(b). Comme...

1). Proof:

Suppose not.

That is, $\exists x \in X^*(t)$, $x' \in X^*(t')$, $t' > t$, $x > x'$

$$f(x,t) \geq f(x',t) \quad \text{since } x \in X^*(t)$$

Then by SSC. $f(x,t') > f(x',t')$

That contradicts that $f(x',t') \geq f(x,t')$ since $x' \in X^*(t')$

So $\forall x \in X^*(t)$, $\forall x' \in X^*(t')$, $t' > t$, ~~we have~~

$x \leq x'$

2). Proof:

Suppose not

$\exists$ selection function $x^*(t)$ s.t. $x^*(t) > x^*(t')$ for some $t' > t$

$$f(x^*(t),t) \geq f(x^*(t'),t) \quad \text{since } \cancel{x \in X^*(t)}\ x^*(t) \in X^*(t)$$

by SSC. $f(x^*(t),t') > f(x^*(t'),t')$

That violates that $x^*(t') \in X^*(t')$

So $\forall$ selection function $x^*(t)$, $x^*(t)$ is non-decreasing in t.

$(x^*(t) \leq x^*(t')$ for $\forall t' \geq t)$

3).

(d) Counterexample:

$f(x;t) = t+1-x^2 \qquad x^*(t)=0 \quad \text{for } \forall t$

~~For~~ This satisfies strict single crossing because.

for $\forall x_1 > x_2$ ~~f~~ $t+1-x_1^2 \geq t+1-x_2^2$

$\Rightarrow \quad t'+1-x_1^2 > t'+1-x_2^2 \qquad \text{for } \forall t' > t.$

but $x^*(t)$ is not strictly increasing.

(c) Counterexample

$$f(x;t) = \begin{cases} \sin x + t & 0 \le x \le 3\pi \\ t & x<0 \text{ or } x > 3\pi \end{cases}$$

Let $x^*(t) = \begin{cases} \frac{\pi}{2} & \text{if } t > \bar{t} \\ \frac{5}{2}\pi & \text{if } t \le \bar{t} \end{cases}$, then $x^*(t)$ is not non-decreasing.

but $f(x;t)$ satisfies single cross property because.

For $\forall x' > x$. $\quad f(x';t) \geq f(x;t)$. there is $\sin x' + t \geq \sin x + t$.

~~then~~ then $\sin x' + t' \geq \sin x + t'$ for $\forall t' > t$

2. (a) $g(f(x,t),t)$ satisfies ID, then $g(f(x,t),t)$ satisfies SCP.

$\forall x'>x. \quad t'>t$ $\qquad g(f(x',t),t) \geq g(f(x,t),t)$

that is $\quad f(x',t) \geq f(x,t)$.

Then by SCP, $\quad g(f(x',t'),t') \geq g(f(x,t'),t')$, that is $\quad f(x',t') \geq f(x,t')$

So $f(x,t)$ satisfies SCP in (x,t)

(b) $g(f(x,t),t)$ satisfies supermodularity, then it satisfies quasi-supermodularity.

$\forall x, x'$

$g(f(x,t),t) \geq g(f(x \wedge x',t),t)$ $\quad$ ~~th~~ that is $\quad f(x,t) \geq f(x \wedge x',t)$

then. $g(f(x \vee x',t),t) \geq g(f(x',t),t)$ $\quad$, that is. $\quad f(x \vee x',t) \geq f(x',t)$.

So f satisfies QSM.

2. (a)

$$\max_{\{x_1, x_2\}} \; p f(x_1, x_2) - (w_1 x_1 + w_2 x_2)$$

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{w_1}{w_2} \qquad \left(\frac{w_1}{w_2} x_1 \frac{\beta}{\alpha}\right)^{\beta} x_1^{\alpha-1} \alpha p = w_1$$

$$x_1^{\alpha+\beta-1} \frac{\left(\frac{\beta}{\alpha}\right)^{\beta} \alpha p}{w_2^{\beta}} = w_1^{1-\beta} w_2^{\beta}$$

$$\begin{aligned} p f_1(x_1, x_2) &= w_1 \\ p f_2(x_1, x_2) &= w_2 \end{aligned} \Rightarrow \begin{cases} p \cdot \alpha x_1^{\alpha-1} x_2^{\beta} = w_1 \\ p\, x_1^{\alpha} \beta x_2^{\beta-1} = w_2 \end{cases} \Rightarrow \begin{aligned} x_1 &= \left(\frac{\beta}{\alpha}\right)^{\frac{-\beta}{\alpha+\beta-1}} (\alpha p)^{-\frac{1}{\alpha+\beta-1}} w_1^{\frac{1-\beta}{\alpha+\beta-1}} w_2^{\frac{\beta}{\alpha+\beta-1}} \\ x_2 &= \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha-1}{\alpha+\beta-1}} (\alpha p)^{-\frac{1}{\alpha+\beta-1}} w_1^{\frac{\alpha}{\alpha+\beta-1}} w_2^{\frac{1-\alpha}{\alpha+\beta-1}} \end{aligned}$$

$\cancel{x_1^* \; w_1}$ $x_1^*(p, w_1, w_2)$ strictly decreases with w_1

$x_2^*(p, w_1, w_2)$ strictly decreases with w_2.

