Professor Howard

September 8, 2010

Exercise 1. Prove that there is no rational number $x \in \mathbb{Q}$ satisfying $x^2 = 2$.

Solution. To get a contradiction, assume there is an $x \in \mathbb{Q}$ such that $x^2 = 2$. If we write $x = a/b$ as a quotient of two integers $a, b \in \mathbb{Z}$ with $b > 0$ then

$$x^2 = 2 \implies (a/b)^2 = 2 \implies a^2 = 2b^2.$$

Now stare at the last equality

$$a^2 = 2b^2 \tag{1}$$

and think about how many times the prime 2 appears in the prime factorization of each side. For any nonzero integer m, the prime 2 appears in the prime factorization of m^2 twice as many times as it appears in the prime factorization of m itself. In particular for any nonzero m the prime 2 appears in the prime factorization of m^2 an *even* number of times. Thus the prime factorization of the left hand side of (1) has an even number of 2's in it. What about the right hand side? The prime factorization of b^2 has an even number of 2's in it, and so the prime factorization of $2b^2$ has an odd number of 2's in it. This shows that the prime factorization of the right hand side of (1) has an odd number of 2's. This is a contradiction, and so no such x can exist. $\square$

Exercise 2. Prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for every $n \in \mathbb{Z}^+$.

Solution. For each $n \in \mathbb{Z}^+$ let $P(n)$ be the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

We will use induction to prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$. First, consider the case $n = 1$. The statement $P(1)$ asserts that

$$1 = \frac{1(1+1)}{2},$$

and this is obviously true. Next we assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$ and deduce that $P(k+1)$ is true. So, suppose that $P(k)$ is true. This means that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2},$$

and adding $k+1$ to both sides and simplifying results in

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Comparing the first and last expressions in this sequence of equalities, we find that $P(k+1)$ is also true. Thus $P(k) \implies P(k+1)$, and so by induction $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$

Exercise 3. Use the Well Ordering Property of $\mathbb{Z}^+$ to prove the Principle of Mathematical Induction.

Solution. Suppose we are given a sequence of statements $P(1), P(2), P(3), \ldots$ with the properties

(a) $P(1)$ is true,

(b) for every $k \in \mathbb{Z}^+$, $P(k) \implies P(k+1)$.

We must show that $P(n)$ is true for every $n \in \mathbb{Z}^+$. To get a contradiction, suppose not. Then there is some $m \in \mathbb{Z}^+$ such that the statement $P(m)$ is false. Consider the set

$$S = \{n \in \mathbb{Z}^+ : P(n) \text{ is false}\}.$$

We know that $P(m)$ is false, and so $m \in S$. In particular $S \neq \emptyset$. By the Well Ordering Property of $\mathbb{Z}^+$, the set S contains a smallest element, which we will call m_0. We know that $P(1)$ is true, and so $1 \notin S$. In particular $m_0 \neq 1$, and so $m_0 > 1$. Thus $m_0 - 1 \in \mathbb{Z}^+$ and it makes sense to consider the statement $P(m_0 - 1)$. As $m_0 - 1 < m_0$ and m_0 is the *smallest* element of S, $m_0 - 1 \notin S$. Of course this implies that $P(m_0 - 1)$ is true. Now taking $k = m_0 - 1$ in the implication $P(k) \implies P(k+1)$ we deduce that $P(m_0)$ is true, and so $m_0 \notin S$. But this is ridiculous, as m_0 is not only in S, it is the smallest element of S. Thus we have arrived at a contradiction. $\square$