EX1.

choose
$\chi_{K_1} = [0, \frac{1}{2}]$
$\chi_{K_2} = [0, \frac{1}{4}]$
...
$\chi_{K_n} = [0, 2^{-n}]$
...

Then $f_{\chi_{K_j}} \to \begin{cases} 0 & x \in (0,1] \\ 1 & x = 0 \end{cases}$

which equal to 0 except except $\{0\}$, which is a measure zero set.

EX2.

if $\limsup_{y \to x} g(y) = g(x) - d < g(x)$ (i.e. $d > 0$), then
$\exists \delta > 0$ s.t. $\sup_{y \in B_\delta(x) \setminus \{x\}} g(y) < g(x) - d/2$, and we can choose $\delta < r$
However, $\sup_{y \in B_\delta(x) \setminus \{x\}} g(y) = \sup_{y \in B_\delta(x) \setminus \{x\}} \sup_{y' \in B_r(y)} f(y')$
$= \sup_{y' \in \cup_{y \in B_\delta(x) \setminus \{x\}} B_r(y)} f(y') < g(x)$

- $B_r(x) \subset \cup_{y \in B_\delta(x) \setminus \{x\}} B_r(y)$

$\implies \sup_{y' \in \cup_{y \in B_\delta(x) \setminus \{x\}} B_r(y)} f(y') \geq \sup_{B_r(x)} f = g(x)$, which make a contradiction

therefore, $g(x)$ is LSC

Besides, we can notice the formula $B_r(x) \subset \cup_{y \in B_\delta(x) \setminus \{x\}} B_r(y)$ also hold for closed balls, therefore the proposition still hold if we replace the open balls by closed.

for the case of $h(x)$, we take $g(x) = -h(x)$, then $g(x) = \inf_{B_r(x)} -f$ and therefore LSC $\implies h = -g$ is USC

EX3.

First, we consider the case E is bounded.

define $g_k(x) = \sup_{1 \le j \le k} f_j(x)$ and $g(x) = \sup_{k \in \mathbb{N}} f(x)$, then $g_k \to g$ pointwisely, and $g(x) \le M_x \forall x$ and thererfore finite.

g_k measurable and finite, therefore by Lusin's theorem $\exists F_k \subseteq E$ is closed and g_k continuous at F_k and $|E - F_k| < \epsilon \times 2^{-k-1}$
Let $F_C = \cap_{k=1}^{\infty} F_k$, then which is closed with $|E - F_C| = |\cup_k (E - F_k)| \le \sum_k |E - F_k| \le \epsilon/2$, and all g_k continuous at F_C

Besides, by Egorov theorem $\exists F_U \subseteq E$ closed, $|E - F| < \epsilon/2$ and $g_k \to g$ uniformly on F_U

Take $F = F_C \cap F_U$, then F will be compact, $|E - F| \le |E - F_C| + |E - F_U| \le \epsilon$ and g_k cont. at F, $g_k \to g$ uniformly.

$\implies g$ is a countinuous function at F
Take $M = \sup_F |g|$, because F is compact and g is continuous at F, $M < \infty$

Therefore, we can find F closed with $|E - F| < \epsilon$, $M < \infty$ s.t. $f_k(x) \le g(x) \le M \forall x \in F$

for the case E unbounded, take $E_N = B_N(0)$, we can make $|E - E_N| < \epsilon/2$ as long as N sufficiently large.

Because E_N finite, we can find $M > 0, F \subseteq E_N$ closed with $|E_N - F| < \epsilon/2$ and $f_k(x) < M \forall x \in F$
Besides, $|E - F| \le |E - E_N| + |E_N - F| \le \epsilon$, therefore such M, F satisfy the request.

EX4.

$f + g$ case

$\lim_{k \to \infty} |\{x : |(f+g)(x) - (f_k + g_k)(x)| > \epsilon\}|$

$\le \lim_{k \to \infty} |\{x : |(f - f_k)(x) + (g - g_k)(x)| > \epsilon\}|$

$\le \lim_{k \to \infty} |\{x : |(f - f_k)(x)| > \epsilon\} \cup \{x : |(g - g_k)(x)| > \epsilon\}|$
$\le \lim_{k \to \infty} |\{x : |(f - f_k)(x)| > \epsilon\}| + |\{x : |(g - g_k)(x)| > \epsilon\}|$

$= 0 + 0 = 0$

$f \times g$ case

fix $\eta > 0$

$|\{|fg - f_k g_k| > \epsilon\}| \leq |\{|f| \times |g - g_k| > \epsilon/2\}| + |\{|g_k| \times |f - f_k| > \epsilon/2\}|$

for any $N > 0$, we have $\{|f| \times |g - g_k| > \epsilon/2\} = \{x : |f(x)| > N, |f| \times |g - g_k| \geq \epsilon/4\} \cup \{x : |f(x)| \leq N, |f| \times |g - g_k| \geq \epsilon/4\}$

$\subseteq \{x : |f(x)| > N, |f| \times |g - g_k| \geq \epsilon/4\} \cup \{x : |f| \times |g - g_k| \geq \epsilon/(4N)\}$

from $|E| < \infty$, we can pick sufficiently large N s.t. $|\{x : |f(x)| > N\}| < \eta/2$

besides, when N fixed, we can choose k sufficiently large s.t. $|\{x : |g - g_k| \geq \epsilon/(4N)\}| < \eta/2$

