$$F = \begin{pmatrix} 0.48 & 0 & -0.24 \\ -0.24 & 0.71 & -0.03 \\ 0.66 & -0.25 & 0.06 \end{pmatrix}$$

$$|F - \lambda I| = \begin{vmatrix} 0.48-\lambda & 0 & -0.24 \\ -0.24 & 0.71-\lambda & -0.03 \\ 0.66 & -0.25 & 0.06-\lambda \end{vmatrix} = (0.48-\lambda)\cdot\left((0.71-\lambda)(0.06-\lambda) - 0.015\right) + (-0.24)\left(0.02 - 0.66\cdot(0.71-\lambda)\right) = 0.$$

$\lambda_1 =$ $\lambda_2 =$ $\lambda_3 =$

$$\begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & 1 & 0 \\ -a_{31} & -a_{32} & 1 \end{pmatrix}\begin{pmatrix} c_t \\ y_t \\ r_t \end{pmatrix} = B_1 \begin{pmatrix} c_{t-1} \\ y_{t-1} \\ r_{t-1} \end{pmatrix} + \nu_t$$

By ~~LDL decomposition and the~~

By chol decomposition.

$$\Omega = \begin{pmatrix} & 0 & 0 \\ & & 0 \\ & & \end{pmatrix}\begin{pmatrix} 0 & & \\ 0 & & \\ 0 & 0 & \end{pmatrix}$$

By LDL decomposition.

$$\cancel{\Omega =} \; LDL' = \begin{pmatrix} & 0 & 0 \\ & & 0 \\ & & \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \end{pmatrix}\begin{pmatrix} 0 & & \\ 0 & \ddots & \\ 0 & & 0 \end{pmatrix} = \Omega$$

$$B_0 = \begin{pmatrix} & 0 & 0 \\ & & 0 \\ & & \end{pmatrix}^{-1}$$

(2) $\cancel{E\, W(1)\, W(2)^2}$

$$E\left(W(1)^2\, W(2)\, W(3)\right)$$

$$= E\left(W(1)^2\, W(2)\left((W(3)-W(2))+W(2)\right)\right)$$

$$= E\left(W(1)^2\, W(2)^2\right) + E\left(W(1)^2\, W(2)\left(W(3)-W(2)\right)\right)$$

by $W(3)-W(2) \perp W(2)$
$W(3)-W(2) \perp W(1)$
$\Rightarrow W(3)-W(2) \perp W(2)\, W(1)^2$

$$E\left(W(1)^2\, W(2)\left(W(3)-W(2)\right)\right) = E\left(W(1)^2 W(2)\right)\cdot E\left(W(3)-W(2)\right)$$

$$= E\left(W(1)^2 W(2)\right)\left(E\,W(3)-E\,W(2)\right) = 0.$$

by $W(2)-W(1) \perp W(1)$

$$E\left(W(1)^2\, W(2)^2\right) = E\,W(1)^2\left(W(2)-W(1)+W(1)\right)^2$$

$$= E\,W(1)^4 + E\,W(1)^2\left(W(2)-W(1)\right)^2 + 2\,E\,W(1)^3\left(W(2)-W(1)\right)$$

$$= E\,W(1)^4 + E\,W(1)^2\, E\left(W(2)-W(1)\right)^2 + 2\,E\,W(1)^3\, E\left(W(2)-W(1)\right)$$

$$= E\,W(1)^4 + E\,W(1)^2\, E\left(W(2)-W(1)\right)^2$$

$W(1) \sim N(0,1)$ $\quad W(2)-W(1) \sim N(0,1)$ $\quad \cancel{W(1)^4}$

$$E\,W(1)^2 = E\left(W(1)^2\right) - \left(E\,W(1)\right)^2 = Var\left(W(1)\right) = 1$$

$$E\left(W(2)-W(1)\right)^2 = E\left(W(2)-W(1)\right)^2 - \left(E\left(W(2)-W(1)\right)\right)^2 = Var\left(W(2)-W(1)\right) = 1$$

$$E\,W(1)^4 = 1^4(4-1) = 3$$

$$E\,W(1)^2\, W(2)^2 = 3 + 1\cdot 1 = 4.$$

$$E\left(W(1)^2\, W(2)\, W(3)\right) = 4.$$

$A \in \{t-2k \mid k \in N,$ k

(2) (a) For given $t \in \{1, 2, \ldots T\}$

① $A \in \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$$P(S_t = A) = \cancel{\left(\frac{1}{2}\right)^{\frac{t-A}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{t+A}{2}}} \quad \left(\frac{1}{2}\right)^{\frac{t-A}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{t+A}{2}} \cdot \binom{t}{\frac{t+A}{2}}$$

$$= \binom{t}{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{\frac{t+A}{2}} = \binom{t}{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{t}$$

