

EX1.

note that $M_0(x) \geq 0 \forall x \in F$

$$M_0(x) < \infty \text{ a.e. in } F \iff \int_F M_0(x) dx < \infty$$

$[\log \frac{1}{\delta(y)}]^{-1} |x - y|^{-1} \geq 0$ and measurable

$$\implies \int_F \int_a^b [\log \frac{1}{\delta(y)}]^{-1} |x - y|^{-1} dy dx = \int_a^b \int_F [\log \frac{1}{\delta(y)}]^{-1} |x - y|^{-1} dx dy$$

$$\begin{aligned} \int_a^b \int_F [\log \frac{1}{\delta(y)}]^{-1} |x - y|^{-1} dx dy &= \int_a^b [\log \frac{1}{\delta(y)}]^{-1} \int_F |x - y|^{-1} dx dy \\ &\leq \int_a^b [\log \frac{1}{\delta(y)}]^{-1} \int_{\delta(y) \leq |x-y| \leq 1} \frac{1}{|x-y|} dx dy \\ &= \int_a^b [\log \frac{1}{\delta(y)}]^{-1} \times 2 \int_{z=\delta(y)}^1 \frac{1}{z} dz dy = \int_a^b [\log \frac{1}{\delta(y)}]^{-1} \times 2 [\log 1 - \log \delta(y)] dy = \\ &\int_a^b 2 dy \leq 2 \end{aligned}$$

$$\implies \int_F M_0 dx \text{ is finite, therefore } M_0 \text{ is finite a.e.}$$

EX2.

$$x \notin F \xrightarrow{F^c \text{ open}} \exists r > 0 \text{ s.t. } I_x = [x - r, x + r] \in F^c$$

$$\implies \Delta = \min\{\delta(x - r), \delta(x + r)\} > 0, \text{ nad obviously } \forall x' \in I_x, \delta(x') \geq \Delta$$

$$\implies M_\lambda(x, F) \geq \int_{I_x} \frac{\Delta^\lambda}{|x-y|^{1+\lambda}} dy \geq 2\Delta^\lambda \int_0^r \frac{1}{z^{1+\lambda}} dz = \infty$$

EX3.

WLOG suppose $f \geq 0$ for convinience, otherwise letting $\hat{f} = |f|$ and the proof can be still finish.

suppose $|\{f > 0\}| > 0$, letting $E_m = \{x : f(x) > \frac{1}{m}\}$, then from $E_m \rightarrow \{f > 0\}$ as $m \rightarrow \infty$ we know that $\exists M$ s.t. $|E_M| > 0$, calling $E = E_M$, and let $r = \sup[\{|e| : e \in E\} \cup \{1\}]$

for E , we have

- $\int_E f dx \geq \frac{|E|}{M} > 0$
- $E \subseteq \{y : |y| \leq r\}, r \geq 1$

for $|x| \geq 1$, letting $Q_x = \{y : |y - x| \leq 2r|x|\}$ is a cube with edges parallel to the coordinate axis and centered at x , therefore

$$f^*(x) \geq \frac{1}{|Q_x|} \int_{Q_x} f dx$$

besides, $y \in E \implies |y| \leq r \implies |y - x| \leq |x| + |y| \leq |x|r + r|x| = 2r|x| \implies y \in Q_x$

$$\implies \int_{Q_x} f dx \geq \frac{|E|}{M}$$

$$\implies f^*(x) \geq \frac{1}{|Q_x|} \frac{|E|}{M} \geq \frac{E}{2r^n M} \frac{1}{|x|^n} \forall |x| \geq 1$$

$$\implies \exists c = \frac{1}{|Q_x|} \frac{|E|}{M} \geq \frac{E}{2r^n M} \text{ s.t. } f^*(x) \geq c \frac{1}{|x|^n}$$

EX4.

Let $M = \sup \phi$

use \otimes to represent convolution

if $|x| > \epsilon$, then $\phi_\epsilon(x) = \underbrace{\epsilon^{-n} \phi(x/\epsilon)}_{\geq 1} = 0$

Let $B_\epsilon = B_\epsilon(0)$, we have

$$f \otimes g(x) - f(x) = \int (f(x - y) - f(x)) \phi_\epsilon(y) dy = \int_{B_\epsilon} (f(x - y) - f(x)) \phi_\epsilon(y) dy$$

$$\implies |f \otimes g(x)| \leq \int_{B_\epsilon} |(f(x - y) - f(x))| \times |\phi_\epsilon(y)| dy \stackrel{\text{symmetry}}{=} \int_{B_\epsilon} |(f(x + y) - f(x))| \times |\phi_\epsilon(y)| dy = M \int_{B_\epsilon + x} |(f(z) - f(x))| dz$$

$B_\epsilon + x \subseteq B_{2\epsilon}(x)$ is shrink reglarly to x

$$\implies M \int_{B_\epsilon + x} |(f(z) - f(x))| dz \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

$$\implies f \otimes \phi_\epsilon(x) \rightarrow f(x) \text{ as } \epsilon \rightarrow 0$$

EX5.

Choose $A \subseteq E_1, B \subseteq E_2$ s.t. $|A| < \infty, |B| < \infty$,

the claim hold if $\exists t$ s.t. $|A \cap B + t| > 0$

$$\text{let } \mu(t) = |A \cap B + t| = \int_{\mathbb{R}} \chi_A(x) \chi_{B+t}(x) dx = \int_{\mathbb{R}} \chi_A(x) \chi_B(x - t) dx \geq 0$$

$\int_{\mathbb{R}} \mu(t) dt = \int_{\mathbb{R}} \chi_A dx \times \int_{\mathbb{R}} \chi_B dx = |A| \times |B| > 0 \implies \mu(t)$ take positive at some points.

$\implies \exists t$ s.t. $|A \cap B + t| > 0 \implies \{a - \hat{b} : a \in A, \hat{b} \in B + t\}$ contain an interval
 $\implies \{x_1 - x_2 : x_1 \in E_1, x_2 \in E_2\} \supseteq \{a - b : a \in A, b \in B\} = \{a - \hat{b} : a \in A, \hat{b} \in B + t\} + t$ contain an interval.