EX1.

From HW3 we know F make an bijection between Cantor set C and $[0,1]$

Let $f_C(x) : [0,1] \to C, f(x) = F^{-1}(x) \in C$, and $f(x) = \begin{cases} f_C(x) & x \in [0,1] \\ 2 & else \end{cases}$

from the monotone proposition, $\{f < a\}$ is an interval within $[0,1]$ for $a \in [0,2)$, and $\{f < a\} = \mathbb{R}$ for $a \geq 2$ and therefore f is measurable.

Let $E \subseteq C$ s.t. $F(E)$ is not measurable and define $\phi(x) = \chi_E(x)$, note that E is measure zero.
$\{\phi < a\} = \mathbb{R} - E$ for $0 \leq a < 1$, $\{\phi < a\}\mathbb{R}$ for $a \geq 1$, therefore ϕ is also measurable.

however, we have

$\phi \circ f(x) = \begin{cases} 1 & f(x) \in E \\ 0 & else \end{cases}$

note that $f(x) \in E \iff x \in F(E)$
$\implies \{\phi \circ f = 1\} = F(E)$ is not measurable, therefore $\phi \circ f$ is not measurable.

EX2.

let $E_1 = \{x : f(x) > a\}, E_2 = \{x : f(Tx) > a\}$
E_1 is measurable by definition, for E_2, we have
$x \in E_2 \iff f(Tx) > a \iff Tx \in E_1 \iff x = T^{-1}y$ for some $y \in E_1$
$\iff x \in T^{-1}(E_1)$

Therefore, $E_2 = T^{-1}E_1$, and by HW3, E_2 is measurable with $|E_2| = |\det(T^{-1})| \times |E_1|$

$\implies \{fT < a\}$ measurable $\forall a \in \mathbb{R}$

$\implies fT$ is measurable.

EX3.

a.

$\limsup_{x \to x_0} f(x) \leq f(x_0)$ and $\limsup_{x \to x_0} g(x) \leq g(x_0)$
$\implies \limsup_{x \to x_0}(f+g)(x) = \limsup_{x \to x_0} f(x) + \limsup_{x \to x_0} g(x) \leq f(x_0) + g(x_0) = (f+g)(x_0)$
$\implies f+g$ is USC at x_0

however, $f-g$ and fg is not necessary continuous:
consider $f(x) = \begin{cases} 0 & x=0 \\ -1 & else \end{cases}, g(x) = \begin{cases} 0 & x=0 \\ -2 & else \end{cases}$
both have $\limsup_{x \to 0} f(x) \leq f(0)$ and $\limsup_{x \to 0} g(x) \leq g(0)$. however, $\limsup_{x \to 0}(f-g) = 1 > (f-g)(0) = 0$, similarly, $\limsup_{x \to 0}(fg)(0) = 2 > (fg)(0) = 0$

to prove the LSC case, we only need to take $f^L(x) = -f(x)$, and the $f-g$ and fg case we take $f_L = -f, g_L = -g$

b.

Let $f(x) = \inf_k f(x)$
$\lim_{x \to x_0} f_k(x) \leq f_k(x_0) \forall k$
$\implies \lim_{x \to x_0} f_k(x) \leq f_k(x_0) \leq f(x_0) \forall k$
$\implies \inf_k \lim_{x \to x_0} f_k(x) = \lim_{x \to x_0} f(x) \leq f(x_0)$

c.

for any $\epsilon > 0$,
$\exists k$ s.t. $|f_k - f|_{\sup} < \epsilon$

besides, $\limsup_{x \to x_0} f_k(x) \geq f_k(x_0)$
$\implies \sup_{|x-x_0|<\delta} f_k(x) > f_k(x) - \epsilon \forall \delta > 0$

