# Lab 2

A $m \times n$ matrix $A$ of real numbers is called *sustainable* if for every pair of rows $i$ and $k$ with $i < k$ and every pair of columns $j$ and $l$ with $j < l$ we have

$$A_{i,j} + A_{k,l} \leq A_{i,l} + Ak, j.$$

In other words, whenever we pick two rows and two columns of a sustainable matrix and consider the four elements at the intersection of the rows and the columns, the sum of the upper-left and lower-right elements is less than or equal to the sum of the lower-left and upper-right elements.

For example, the following matrix is sustainable:

```
10 17 13 28 23
17 22 16 29 23
24 28 22 34 24
11 13  6 17  7
45 44 32 37 23
36 33 19 21  6
75 66 51 53 34
```

> *Hint:* write out a $5 \times 5$ matrix on paper, circle two of the rows, circle two of the columns, and observe the entries at the four intersections. If your matrix is sustainable, then no matter what two rows and what two columns you circled, (upper-left intersection plus lower-right intersection) will be less than or equal to (upper-right intersection plus lower-left intersection)
>
> *Check your understanding:* Is an identity matrix (ones along the main diagonal, zeros everywhere else) sustainable? Is a matrix where all entries are the same number sustainable?

For Lab 2, answer the following questions. Post your answers, and your source code, in the assignments page on Canvas. You may work alone or with a partner. If you choose to work with a partner, both people turn in the assignment and both people put both names on the assignment (this does mean you'll each turn in a copy of the same thing, but this makes it easier for me to see who's turned things in on Canvas).

1. Prove that an $m \times n$ matrix is sustainable if and only if for all $i = 1, 2, \ldots, m-1$ and $j = 1, 2, ..., n-1$, we have

   $$A_{i,j} + A_{i+1,j+1} \leq A_{i,j+1} + A_{i+1,j}.$$

   > *Hint:* for the "if" part, use induction separately on rows and columns.

2. The following matrix is not sustainable. Change one element in order to make it sustainable (Hint: Use part (a))

```
37 23 22 32
21  6  7 10
53 34 30 31
32 13  9  6
43 21 15  8
```

3. Let $f(i)$ be the index of the column containing the leftmost minimal element of row $i$. Prove that $f(1) \leq f(2) \leq \cdots \leq f(m)$.

4. Here is a description of a divide-and-conquer algorithm that computes the left-most minimum element in each row of an mxn sustainable matrix A:

   > Construct a submatrix $A'$ of $A$ consisting of the even-numbered rows of $A$. Recursively determine the leftmost minimum for each row of $A'$. Then compute the leftmost minimum in the odd-numbered rows of $A$.

   Explain how to compute the leftmost minimum in the odd-numbered rows of $A$ (given that the leftmost minimum of the even-numbered rows is known) in $O(m + n)$ time.

   Implement this algorithm in the programming language of your choice, and verify that it gives correct results for the sustainable matrix above. Does it give correct results for the example non-sustainable matrix above?

5. Write the recurrence describing the running time of the algorithm described in Part 5. Show that its solution is $O(m + nlogm)$.