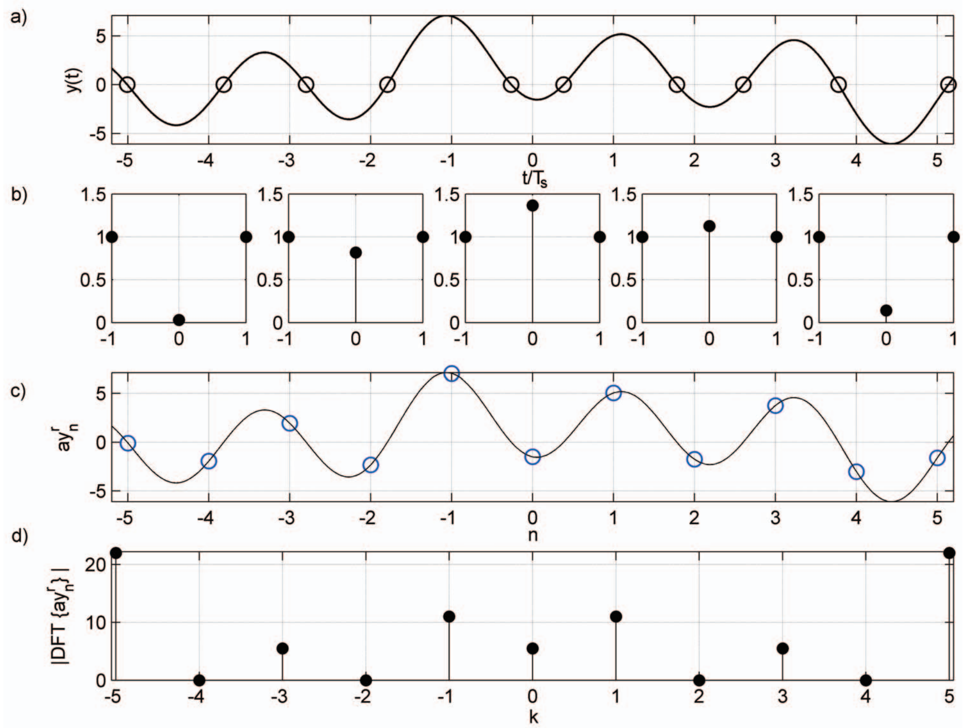

Fig. 3. Signal reconstruction from implicit sampling by the 3 points DFTs: a) signal $y(t)$ (black continuous line) with all real zeros (black circles), b) 3 points DFTs calculated from successive pairs of $y(t)$ zeros, c) signal y^r_n reconstructed from 3 points IDFTs, compare to Fig. 3(a), d) N point DFT of scaled reconstructed signal y^r_n, compare to Fig. 2(c).

Discrete-time sampling sinusoid is defined as

$$s_n = 4\cos(5\omega_N\, n) \tag{11}$$

and its continuous–time counterpart signal $s(t)$ is depicted in Fig. 2(a). The signal y_n with real zeros, defined by (7), is

$$y_n = 0.5 + 2\cos(\omega_N\, n) + \cos(3\omega_N\, n + \pi/2) - 4\cos(5\omega_N\, n) \tag{12}$$

and its continuous–time counterpart signal $y(t)$ is depicted in Fig. 2(b). The zeros of $y(t)$ are shown with the circle markers in Fig. 2(b). It is observed in Fig. 2(a, b), that when the value of analyzed signal $x(t)$ equals the value of sampling sinusoid $s(t)$ then $y(t)$ equals 0. In implicit sampling we only measure the times when $y(t)$ equals 0.