$\implies \sup_{|x-x_0|<\delta} f(x) \geq \sup_{|x-x_0|<\delta}[f(x) - f_k(x) + f_k(x)]$
$\geq -|f - f_k|_{\sup} + \sup_{|x-x_0|<\delta} f_k(x) \geq -\epsilon + f_k(x_0) \geq -\epsilon + f(x_0) - \epsilon = f(x_0) - 2\epsilon \forall \delta > 0$

take $\delta \to 0$

$\implies \limsup_{x\to x_0} f(x) \geq f(x_0) - 2\epsilon \forall \epsilon > 0$

$\implies \limsup_{x\to x_0} f(x) \geq f(x_0)$, f is USC

EX4.

a.

if $< f_k >$ is a sequence of decreasing function, then $\inf_k f_k(x) = \lim_{k\to\infty} f_k(x)$
by EX3-b, $f = \lim_{k\to\infty} f_k$ is USC at x_0
Besides, the sequence of continuous function at x_0 is also a sequence of USC at x_0 and the limit therefore USC at x_0

b.

(ref. https://math.stackexchange.com/questions/4740040/upper-semi-continuity-approximation-continuous-functions)

define $f_k(x) = \sup_{y\in[a,b]}\{f(y) - k|x-y|\}$

$|f_k(x_1) - f_k(x_2)| = \left|(\sup_{y_1\in[a,b]} f(y_1) - k|y_1 - x_1|) - (\sup_{y_2} f(y_2) - k|y_2 - x_2|)\right|$
$\leq \sup_{y_1,y_2\in[a,b]} |f(y_1) - f(y_2) - k[|y_1 - x_1| - |y_2 - x_2|]|$
$\leq \sup_{y_2=y_1\in[a,b]} |f(y_1) - f(y_2) - k[|y_1 - x_1| - |y_2 - x_2|]|$
$= k\,||y - x_1| - |y - x_2|| \leq k|x_1 - x_2|$

$\implies f_k$ is Lipchitz and therfore continuous.

besides, $f_k(x) - f_{k+1}(x)$
$= \sup_{y_1} f(y_1) - k|y_1 - x| - \sup_{y_2}(f(y_2) - (k+1)|x - y_2|)$
assume $f_{k+1}(x) = f(y) - (k+1)|x-y|$, then
$f_k(x) - f_{k+1}(x) \geq f(y) - k|y-x| - [f(y) - (k+1)|y-x|] = |y-x| \geq 0$

$\implies f_k$ is decreasing sequence.

Finally, we have $\lim_{k\to\infty} f_k(x) = f(x)$:

- first, from the proposition of compact set, assume $f_k(x) = f(y_k) - k|x - y_k|$

(it there're multiple, pick one arbitrary), then from $\limsup_{k\to\infty} k|x-y_k|$ finite we can find $y_k \to x$ as $k \to \infty$.
besides,

- second, assume $M = \limsup_{k\to\infty} k|x-y_k|$, we claim $M = 0$
  - if not, then $\forall K\ \exists k > K$ s.t. $k|x-y_k| \geq M > 0$
    $\implies$ for any large K, we can find $k \geq K$ s.t. $f_k(x) = f(y_k) - k|x-y_k| \geq f(y_k) - M/2 \geq f(x)$
    however, y_k can arbitrary close to x, and therefore $f(y_k) < f(x) + M/4$ for y_k sufficiently close to x, this make a contradiction

$\implies \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} f(y_k) + k|y_k - x| = \lim_{k\to\infty} f(y_k)$, and from $f_k(x) \geq f(x)$ and $f(x) \geq \limsup_{y\to x} f(y)$ we have $\lim_{k\to\infty} f_k(x) = f(x)$

EX5.

Let V be the Vitali set in $[0,1]$, and $f(x) = \begin{cases} x & x \in V \\ (x+2)^2 & x \notin V \end{cases}$

Then obviously, $\{f = c\}$ have at most 2 elements, therefore is measure zero.
However, $\{0 \leq f \leq 1\} = V$ is not measurable, therefore f is not measurable.