② $A \notin \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$P(S_t = A) = 0$

(b) ① $A \in \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(S_t = A) - P(S_t = A, \min_{t' \le t} S_{t'} > B)$$

since $\min_{t' \le t} S_{t'} \le S_t = A \le B$, $\cancel{P\ S_t = A, \min_{t' \le t} S_{t'}}$

$\cancel{P \quad S_t = A}$

$$P(\min_{t' \le t} S_{t'} \le B \mid S_t = A) = 1$$

$$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(\min_{t' \le t} S_{t'} \le B \mid S_t = A)\, P(S_t = A)$$

$$= P(S_t = A) = \binom{t}{\frac{t-A}{2}} \cdot \left(\frac{1}{2}\right)^{t}$$

③ $A \notin \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$P(S_t = A) = 0 \Rightarrow P(S_t = A, \min_{t' \le t} S_{t'} \le B) = 0.$

(c) ① $A \in \{t-2k \mid k \in N, \quad k$

$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(S_t = A) - P(S_t = A, \min_{t' \le t} S_{t'} > B)$

$= P(S_t = A) - P(\min_{t' \le t} S_{t'} > B \mid S_t = A)\, P(S_t = A)$

$P(\min_{t' \le t} S_{t'} > B \mid S_t = A) = P(S_0 > B, S_1 > B, S_2 > B \cdots S_t > B \mid S_t = A)$

$= P(S_0 > B, S_1 > B \cdots S_{t-1} > B \mid S_t = A)$

~~$= P \; t^*$~~

$= \begin{cases} 0 & B \geq 0 \\ P(t^* \geq t \mid S_t = A) & B < 0 \end{cases}$ (t^* is the first time $S_t = B$.)

$= \begin{cases} 0 & B \geq 0 \\ P(t^* \geq t+1 \mid S_t = A) & B < 0 \end{cases}$

$= \begin{cases} 0 & B \geq 0 \\ P(S_{t+1} \geq B \mid S_t = A) & B < 0 \end{cases}$

$\cancel{(\frac{1}{2})^m \cdot (\frac{1}{2})^{m+B}}$

$(\frac{1}{2})^k$

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$t - t^* + |s_t^\circ - s_{t^*}|$

(c) Define $\hat{S}_t = \begin{cases} S_t & \text{if } t \le t^* \\ 2B - S_t & \text{if } t > t^* \end{cases}$

where $t^* = \min\{t \mid S_t = B\}$

For $t > t^*$

$\hat{S}_t = 2B - S_t = \cancel{S_{t^*}} + B - S_t = B + \sum_{i=1}^{t^*} \varepsilon_i - \sum_{i=1}^{t} \varepsilon_i$

$= \cancel{B + \sum}$

$= B - \sum_{i=t^*+1}^{t} \varepsilon_i$

$= S_{t^*} - \sum_{i=t^*+1}^{t} \varepsilon_i$

while $S_t = S_{t^*} + \sum_{i=t^*+1}^{t} \varepsilon_i$

Let ε_t° be the ~~non~~ realization of ε_t. S_t° be the realization of S_t

$\cancel{P(S_t^\circ \mid S_{t^*}^\circ)}$

$P(S_t = S_t^\circ \mid S_{t^*} = S_{t^*}^\circ) = \cancel{P\left(\varepsilon_{t^*+1} \ldots\right)}$

$\binom{t-t^*}{\frac{t-t^*-|S_t^\circ - S_{t^*}^\circ|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*-|S_t^\circ - S_{t^*}^\circ|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*+(S_t^\circ - S_{t^*}^\circ)}{2}}$

$P(\hat{S}_t = \hat{S}_t^\circ \mid S_{t^*} = S_{t^*}^\circ) = \binom{t-t^*}{\frac{t-t^*-|\hat{S}_t^\circ - S_{t^*}^\circ|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*-|\hat{S}_t^\circ - \hat{S}_{t^*}^\circ|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*+|\hat{S}_t^\circ - \hat{S}_{t^*}^\circ|}{2}}$

