$A \in \{t-2k \mid k \in N,\ k$

(2) (a) For given $t \in \{1, 2, \dots T\}$

① $A \in \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$$P(S_t = A) = \cancel{\left(\tfrac{1}{2}\right)^{\frac{t-A}{2}} \cdot \left(\tfrac{1}{2}\right)^{\frac{t+A}{2}}} \quad \left(\tfrac{1}{2}\right)^{\frac{t-A}{2}} \cdot \left(\tfrac{1}{2}\right)^{\frac{t+A}{2}} \cdot \binom{t}{\frac{t+A}{2}}$$

$$= \binom{t}{\frac{t-A}{2}} \left(\tfrac{1}{2}\right)^{\frac{t-A}{2}} \left(\tfrac{1}{2}\right)^{\frac{t+A}{2}} = \binom{t}{\frac{t-A}{2}} \left(\tfrac{1}{2}\right)^{t}$$

② $A \notin \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$P(S_t = A) = 0$

(b) ① $A \in \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(S_t = A) - P(S_t = A, \min_{t' \le t} S_{t'} > B)$$

since $\min_{t' \le t} S_{t'} \le S_t = A \le B$, $\cancel{P\ S_t = A, \min_{t' \le t} S_{t'}}$

$$P(\min_{t' \le t} S_{t'} \le B \mid S_t = A) = 1$$

$$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(\min_{t' \le t} S_{t'} \le B \mid S_t = A)\ P(S_t = A)$$

$$= P(S_t = A) = \binom{t}{\frac{t-A}{2}} \cdot \left(\tfrac{1}{2}\right)^{t}$$

③ $A \notin \{t-2k \mid 0 \le k \le t\}$ $k \in N$

$P(S_t = A) = 0 \Rightarrow P(S_t = A, \min_{t' \le t} S_{t'} \le B) = 0.$

$t-t^* + |s_t^0 - s_{t^*}^0|$

(c) Define $\hat{S}_t = \begin{cases} S_t & \text{if } t \leq t^* \\ 2B - S_t & \text{if } t > t^* \end{cases}$

where $t^* = \min\{t \mid S_t = B\}$

For $t > t^*$

$$\hat{S}_t = 2B - S_t = S_{t^*} + B - S_t = B + \sum_{i=1}^{t^*} \xi_i - \sum_{i=1}^{t} \xi_i$$

$$= \cancel{B + \sum}$$

$$= B - \sum_{i=t^*+1}^{t} \xi_i$$

$$= S_{t^*} - \sum_{i=t^*+1}^{t} \xi_i$$

while $S_t = S_{t^*} + \sum_{i=t^*+1}^{t} \xi_i$

If $B > 0$, then

~~$P(S_t = A)$~~ $P(S_t = A, \min_{t' \leq t} S_{t'} \leq B)$

$= P(S_t = A)$

so we ~~only~~ consider $B \leq 0$

Let ξ_t^0 be the ~~ren~~ realization of ξ_t. s_t^0 be the realization of S_t

$\cancel{P(S_t^0 \mid S_{t^*}^0)}$

$$P(S_t = s_t^0 \mid S_{t^*} = s_{t^*}^0) = \cancel{P\left(\xi_{t^*+1} \cdots\right)}$$

$$\binom{t-t^*}{\frac{t-t^* - |s_t^0 - s_{t^*}^0|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^* - |s_t^0 - s_{t^*}^0|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^* + (s_t^0 - s_{t^*}^0)}{2}}$$

$$P(\hat{S}_t = \hat{s}_t^0 \mid S_{t^*} = s_{t^*}^0) = \binom{t-t^*}{\frac{t-t^* - |\hat{s}_t^0 - s_{t^*}^0|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^* - |\hat{s}_t^0 - \hat{s}_{t^*}^0|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^* + |\hat{s}_t^0 - \hat{s}_{t^*}^0|}{2}}$$

$$= \binom{t-t^*}{\frac{t-t^* - |s_t^0 - s_{t^*}^0|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^* - |s_t^0 - s_{t^*}^0|}{2}} \left(\frac{1}{2}\right)^{\frac{t-t^* + |s_t^0 - s_{t^*}^0|}{2}}$$

$$= P(S_t = s_t^0 \mid S_{t^*} = s_{t^*}^0)$$

$$P(\hat{S}_t = S_{t'} \mid S_{t^*} = S^{\circ}_{t^*}) = \binom{t-t^*}{\frac{t-t^*-|S_{t'}-S^{\circ}_{t^*}|}{2}} \left(\tfrac{1}{2}\right)^{\frac{t-t^*-|S_{t'}-S^{\circ}_{t^*}|}{2}} \left(\tfrac{1}{2}\right)^{\frac{t-t^*+|S^{\circ}_t - S_{t^*}|}{2}}$$

