A SHORT METHOD FOR THE DISCOVERY OF NEPTUNE

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Summary

The mathematical basis is described of a method for the discovery of Neptune that involves far less numerical calculation. The time of heliocentric conjunction can be found solely from considerations of the discrepancy in the longitude of Uranus. This information alone makes prediction possible within less than 15° on the basis of Bode's law. By finding the distance in a circular orbit appropriate to the best fit of the observations a prediction can be made comparable in accuracy with that achieved by Le Verrier. It is shown how by suitable combination of the equations of condition the number of unknowns can be reduced to three (as compared with eight in the original methods) for any assumed mean distance, and the same process removes certain awkward features that would otherwise enter for orbits near the 2 : 1 resonance.

1. *Introduction.*—It is an outstanding question of interest in relation to the discovery of Neptune whether in fact there could have been devised any much less laborious method that would have achieved the prediction with an accuracy comparable with that attained by Adams ($2\frac{1}{2}°$) and Le Verrier (1°). Such a method is described in the present paper as to its theoretical basis; the details of its numerical application have been given in full elsewhere*. The whole of the present investigation was inspired by discussions between the author and J. E. Littlewood following the 1946 centenary celebrations of the original discovery, and the procedure here described in Section 2 for estimating the time of heliocentric conjunction, which is the first step of the present method, is due principally to him†.

It is helpful first to explain briefly why the original methods were so extensive, and for this the one used by Le Verrier will be sufficient‡. All perturbations by known planets having been dealt with, the unknowns that have to be solved for consist of the four corrections necessary to the existing elements assigned to the orbit of Uranus, δa (or δn), δe, $\delta \epsilon$, and $\delta \varpi$; the four elements of the (coplanar) orbit of Neptune, a', e', ϵ', and ϖ'; and also the mass m' of the unknown planet; making apparently *nine* in all. The quantities e' and ϖ' can be replaced by $h' = e' \cos \varpi'$, $k' = e' \sin \varpi'$, as is usual, and h' and k' then occur only *linearly* in the perturbations in longitude of Uranus§. But a' and ϵ' do not occur linearly, while m' multiplies everything throughout. The mean distance a' was assigned (with considerable inaccuracy, it will be recalled) by means of Bode's law, but ϵ' occurs trigonometrically as $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\,\epsilon'$, $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\,2\epsilon'$, etc. To determine a first

* *Vistas in Astronomy*, Vol. III, 1959, Pergamon Press, London.

† For Prof. Littlewood's own account of the problem see his *A Mathematician's Miscellany*, pp. 117–134, Methuen, 1953.

‡ For a more detailed account of Le Verrier's method see: F. Tisserand, *Mécanique Céleste*, I, Ch. 23, Paris 1889; W. M. Smart, *Celestial Mechanics*, p. 259, Longmans, 1953.

§ Tisserand, *Mécanique Céleste*, Vol. I, p. 378.

approximation to ϵ', with a view eventually to having linear equations, Le Verrier took in turn 40 values for ϵ' spaced at 9° intervals covering the whole possible range from 0° to 360°, and for each value solved by least squares 18 consolidated equations of condition relating to 1690–1845, now in seven variables. It was then possible to select that value of ϵ' giving the best fit to the whole series of observations. Using this, he then proceeded to improve the solution to arrive finally at a set of "best" elements for Neptune.

The quantity ϵ' is closely associated with the longitude of the unknown planet, and the first step of the method to be described in this paper leads to an equivalent but even more specific piece of information concerning the unknown planet, namely the time of heliocentric conjunction with Uranus.

2. *Determination of the time of conjunction.*—The fact that the rate of change of angular momentum, dh/dt, must vanish at conjunction would enable the instant to be found were it not that $h=r^2\,dv/dt$ is not itself sufficiently well determined because of its strong dependence on the radius vector r. But an alternative procedure is possible that depends only on considerations of the longitude, and is one that involves little calculation in application.