$= \binom{t-t^*}{\frac{t-t^*-|S_t^\circ - S_{t^*}^\circ|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*-|S_t^\circ - S_{t^*}^\circ|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*+|S_t^\circ - S_{t^*}^\circ|}{2}}$

$= P(S_t = S_t^\circ \mid S_{t^*} = S_{t^*}^\circ)$

if $B > 0$, then

$\cancel{P(S_t = A)}$ $P(S_t = A, \min_{t' \le t} S_{t'} \le B)$
$= P(S_t = A)$

so we ~~only~~ consider $B \le 0$

$$P(\hat{S}_t = s_t^{\circ} \mid S_{t^*} = s_{t^*}^{\circ}) = \binom{t-t^*}{\frac{t-t^*-|s_t^{\circ}-s_{t^*}^{\circ}|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*-|s_t^{\circ}-s_{t^*}^{\circ}|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^*+|s_t^{\circ}-s_{t^*}^{\circ}|}{2}}$$

$$= P(S_t = s_t^{\circ} \mid S_{t^*} = s_{t^*}^{\circ})$$

Since $\hat{S}_t = S_t$ for $t \le t^*$

We have ① $\hat{S}_t$ has symmetric distribution across B
② $\hat{S}_t$ has the ~~same~~ same distribution as S_t.

$$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(\hat{S}_t = 2B - A)$$

$$= \begin{cases} \binom{t}{\frac{t-|2B-A|}{2}} \left(\frac{1}{2}\right)^{\frac{t-|2B-A|}{2}} \left(\frac{1}{2}\right)^{\frac{t+|2B-A|}{2}} & \text{if } \frac{t-|2B-A|}{2} \in \{0,1,\dots,t\} \\ 0 & \text{if } \frac{t-|2B-A|}{2} \notin \{0,1,\dots,t\} \end{cases}$$

(d). $P(\min_{t' \le t} S_{t'} \le B) = P(S_t > B, \min_{t' \le t} S_{t'} \le B) + P(S_t \le B, \min_{t' \le t} S_{t'} \le B)$

$$= P(\hat{S}_t < B) + P(S_t \le B) \quad (*)$$

$$= P(S_t < B) + P(S_t \le B)$$

$$= 2P(S_t \le B) - \cancel{2P(S_t=B)}\ P(S_t = B)$$

(*) is because $\hat{S}_t$ has the same distribution as S_t by definition

and $P(S_t \le B, \min_{t' \le t} S_{t'} \le B) = P(S_t \le B)$

(e) the distribution of S_t is that.

$$P(S_t = A) = \begin{cases} \binom{t}{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{\frac{t+A}{2}} = \left(\frac{1}{2}\right)^t \binom{t}{\frac{t-A}{2}} & A \in \{-t, -(t-1), \cdots 0, 1, \cdots t-1, t\} \\ 0 & A \notin \{-t, -(t-1), \cdots 0, 1, \cdots t-1, t\} \end{cases}$$

$$\frac{1}{\sqrt{T}} S_{[T\gamma]} = \frac{\sqrt{[T\gamma]}}{\sqrt{T}} \cdot \sqrt{[T\gamma]} \frac{1}{[T\gamma]} S_{[T\gamma]}$$

$$= \frac{\sqrt{[T\gamma]}}{\sqrt{T}} \sqrt{[T\gamma]} \frac{1}{[T\gamma]} (\varepsilon_1 + \varepsilon_2 \cdots \varepsilon_{[T\gamma]})$$

ε_i are iid, $E(\varepsilon_i) = 0$, $Var(\varepsilon_i) = 1 < \infty$, by CLT.

$$\sqrt{[T\gamma]} \frac{1}{[T\gamma]} (\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_{[T\gamma]}) \xrightarrow[T\to\infty]{d} N(0,1)$$

$$\frac{\sqrt{[T\gamma]}}{\sqrt{T}} \xrightarrow[T\to\infty]{} \sqrt{\gamma}$$

so $\frac{1}{\sqrt{T}} S_{[T\gamma]} \xrightarrow{d} \sqrt{\gamma}\, N(0,1) = N(0,\gamma)$

(f) ~~given~~ given sample $\{\varepsilon_1, \varepsilon_2 \cdots \varepsilon_T\}$

$$\frac{1}{\sqrt{T}} S_{[T\gamma]} \xrightarrow{d} N(0,\gamma) = W(\gamma)$$

Let $F(g) = \min\limits_{0 \le t \le r} g(t)$

① F is continuous

Proof: $\forall \varepsilon > 0$.