

# Supplement: Choice of Saccade Endpoint Under Risk

Our ideal observer is adapted from that of Najemnik and Geisler (2005). For a search task in which one of a set of orthogonal targets is displayed embedded in normally distributed white noise, and the precise forms and locations of potential targets are known to the observer, the optimal strategy is template matching (Green & Swets, 1966). If all signals have equal power, equal prior probability and equal payoffs, the ideal strategy is to cross-correlate the stimulus with each potential target and select the target resulting in maximum response. The expected value of the template response to the target is proportional to the contrast of the target. The variance of the template response is the sum of variances due to external display noise and any internal noise. The ratio of the mean to the standard deviation of the template response is equivalent to  $d'$  as determined by the empirically derived visibility map.

In our experiment, observers are given two fixations on the stimulus at locations,  $F_1$  (the initial fixation on the center of the stimulus) and  $F_2$  (the endpoint of the saccade). At each potential stimulus location  $i$  there are two template responses,  $W_{i,1}$  and  $W_{i,2}$ , one for each fixation. We scale the template response to stimulus location  $i$  during fixation  $j$  so that

$$E(W_{i,j}) = \begin{cases} .5 & \text{if } i = \text{target location} \\ -.5 & \text{otherwise} \end{cases} \quad (1)$$

The detectability  $d'_{i,j}$  of the target at stimulus location  $i$  may be determined by centering the visibility map at the current fixation  $F_j$  and looking up the  $d'$  value corresponding to location  $i$ . Thus, random variable  $W_{i,j}$  has standard deviation  $\sigma_{i,j} = 1/d'_{i,j}$ .

By Bayes' rule, the posterior probability that the target is at position  $i$  out of the  $n$  possible positions after the initial fixation  $F_1$  is

$$p(i|\mathbf{w}_1) = \frac{p(i)p(\mathbf{w}_1|i)}{\sum_{j=1}^n p(j)p(\mathbf{w}_1|j)}, \quad (2)$$

where  $\mathbf{w}_1 = (w_{1,1}, \dots, w_{n,1})$  is the set of measured template responses. We assume conditional independence of template responses, so that

$$p(i|\mathbf{w}_1) = \frac{p(i) \prod_{k=1}^n p(\mathbf{w}_{k,1}|i)}{\sum_{j=1}^n p(j) \prod_{k=1}^n p(\mathbf{w}_{k,1}|j)}. \quad (3)$$

By further assuming normally distributed template responses, we have

$$p(i|\mathbf{w}_1) = \frac{p(i) \frac{1}{\sqrt{2\pi}\sigma_{i,1}} \exp[-(w_{i,1} - .5)^2 / 2\sigma_{i,1}^2] \prod_{k \neq i} \frac{1}{\sqrt{2\pi}\sigma_{k,1}} \exp[-(w_{k,1} + .5)^2 / 2\sigma_{k,1}^2]}{\sum_{j=1}^n p(j) \frac{1}{\sqrt{2\pi}\sigma_{j,1}} \exp[-(w_{j,1} - .5)^2 / 2\sigma_{j,1}^2] \prod_{k \neq j} \frac{1}{\sqrt{2\pi}\sigma_{k,1}} \exp[-(w_{k,1} + .5)^2 / 2\sigma_{k,1}^2]}. \quad (4)$$

After algebraic manipulation, substitution of  $d'_{i,1} = 1/\sigma_{i,1}$ , and assuming equal priors ( $p(i) = 1/n$ ) as was the case in our experiment, we find that

$$p(i|\mathbf{w}_1) = \frac{\exp(d'_{i,1}{}^2 w_{i,1})}{\sum_{j=1}^n \exp(d'_{j,1}{}^2 w_{j,1})}. \quad (5)$$

In our task, there are rewards  $V_i$  for correctly choosing each target  $i$ , but no penalties for incorrect answers. If forced to choose a target location at this point, the ideal observer chooses the location that leads to maximum expected gain:

$$I = \operatorname{argmax}_i (p(i|\mathbf{w}_1)V_i). \quad (6)$$

In our task, observers do not choose their response based on the glance from the first fixation alone. Rather, they choose a saccade target, make the saccade, and then choose a response based on that additional glance as well. The ideal observer chooses a saccade endpoint so as to maximize expected gain:

$$F_2^{opt} = \operatorname{argmax}_{F_2} \left( \sum_{i=1}^n p(c|i, F_2, \mathbf{w}_1) p(i|\mathbf{w}_1) V_i \right), \quad (7)$$

where  $p(c|i, F_2, \mathbf{w}_1)$  is the expected probability of being correct given pre-saccade template responses  $\mathbf{w}_1$ , having selected saccade endpoint  $F_2$ , and assuming the target is located at position  $i$ .