In undisturbed elliptic motion we have for the heliocentric longitude, in standard notation,

$$v=nt+\epsilon+2e\sin(nt+\epsilon-\varpi), \tag{1}$$

to the first order in the eccentricity. We note that for Uranus $e=0{\cdot}047$. If the elements are in error as a result of small perturbations depending on the mass m' of the unknown planet (Sun $=1$), the resulting error in v will be given by

$$\Delta v=t\Delta n+\Delta\epsilon+2\Delta e\sin(nt+\epsilon-\varpi)+2e(\Delta\epsilon-\Delta\varpi)\cos(nt+\epsilon-\varpi)$$
$$+2et\Delta n\cos(nt+\epsilon-\varpi), \tag{2}$$

and on the right-hand side, from consideration of the equations of motion, the first four terms are of order m', while the last term is of order em' and so is much smaller than the others. Hence if we agree to neglect this last term we can write, with an obvious notation,

$$\Delta v=m'(a_0+bt+c\cos nt+d\sin nt) \tag{3}$$

for any discrepancy in longitude due to perturbations, correct to the present order.

Consider now the elliptic orbit E best fitting the observations of Uranus (after allowing for all known perturbations). The observations will not in fact be properly representable by E because they contain effects of the unknown body m'. Writing $v_E(t)=$ calculated heliocentric longitude of Uranus in ellipse E, $v(t)=$ actual heliocentric longitude determined from observation, then the well-known discrepancies in longitude, which constitute the observational material to be explained, are given by

$$\delta(t)=v(t)-v_E(t). \tag{4}$$

The values of $\delta(t)$ from 1690–1840 were the quantities utilized by Adams* and have been used in applying the present method. Le Verrier had observational data from 1690–1845.

Next, denoting by t_0 the instant of heliocentric conjunction that we wish to find, let E_0 denote the instantaneous, or osculating, elliptic orbit of Uranus

* *Coll. Works*, I, p. 11, Cambridge, 1896.

corresponding to t_0, and $v_0(t)$ the longitude of Uranus in this instantaneous orbit. Also let the perturbations produced by m' since time t_0 be denoted by $w(t)$, so that

$$w(t)=v(t)-v_0(t).$$

Then we have from (4)

$$\delta(t)=(v_0-v_E)+w, \tag{5}$$

and in this both terms are of order m'. Since we are proposing to neglect all terms of order em', then in calculating $w(t)$ both orbits E and E_0 can be regarded as circular, and therefore $w(t)$ will take equal and opposite values at times equally separated on the two sides of conjunction. So if τ measures time from conjunction $w(t_0-\tau)=-w(t_0+\tau)$, or writing

$$W(\tau)=w(t)=w(t_0+\tau) \tag{6}$$

then $W(\tau)$ is an odd function of τ. The analytical form of W is given below by (11) but is not required for the present purpose of finding t_0.

Now to the first term on the right in (5), the form (3) will clearly be applicable, and so

$$\delta(t)=m'\{a_0+bt_0+b\tau+c\cos(nt_0+n\tau)+d\sin(nt_0+n\tau)\}+W(\tau),$$

which may be written, introducing new constants, in the form

$$\delta(t_0+\tau)=A+B(1-\cos n\tau)+\{C\tau+D\sin n\tau+W(\tau)\}. \tag{7}$$

Since $W(\tau)$ is an odd function, we also have

$$\delta(t_0-\tau)=A+B(1-\cos n\tau)-\{C\tau+D\sin n\tau+W(\tau)\}.$$

Hence, since $\delta(t_0)=A$, it follows that neglecting terms of order em' the function given by

$$\rho(\tau)=\frac{\delta(t_0+\tau)-2\delta(t_0)+\delta(t_0-\tau)}{1-\cos n\tau}=B \tag{8}$$

will be constant.