$$= P(S_t = S^{\circ}_t \mid S_{t^*} = S^{\circ}_{t^*})$$

Since $\tilde{S}_t = S_t$ for $t \le t^*$

We have $\begin{cases} ① \ \hat{S}_t \text{ has symmetric distribution across } B \\ ② \ \hat{S}_t \text{ has the } \text{same} \text{ distribution as } S_t. \end{cases}$

$$P(S_t = A, \min_{t' \le t} S_{t'} \le B) = P(\hat{S}_t = 2B - A)$$

$$= \begin{cases} \binom{t}{\frac{t-|2B-A|}{2}} \left(\tfrac{1}{2}\right)^{\frac{t-|2B-A|}{2}} \left(\tfrac{1}{2}\right)^{\frac{t+|2B-A|}{2}} & \text{if } \frac{t-|2B-A|}{2} \in \{0, 1, \dots t\} \\ 0 & \text{if } \frac{t-|2B-A|}{2} \notin \{0, 1, \dots t\} \end{cases}$$

(d). $P(\min_{t' \le t} S_{t'} \le B) = P(S_t > B, \min_{t' \le t} S_{t'} \le B) + P(S_t \le B, \min_{t' \le t} S_{t'} \le B)$

$$= P(\hat{S}_t < B) + P(S_t \le B) \quad (*)$$

$$= P(S_t < B) + P(S_t \le B)$$

$$= 2P(S_t \le B) - \cancel{2P(S_t = B)}\ P(S_t = B)$$

(*) is because $\hat{S}_t$ has the same distribution as S_t by definition

and $P(S_t \le B, \min_{t' \le t} S_{t'} \le B) = P(S_t \le B)$

(e) the distribution of S_t is that.

$$P(S_t = A) = \begin{cases} \binom{t}{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{\frac{t-A}{2}} \left(\frac{1}{2}\right)^{\frac{t+A}{2}} = \left(\frac{1}{2}\right)^t \binom{t}{\frac{t+A}{2}} & A \in \{-t, -(t-1), \cdots 0, 1, \cdots t-1, t\} \\ 0 & A \notin \{-t, -(t-1), \cdots 0, 1, \cdots t-1, t\} \end{cases}$$

$$\frac{1}{\sqrt{T}} S_{[T\gamma]} = \frac{\sqrt{[T\gamma]}}{\sqrt{T}} \cdot \sqrt{[T\gamma]} \frac{1}{[T\gamma]} S_{[T\gamma]}$$

$$= \frac{\sqrt{[T\gamma]}}{\sqrt{T}} \sqrt{[T\gamma]} \frac{1}{[T\gamma]} (\varepsilon_1 + \varepsilon_2 \cdots \varepsilon_{[T\gamma]})$$

ε_i are iid, $E(\varepsilon_i) = 0$, $Var(\varepsilon_i) = 1 < \infty$, by CLT.

$$\sqrt{[T\gamma]} \frac{1}{[T\gamma]} (\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_{[T\gamma]}) \xrightarrow[T\to\infty]{d} N(0,1)$$

$$\frac{\sqrt{[T\gamma]}}{\sqrt{T}} \xrightarrow[T\to\infty]{} \sqrt{\gamma}$$

so $\frac{1}{\sqrt{T}} S_{[T\gamma]} \xrightarrow{d} \sqrt{\gamma} N(0,1) = N(0,\gamma)$

(f) ~~given~~ given sample $\{\varepsilon_1, \varepsilon_2 \cdots \varepsilon_T\}$

$$\frac{1}{\sqrt{T}} S_{[T\gamma]} \xrightarrow{d} N(0,\gamma) = W(\gamma)$$

Let $F(g) = \min_{0 \le \tau \le r} g(\tau)$

① F is continuous

Proof: $\forall \varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{2}$, for $\forall g_1, g_2$ $d_\infty(g_1, g_2) \le \delta$, $g_1, g_2 \in C[0,\gamma]$ ~~(B)~~

(a) Let $g_1(\tau_1^*) = \min_{0 \le \tau \le r} g_1(\tau)$ $g_2(\tau_2^*) = \min_{0 \le \tau \le r} g_2(\tau)$ $0 \le \tau_1^* \le r$, $0 \le \tau_2^* \le r$

