Viaje a la luna (Teo)

c. Sea $\vec{R}_L(\phi, t)$ el vector de posición de la luna vista desde la tierra, dado por:

$$\vec{R}_L = d_{TL}\cos(\omega t)\,\hat{\imath} + d_{TL}\,\text{sen}(\omega t)\,\hat{\jmath}$$

donde d_{TL} es la distancia de la tierra a la luna, que asumimos constante.

Ahora, definimos $\vec{r}_L$ como el vector de posición de la luna vista desde el cohete. Si $\vec{r}$ es el vector de posición del cohete, podemos definir $\vec{r}_L$ así:

$$\vec{r}_L = \vec{R}_L - \vec{r} = d_{TL}\cos(\omega t)\,\hat{\imath} + d_{TL}\,\text{sen}(\omega t)\,\hat{\jmath} - r(t)\cos\phi(t)\,\hat{\imath} - r(t)\,\text{sen}\,\phi(t)$$

Calculando la norma de este vector se obtiene:

$$\left[(d_{TL}\cos(\omega t) - r(t)\cos\phi(t))^2 + (d_{TL}\,\text{sen}(\omega t) - r(t)\,\text{sen}\,\phi(t))^2\right]^{1/2}$$

$$= \left[r(t)^2 + d_{TL}^2 - 2r(t)\cos\phi(t)\,d_{TL}\cos(\omega t) - 2r(t)\,\text{sen}\,\phi(t)\,d_{TL}\,\text{sen}(\omega t)\right]^{1/2}$$

$$= \left[r(t)^2 + d_{TL}^2 - 2r(t)\,d_{TL}\left(\cos\phi(t)\cos(\omega t) + \text{sen}\,\phi(t)\,\text{sen}(\omega t)\right)\right]^{1/2}$$

Usando la relación trigonométrica: $\cos(A-B) = \cos A\cos B + \text{sen}A\,\text{sen}B$, queda:

$$= \left[r(t)^2 + d_{TL}^2 - 2r(t)\,d_{TL}\cos(\phi(t) - \omega t)\right]^{1/2}$$

d. Primero calculamos el Laplaciano del sistema, dado por

$L = T - U$.

Sabemos que $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ corresponde a la energía cinética. Para encontrar su equivalente en coordenadas polares, se realiza lo siguiente:

$$x(t) = r(t)\cos\phi(t) \Rightarrow \dot{x}(t) = \dot{r}\cos\phi - r\,\text{sen}\phi\,\dot{\phi}$$

$$y(t) = r(t)\,\text{sen}\,\phi(t) \Rightarrow \dot{y}(t) = \dot{r}\,\text{sen}\phi + r\cos\phi\,\dot{\phi}$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = (\dot{r}\cos\phi - r\,\text{sen}\phi\,\dot{\phi})^2 + (\dot{r}\,\text{sen}\phi + r\cos\phi\,\dot{\phi})^2$$

$$= \dot{r}^2 - 2r\dot{r}\,\text{sen}\phi\,\dot{\phi}\cos\phi + 2r\dot{r}\,\text{sen}\phi\cos\phi\,\dot{\phi} + r^2\dot{\phi}^2$$

$$= \dot{r}^2 + r^2\dot{\phi}^2$$

Luego: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

La energía potencial, U, será la suma de la energía potencial gravitacional ocasionada por la tierra y por la luna, es decir:

$$U = -G\frac{mM_T}{r} - G\frac{mM_L}{r_L}$$

donde $r = |\vec{r}|$ y $r_L = |\vec{r}_L|$.

En este caso, las coordenadas generalizadas corresponden a r y ϕ. Por ende, los momentos generalizados son los siguientes:

$$P_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + G\frac{mM_T}{r} + G\frac{mM_L}{r_L}\right) = m\dot{r}$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

