# Real Analysis Homework 01

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I use $A - B$ to represent the notation $A \backslash B := \{x | B \in A \cap x \notin B\}$ at the course, and the notation $A^c$ is represent the complement of $A$

Besides, if we type "Let $I = \{I_k\}_{k=1}^{\infty}$ be a collection of rectangles which covering $E$ ", it implies $E \subseteq \cup_{k=1}^{\infty} I_k$, not $E \subseteq I_k \forall k$

# EX1.

Let the set be $S$, and $S_n$ is the set after the $n^{th}$ construction step.

## 1. perfect

If a node is an isolated point in $S$, WLOG suppose this node is not at the right or upper edge(otherwise we can flip the entire graph vertically or horizentally), then the square regions of its right, upper, and upper right corner squares are removed, these are a continuous polygon.
By the proposition of construction step, these three regions are removed at the same time.
However, it implies this node is the upper right corner node of an block in a certain step, and therefore it should not be an isolated point(consider the $\lceil -\log_2 \epsilon \rceil + 1$ step, it make sure there's at least one node close enouth with corner node of distance $< \epsilon$).

## 2. measure zero

After the $n^{th}$ step of construction, the area of the set is $|S_n| = (\frac{4}{9})^n$.

$$|S|_e = | \cap_{n=1}^{\infty} S_n|_n \implies |S|_e \leq |S_n| = (\frac{4}{9})^n \forall n$$
$$\implies |S|_e = 0$$

## 3. equal $C \times C$

consider the mapping $\mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ by $\phi((x, y)) = (x, y)$, it will be a bijection between $S$ and $C \times C$

# EX2.

Let such set be $C_\delta$, and $C_\delta^k$ be the set after construction step $n$.

## 1. perfect

Obviously, $0$ and $1$ is impossible to isolated, therefore we only discuss the nodes in $(0,1)$
if a node $a$ is isolated, then there's an interval $(a,b) \subseteq [0,1]$ is disjoint with $C_\delta$
By the proposition of Cantor-type set, $a$ is a rightmost point of an interval in a centain step of construction.

therefore, $a$ is impossible to be isolated, contradiction.

## 2. measure

at the $n^{th}$ construction step, the volume of the set $C_\delta^n$ is $|C_\delta^n| = 1 - \sum_{k=1}^{n} 2^{k-1} \times (\delta 3^{-k})$
$= 1 - \frac{1}{2}\delta \sum_{k=1}^{n} (\frac{2}{3})^k$

Then from the nested proposition of the $C_\delta^n$, we have
$C_\delta = \liminf_{n\to\infty} C_\delta^n = \limsup_{n\to\infty} C_\delta^n$

With following proposition, we know that $|C_\delta| = \lim_{n\to\infty} |C_\delta^n| = 1 - \delta$

- if $A_1 \supseteq A_2 \supseteq ...$ are nested measurable sets, then $|\limsup_{n\to\infty} A_n| = \lim_{n\to\infty} |A_n|$:
    - first, $\cup_{k=n}^{\infty} A_k = A_n$ is measurable, therefore $\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ is measurable.
    - besides, $A_n - A = (A_n - An + 1) \cup (A_{n+1} - A) = ... = \cup_{k=n}^{\infty} (A_n - A_{n+1})$
    - $\implies |A_n - A| \leq \sum_{k=n}^{\infty} |A_n - A_{n+1}|$
    - From $\infty > |A_1 - A| \geq |A_1 - A_2| + |A_2 - A_3| + ... + |A_{n-1} - A_n|$ we can find that $\sum_{k=n}^{\infty} |A_k - A_{k+1}| \to 0$ as $n \to \infty$, therefore $|A_n - A| \to 0$ as $n \to \infty$
      $\implies |A_n| - |A| < |A_n - A| \to 0$ as $n \to \infty$

### 3. contain no intervals

We can easily seen that after the $n^{th}$ step, there's no any continuous intervals with length greater than $2^{-n}$ in $C_\delta^n$

Therefore, the length of the intervals in $C$ will not greater than $\inf_n 2^{-n} = 0$

i.e. there's no interval contained in $C_\delta$

# EX3.

From the description,

- after the first step, there's $2$ intervals of length $\frac{1-\theta_1}{2}$
- after the second step, there's $4$ intervals of length $\frac{1-\theta_1}{2} \times \frac{1-\theta_2}{2}$
  ...
- after the $n^{th}$ step, there's $2^n$ intervals of length $\Pi_{k=1}^n \frac{1-\theta_k}{s2} = 2^{-n} \times \Pi_{k=1}^n (1-\theta_k)$

The remainder has nonzero measure
$\iff \limsup_{n\to\infty} 2 \times 2^{-n} \times \Pi_{k=1}^n (1-\theta_k) \overset{each\ terms \in (0,1)}{=} \Pi_{n=1}^\infty (1-\theta_n)$ converge to a nonzero value
$\iff \sum_{n=1}^\infty \log(1-\theta_k)$ converge
( $\iff \sum_{n=1}^\infty -\log(1-\theta_k)$ converge)