This result provides the following simple rule for determining the approximate time of conjunction: *Select an instant t_0, calculate the second difference $\delta(t_0+\tau)-2\delta(t_0)+\delta(t_0-\tau)$ for a number of values of τ, divide each by $(1-\cos n\tau)$, and the resulting quantity will be (approximately) constant if t_0 has been selected at conjunction.*

The result of applying this rule to the observational data $\delta(t)$ (Adams, *loc. cit.*) is shown in Fig. 1. The quantity $\rho(\tau)$ turns out to be most nearly constant for a value of t_0 of about 1822·3. This time for the heliocentric conjunction in longitude differs by only just over six months from the actual value 1821·74.

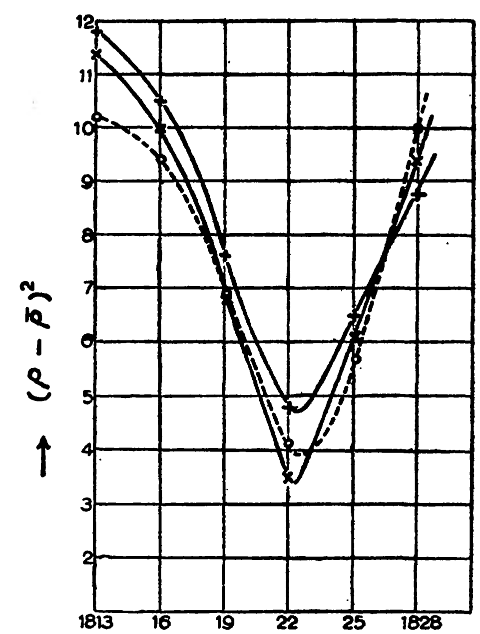

FIG. 1.—*Graph showing degree of constancy of $\rho(\tau)$. The ordinate represents the root mean value of $(\rho-\bar{\rho})^2$ plotted at t_0* =1813, 1816, 1819, 1822, 1825, and 1828. *The different curves correspond to different weightings. For each the minimum occurs at about* 1822·3.

+ equal weightings of $\rho(\tau)$;
× weightings increasing with τ;
○ weights increasing and then decreasing.

3. *Prediction purely from a knowledge of conjunction.*—On the basis of the unknown planet being in a circular orbit, it is a simple matter once the time of conjunction is available to calculate its longitude for any assumed size of orbit. Even on the crude assumption of Bode's law, viz. $a'/a=2$, the longitude so predicted for the date of discovery is only about 13° behind the actual position. The actual planet would have been well inside a zodiacal belt 30° long by 10° wide centred on this place, which was the kind of region Airy suggested should be searched by Challis.

The foregoing procedure can be carried out with a minimum of arithmetical labour, taking at most a few hours, but the question arises, in view of the serious failure of Bode's law, which had it held would have made discovery almost triflingly easy by the present method, whether any better estimate of a'/a can be reached without unduly extensive calculation. For this purpose, Bode's law is dropped and instead we assume always a circular orbit for the unknown body, which is at least an equally valid assumption to make for any planetary orbit.

4. *Equations for finding an improved value of a'.*—On this basis then, the perturbations in longitude of Uranus are given by*

$$P(t) = -m' \sum_{1}^{\infty} F_i \sin i\{(n'-n)t+\epsilon'-\epsilon\} + m'e \sum_{-\infty}^{+\infty} G_i \sin\left[i\{(n'-n)t+\epsilon-\epsilon\}+nt+\epsilon-\varpi\right] \tag{9}$$

wherein

$$F_i = \frac{i(z_i^2+3)}{z_i^2(1-z_i^2)}\, aA^{(i)} + \frac{2z_i}{z_i^2(1-z_i^2)}\, a^2 \frac{\partial A^{(i)}}{\partial a},$$

$z_i = i(1-n'/n)$, and $A^{(i)}$ is a certain infinite series in a/a'; while a similar form gives G_i. Unless a definite numerical value is adopted for a/a' there is clearly therefore no possibility of arriving at linearized equations, and this plainly must have been the great attraction of Bode's law.