Por ende, la expresión para el Hamiltoniano sería la siguiente:

$$H = P_r\dot{r} + P_\phi\dot{\phi} - L = m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - G\frac{mM_T}{r} - G\frac{mM_L}{r_L}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - G \frac{m M_T}{r} - G \frac{m M_L}{r}$$

$$= \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} - G \frac{m M_T}{r} - G \frac{m M_L}{r_L}$$

e. Siguiendo las ecuaciones estándar en un sistema Hamiltoniano $\left(\frac{\partial H}{\partial q_i} = -\dot{P}_i ; \frac{\partial H}{\partial P_i} = \dot{q}_i\right)$, se puede concluir lo siguiente:

$$\dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r}{m} \quad ; \quad \dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mr^2}$$

$$\dot{P}_r = -\frac{\partial H}{\partial r} = -\left(\frac{-P_\phi^2}{mr^3} + G \frac{m M_T}{r^2} - G m M_L \cdot \frac{\partial}{\partial r}\left(\frac{1}{r_L}\right)\right)$$

$$\dot{P}_\phi = -\frac{\partial H}{\partial \phi} = -\left(-G m M_L \cdot \frac{\partial}{\partial \phi}\left(\frac{1}{r_L}\right)\right)$$

Calculamos las derivadas correspondientes:

$$\frac{\partial}{\partial r}\left(\frac{1}{r_L}\right) = \frac{\partial}{\partial r}\left(\left[r^2 + d_{TL}^2 - 2r d_{TL} \cos(\phi - \omega t)\right]^{-1/2}\right)$$

$$= -\frac{1}{2}\left[\frac{1}{r_L^3}\right] \cdot (2r - 2 d_{TL} \cos(\phi - \omega t)) = \frac{-1}{r_L^3}(r - d_{TL} \cos(\phi - \omega t))$$

$$\frac{\partial}{\partial \phi}\left(\frac{1}{r_L}\right) = -\frac{1}{2}\left[\frac{1}{r_L^3}\right] \cdot (2 r d_{TL} \operatorname{sen}(\phi - \omega t)) = \frac{-1}{r_L^3} \cdot r d_{TL} \operatorname{sen}(\phi - \omega t)$$

Así, concluimos:

$$\dot{P}_r = \frac{P_\phi^2}{mr^3} - G \frac{m M_T}{r^2} - G \frac{m M_L}{r_L^3}(r - d_{TL} \cos(\phi - \omega t))$$

$$\dot{P}_\phi = -G \frac{m M_L}{r_L^3} r d_{TL} \operatorname{sen}(\phi - \omega t)$$

F. Usando las ecuaciones del ejercicio anterior, obtenemos:

$$\dot{\tilde{r}} = \frac{\dot{r}}{d_{TL}} = \frac{p_r}{m d_{TL}} = \tilde{p}_r$$

$$\dot{\phi} = \frac{p_\phi}{mr^2} = \frac{p_\phi}{mr^2} \cdot \frac{d_{TL}^2}{d_{TL}^2} = \frac{\tilde{p}_\phi}{r^2} \cdot d_{TL}^2 = \frac{\tilde{p}_\phi}{\tilde{r}^2}$$

$$\dot{\tilde{p}}_r = \frac{\dot{p}_r}{m d_{TL}} = \frac{1}{m d_{TL}} \left(\frac{p_\phi^2}{mr^3} - \frac{G m M_T}{d_{TL}^2} \left(\frac{1}{\tilde{r}^2} + \frac{\mu}{\frac{r_L^3}{d_{TL}^3}} \cdot \left(\tilde{r} - \cos(\phi - \omega t) \right) \right) \right) *$$

$$= \frac{1}{m d_{TL}} \left(\frac{(m d_{TL}^2)^2}{d_{TL}^3} \frac{\tilde{p}_\phi^2}{m \tilde{r}^3} \right) - \Delta \left(\frac{1}{\tilde{r}^2} + \frac{\mu}{\tilde{r}'^3} \left(\tilde{r} - \cos(\phi - \omega t) \right) \right)$$

$$= \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - \Delta \left(\frac{1}{\tilde{r}^2} + \frac{\mu}{\tilde{r}'^3} \left(\tilde{r} - \cos(\phi - \omega t) \right) \right)$$

$$\dot{\tilde{p}}_\phi = \frac{\dot{p}_\phi}{m d_{TL}^2} = \frac{1}{m d_{TL}^2} \left(-G \frac{m M_L}{r_L^3} r d_{TL} \sin(\phi - \omega t) \right) = -\left(\Delta \frac{\mu}{r_L^3} \tilde{r} \cdot d_{TL}^3 \sin(\phi - \omega t) \right)$$

$$= -\left(\frac{\Delta \mu \tilde{r}}{\frac{r_L^3}{d_{TL}^3}} \sin(\phi - \omega t) \right) = -\frac{\Delta \mu \tilde{r}}{\tilde{r}'^3} \sin(\phi - \omega t)$$

