EX1.

we will directly prove the case that y-dimension is the increasing function $f(h)$:

Let H be the set of all heights, and the set $K = \{Rect(h, f(h)) : h \in H\}$ cover E, where $Rect(a, b)$ represent the rectangle of width a and height b(omit the axis)

let $h_1^* = \sup H, f_1^* = \sup_{h \in H} f(h)$, and pick $h_1 \in H$ s.t. $f(h_1) \geq f_1^*$ and $h_1 \geq h_1^*$, which can be chosen by following process

- pick $h_1^x \in H$ s.t. $h_1^x \geq \frac{1}{2} h_1^*$
- pick $h_1^y \in H$ s.t. $f(h_1^y) \geq \frac{1}{2} f_1^*$
- let $h_1 = \max(h_1^x, h_1^y)$

let $Q_1 = Rect(h_1, f(h_1))$

split K into K_1, K_1', where all cube within K_1 are disjoint with Q_1, and all cube in K_1' are intersect with Q_1.

We claim that K_1' is covered by $5Q_1$:

- WLOG suppose the origin located at the center of Q_1, i.e. the upper right corner of Q_1 is $(\frac{h_1}{2}, \frac{1}{2} f(h_1))$
  For any $Rect(h', f(h')), h' \neq h_1$, we have $h' \leq 2h_1$ and $f(h') \leq f_1^* \leq 2f(h_1)$
  $\implies$ if $Rect(h', f(h'))$ is intersect with Q_1,the right upper corner of $Rect(h', f(h'))$ is at the lower-left of $(\frac{5}{2} h_1, \frac{5}{2} f(h_1)) \implies Rect(h', f(h')) \subseteq Q_1$

Therefore, the set K_1, K_1' have the same property as in Lemma 7.4, therefore we can prove Lemma 7.4 is still hold with $\beta = 5^{-2}$

EX2.

by the property of Riemann-Stieltjes integral, $\int_a^b \phi df = \int_a^b \phi d(g + h) = \int_a^b \phi dg + \int_a^b dh$

g is abs. continuous $\implies$ g is B.V. $\implies$ $\int_a^b \phi dg$ is exist, and by thm 7.32 is equal to $\int_a^b \phi g' dx$

$\implies \int_a^b \phi df = \int_a^b \phi dh + \int_a^b \phi g' dx$

EX3.

The $\rightarrow$ part is trivial, so we only prove the converse part

fix $\epsilon > 0$
Let $\delta > 0$ s.t. for any finite collection of non-overlapping interval $\{[a_i, b_i]\}_{i=1}^N$ with $\sum_{k=1}^N |b_k - a_k| < \delta$, the sum
$\sum_{j=1}^N |f(b_j) - f(a_j)| < \epsilon$

For any countable collections of nonoverlapping intervals $\{[A_j, B_j]\}$, we have

- if $\sum_{j=1}^\infty |B_j - A_j| < \delta$, then $\sum_{j=1}^N |B_j - A_j| < \delta \forall N \in \mathbb{N}$
  $\implies \sum_{j=1}^N |f(B_j) - f(A_j)| < \epsilon \forall N \in \mathbb{N}$
  $\implies \lim_{N\to\infty} \sum_{j=1}^N |f(B_j) - f(A_j)| \le \epsilon < 2\epsilon$

$\implies \sum_{j=1}^\infty \sum_{j=1}^N |f(B_j) - f(A_j)| < 2\epsilon$ if $\sum_{j=1}^\infty |B_j - A_j| < \delta$
$\implies f$ is abs. cont.

EX4.

First, we prove the case that $|\cup_{B\in\mathcal{F}} B|_e < \infty$:

Let $K_1 = \mathcal{F}$,
take $R_1^* = \sup\{rad(B) : B \in K_1\}$, choose B_1 from $\mathcal{F}$ s.t. $rad(B_1) > \frac{1}{2} R_1^*$

split $K_1 = K_2 \cup K_2'$ s.t. all elements in K_2 disjoint with B_1 and all elements in K_2' has nonempty intersection with B_1

if $B' \in K_2'$, then $B' \subseteq 5B_1$:

- WLOG suppose B_1 centered at origin
- suppose B' centered at c, radius $R' < 2rad(B_1)$, i.e. $|c - 0| \le R' + rad(B_1)$

$x \in B' \iff |x-c| < R' \implies |x| \leq |x-c| + |c| \leq R' + R' + rad(B_1) < 5rad(B_1) \implies x \in 5B_1$

Therefore, $\cup_{B' \in K_2'} B' \subseteq 5B_1$.

continue this process, we obtain $B_2, B_3, ...$, $K_2, K_3, ...$ and $K_2', K_3', ...$, because the finiteness of $|\cup_{B \in \mathcal{F}} B|$, $rad(B_k) \to 0$ as $k \to \infty$, and therefore $5B_1 \cup 5B_2 \cup ...$ will cover the entire $\mathcal{F}$, and obviously the propositions hold

if $|\cup_{B \in \mathcal{F}} B| = \infty$, first only consider balls within $B_k(0)$ and the requests hold, then take $k \to \infty$ we can obtain the same conclusion.

EX5.

fix $\epsilon > 0$
V is abs. cont. $\implies \exists \delta > 0$ s.t.
$\forall \{[a_j, b_j]\}$ with $\sum_j b_i - a_j < \delta$, we have $\sum_j |V(b_j) - V(a_j)| < \epsilon$

by the proposition of V, $|V(b_j) - V(a_j)| \geq |f(b_j) - f(a_j)|$

$\implies \sum_j |f(b_j) - f(a_j)| < \epsilon$

$\implies f$ is abs. cont.