(b)

$$\max_{\{x_1, \dots x_n\}} \; p f(x_1, x_2 \dots x_n) - (w_1 x_1 + w_2 x_2 + \dots w_i x_i + \dots w_n x_n)$$

$x_i^*(w_i)$ is non-increasing in w_i

Proof: ~~① $\hat{x}_i = -x_i$, $\tilde{\pi}(x_i, w_i)$~~

① define $\tilde{\pi}(x_i, w_i) = \max_{x_j,\, j \neq i} \; p f(x_1, x_2 \dots x_n) - (w_1 x_1 + w_2 x_2 + \dots w_n x_n)$

Let $\hat{x}_i = -x_i$, then for $\forall\, \hat{x}_i' > \hat{x}_i$, $w_i' > w_i$ $(x_i' < x_i)$

(Let $\tilde{\tilde{\pi}}(\hat{x}_i, w_i) = \tilde{\pi}(-\hat{x}_i, w_i)$)

$\because$ $\tilde{\pi}(\hat{x}_i', w_i) \geq \tilde{\pi}(\hat{x}_i', w_i') \Rightarrow$

~~$\max_{x_j, j \neq i} p f(\hat{x}_1', \hat{x}_2' \dots \hat{x}_n') = (w_1 \hat{x}_1' + w_2 \hat{x}_2' + \dots w_n \hat{x}_n') \geq \max_{x_j, j \neq i} p f(\hat{x}_1, \hat{x}_2, \dots \hat{x}_n) - (w_1 \hat{x}_1 + w_2 \hat{x}_2 + \dots w_n \hat{x}_n)$~~

$$\max_{\{x_j\}_{j \neq i}} p f(x_1, \dots x_i' \dots x_n) - (w_1 x_1 \dots + w_i x_i' + \dots w_n x_n) \geq \max_{\{x_j\}_{j \neq i}} p f(x_1, \dots x_i' \dots x_n) - (w_1 x_1 \dots w_i' x_i' + \dots w_n x_n)$$

$$\max_{\{x_j\}_{j \neq i}} p f(x_1 \dots x_i' \dots x_n) - (w_1 x_1 \dots w_i x_i' \dots w_n x_n) = \left(\max \left(p \cdot f(x_1, \dots x_i' \dots x_n) - \sum_{j=1, j \neq i} w_j x_j\right)\right) - w_i x_i'$$

$$\max_{\{x_j\}_{j \neq i}} p \cdot f(x_1, \dots x_i \dots x_n) - (w_1 x_1 \dots w_i x_i \dots w_n x_n) = \left(\max \left(p f(x_1, \dots x_i, \dots x_n) - \sum_{j=1, j \neq i} w_j x_j\right)\right) - w_i x_i$$

$$x_i' < x_i \Rightarrow (w_i' - w_i)\, x_i' < (w_i' - w_i)\, x_i$$

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$$\left(\max_{\{x_j\}_{j\neq i}} p f(x_1, \ldots x_i \ldots x_n)\right) - \sum_{j=1, j\neq i}^{n} w_j x_j) - w_i x_i' \geq \max\left(p f(x_1, \ldots x_i \ldots x_n) - \sum_{j=1, j\neq i}^{n} w_j x_j\right) - w_i x_i$$

$$\Rightarrow \max_{\{x_j\}_{j\neq i}} \left(p f(x_1, \ldots x_i \ldots x_n) - \sum_{j=1, j\neq i}^{n} w_j x_j\right) - w_i' x_i' \geq \max\left(p f(x_1 \ldots x_n) - \sum_{j=1, j\neq i}^{n} w_j x_j\right) - w_i' x_i$$

$$\Rightarrow \hat{\pi}(\hat{x}_i', w_i') \geq \hat{\pi}(\hat{x}_i, w_i')$$

So $\hat{\pi}$ is single crossing in $(\hat{x}_i, w_i)$

So $\hat{\pi}$ $\hat{x}_i^*(w_i)$ is nondecreasing in w_i

So $x_i^*(w_i)$ is non increasing in w_i

(c) $x_i^*(w_j)$ ~~doesn't~~ isn't necessarily ~~non-decr~~ non-increasing in w_j

Additional assumption: given ~~x_j~~ x_k. $k \neq i, j$.

f is supermodular in (x_i, x_j)

By Corollary 2.23 x_i and x_j are complements

$x_j^*(w_j)$ is non-increasing in w_j.

and $x_i^*(w_j)$ is non-increasing in w_j.