* Note que: $\frac{r_L^3}{d_{TL}^3} = \frac{\left(r^2 + d_{TL}^2 - 2 r d_{TL} \cos(\phi - \omega t) \right)^{3/2}}{d_{TL}^3} =$

$$= \left[\frac{1}{d_{TL}^2} \left(r^2 + d_{TL}^2 - 2 r d_{TL} \cos(\phi - \omega t) \right) \right]^{3/2} = \left(\tilde{r}^2 + 1 - 2 \tilde{r} \cos(\phi - \omega t) \right)^{3/2}$$

$$= \tilde{r}'^3$$

9. Por definición, se tiene que:

$$\tilde{p}_r = \frac{p_r}{m d_{TL}} = \frac{m}{m d_{TL}}\dot{r} = \frac{1}{d_{TL}}\frac{d}{dt}\sqrt{x^2+y^2} = \frac{1}{2d_{TL}}(x^2+y^2)^{-1/2}(2x\dot{x}+2y\dot{y})$$

$$= \frac{1}{d_{TL}}\cdot\frac{1}{r}(x\dot{x}+y\dot{y}) = \frac{x v\cos\theta + y v \operatorname{sen}\theta}{d_{TL}\, r} = \frac{r\cos\phi\, v\cos\theta + r\operatorname{sen}\phi\, v\operatorname{sen}\theta}{d_{TL}\, r}$$

$$= \frac{r v(\cos\theta\cos\phi + \operatorname{sen}\theta\operatorname{sen}\phi)}{r\cdot d_{TL}} = \frac{v}{d_{TL}}\cos(\theta-\phi) = \tilde{v}\cos(\theta-\phi)$$

$$\Rightarrow \tilde{p}_r^{\,0} = \tilde{v}_0\cos(\theta-\phi)$$

$$\tilde{p}_\phi = \frac{p_\phi}{m d_{TL}^2} = \frac{m r^2\dot{\phi}}{m d_{TL}^2} = \tilde{r}^2\frac{d\phi}{dt} = \tilde{r}^2\frac{d}{dt}\left[\arctan\left(\frac{y}{x}\right)\right]$$

$$= \frac{\tilde{r}^2}{1+\frac{y^2}{x^2}}\cdot\frac{d}{dt}\left(\frac{y}{x}\right) = \frac{\tilde{r}^2}{1+\frac{y^2}{x^2}}\left(\dot{y}x^{-1} - yx^{-2}\dot{x}\right) = \frac{x^2}{x^2}\cdot\frac{\tilde{r}^2}{1+\frac{y^2}{x^2}}\left(\dot{y}x^{-1} - yx^{-2}\dot{x}\right)$$

$$= \frac{\tilde{r}^2}{x^2+y^2}(\dot{y}x - y\dot{x}) = \frac{\tilde{r}^2}{r^2}(\dot{y}x - y\dot{x}) = \frac{\tilde{r}^2}{r^2}(v\sin\theta\, r\cos\phi - r\sin\phi\, v\cos\theta)$$

$$= \frac{\tilde{r}^2}{r}\cdot v(\sin(\theta-\phi))^{*} = \frac{\tilde{r}^2}{\tilde{r}\cdot d_{TL}}\cdot v\cdot\sin(\theta-\phi) = \tilde{r}\cdot\frac{v}{d_{TL}}\cdot\sin(\theta-\phi)$$

$$= \tilde{r}\,\tilde{v}\sin(\theta-\phi)$$

$$\Rightarrow \tilde{p}_\phi^{\,0} = \tilde{r}_0\tilde{v}_0\sin(\theta-\phi)$$

$* \sin\theta\cos\phi - \sin\phi\cos\theta = \sin\theta\cos\phi - \cos\theta\sin\phi = \sin(\theta-\phi)$