Fig. 2(c) shows the DFT of y_n calculated from z_i by expanding polynomial (4), i.e. at first time instances of zero-crossings n_i are measured, then z_i is computed by (5) and finally polynomial (4) is expanded from its roots. The coefficients by the powers of variable z_n are DFT coefficients. Sampling sinusoid s_n is present in the DFT bins Y_{-5} and Y_5. Assuming periodicity of the signal y_n and the knowledge of s_n amplitude A_{sin} the scaling factor a (4) may be computed from Y_{-M} or Y_M as

$$a = A_{\sin}\frac{N}{2\,|\,Y_{-M}\,|}. \tag{13}$$

By setting Y_{-M}=0 and Y_M=0 and calculating the IDFT uniformly sampled signal x_n is reconstructed.

In Fig. 2(d) the samples of uniformly sampled (red star markers) and non-uniformly sampled (green diamond markers) signal are compared. The samples of non-uniformly sampled signal equal the values of sampling sinusoid for zero-crossings of $y(t)$. For the case of non-uniform sampling the positions of samples on the time (index) n axis are for n_i being real, e.g. the first non-uniform sample in Fig. 2(d) is placed in $n_i \approx$-5.009). The range of n_i may by slightly broader than from -M to M, in the example the last non-uniform sample has an index $n_i \approx$5.137.

III. Efficient Signal Reconstruction

By measuring n_i, computing z_i (5), and expanding polynomial (4) we obtain the DFT coefficients Y_k of uniformly sampled signal y_n. Computing Y_k from z_i requires $N^2/2$ multiplications [3], and if y_n or x_n is needed then the IDFT must be used which is additional $N\log_2 N$ multiplications. In the following we explain how to reconstruct y_n from z_i with no more than 5(N-1) multiplications.

Signal flow in the proposed reconstruction algorithm is depicted in Fig. 3. The idea is to calculate the 3 points DFTs from each successive pair of signal zeros. Exemplary all real zeros signal $y(t)$ for the previously defined test signal y_n (12) is shown again in Fig. 3(a). In Fig. 3(b) absolute values of the consecutive 3 points DFTs computed by expanding roots of the second order polynomials (4), are shown. By computing the IDFTs of this DFTs we obtain 3 points fragments of the reconstructed signal y_n, denoted by y^r_n, that overlap by one sample. This overlapping sample is used for scaling correction. The current 3 sample fragment of the y^r_n is rescaled so that the first sample in this fragment has the same value as the last sample in the previous fragment. Reconstructed signal y^r_n differs from the original y_n only by a scaling factor a (it may happen that a is negative). Fig. 3(c) shows reconstructed signal y^r_n scaled by a for comparison with Fig. 3(a). Scaled reconstructed signal ay^r_n, Fig. 3(c), is the same as the test signal y_n (12) within numerical precision. Fig. 3(d) presents the DFT of scaled reconstructed signal ay^r_n which is the correct DFT of the test signal defined by (12).

Computational complexity of the proposed reconstruction algorithm is as follows. Expanding the 2nd order polynomial from its roots takes 1 complex addition and 1 complex multiplication, and calculating the 3 points IDFT costs 6 additions and 9 multiplications. The signal of length N has N-1 zeros, and the reconstruction of length N signal requires (N-1)/2 2nd order polynomial expansions and (N-1)/2 3 points DFTs computations. Overall computational complexity is 7(N-1)/2 complex additions and 5(N-1) complex multiplications.

The proposed signal reconstruction algorithm is expected to be robust against inaccuracies in $y(t)$ zeros measurement. The signal y_n is reconstructed from three points sections based only on two successive zeros of $y(t)$. Thus measurement errors influence only the current section, i.e. do not propagate to the remaining samples of y_n.

IV. Conclusion

Implicit non-uniform sampling may be attractive because of possible hardware realization. The non-uniform ADC is mainly composed from zero crossing detector and Time to Digital Converter (TDC) and only the precise time measurement is carried on. There is no need for sample and hold and amplitude quantization circuits.

For the considered periodic test signal the difference between original and scaled reconstructed signal was negligible small however the accuracy of signal reconstruction from implicit non-uniform sampling may suffer from spectral leakage problems for non-periodic signals.

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