EX1.

(ref. https://math.stackexchange.com/questions/4797313/if-f-k-oversetm-longrightarrowf-and-and-int-e-f-kp-le-m-then-int)
$\int_E |f - f_k|^p = \int_0^\infty p\alpha^{p-1}\omega_{|f-f_k|}(\alpha)d\alpha \to 0$ as $k \to \infty$
$\implies \lim_{k\to\infty} \alpha^{p-1}\omega_{|f-f_k|}(\alpha) = 0$ a.e. on $[0, \infty)$
$\implies \lim_{k\to\infty} \omega_{|f-f_k|}(\alpha) = 0$ a.e. on $[0, \infty)$
$\implies \lim_{k\to\infty} |\{|f_k - f| > \alpha\}| = 0 \forall \alpha > 0$
$\implies f_k \overset{m}{\to} f$
$\implies |f_k|^p \overset{m}{\to} |f|^p$

$\implies \exists\{f_{n_K}\}$ s.t. $f_{n_k} \to f$ a.e.

$\implies \int_E |f|^p = \int_E \liminf_{k\to\infty} |f_{n_k}|^p \leq \liminf_{k\to\infty} \int_E |f_{n_k}|^p \leq M$ by Fatou's Lemma

EX2.

Let $f(x) = \frac{1}{\log(x+5)}$, then $\lim_{x\to\infty} f(x) = 0$, but
$\log(x+5)^p \in o(x+5) \forall p > 0 \implies \exists R$ s.t. $\log(x+5)^p < x + 5 \forall x > R$

$\implies \int_0^\infty f(x)^p dx \geq \int_R^\infty f(x)^p dx \geq \int_R^\infty \frac{1}{x} dx = \infty$

therefore, $f(x) \notin L^p(0, \infty)$

EX3.

a.

$|f_k|$ are positive
$\implies \sum \int_E |f_k(x)| = \int_E \sum |f_k(x)| < \infty \forall x$
$\implies \sum |f_k(x)|$ is finite a.e.

$\implies \sum f_k$ converge absolutely a.e.

b.

We can claim $\sum_k \int_{[0,1]} |a_k| \times |x - r_k|^{-\frac{1}{2}} < \infty$ and therefore $\sum_k a_k \times |x - r_k|^{-\frac{1}{2}}$ converge absolutely by subproblem a:

$|\int_0^1 \frac{a_k}{\sqrt{|x-r_k|}} dx| = |a_k[\int_0^{r_k} \frac{1}{\sqrt{r_k - x}} dx + \int_{r_k}^1 \frac{1}{\sqrt{x - r_k}} dx]| = |a_k[2\sqrt{r_k} + 2\sqrt{1 - r_k}]| \leq 4|a_k|$

$\implies \sum_k \int_{[0,1]} |a_k| \times |x - r_k|^{-\frac{1}{2}} \leq \sum_k 4|a_k| < \infty$, and the claim hold.

EX4.

Let $P_n = \{f > \frac{1}{n}\}, N_n = \{f < -\frac{1}{n}\}$, both measurable
$0 = \int_{P_n} f \geq \inf_{P_n} f \times |P_n| \geq \frac{1}{n} \times |P_n| \implies |P_n| = 0$
$0 = \int_{N_n} f \leq \sup_{N_n} f \times |N_n| \leq -\frac{1}{n}|N_n| \implies |N_n| = 0$

$\implies |\{f > 0\}| = |\cup_n P_n| = \lim_{n\to\infty} |P_n| = 0$, and so as N_n

$\implies |\{f \neq 0\}| \leq |\{f > 0\}| + |\{f < 0\}| = 0$, $f = 0$ a.e.

EX5.

(ref. https://math.stackexchange.com/questions/3984200/f-in-lp-iff-sum-k-in-mathbbz2kp-lambda-f2k-infty)

if $f \in L^p$, then
$\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) = \sum_{k=-\infty}^{\infty} \frac{p}{1-2^{-p}} \int_{2^{k-1}}^{2^k} \alpha^{p-1} d\alpha \omega(2^k) \leq \frac{p}{1-2^{-p}} \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \alpha^{p-1}\omega(\alpha) d\alpha = \frac{1}{1-2^{-p}} \int_{-\infty}^{\infty} p\alpha^{p-1}\omega(\alpha) d\alpha = \frac{1}{1-2^{-p}} \int f^p < \infty$

For the converse side,
$\sum_{k=-\infty}^{\infty} 2^{pk}\omega(2^k) < \infty \implies \omega(\alpha) < \infty \forall \alpha > 0 \implies f \in L^p$ by Exercise 16.