Then we can find that

- if $\sum_{n=1}^\infty \log(1-\theta_k)$ converge, from $0 \le \theta_n \le -\log(1-\theta_n)$, $\sum_{n=1}^\infty \theta_n$ converge.
- if $\sum_{n=1}^\infty \theta_n$ converge, then $\lim_{n\to\infty} \theta_n = 0$
  from $\log(1-x) \simeq -x$ for small $N$, we can find that $\sum_{n=1}^\infty \log(1-\theta_n)$ converge

Therefore, the remainder has measure zero $\iff \sum_{n=1}^\infty \theta_n$ not converge, i.e. $\sum_{n=1}^\infty \theta_n = \infty$

# EX4.

Let $E = \limsup_{n\to\infty} E_n = \cap_{n=1}^{\infty} \cup_{k\geq n} E_k$

By definition, $|E|_e = |\cap_{n=1}^{\infty} \cup_{k\geq n} E_k| \leq |\cup_{k\geq n} E_k|_e \leq \sum_{k=n}^{\infty} |E_k|_e \forall n \in \mathbb{N}$

because $|E_k|_e \geq 0 \forall k$, therefore $\sum_{k=1}^{\infty} |E_k|_e < \infty$ implies $\sum_{k=1}^{\infty} |E_k|_e$ converge
$\implies \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\sum_{k=n}^{\infty} |E_k|_e < \epsilon \forall n \geq N$

$\implies |E|_e < \epsilon \forall \epsilon > 0 \implies |E|_e = 0$
therefore, $E = \limsup_{n\to\infty} E_n$ has measure zero.

for the $\liminf$ case, define $I = \liminf_{n\to\infty} E_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k$ and fix $\epsilon > 0$,

$|I|_e = |\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k| \leq \sum_{n=1}^{\infty} |\cap_{k=n}^{\infty} E_k|_e$

note that $|\cap_{k=n}^{\infty} E_k| \leq |E_m| \forall m \geq n$
For each $n$, choose $N_n$ s.t. $|E_k| < \epsilon \times 2^{-n} \forall k \geq N_n$ (it always exist from the fact $\sum_{n=1}^{\infty} |E_k|_e$ converge)
then we have $|\cap_{k=n}^{\infty} E_k| \leq \epsilon \times 2^{-k}$
$\implies |I|_e \leq \sum_{n=1}^{\infty} \epsilon 2^{-n} = \epsilon \forall \epsilon > 0$

$\implies |I|_e = 0$, i.e. $\liminf_{n\to\infty} E_n$ has measure zero.

# EX5.

first, consider the situation that $E_1, E_2$ are disjoint.
$|E_1 \cup E_2| \leq |E_1| + |E_2|$ is obvious, then we will prove the part $|E_1 \cup E_2| \geq |E_1| + |E_2|$

By EX7-ii-1, $\exists A \subseteq E_1, B \subseteq E_2$ s.t. $A, B$ are closed and $|A|_e \geq |E_1| - \epsilon, |B|_e \geq |E_2| - \epsilon$

$A, B$ closed and dosjoint $\implies d(A, B) > 0$

Therefore, $|E_1 \cup E_2| \geq |A \cup B| = |A| + |B| \geq |E_1| + |E_2| - 2\epsilon$

$\implies |E_1 \cup E_2| \geq |E_1| + |E_2| \implies |E_1 \cup E_2| = |E_1| + |E_2|$

For $E_1 \cap E_2 \neq \emptyset$, because $E_2 - E_1, E_1 \cap E_2$ are measurable by proposition of measurability 2 and 3, therefore

- $|E_1 \cup E_2| = |E_1 \cup (E_2 - E_1)| = |E_1| + |E_2 - E_1|$
- $|E_2| = |(E_2 - E_1) \cup (E_2 \cap E_1)| = |E_2 - E_1| + |E_2 \cap E_1|$

$\implies |E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2 - E_1| + |E_2| - |E_2 - E_1| = |E_1| + |E_2|$

# EX6.

fix $\epsilon > 0$
Let $G_1, G_2$ are open sets covering $E_1, E_2$ respectively, $|G_1 - E_1|_e < \epsilon, |G_2 - E_2|_e < \epsilon$

Let $I^1 = \{I^1_k\}, I^2 = \{I^2_k\}$ be collection of intervals which covering $G_1 - E_1, G_2 - E_2$ respectively with $\sum_{A \in I^1} |A| < \epsilon, \sum_{B \in I^2} |B| < \epsilon$

Take $G = G_1 \times G_2$, the direct product of open sets is open, and $I^1 \times I^2 = \{A \times B : A \in I^1, B \in I^2\}$ can cover $G - E_1 \times E_2$

Therefore, we have
$|G - E_1 \times E_2|_e \leq |I^1 \times I^2| = \sum_{A \in I^1, B \in I^2} |A \times B| = \sum_{A \in I^1} |A| \times \sum_{B \in I^2} |B| \leq \epsilon^2$

hence, we can always find $G \supseteq E_1 \times E_2$ s.t. $|G - E_1 \times E_2|_e < \epsilon^2 \forall \epsilon^2 > 0$, by definition $E_1 \times E_2$ is measurable.