$\implies \{|f| \times |g - g_k| > \epsilon/2\} \leq |\{x : |f(x)| > N, |f| \times |g - g_k| \geq \epsilon/4\}| + |\{x : |f(x)| \leq N, |f| \times |g - g_k| \geq \epsilon/4\}| \leq \eta$ as k sufficently large

replace the role of f, g_k, we can also prove $|\{|g_k| \times |f - f_k| > \epsilon/2\}| < \eta$ as k sufficeintly large

$\implies |\{x : |f(x)g(x) - f_k(x)g_k(x)| > \epsilon\}| \to 0$ as $k \to \infty$, therefore $f_k g_k \to fg$ in measure as $|E| < \infty$

f/g case

Because we have prove the $f \times g$ case, it is sufficient to show $\frac{1}{g_k} \xrightarrow{m} \frac{1}{g}$

$g \neq 0$ a.e. $\implies |\{|g| < \frac{1}{k}\}| \to 0$ as $k \to \infty$

fix $\eta > 0$
$\{|\frac{1}{g} - \frac{1}{g_k}| > \epsilon\} = \{|\frac{1}{g} - \frac{1}{g_k}| > \epsilon, |g| < \frac{1}{N}\} + \{|\frac{1}{g} - \frac{1}{g_k}| > \epsilon, |g| > \frac{1}{N}\}$
$\subseteq \{|g| \leq \frac{1}{N}\} \cup \{|g| \geq \frac{1}{N}, |\frac{g - g_k}{g_k g}| < \epsilon\}$

choose N sufficiently large s.t. $|\{|g| \leq \frac{1}{N}\}| < \eta/2$, and by Egorov theorem choose $F \subseteq E$ is closed, $K \in \mathbb{N}$ s.t.

- $|E-F|<\eta/2$
- $|g(x)-g_k(x)|<\min\{\frac{1}{2N},\frac{\epsilon}{2N^2}\}\forall k\geq K, x\in F$

then for $k\geq K$, we have

- $|\{|g|\leq\frac{1}{N}\}|\leq\eta/2$
- $\{|g|\geq\frac{1}{N},|\frac{g-g_k}{g_k g}|<\epsilon\}\subseteq\{|\frac{g-g_k}{\frac{1}{N}\times\frac{1}{2N}}|<\epsilon\}\subseteq\{|g-g_k|\leq\frac{\epsilon}{2N^2}\}\subseteq E-F$

$\implies |\{|\frac{1}{g}-\frac{1}{g_k}|>\epsilon\}|\leq|\{|g|\leq\frac{1}{N}\}|+|\{|g|\geq\frac{1}{N},|\frac{g-g_k}{g_k g}|<\epsilon\}|$
$\leq\eta+|E-F|\leq\eta$

$\implies \forall\eta>0,\exists K$ s.t. $|\{|\frac{1}{g}-\frac{1}{g_k}|>\epsilon\}|<\epsilon\forall k\geq K$

$\implies \frac{1}{g_k}\xrightarrow{m}\frac{1}{g}$

and by the $f\times g$ case, $\frac{f_k}{g_k}\xrightarrow{m}\frac{f}{g}$

EX5.

fix $a\in E$
if $f_k\to f$, let $A_k=\{x:f_k(x)>a\}$ and $A=\{x:f(x)>a\}$

from the increasing property of f_k, we know that A_k will increase to A, i.e. $A=\limsup_{k\to\infty}A_k$, therefore $f(a)=|A|=\lim_{k\to\infty}|A_k|=\lim_{k\to\infty}f_k(a)$

if $f_k\xrightarrow{m}f$, fix any $\epsilon>0$, we claim
$\omega_f(a+\epsilon)\leq\liminf_{k\to\infty}\omega_{f_k}(a)\leq\limsup_{k\to\infty}\omega_{f_k}(a)\leq\omega_f(a-\epsilon)$

- for any $\eta>0$, exist K and a set F s.t. $|E-F|<\eta$ and $|f_k(x)-f(x)|<\epsilon\forall x\in F,k\geq K$
  therefore, $f_k(x)>a\implies f(x)>a-\epsilon\forall x\in F,k\geq K$
  $\implies \{f_k(x)>a\}\subseteq(E-F)\cup\{f(x)>a-\epsilon\}$
  $\implies \omega_{f_k}(a)\leq|E-F|+\omega_f(a-\epsilon)=\eta+\omega_f(a-\epsilon)\forall k\geq K$
  $\implies \limsup_{k\to\infty}\omega_{f_k}(a)\leq\omega_f(a-\epsilon)$
- with the symmetric step, we can also prove $\omega_f(a+\epsilon)\leq\liminf_{k\to\infty}\omega_{f_k}(a)$

ω_f continuous at $a\iff\lim_{\epsilon\to0}\omega_f(a+\epsilon)=\lim_{\epsilon\to0}\omega_f(a-\epsilon)=\omega_f(a)$

Therefore, at a which ω_f is continuous, $\lim_{\epsilon\to 0}\omega_f(a+\epsilon) = \liminf_{k\to\infty}\omega_k(a) = \limsup_{k\to\infty}\omega_k(a) = \lim_{\epsilon\to 0}\omega_f(a-\epsilon) = \omega_f(a)$

$\implies \lim_{k\to\infty}\omega_{f_k}(a) = \omega_f(a)$