3.1、 the Intertemporal Euler Equation.

$$u_c(c_t) = \beta u_c(c_{t+1})(1+r) \Rightarrow u_c(c_t) = u_c(c_{t+1}) \Rightarrow c_t = c_{t+1}.$$

$$\max_{\{c_1, c_2\}} u(c_1) + \beta E\, u(c_2)$$

$$c_1 + a_2 = w$$

$$c_2 = w + a_2(1+r)$$

$$L(c_1, c_2, \lambda_1, \lambda_2) = u(c_1) + \beta E\, u(c_2) + \lambda_1(w - c_1 - a_2) + \lambda_2(w + a_2(1+r) - c_2).$$

$$u_c(c_1) = \lambda_1$$

$$\beta u_c(c_2) = \lambda_2 \qquad \Rightarrow \qquad u_c(c_1) = \beta(1+r)\, u_c(c_2)$$

$$-\lambda_1 + (1+r)\lambda_2 = 0$$

$$\Rightarrow c_1 = c_2 \Rightarrow a_2 = 0$$

2、

$$\max_{\{c_1, c_2\}} u(c_1) + \beta E\, u(c_2)$$

$$c_1 + a_2 = w \qquad \Rightarrow \quad EE: \quad u_c(c_1) = \beta E(u_c(c_2)(1+r))$$

$$c_2 = X + (1+r)a_2 \qquad\qquad = E u_c(c_2)$$

$$L(c_1, c_2, \lambda_1, \lambda_2) = u(c_1) + \beta E\Big(u(c_2) + \lambda_1(w - c_1 - a_2) + \lambda_2(\cancel{w} X + (1+r)a_2 - c_2)\Big)$$

$$= u(c_1) + \lambda_1(w - c_1 - a_2) + \beta E\Big(u(c_2) + \lambda_2(X + (1+r)a_2 - c_2)\Big).$$

FOC:

$$\begin{cases} u_c(c_1) = \lambda_1 \\ E u_c(c_2) = E\lambda_2 \\ -\lambda_1 + \beta E(\lambda_2(1+r)) = 0 \end{cases} \Rightarrow u_c(c_1) = \beta(1+r) E(u_c(c_2)) = E(u_c(c_2))$$

(*) $E(u_c(X)) < u_c(w)$ a sufficient condition is $u_{ccc}(X) < 0$.
under the condition that $u_{cc} < 0$. we have ~~...~~

~~Proof. $u_c(c_1) = E(u_c(X + \ldots))$~~

Proof: $u_c(c_1) - E\, u_c(c_2)$ is ~~decreasing~~ strictly increasing in a_2.

because $\quad u_c(c_1) - E u_c(c_2) = u_c(w-a_2) - E u_c(x + a_2(1+r))$

$u_c(w-a_2)$ strictly increases as a_2 increases.

$u_c(x + a_2(1+r))$ strictly decreases as a_2 increases for $\forall x$.

$\Rightarrow E u_c(x + a_2(1+r))$ strictly decreases as a_2 increases.

$a_2 = 0 \Rightarrow \quad u_c(c_1) - E u_c(c_2) = u_c(w) - E(u_c(x)) < 0$

so $a_2^* > 0$

3. $u(c)$ satisfies this condition

proof: $u_c(c) = c^{-r}$

$u_{cc}(c) = -r c^{-r-1}$

~~$u_{ccc}(c) = r(r+1) c^{-r-2}$~~

$$E u_c(x) < u_c(w) \Rightarrow E(x^{-r}) < w^{-r}$$

$$\Rightarrow \frac{1}{2}\left((w+b)^{-r} + (w-b)^{-r}\right) < w^{-r}$$

$$\Rightarrow \frac{1}{2}\left(\left(1+\frac{b}{w}\right)^{-r} + \left(1-\frac{b}{w}\right)^{-r}\right) < 1$$

$\Rightarrow$

~~when ... however~~

when $r > 0$, $\frac{1}{2}\left((1+\varepsilon)^{-r} + (1-\varepsilon)^{-r}\right) > 1$ for $\forall \varepsilon > 0$, $|\varepsilon| < 1$.

$$\Rightarrow \frac{1}{2}\left(\left(1+\frac{b}{w}\right)^{-b} + \left(1-\frac{b}{w}\right)^{-b}\right) > 1$$

$$\Rightarrow E(x^{-r}) > w^{-r} \Rightarrow u_c(w) - E u_c(x) < 0.$$

$$\Rightarrow a_2^* > 0$$

3.2

(1) $u_c(c) = \alpha_1 - 2\alpha_2 c$.

