EX1.

(ref. https://math.stackexchange.com/questions/2115139/is-lebesgue-outer-measure-continuous-from-above-for-sets-with-finite-lebesgue-ou?utm_source=chatgpt.com)

From Theorem 3.38, we know that the Vitari set V, which form by selecting exactly one element from each equality class of irrational numbres in $[-1,1]$ separately, is nonmeasurable.

Define $\{r_k\}_{k=1}^{\infty}$ be all rational numbres at $[0,1]$, let $V_k = \{v + r_k : v \in V\}$, we can find that each V_k is disjoint

- otherwise, $V_m \cap V_n \neq \emptyset \implies v_1 + r_m = v_2 + r_n$ for some $v_1, v_2 \in V$
  $\implies v_1 - v_2 = r_2 - r_1 \in \mathbb{Q} \implies v_1 = v_2 \implies r_m = r_n, m = n$, contradiction

Besides, it's from definition that $\cup_{k=1}^{\infty} V_k \subseteq [-1,2]$

define $E_k = \cup_{j=k}^{\infty} V_k$, then we can find that

- E_k is decreasing to $E = \emptyset$, which from the proposition that V_k is disjoint.
  ($a \in [0,2] \implies a$ belong to at most one V_k)
- $|E_k|_e \geq |V|_e$
  Because V have nonzero measure, and V_k is just the translate of V, therefore they have the same outer measure
  $\implies |E_k|_e \geq |V_k|_e = |V|_e$
  $\implies \lim_{k\to\infty} |E_k|_e \geq |V|_e > 0 = |E|_e$

EX2.

If $|E| = \infty$, then no matter A is finite or $|A|_e = \infty$, both side of the equation are ∞ and therefore trivial

Following we assume $|E| < \infty$

First, suppose $B_k \subseteq A$ is closed and $|B_k| \geq |A|_i - \frac{1}{k}$, $\lim_{k\to\infty} |B_k| = |A|_i$

$|E| = |B_k| + |E - B_k| \geq |A|_i - \frac{1}{k} + |E - B_k| \geq |A|_i + |E - A|_e - \frac{1}{k} \forall k$

$\implies |E| \geq |A|_i + |E - A|_e$

For the converse part, assume $|E| > |A|_i + |E - A|_e$, i.e.
$|E| - |A|_i - d > |E - A|_e$ for some d

Then there exist $G \supseteq E - A$, G is open, and $|G| \leq |E - A|_e + \epsilon$ for some $0 < \epsilon < d/2$

Take $H = E - G$, we have $|H| = |E| - |E \cap G|$
$\geq |E| - |E - A|_e - \epsilon \geq |E| - (|E| - |A|_i - d) - d/2 = |A|_i + d/2$

Therefore, there's an measurable set $H \subseteq A$ with $|H| > |A|_i$, this cause a contradiction from $|H| \leq \sup_{H': H'\ is\ F_\sigma, H' \subseteq A} |H'| = |A|_i$

$\implies |E| \geq |A|_i + |E - A|_e$, and thus we obtain the equality.

EX3.

(motivated by https://www.facebook.com/groups/120223891488/posts/10161752568386489/)

Let f be the Cantor Lebesgue function, for each x in Cantor set, $x = \sum_{k=1}^{\infty} a_k 3^{-k}$, where $a_k \in \{0, 2\} \forall k$, f will map x to $y = \sum_{k=1}^{\infty} b_k 2^{-k}$,
where $b_k = \begin{cases} 0 & if\ a_k = 0 \\ 1 & if\ a_k = 2 \end{cases}$

Therefore, the image of the Cantor set will be $\{\sum_{k=1}^{\infty} b_k 2^{-k} | b_k \in \{0, 1\}\} = [0, 1]$

From Corollary 3.39, $[0, 1]$ contain a nonmeasurable subset U, and the preimage of U, i.e. $f^{-1}(U)$ is a subset of Cantor set and thus has measure zero.

Therefore, $f^{-1}(U)$ is a measurable set, and the function f map it onto a nonmeasurable set.

EX4.

for any collection of intervals $I = \{I_k\}_{k=1}^{\infty}$, define $I_h = \{I_k^h\}_{k=1}^{\infty}$, where $I_k^h = \{a + h\}$, we can easily to see

- I covers $E \iff I_h$ covers E_h

Therefore, there's an one-to-one relation between $\mathbb{I} = \{I : I\ cover\ E\}$ and $\mathbb{I}_h = \{I_h : I_h\ cover\ E_h\}$ and $\sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} |I_k^h|$ for each one-to-one pair.

Therefore, $|E|_e = \sup_{I \in \mathbb{I}} \sum_{k=1}^{\infty} |I_k| = |E_h|_e = \sup_{I_h \in \mathbb{I}_h} \sum_{k=1}^{\infty} |I_h^k|$

samely, for any F is closed, define $F_h = \{x + h : x \in F\}$, we have $F \subseteq E \iff F_h \subseteq E_h$

Therefore, $\sup_{F \subseteq E, F\ closed} |F| = \sup_{F \subseteq E, F\ closed} |F|_e = \sup_{F_h \subseteq E_h, F_h\ closed} |F_h|$
$\implies |E|_i = |E_h|_i$

Therefore, $|E|_i = |E|_e \iff |E_h|_i = |E_h|_e$
E measurable iff E_h measurable.