The assumption of a circular orbit and knowledge of the time of conjunction immediately reduce the number of unknowns associated with Neptune to *two*, namely m' and a', and also remove altogether the awkward feature otherwise produced by the appearance of ϵ' trigonometrically, for once conjunction is known t can be measured from it. Also since e is small, the terms in $m'e$ in (9) are small compared with those solely in m', and the perturbations can be calculated as if Uranus itself moved in a circular orbit. The principal terms in $m'e$ can however readily be included for higher accuracy. Accordingly, if from now on the time is measured from conjunction, the whole expression for the discrepancies in heliocentric longitude will be of the form

$$\delta v = \delta\epsilon + t\delta n + \delta\alpha \sin nt + \delta\beta \cos nt + P(t) \tag{10}$$

wherein now

$$P(t) = -m' \sum_{1}^{\infty} F_i \sin i(n-n')t = -m' \sum_{1}^{\infty} F_i \sin iD, \tag{11}$$

say. If, for any assumed value of a/a', direct comparison of (10) with the observations is made, the variables are now only *five* in number, $\delta\epsilon$, δn, $\delta\alpha$, $\delta\beta$, and m', already a considerable reduction on the number used by Adams and

* Tisserand, Vol. I, p. 365.

Le Verrier, though practical application would require solution for a number of values of a/a'. But it is possible to eliminate the variables $\delta\alpha$ and $\delta\beta$, as we shall next explain.

5. *Further reduction of the number of unknowns.*—In view of the unavoidable feature that a series of values of a/a' must be taken, it is convenient to take advantage of a simple regrouping of the equations of condition that has the effect of removing $\delta\alpha$ and $\delta\beta$, as well as a certain other important consequence. For this, we note the simple identities

$$\frac{\sin}{\cos}(x+\theta)+\frac{\sin}{\cos}(x-\theta)-\frac{\sin}{\cos}x=(2\cos\theta-1)\frac{\sin}{\cos}x \qquad (12)$$

in which, if $\theta=60°$, the right-hand sides vanish.

Now for Uranus $2\pi/n=84{\cdot}015$ years, and accordingly $60°$ of longitude is described by the planet in almost exactly 14 years. Hence if instead of a particular observation (discrepancy in longitude), $f(t)$ say, we adopt

$$f(t+14)+f(t-14)-f(t),$$

where t is now measured in years, then the terms in $\delta\alpha$ and $\delta\beta$ will disappear.

This device is of course no more than a regrouping of the equations of condition, incidentally reducing their number since, to some extent, 14 years are lost at each end of the range of available observations. But as will be shown later it simultaneously has the important effect of making the method safely applicable near the resonance $n/n'=2$ (for which $a/a'=0{\cdot}63$ approximately) where in fact the coefficient F_2 can become very large.

Under this regrouping, the terms $\delta\epsilon+t\delta n$ come out the same, but the terms in $P(t)$ have to be adjusted as to their coefficients. For the term in $\sin D$, for instance,

$$\begin{aligned} D+\theta &= (n-n')(t+14) \qquad \text{where} \quad n\,.\,14=60° \\ &= D+(1-n'/n)60°. \end{aligned}$$

So, writing $\nu=n'/n$, we have $\theta=(1-\nu)60°$, and hence to transform $P(t)$ the coefficient F_1 has to be multiplied by $2\cos\{(1-\nu)60°\}-1$, the coefficient F_2 has to be multiplied by $2\cos\{(1-\nu)120°\}-1$, and so on. With these conversion factors applied, we can write, denoting the new coefficients by f_i,

$$Q(t)\equiv P(t+14)+P(t-14)-P(t)=m'\sum f_i\sin iD, \qquad (13)$$

and the revised equations of condition take the form

$$\delta\epsilon+t\delta n+Q(t)=\delta v(t+14)+\delta v(t-14)-\delta v(t), \qquad (14)$$

and these, for each selected a/a', involve now only *three* unknowns, namely $\delta\epsilon$, δn, and m'.

