EX1.

(ref. https://math.stackexchange.com/questions/215215/showing-a-function-of-two-variables-is-measurable)

Let $f_n(x,y) = f(x, \frac{k}{n})$, where $\frac{k}{n} \le y < \frac{k+1}{n}$

$\{f_n < a\} = \cup_{k=0}^{n} \{x : f(x, \frac{k}{n}) < a\} \times [\frac{k}{n}, \frac{k+1}{n})$

From the continuity of x, $\{x : f(x, \frac{k}{n}) < a\}$ is measurable, and therefore \cup_{k=0}^n{ x:f(x,\frac kn)< a } \times [\frac kn,\frac{k+1}n)$measurable,$f_n$ is a measurable function.

From the continuity of y, $\lim_{n\to\infty} f_n = f$, and then from Theroem 4.12 f is measurable.

EX2.

Let $C_n = \{a_1 f_1 + a_2 f_2 + ... + a_N f_N | N \in \mathbb{N}, a_k \ge 0, f_k \in U_n \forall k, \sum_{k=1}^{N} a_k = 1\}$ be the minimum convex set contain U_n,
first, we prove any simple function is within C_n:

- first, for any set $E \subseteq [0,1), c \in \mathbb{R}$, we have $c\chi_E \in C_n$:
  - if $c \le 0$, then the solution is trivial
  - if $|E| < \frac{1}{n}$, then obviously $c \times \chi_E \in C_n$
  - if $|E| \ge \frac{1}{n}$, Let $E_k = E \cap [\frac{k-1}{n+1}, \frac{k}{n+1})$, then $E = \cup_{k=1}^{n+1} E_k$ and $|E_k| \le \frac{1}{n+1} < \frac{1}{n}$
    $\implies (n+1)c\chi_{E_k} \in C_n \forall k = 1, 2, ..., n+1$
    $\implies \frac{1}{n+1}[c(n+1)\chi_{E_1}] + \frac{1}{n+1}[c(n+1)\chi_{E_2}] + ... + \frac{1}{n+1}[c(n+1)\chi_{E_{n+1}}] \in C_n$, which is equal to $c\chi_E$
- second, for any disjoint sets $E_1, ..., E_N$ and constants $a_1, ..., a_N$,
  $\sum_{k=1}^{N} a_k \chi_{E_k} = \sum_{k=1}^{N} \frac{1}{N} \underbrace{[a_k N \chi_{E_k}]}_{within\ C_n} \in C_n$

Therefore, all simple functions are within C_n

Now assume f is a measurable functions, then by above proposition we can find a sequence $\{f_1, f_2, ...\}$ s.t. $\lim_{k\to\infty} f_k = 2f$

Besides, by Egorov theorem we can find $F \subseteq [0,1)$ and K s.t. $\sup_F |f_k - 2f| < \frac{1}{2n} \forall k \geq K$ and $|E - F| < \frac{1}{2n}$

$\implies \sup_F |2f - f_{K+1}| < \frac{1}{2n} \implies |\{x : (f - f_K)(x) > \frac{1}{n}\}| \leq |E - F| < \frac{1}{n}$

$\implies 2f(x) - f_K(x) \in U_n \subseteq C_n$, and we know $f_K \in C_N$ by previous proposition

$\implies \frac{1}{2}(2f - f_K) + \frac{1}{2}(f_K) = f \in C_n$

$\implies S \subseteq C_n$ and obviously $C_n \subseteq S$, therefore $S = C_n \forall n$

EX3.

use $\partial_x f$ to represent $\frac{\partial}{\partial x} f$, and Let M be the upper bound of $\partial_x f$ in $[0,1]$

$\frac{d}{dx}\int_0^1 f(x,y)dy = \lim_{\delta\to 0} \frac{\int_0^1 f(x+\delta,y)dy - \int_0^1 f(x,y)dy}{\delta}$
$= \lim_{\delta\to 0}\int_0^1 \frac{f(x+\delta,y)-f(x,y)}{\delta}dy$

Therefore, $|\frac{d}{dx}\int_0^1 f(x,y)dy - \int_0^1 \partial_x f(x,y)dy| = \lim_{\delta\to 0} |\int_0^1 \frac{f(x+\delta,y)-f(x,y)}{\delta} - \partial_x f(x,y)dy| \leq \lim_{\delta\to 0}\int_0^1 |\frac{f(x+\delta,y)-f(x,y)}{\delta} - \partial_x f(x,y)|dy$
$\leq \lim_{k\to\infty}\int_0^1 |\frac{f(x+\frac{1}{k},y)-f(x,y)}{\frac{1}{k}} - \partial_x f(x,y)|dy$

Let $f_k(x,y) = \frac{f(x+\frac{1}{k},y)-f(x,y)}{\frac{1}{k}}$, then $f_k(x,y) \to \partial_x f(x,y)$ as $k\to\infty$
therefore, fix any $\eta > 0$, $\exists F \subseteq [0,1], K$ s.t. $|E - F| < \eta/2$ and $|f_k(X,y) - f(x,y)| \leq \eta/2 \forall k \geq K$

$\implies$ for $k \geq K$, $\int_0^1 |\frac{f(x+\frac{1}{k},y)-f(x,y)}{\frac{1}{k}} - \partial_x f(x,y)|dy = \int_0^1 |f_k(x,y) - \partial_x f(x,y)|dy$
$\leq \int_F |f_k(x,y) - \partial_x f(x,y)|dy + \int_{E-F} |f_k(x,y) - \partial_x f(x,y)|dy$
$\leq \int_F \eta/2 dy + M|E - F| \leq \eta$

EX4.

for any function $F : E \to \mathbb{R}$, define $F^+ = \max(0, F)$ and $F^- = \max(0, -F)$

Let $f = f^+ - f^-$, $f \in L(0,1) \implies \int_0^1 f^+ dx \geq 0$ and $\int_0^1 f^- dx \geq 0$ finite

$\implies \int_0^1 x^k f^+ dx = \int_0^1 (x^k f)^+ dx \leq \int_0^1 f^+ dx$ and $\int_0^1 x^k f^- dx = \int_0^1 (x^k f)^- dx \leq \int_0^1 f^- dx$ finite

$\implies$ By definition $\int_0^1 x^k f dx$ exist and finite, hence $x^k f \in L(0,1)$

Besides, $x^k f \leq |f|$ in $[0,1]$, therefore by Fatou's Lemma we have
$\lim_{k\to\infty} \int_0^1 |x^k f| dx \leq \int_0^1 |\lim_{k\to\infty} x^k f| dx = \int_0^1 0 dx = 0$
$\implies \int_0^1 x^k f dx \to 0$ as $k \to \infty$

EX5.

fix $\epsilon > 0$,

$\exists F \subseteq E$ s.t. $|E - F| < \frac{\epsilon}{3M}$ and $f_k \to f$ uniformly on F.

Therefore, we can choose sufficiently large K s.t. $\sup_F |f_k - f| < \frac{\epsilon}{3M} \forall k \geq K$

$\implies \forall k \geq K$, we have
$|\int_E f_k dx - \int_E f dx| \leq |\int_E f_k dx - \int_F f_k dx| + |\int_F f_k dx - \int_F f dx| + |\int_F f dx - \int_E f dx|$
$\leq |E - F| M + |F| \times \frac{\epsilon}{3M} + |E - F| M < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

$\implies \int_E f_k dx \to \int_E f dx$ as $k \to \infty$ by definition.