(d) $x_i^*(w_j)$ isn't necessarily non-decreasing in w_j.

Additional Assumptions f is ~~sup~~ submodular in (x_1, x_2)

By Corollary 2.23. x_1 and x_2 are substitutes.

$x_j^*(w_j)$ is ~~non-incr~~ non-increasing in w_j

$x_i^*(w_j)$ is non-decreasing in w_j.

(e) $f(x_1, x_2) = \frac{1}{2}(x_1 + x_2)^{\frac{1}{2}}$

fix w_1, $x_2^* = \begin{cases} (2w_2)^{-2} & w_2 < w_1 \\ [0, (2w_2)^{-2}] & w_2 = w_1 \\ 0 & w_2 > w_1 \end{cases}$ $\rightarrow$ $x_1^*(w_2)$ is non-decreasing in w_2.

4.

(a) $U(m,x) = w - px + v(x)$. by monotonicity in m.

Yes. $\forall$ $x^*(p)$ is non-increasing in p.

Proof:

Let $U(m,x) = w - px + v(x) = u(x;p)$

$\underline{\hat{u}(x;-\hat{p})} = u(x;-\hat{p})$

For $\forall$ $x' > x$. $\hat{p} < \hat{p}'$ $\quad p = -\hat{p} > -\hat{p}' = p'$

$\hat{u}(x';\hat{p}) \geq \hat{u}(x;\hat{p})$

$\Rightarrow$ $w - px' + v(x') \geq w - px + v(x)$

$p - p' > 0 \Rightarrow (p-p')x' \geq (p-p')x$

so $w - p'x' + v(x') \geq w - p'x + v(x)$.

so $\hat{u}(x';\hat{p}') \geq \hat{u}(x;\hat{p}')$

~~$\hat{u}(x,\hat{p})$~~ $x^*(\hat{p})$ is non-decreasing in $\hat{p}$.

$x^*(p)$ is non-increasing in p.

(b) ~~$v'' \geq 0$~~ ~~the~~ condition is that $v'\left(\frac{w-m^*(p)}{p}\right) + \frac{w-m^*(p)}{p} v''\left(\frac{w-m^*(p)}{p}\right) \geq 0$.

~~Let $\hat{u}(m,\hat{x};p) = m, u($~~

~~Let $\hat{u}(m,\hat{x};p) = U(m,\hat{x};p)$~~

$U(m,x) = m + v\left(\frac{w-m}{p}\right) = \hat{u}(m;p)$.

~~$\frac{\partial \hat{u}(m,p)}{\partial m \partial p} = \cdots v''\left(\frac{w-m}{p}\right)(w-m)p^{-2} \geq 0$~~

$$\frac{\partial \hat{u}(m,p)}{\partial m} = \left(-\frac{1}{p}\right)v'\left(\frac{w-m}{p}\right) + 1.$$

$$\frac{\partial \hat{u}(m,p)}{\partial m \partial p} = \frac{1}{p^2} v'\left(\frac{w-m}{p}\right) - \frac{1}{p} v''\left(\frac{w-m}{p}\right)(-m)\cdot(-p^{-2})$$

$$= \frac{1}{p^2} v'\left(\frac{w-m}{p}\right) + \frac{w-m}{p^3} v''\left(\frac{w-m}{p}\right) \geq 0 \quad \text{for } m = m^*(p)$$

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$\tilde{u}$ is ID in (m, p)

$\Rightarrow$ $m^*(p)$ is non-decreasing in p.

(c)

$$V'\left(\frac{w-m^*(p)}{p}\right) + \frac{w-m^*(p)}{p} V''\left(\frac{w-m^*(p)}{p}\right)$$

$$= \left(\frac{w-m^*(p)}{p}\right)^{-1} - \frac{w-m^*(p)}{p}\left(\frac{w-m^*(p)}{p}\right)^{-2} = 0$$

$m^*(p)$ is non-decreasing in p.

$$\hat{u}(m,p) = m + V\left(\frac{w-m}{p}\right) = m + \ln\left(\frac{w-m}{p}\right)$$

$$\frac{\partial \hat{u}(m,p)}{\partial m} = 1 - \frac{p}{w-m} = 0 \Rightarrow m = w-p.$$

$$m^*(p) = w-p$$

(d) ① $V(x) = x^{\frac{1}{2}}$, then. $m^*(p)$ is strictly increasing in p.

