3.

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \alpha^2 \nabla^2 u \qquad \text{(Cartesianas)}$$

Para coordenadas polares esto sería, usando la expresión encontrada para el Laplaciano en las notas*:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\left(u_{\ell+1,i,j} - 2u_{\ell,i,j} + u_{\ell-1,i,j}\right)}{(\Delta t^2)} = \alpha^2 \nabla^2 u$$

$$\Rightarrow u_{\ell+1,i,j} = (\Delta t\,\alpha)^2 \nabla^2 u + 2u_{\ell,i,j} - u_{\ell-1,i,j}$$

$$= (\Delta t\,\alpha)^2\left(\frac{u_{\ell,i+1,j} - 2u_{\ell,i,j} + u_{\ell,i-1,j}}{(\Delta\rho)^2} + \frac{1}{\rho[i]}\left(\frac{u_{\ell,i,j} - u_{\ell,i-1,j}}{\Delta\rho}\right)\right.$$

$$\left. + \frac{1}{\rho[i]^2}\left(\frac{u_{\ell,i,j+1} - 2u_{\ell,i,j} + u_{\ell,i,j-1}}{(\Delta\phi)^2}\right)\right) + 2u_{\ell,i,j} - u_{\ell-1,i,j}$$

$$\Rightarrow u_{\ell+1,i,j} = \nu^2\left[u_{\ell,i+1,j} - 2u_{\ell,i,j} + u_{\ell,i-1,j} + \frac{\Delta\rho}{\rho[i]}\left(u_{\ell,i,j} - u_{\ell,i-1,j}\right)\right.$$

$$\left. + \frac{\lambda^2}{\rho[i]^2}\left(u_{\ell,i,j+1} - 2u_{\ell,i,j} + u_{\ell,i,j-1}\right)\right] + 2u_{\ell,i,j} - u_{\ell-1,i,j}$$

donde $\nu = \dfrac{\alpha\,\Delta t}{\Delta\rho}$ y $\lambda = \dfrac{\Delta\rho}{\Delta\phi}$

*Esta ecuación es una consecuencia directa del Laplaciano para coordenadas polares:

$$\nabla^2 u = \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}$$