For the case $|E_1| = \infty, |E_2| = \infty$, the equality is trivial.

For the cases $|E_1|, |E_2| < \infty$,
Let $G_1 = \cup_{k=1}^{\infty} U_k, G_2 = \cup_{k=1} V_k$, where each $U_k, V_k$ are closed intervals, then $G = \{U_i \times V_j : i, j \in \mathbb{N}\}$, and each $U_i, V_j$ are disjoint except the edge.

Because the edge of the rectangle has measure zero, therefore
$|G| = \sum_{i,j \in \mathbb{N}} |U_i| \times |V_j| \overset{\text{both positive and converge}}{=} (\sum_{i=1}^{\infty} |U_i|)(\sum_{j=1}^{\infty} |V_j|) = |G_1| \times |G_2|$

$|E_1| \times |E_2| \leq |G_1| \times |G_2| \leq |E_1| \times |E_2| + (|E_1| + |E_2|)\epsilon + \epsilon^2$

$\implies |G| - |G - E_1 \times E_2| \leq |E_1 \times E_2| \leq |G|$

$\implies |E_1| \times |E_2| - \epsilon^2 \leq |E_1 \times E_2| \leq |E_1| \times |E_2| + \epsilon(\epsilon + 2|E_1| + 2|E_2|) \forall \epsilon > 0$

$\implies |E_1 \times E_2| = |E_1| \times |E_2|$

For the case $|E_1| = 0, |E_2| = \infty$, fix $\epsilon > 0$, Let $I^n = \{I_k^n\}_{k=1}^{\infty}$ be a collection of intervals which covering $E_1$ and $\sum_{k=1}^{\infty} |I_k^n| < \frac{\epsilon}{n^2}$

Besides, Let $J = \{J_k\}_{k=1}^{\infty}$ with $J_k = [-k-1, -k] \cup [k, k+1]$, then $J$ will covering $\mathbb{R} \supseteq E_2$

Let $I = \{I_k^n \times J_n | n \in \mathbb{N}, k \in \mathbb{N}\}$, then $I$ will be a collection of rectangles, covering $E_1 \times E_2$, and therefore
$|E_1 \times E_2| \leq \sum_{A \in I} |A| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |I_k^n \times J_n| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |I_k^n| \leq \sum_{n=1}^{\infty} \frac{\epsilon}{n^2} = \epsilon \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2\epsilon$ for any $\epsilon > 0$

Therefore, $|E_1 \times E_2| = 0$

# EX7.

## i.

$\forall \epsilon > 0$, choose $H \in G_\delta$ s.t. $H \supset E$ and $|E|_e = |H|_e$, from definition $F \subset H \forall F \subset E$
$\implies |F| < |H| \forall F \subset E$
$\implies \sup_{F: F \subset E, F\ closed} |F| \leq |H|_e = |E|_e$

## ii.

We separate the proof into two parts,

### ii-1

$E$ measurable $\implies$ $E^c$ measurable $\implies \forall \epsilon > 0 \exists G \supseteq E^c$ s.t. $|G - E^c|_e < \epsilon$
$G$ open $\implies$ $G^c$ closed.
besides, $|E - G^c|_e = |E \cap (G^c)^c|_e = |G \cap (E^c)^c| = |G - E|_e < \epsilon$, $G^c \subset (E^c)^c = E$
$\implies$ $G^c$ closed, contained in $E$ and $|G^c|_e \geq |E| - |E - G^c|_e \geq |E| - \epsilon$

$\implies \sup_{A \subseteq E, A\ closed} |A|_e = |E|$

**ii-2**

(Ex5 is depend on ii-1, but ii-1 does not depend on ii-2, so it would not cause the circular justification)

Fix $\epsilon > 0$

$|E|_i = |E|_e = v \implies \exists F \subseteq E$ closed and $G \supseteq E, G \in G_\delta$ open s.t. $|F| > v - \epsilon, |G|_e < v + \epsilon$

$\implies |E - F| \geq |E| - |F| \leq \epsilon, |G - E| \geq$

$F, G$ measurable $\implies G - F$ measurable and $F, G - F$ disjoint

$\implies |G - F|_e = |G|_e - |F|_e \leq v + \epsilon - (v - \epsilon) = 2\epsilon$

$G - E \subset G - F \implies |G - E|_e > |G - F|_e = 2\epsilon$

$\implies E$ is measurable