The procedure adopted was therefore to set up these modified equations of condition for each of a short series of values of a/a', and solve by least squares to find what value of the ratio of axes gives the best fit to the observations.

6. *Effect of the transformation on the coefficient F_2.*—The following table shows the values of F_1, F_2, F_3, F_4 for an appropriate set of values of a/a'.

TABLE I

Coefficients of terms in $P(t) = m'\Sigma F_i \sin iD$

a/a'	$\sin D$	$\sin 2D$	$\sin 3D$	$\sin 4D$
0·50	− 5·738	1·429	0·141	0·031
0·55	− 7·432	3·380	0·299	0·068
0·60	− 9·911	15·712	0·672	0·154
0·65	−13·718	−38·422	1·692	0·366
0·70	−19·944	−18·655	5·461	0·967

It will be noticed that F_2 becomes very large in magnitude as the resonance $n'/n = 2$, or $a/a' = 0{\cdot}63$, is approached from either side. The expression (9), or (11), for the perturbations in longitude is not of course applicable for exact resonance, at which the coefficient F_2 would become infinite. Now to avoid excessive calculation of the various F_i for different a/a', it would obviously be convenient if we could interpolate within the above table, but in the case of F_2 interpolation from only a few values is not possible with much accuracy because of the resonance. (All these coefficients are nowadays available in tabular form*, but as they were not so to Adams and Le Verrier any comparison of methods must suppose the F_i to be calculated as part of the work.)

But the conversion factor $2\cos\theta - 1$ vanishes at the resonance when the coefficient F_2 has its infinity through the factor $1 - z_i = 2\nu - 1$ in its denominator. But it is easily seen that the expression

$$\frac{2\cos\{(1-\nu)120^\circ\} - 1}{2\nu - 1}$$

has limit $\pi/\sqrt{3}$ as $\nu \to \frac{1}{2}$, and the coefficient f_2 passes smoothly through the resonance value without singularity, as the following table of the transformed coefficients shows:

TABLE II

Coefficients of terms in $Q(t) = m'\Sigma f_i \sin iD$

a/a'	$\sin D$	$\sin 2D$	$\sin 3D$	$\sin 4D$
0·50	− 3·208	−0·814	−0·266	−0·087
0·55	− 4·664	−1·290	−0·469	−0·176
0·60	− 6·877	−2·052	−0·821	−0·345
0·65	−10·382	−3·299	−1·436	−0·666
0·70	−16·165	−5·471	−2·558	−1·284

The smooth trend of these coefficients makes sufficiently accurate interpolation for other intermediate values of a/a' easily possible, for it is mainly the general form of the curve represented by $Q(t)$ as a/a' changes that matters, and our aim is to find that value of a/a' giving the closest fit to the (adjusted) observations. Another feature of this table is that the coefficients corresponding to $\sin 5D$ and $\sin 6D$ can also be roughly estimated for any given a/a' from inspection of the general run of the earlier coefficients.

7. *The solution giving the best fit.*—The equations of condition (14) have been set up and solved for a series of seven different values of a/a' ranging from 0·50 (Bode's law) to 0·65. The mean square value of the resulting sets of residuals for each solution is shown plotted against a'/a in Fig. 2, from which it is seen that there is a pronounced minimum at just about $a'/a = 1{\cdot}6$, and there is obviously

* *Yale Trans.*, VI and VII, Cambridge, 1932.

little point in endeavouring to improve on the round figure of 1·6 for the ratio of axes. This corresponds to a mean distance for the unknown planet of 30·71 a.u. compared with the true value of 30·07.

The longitude thereby predicted for the date of discovery is readily found to be 329°·4 compared with the actual value of 328°·4. The error of the present method is therefore about 1° ahead of the true position, whereas that of Le Verrier was just under 1° behind.