(1) suppose $g_2(\tau_2^*) < g_1(\tau_1^*) - \delta$, then

$$g_2(\tau_2^*) < g_1(\tau_1^*) - \delta \le g_1(\tau_2^*) - \delta \Rightarrow g_1(\tau_2^*) - g_2(\tau_2^*) > \delta$$

~~Contr~~ that ~~contradicts that~~ $d_\infty(g_1, g_2) \le \delta$

$\cdots S_t = 2B - A)$

(2) Suppose $g_2(\tau_2^*) > g_1(\tau_1^*) + \delta$, then

$g_2(\tau_1^*) \geq g_2(\tau_2^*) > g_1(\tau_1^*) + \delta \Rightarrow g_2(\tau_1^*) - g_1(\tau_1^*) > \delta$

that contradicts that ~~of~~ $\cancel{d_\infty(g_1, g_2) < \delta}$

$d_\infty(g_1, g_2) \leq \delta$

So $\| g_1(\tau_1^*) - g_2(\tau_2^*) \| \leq \delta \Rightarrow \| \min_{0 \leq \tau \leq r} g_1(\tau) - \min_{0 \leq \tau \leq r} g_2(\tau) \| \leq \delta = \frac{\varepsilon}{2} < \varepsilon$.

$\Rightarrow \| F(g_1(\tau)) - F(g_2(\tau)) \| < \varepsilon$

③ F is bounded.

$\exists M > 1$

$\| F(g) \| = \left| \min_{0 \leq \tau \leq r} g(\tau) \right| \leq \|g\| = \sup_{0 \leq \tau \leq r} |g(\tau)|$

$\frac{1}{\sqrt{T}} S_{[T\tau]} \xrightarrow[T \to \infty]{d} W(\tau)$, by FCLT

$\min_{0 \leq \tau \leq r} \frac{1}{\sqrt{T}} S_{[T\tau]} \xrightarrow[T \to \infty]{d} \min_{0 \leq \tau \leq r} W(\tau)$

$P\left(\min_{0 \leq \tau \leq r} \frac{1}{\sqrt{T}} S_{[T\tau]} \leq B \right) \xrightarrow[T \to \infty]{} P\left(\min_{0 \leq \tau \leq r} W(\tau) \leq B \right)$.

$$
\begin{aligned}
P\left(\min_{0 \leq \tau \leq r} \frac{1}{\sqrt{T}} S_{[T\tau]} \leq B \right) &= P\left(\min_{0 \leq \tau \leq r} S_{[T\tau]} \leq B \right) \\
&= P\left(\min_{0 \leq t \leq [Tr]} S_t \leq B \right) \\
&= 2P\left(S_{[Tr]} \leq B \right) - P\left(S_{[Tr]} = B \right) = \cancel{2P\left(\frac{1}{\sqrt{T}} S_{[Tr]} \leq B \right) -} \\
&= \cancel{2 \sum_{A=-[Tr]}^{[Tr]} \left(\tfrac{1}{2}\right)^{[Tr]} \binom{t}{\frac{t-A}{2}} - \left(\tfrac{1}{2}\right)^{t} \binom{t}{\frac{t-B}{2}}} \quad \cancel{P\left(\frac{1}{\sqrt{T}} S_{[Tr]} = B \right)} \\
&= 2 \sum_{A=-[Tr]}^{B} \left(\tfrac{1}{2}\right)^{[Tr]} \binom{[Tr]}{\frac{[Tr]-A}{2}} - \left(\tfrac{1}{2}\right)^{[Tr]} \binom{[Tr]}{\frac{[Tr]-B}{2}}
\end{aligned}
$$

$\cancel{\xrightarrow[T \to \infty]{} 2 \int_{-\infty}^{B} \frac{1}{\sqrt{2\pi t}}}$

$S_t = A$ min ($\le B$) - $P($ ...

$2P$

$$2P\left(S_{[Tr]} \le B\right) - P\left(S_{[Tr]} = B\right) = 2P\left(\frac{1}{\sqrt{T}} S_{[Tr]} \le b\right) - P\left(\frac{1}{\sqrt{T}} S_{[Tr]} = b\right).$$

$$\xrightarrow[T\to\infty]{} 2P\left(W(r) \le b\right) - P\left(W(r) = b\right) \qquad (W(r) \sim N(0, r))$$

$$= 2 \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi r}} e^{-\frac{x^2}{2r}} dx - 0.$$

$$P\left(\min_{0 \le \tau \le r} W(\tau) \le b\right) = 2 \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi r}} e^{-\frac{x^2}{2r}} dx$$