$u_c(w) - E u_c(x) = 0$. prudence is not satisfied.

(2)

$$\max_{\{c_1, c_2, c_3\}} u(c_1) + E(\beta u(c_2) + \beta^2 u(c_3))$$

s.t. $c_1 + a_2 = w$

$c_2 + a_3 = x + (1+r) a_2$

$c_3 = w + a_3(1+r)$

$$L(c_1, c_2, c_3, a_2, a_3) = u(c_1) + E\Big(\beta u(c_2) + \beta^2 u(c_3) + \lambda_1 (w - a_2 - c_1) + \lambda_2 (x + (1+r)a_2 - c_2 - a_3) + \lambda_3 (w + a_3(1+r) - c_3)\Big)$$

~~FOC: $u_c(c_1) = \lambda_1$~~

$$= u(c_1) + \lambda_1 (w - a_2 - c_1) + E\Big(\beta u(c_2) + \beta^2 u(c_3) + \lambda_2 (x + (1+r) a_2 - c_2 - a_3) + \lambda_3 (w + a_3(1+r) - c_3)\Big)$$

FOC: $u_c(c_1) = \lambda_1$

$E(\beta u_c(c_2)) = E(\lambda_2)$

$E(\beta^2 u_c(c_3)) = E(\lambda_3)$ $\Rightarrow$ $u_c(c_1) = E u_c(c_2) = E u_c(c_3)$

$-\lambda_1 + (1+r) E\lambda_2 = 0$

$-E\lambda_2 + (1+r) E\lambda_3 = 0$

$\Rightarrow$ ~~$c_2 = c_3$~~

$c_1 = E(c_2) = E(c_3)$

$\Rightarrow \overset{w - a_2}{c_1} = w + a_2(1+r) - E(a_3)$

$= w + (1+r) E(a_3)$

$E(a_3) = \frac{1+r}{2+r} a_2$

$\Rightarrow w - a_2 = w + \left(\frac{1}{2+r} + 1\right) a_2$

$\Rightarrow a_2 = 0, \; E(a_3) = 0.$

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3、 $L(c_1, c_2, c_3; a_2, a_3) = u(c_1) + E\big(\beta u(c_2) + \beta^2 u(c_3) + \lambda_1 (W - a_2 - c_1) + \lambda_2 (X + a_2(1+r) - a_3 - c_2) + \lambda_3 (W + a_3(1+r) - c_3) + \lambda_4 \cdot a_3\big)$

$u_c(c_1) = \lambda_1$

$E\beta u_c(c_2) = E\lambda_2$

$E\beta^2 u_c(c_3) = E\lambda_3$ $\Rightarrow$ $u_c(c_1) = E u_c(c_2) \geq E u_c(c_3)$

$-\lambda_1 + E\lambda_2 (1+r) = 0$

$-E\lambda_2 + (1+r) E\lambda_3 + E\lambda_4 = 0$ $\Rightarrow$ $c_1 = E(c_2) \leq E(c_3)$

$\Rightarrow W - a_2 = E\, a_2(1+r) + W - E(a_3) \leq W + (1+r)E(a_2)$

$\Rightarrow E(a_3) \geq \frac{1+r}{2+r} a_2$ $\quad E(a_3) = (2+r) a_2$

$\Rightarrow a_2 \geq 0.$

Intertemporal substitution from period (2) given a_2, X.

$\max_{\{c_2, c_3\}} u(c_2) + \beta u(c_3)$

s.t. $c_2 + a_3 = \underline{X} + \underline{a_2}(1+r)$

$c_3 = a_3(1+r) + W$

$u_c(c_2) = \beta(1+r) u_c(c_3) \Rightarrow u_c(c_3) \Rightarrow c_2 = c_3,$

$\Rightarrow a_3 = \frac{X + a_2(1+r) - W}{2+r}$

When $6 = 0 \Rightarrow X = W$, $a_3 = \frac{1+r}{2+r} a_2. \Rightarrow a_2 = 0.$

When $6 > 0$ but small. If $a_2 = 0 \Rightarrow a_3 = 0$. Contradicts that $a_3 = \frac{X + a_2(1+r) - W}{2+r}$
$= \frac{X - W}{2+r} = \pm\frac{6}{2+r}$

so $a_2 > 0.$