Consistently with the closeness of a' to the true value, the present method also gives a closer value of the mass of the unknown body than was obtained by either Le Verrier or Adams. The value found is $m'/\odot = 1/25\,000$ in round figures.

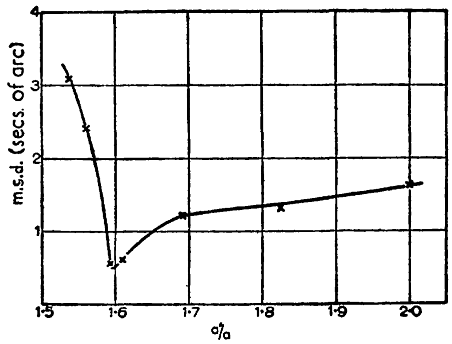

FIG. 2.—*Curve of mean square residuals computed for a number of values of a'/a. The best fit is seen to occur at about $a'/a = 1{\cdot}6$.*

8. *Prediction at a date other than* 1846.—The orbital elements (and mass) obtained by the various methods, adopting the better of the two solutions given by Adams, are as follows.

TABLE III

	Adams	Le Verrier	Present method	Neptune
a	37·25	36·15	30·71	30·07
e	0·1206	0·1076	0	0·0086
ϖ	299°	285°	—	44°
$\odot/m'$	6670	9350	25500	19300
Longitude at discovery (equinox 1950)	330°·9	327°·4	329°·4	328°·4

It is a simple matter to find the longitude that these solutions would predict at any other time than 1846·73, the time of discovery. The three curves of Fig. 3 show the difference between the longitude calculated in turn from the above values and the actual longitude of Neptune. It is evident from the curves that some element of good fortune entered into the predictions to make them as good as they were. Le Verrier's solution appears superior to that of Adams, and his prediction would have been better still at a somewhat earlier date (supposing the same material to have been available, which could not in fact have been so) since the error changes sign at about 1842. Both solutions place the unknown planet *ahead* of the true position in the earlier part of the range, so that with a'/a assumed nearly equal to 2, the hypothetical planet will move more slowly than the actual one and therefore must eventually coincide with it in longitude. For Adams' solution this does not occur till as late as 1856. But their orbits were intended to fit the observations prior to 1840 in Adams' case, and 1845 in Le Verrier's, so comparison can only properly be made for times earlier than discovery. This shows that the errors implied by Adams' solution are on the whole roughly twice as great as those of Le Verrier's. (A possible reason for this is discussed below.) On the other hand the circular solution arrived at by the present method shows an accuracy comparable with that of Le Verrier's orbit, but persisting over a much longer range in time. The three curves show that $\pm 1^\circ$ accuracy would be obtained by Adams' solution only for about 5 years, by Le Verrier's solution for about 12 years, while for the present solution it would obtain for about 24 years.

The orbits arrived at by Adams and Le Verrier both exhibit the curious feature that their difference in longitude from the true position is at its least fairly near the time of prediction—six years earlier for Le Verrier and nine years later for Adams. Why this place of "best fit" should have come near the crucial time seems something of a puzzle when the observations utilized extended back some 150 years. It may well be that the latest observations were of particularly strong influence, but in any event it would seem that some element of good fortune attended the degree of closeness of the actual prediction.