$$\cancel{\frac{\partial \hat{u}(m,p)}{\partial m \, \partial p}} \quad \frac{\partial \hat{u}(m^*(p), p)}{\partial m \partial p} = \frac{1}{p^2}\left(V'\left(\frac{w-m^*(p)}{p}\right) + \frac{w-m^*(p)}{p} V''\left(\frac{w-m^*(p)}{p}\right)\right).$$

$$= \frac{1}{p^2} \frac{1}{4}\left(\frac{w-m^*(p)}{p}\right)^{-\frac{1}{2}} > 0$$

$$\hat{u}(m,p) = m + \left(\frac{w-m}{p}\right)^{\frac{1}{2}} \qquad m^*(p) = w - \frac{1}{4}p^{-1} \quad m^*(p) \text{ strictly increases in } p.$$

② $V(x) = -x^{-1}$, then. $m^*(p)$ is strictly decreasing in p.

$$\frac{\partial \hat{u}(m^*(p), p)}{\partial m \partial p} = \frac{1}{p^2}\left(-\left(\frac{w-m^*(p)}{p}\right)^{-2}\right) < 0.$$

$$\hat{u}(m,p) = \cancel{m + \frac{p}{w-m}} \; m - \frac{p}{w-m} = \cancel{m + m} \qquad m^*(p) = w - \sqrt{p}$$

$m^*(p)$ strictly decreases in p.

(e) ~~Yes~~ No. $X^*(p)$ is non increasing in p under additional assumption that V is supermodular in X. that is $\frac{\partial V(x)}{\partial x_i \partial x_j} > 0$ for $\forall i,j$.

$U(m,X) =$ ~~...~~ $w - p\cdot x + v(x)$ (Monotonicity in m)

$= V(x) - p\cdot x + w = u(x;p)$

~~Let G~~

Let $\hat{u}(\hat{x};p) = u(-\hat{x};p)$, then

for $\forall \hat{x}' > \hat{x}$, ~~...~~ $p' > p$. $\hat{x}' = -x'$ $\hat{x} = -x$

if $\hat{u}(\hat{x}';p) \geq \hat{u}(\hat{x},p)$, then.

~~$V(\hat{x}) \ p\cdot\hat{x} + w$~~

$V(-\hat{x}') + p\cdot\hat{x}' + w \geq V(-\hat{x}) + p\cdot\hat{x} + w$

~~$V(x)$~~

$V(x') - p\cdot x' + w \geq V(x) - p\cdot x + w$

~~since $x' < x$~~ $x' < x$, $\Rightarrow$ $(p'-p)\cdot x' < (p'-p)\cdot x$

$\Rightarrow$ $V(x') - p'\cdot x' + w \geq V(x) - p'\cdot x + w$.

$\Rightarrow$ ~~$V(\hat{x}')$~~ $\hat{u}(\hat{x}',p') \geq \hat{u}(\hat{x},p')$

So $\hat{u}(\hat{x},p)$ is single crossing in $(\hat{x},p)$

~~So together with~~.

Since $\frac{\partial V(x)}{\partial x_i \partial x_j} > 0$ for $\forall i,j$ $\frac{\partial \hat{u}(\hat{x};p)}{\partial \hat{x}_i \partial \hat{x}_j} = \frac{\partial u(x,p)}{\partial x_i \partial x_j} = \frac{\partial V(x,p)}{\partial x_i \partial x_j} > 0$ for $\forall i,j$

$\hat{u}$ is ~~supa~~ supermodular in $\hat{x}$. $\Rightarrow$ $\hat{u}$ is quasisupermodular in $\hat{x}$

So $\hat{x}^*(p)$ is non-decreasing in p.

$x^*(p)$ is non-increasing in p.

8. Proof:

① Suppose $\bar{y} > \bar{z}$,

then for $\bar{y} \in Y$, $\bar{z} \in Z$.

$\max(\bar{y}, \bar{z}) \in Z \Rightarrow \bar{y} \in Z$. $\bar{y} > \bar{z}$

contradicts that $\bar{z}$ is the greatest ~~solution~~ element of Z.

so $\bar{y} \le \bar{z}$

② Suppose $\underline{y} > \underline{z}$

for $\underline{y} \in Y$, $\underline{z} \in Z$

$\min(\underline{y}, \underline{z}) \in Y \Rightarrow \underline{z} \in Y$ $\underline{z} < \underline{y}$

contradicts that $\underline{y}$ is the least element of Y.

so $\underline{y} \le \underline{z}$