In determining the computation of the expected probability correct, we must

look ahead to the situation that will occur *after* the saccade. Once the eye arrives at the new location, a new set of template responses will be measured,  $\mathbf{W}_2 = (W_{1,2}, \dots, W_{n,2})$ . Here, we use capital letters because,  $\mathbf{W}_2$  is a random vector prior to gathering the measurements. Suppose position  $i$  contains the target. The observer will be correct if position  $i$  is chosen, which will occur if that choice has maximum expected value, i.e., if  $p(i|\mathbf{w}_1, F_2, \mathbf{W}_2)V_i > p(j|\mathbf{w}_1, F_2, \mathbf{W}_2)V_j$  for all  $j \neq i$ . A similar series of substitutions and algebra as was used to derive Eq. 5 yields

$$p(c|i, F_2, \mathbf{w}_1) = p \left( \frac{\exp [d_{i,1}'^2 w_{i,1} + d_{i,2}'^2 W_{i,2}] V_i}{\exp [d_{j,1}'^2 w_{j,1} + d_{j,2}'^2 W_{j,2}] V_j} > 1 \quad \forall j \neq i \right). \quad (8)$$

Taking the log of the inequality and rearranging terms yields

$$p(c|i, F_2, \mathbf{w}_1) = p \left( \frac{d_{i,1}'^2 w_{i,1} + d_{i,2}'^2 W_{i,2} - d_{j,1}'^2 w_{j,1} + \ln(V_i/V_j)}{d_{j,2}'^2} > W_{j,2} \quad \forall j \neq i \right). \quad (9)$$

The terms for different values of  $j$  are not statistically independent, but become so if we condition on the value of  $W_{i,2}$ , we obtain

$$p(c|i, F_2, \mathbf{w}_1) = \int p(w_{i,2}) \prod_{j \neq i} p \left( \frac{d_{i,1}'^2 w_{i,1} + d_{i,2}'^2 w_{i,2} - d_{j,1}'^2 w_{j,1} + \ln(V_i/V_j)}{d_{j,2}'^2} > W_{j,2} \right) dw_{i,2}. \quad (10)$$

Expressing this in terms of the standard normal distribution:

$$p(c|i, F_2, \mathbf{w}_1) = \int d'_{i,2} \phi(d'_{i,2}(w_{i,2} - .5)) \prod_{j \neq i} \Phi\left(d'_{j,2} \left[ \frac{d'^2_{i,1} w_{i,1} + d'^2_{i,2} w_{i,2} - d'^2_{j,1} w_{j,1} + \ln(V_i/V_j)}{d'^2_{j,2}} + .5 \right]\right) dw_{i,2}. \quad (11)$$

where  $\phi$  and  $\Phi$  are the standard normal density and cumulative distribution functions, respectively. Finally, we introduce a change of variable,  $z = d'_{i,2}(w_{i,2} - .5)$ , so that

$$p(c|i, F_2, \mathbf{w}_1) = \int \phi(z) \prod_{j \neq i} \Phi\left(\frac{d'^2_{i,1} w_{i,1} + d'_{i,2} z + .5 d'^2_{i,2} + .5 d'^2_{j,2} - d'^2_{j,1} w_{j,1} + \ln(V_i/V_j)}{d'_{j,2}}\right) dz. \quad (12)$$

A single search trial is simulated by first sampling from the eight template distributions to generate a random vector  $\mathbf{w}_1$ . For each such simulated  $\mathbf{w}_1$  vector, each target location  $i$ , and potential saccade locations  $F_2$  (over a dense grid of possible saccade locations), Eq. 12 is evaluated by numerical integration. Finally, Eq. 7 is used to determine  $F_2^{opt}$ . Fig. 5 shows the spatial distribution of optimal fixation locations based on 1000 simulated trials per panel.

Following a saccade to position  $F_2^{opt}$ , the ideal observer now measures a second

set of template responses ( $\mathbf{w}_2$ ). It computes updated posterior probabilities:

$$p(i|\mathbf{w}_1, F_2^{opt}, \mathbf{w}_2) = \frac{p(i)\exp [d_{i,1}'^2 w_{i,1} + d_{i,2}'^2 w_{i,2}]}{\sum_{j=1}^n p(j)\exp [d_{j,1}'^2 w_{j,1} + d_{j,2}'^2 w_{j,2}]}, \quad (13)$$

where the values of  $d_{i,2}'$  are based on the visibility map centered on the eye position after the second saccade. Finally, the ideal observer selects the location,  $I$ , for which expected gain is maximized:

$$I = \underset{i}{\operatorname{argmax}} (p(i|\mathbf{w}_1, F_2^{opt}, \mathbf{w}_2) V_i). \quad (14)$$

## References

- Green, D. & Swets, J. (1966). *Signal Detection Theory and Psychophysics*. New York: Kreiger Publihing Co.
- Najemnik, J. & Geisler, W. (2005). Optimal eye movement strategies in visual search. *Nature*, 434, 387–391.