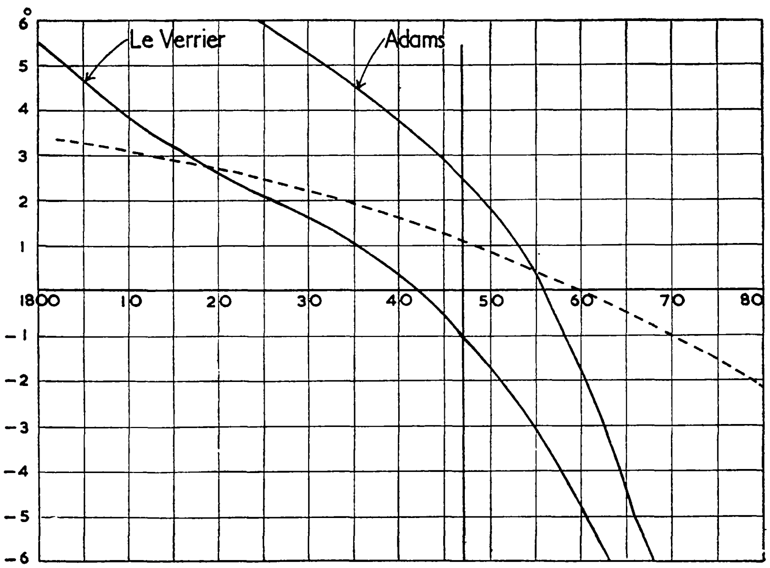

FIG. 3.—*Showing the amounts by which the orbits of Adams, Le Verrier, and the circular orbit solution differ from the true position of Neptune.*

9. *Other points.*—The weaker agreement of Adams' orbit with the actual one seems very probably to arise from what appears to be an invalid step made by him at the very outset of his work. Adams says (*loc. cit.*, I, p. 11). "It is easily seen that the series expressing the corrections of mean longitude in terms of the corrections applied to the elements of the orbit, is more convergent than that which gives the correction of the *true* longitude, and the same thing is true for the perturbations of the mean longitude, as compared with those of the true. The corrections found above were accordingly converted into corrections of mean longitude by multiplying each of them by the factor r^2/ab." Now whereas such a relation as that here claimed subsists between the true longitude and the mean longitude in an *undisturbed* orbit in virtue of the equation

$$r^2 \frac{dv}{dt} = h = nab,$$

which may be written

$$(r^2/ab)\, dv = n\, dt = dl, \tag{15}$$

there would seem no validity in supposing that in perturbed motion the variation in true longitude Δv and that in mean longitude Δl are similarly related.

No such relation seems known in celestial mechanics. We have in any case that

$$v = \int hr^{-2}\,dt, \qquad l = nt + \epsilon,$$

and so for a variation Δ, in which all the elements are changed,

$$\Delta v = \int (r^{-2}\Delta h - 2hr^{-3}\Delta r)\,dt, \qquad \Delta l = t\Delta n + \Delta\epsilon,$$

and that the ratio of these cannot reduce to a form independent of the variations of the elements, such as (15), seems evident. But the actual case is more complex still because the discrepancies involve also the corrections to the orbit of Uranus. The validity of this step has also been queried by E. W. Brown*.

In his paper E. W. Brown also was concerned with the possibility of finding some simple criterion for the prediction of an unknown planet, and the method he proposed therein rests on consideration of the series $-m' \sum F_i \sin iD$ for the principal perturbations in longitude, naturally enough, as here. Brown's discussion shows how the general trend of this function (with changing a/a'), rather than its precise value at all stages, is sufficient to estimate the position of the unknown planet, though he applies the method on the basis that the first three terms of this function are adequate. Independent preliminary determination of the instant of conjunction is not included in his method, and one gets the impression that this could be a considerable defect when it comes to application. There is difficulty in being certain, however, from the actual work given in his paper, since the present author has not found it possible to recover the numerical values he gives for several of the various coefficients. At all events, Brown himself reaches the conclusion that his method predicts conjunction as occurring *close to* 1840, which in fact is almost 90° from the true position, and would lead to a longitude at the time of discovery some 27° ahead of the actual place of Neptune, even assuming a circular orbit at exactly the right distance. Nevertheless, the general theory behind Brown's method would appear to be sound, though, handicapped as it would be in detailed application by lack of knowledge of conjunction, to reach a prediction of accuracy comparable with those of Adams and Le Verrier might very well require considerable numerical labour.

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* *M.N.*, **92**, 93, 1931. See Section 14.