$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}$$

$$\hat{\gamma}(k) = \frac{1}{T}\sum_{t=1}^{T-k}(x_t - \bar{x})(x_{t+k} - \bar{x})$$

$$\hat{\gamma}(0) = \frac{1}{T}\sum_{t=1}^{T}(x_t - \bar{x})^2$$

$$\bar{x} = \frac{1}{T}\sum_{t=1}^{T} x_t = a + b\frac{T+1}{2}$$

$$\cancel{\hat{\gamma}(k) = \frac{1}{T}\sum_{t=1}^{T-k} b\frac{T-1}{2}} \qquad x_j - \bar{x} = b\left(j - \frac{T+1}{2}\right)$$

$$\frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} = \frac{\sum_{j=1}^{T-k}\left(j - \frac{T+1}{2}\right)\left(j+k-\frac{T+1}{2}\right)}{\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2} = \frac{\sum_{j=1}^{T-k}\left(j-\frac{T+1}{2}\right)^2 + k\sum_{j=1}^{T-k}\left(j-\frac{T+1}{2}\right)}{\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2}$$

$$\frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} - 1 = \frac{-\frac{k^2}{2}(T-k) - \sum_{j=T-k+1}^{T}\left(j-\frac{T+1}{2}\right)^2}{\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2}$$

$$= -\frac{\frac{k^2}{2}T - \frac{k^3}{2} + \sum_{j=T-k+1}^{T}\left(j-\frac{T+1}{2}\right)^2}{\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2} \leq 0. \Rightarrow \lim_{T\to\infty} \frac{\hat{\rho}(k)}{\hat{\rho}(0)} - 1 \leq 0$$

$$\cancel{\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2 = \frac{1}{4}\sum_{j=1}^{T}\left(2j-(T+1)\right)^2}$$

$$\cancel{\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2 \geq T\left(\frac{1}{T}\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)\right)^2}$$

~~x^2 is convex. so we have. by convexity~~

$$\sum_{j=1}^{T}\left(j-\frac{T+1}{2}\right)^2 = \sum_{j=1}^{T} j^2 - (T+1)\left(\sum_{j=1}^{T} j\right) + \left(\frac{T+1}{2}\right)^2 T.$$

$$= \frac{1}{6}\cdot T(T+1)(2T+1) - (T+1)\frac{(T+1)}{2}T + \left(\frac{T+1}{2}\right)^2 T.$$

$$= \frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{6}.$$

$$\cancel{\frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} - 1 = -\frac{}{\frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{6}}}$$

$$\sum_{j=T-k+1}^{T} j^2 + \left(\sum_{j=T-k+1}^{T} j\right)(T+1) + \frac{(T+1)^2}{4}k.$$

$$= \frac{T(T+1)(2T+1)}{6} - \frac{(T-k)(T-k+1)(2T-2k+1)}{6} + \frac{(2T-k+1)k}{2}(T+1) - \frac{k}{4}(T+1)^2$$

$$= \cancel{(3+6k)T^2} \quad \frac{6kT^2 + (6k^2-6(k+1))T + k(k-1)(2k-1)}{6} + \cancel{\frac{3}{4}T^2} \; \frac{3}{4}kT^2 \quad \frac{(2-k)k}{2}T + \left(\frac{1}{2}-k\right)\frac{k}{2}$$

$$\text{So } \lim_{T\to\infty} \frac{\frac{7}{4}kT^2 + \cancel{k}\left(\frac{k^2}{2}+\frac{1}{6}\right)T + \frac{k(k-1)(2k-1)}{6} + \frac{k}{2}\left(\frac{1}{2}-k\right)}{\cancel{\frac{T^3}{3}}\ \frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{6}} = 0.$$

$$\text{So } \lim_{T\to\infty} \frac{\hat{r}(k)}{\hat{r}(0)} - 1 = 0 \Rightarrow \lim_{T\to\infty} \frac{\hat{r}(k)}{\hat{r}(0)} = 1.$$

(4)

$\{X(i)\}_{i=1}^{n}$ are iid.