EX1.

fix $\epsilon > 0$.

f is a.c.
$\implies \exists\delta > 0$ s.t. for any $\{(a_k, b_k)\}$ with $\sum_k |b_k - a_k| < \delta$, we have $\sum_k |f(b_k) - f(a_k)| < \epsilon$

$|Z| = 0 \implies \exists I = \{[\ell_i, r_i]\}_{i=1}^{\infty}$ cover Z with $\sum_i (r_i - \ell_i) < \delta$

Let $m_i, M_i \in [\ell_i, r_i]$ s.t.

- $f(m_i) = \inf_{[\ell_i, r_i]} f$
- $f(M_i) = \sup_{[\ell_i, r_i]} f$

both exists because $[\ell_i, r_i]$ closed.

then $f([\ell_i, r_i]) \subseteq [f(m_i), f(M_i)]$

$\implies f(Z) \subseteq f(\cup_i [\ell_i, r_i]) \subseteq \cup_i [f(m_i), f(M_i)]$

Besides, $|\cup_i [f(m_i), f(M_i)]| \leq \sum_i (f(M_i) - f(m_i)) < \epsilon$ from $\sum_i |M_i - m_i| \leq \sum_i (r_i - \ell_i) \leq \delta$

$\implies |f(Z)| \leq \epsilon \forall \epsilon > 0$

$\implies |f(Z)| = 0$

EX2.

first, assume f is the characteristic function of an interval $[\ell, r] = [g(u), g(v)] \subseteq [a, b]$, then

$$LHS = r - \ell = g(v) - g(u) = \int_u^v g'(t)dt = \int_\alpha^\beta f(g(t))g'(t)dt \\ = RHS$$

second, if $f = \chi_G$ with some opened G, because G is the countable union of non-overlapping intervals, therefore the equation still hold.---(2)

next, we assume f_k be a sequence of simple measurable functions uniformly apprimate to f, then from (2), each simple function is the linear combination of χ_G, and therefore it hold for any f_k, and then for f

EX3.

ϕ is convex
$\iff \forall u < v$ in (a, b) we have
$\phi(\frac{a+b}{2}) \leq \frac{\phi(u)+\phi(v)}{2}$

$\iff \int_a^{\frac{u+v}{2}} fdx \leq \frac{1}{2}[\int_a^u fdx + \int_a^v fdx]$

$\iff 0 \leq \int_a^u fdx - 2\int_a^{\frac{u+v}{2}} fdx + \int_a^v fdx$
$\iff 0 \leq \int_{\frac{u+v}{2}}^v fdx - \int_u^{\frac{u+v}{2}} fdx$
$\iff 0 \leq \int_0^{\frac{v-u}{2}} f(t + \frac{u+v}{2}) - f(t+u)dx$

f monotone increasing $\implies f(t + \frac{u+v}{2}) - f(t+u) \geq 0 \implies$ the inequality always hold

$\implies \phi$ is convex

EX4.

Let f be the Cantor-Lebesgue function on $[0, 1]$ and 0 elsewhere, and $g(x) = 0$, then

- f is continuous and monotone increasing, therefore f is of bounded variation
- $g \in C^\infty$

However,
$\int_{-\infty}^{\infty} f'gdx = \int_{-\infty}^{\infty} 0xdx = 0$

$\int_{-\infty}^{\infty} fg'dx = \int_{-\infty}^{\infty} f(x)dx > 0$

EX5.

Consider the case $f = 0$

- if not, then let $g = 0, g_k = f_k - f$ and perform the same discussion on g_k

Let $F_k(E) = \int_E f_k dt$ be the set functions

fix $\epsilon > 0$

$F_k(x)$ are u.a.c. $\implies \exists\delta$ s.t. $\forall F \subseteq [0,1]$ with $|F| < \delta$, we have $|F_k(F)| = |\int_F f_k(x)dx| < \epsilon \forall k$

By the lemma of Egorov theorem, $\exists E \subseteq [0,1]$ s.t. $|[0,1] - E| < \delta$ and $f_k \to f$ uniformly on E

$\implies \int_0^1 |f - f_k|dx = |F_k([0,1])| \leq |F_k(E)| + |F_k([0,1] - E)| \leq \int_E |f_k|dx + \epsilon \to 0 + \epsilon$ as $k \to \infty$ by the proposition of uniform convergence.

Now assume $\int_0^1 |f_k|dx \to 0$, then $\exists N$ s.t. $\int_0^1 |f_k|dx < \epsilon/2 \forall k \geq N$

Besides, foreach $k < N$ we can find δ_k s.t. $\int_E |f|dx < \epsilon/2$ for any $|E| < \delta$

Let $\delta = \min\{\delta_1, ..., \delta_{N-1}\}$, then no matter $k \geq N$ or not, we have $\int_E |f_k|dx < \epsilon$ for $|E| < \delta$

$\implies \{f_k\}$ are u.a.c