7(1)

$$\frac{1}{\sqrt{T}} Y_{[T\tau]} = \frac{1}{\sqrt{T}} \max\{0, Y^*_{[T\tau]}\}$$

$$= \max\{0, \frac{1}{\sqrt{T}} Y^*_{[T\tau]}\}$$

$$= \max\{0, \frac{1}{\sqrt{T}} \sum_{i=1}^{[T\tau]} \varepsilon_i\} = \max\{0, S_T(\tau)\}$$

$$\cancel{= \max\{0, \frac{1}{\sqrt{T}} S_T(\tau)\}} \qquad S_T(\tau) = \frac{1}{\sqrt{T}} \sum_{i=1}^{[T\tau]} \varepsilon_i$$

Since $\varepsilon_t \sim iid(0, \sigma^2)$, by Functional CLT

$$S_T(\tau) \xrightarrow[T\to\infty]{d} \sigma W(\tau)$$

$$\max\{0, S_T(\tau)\} \xrightarrow[T\to\infty]{d} \max\{0, \sigma W(\tau)\} = \sigma \max\{0, W(\tau)\}$$

Note that ~~$F = \max\{0,$~~

$F(S_T(\tau)) = \max\{0, S_T(\tau)\}$ is a continuous and bounded Function.

Let $F(f(\tau)) = \max\{0, f(\tau)\}$ this transformation is a continuous transformation from $f(\tau)$ to $Ff(\tau)$

$\|F(f(\tau))\| \le \|f(\tau)\| < \infty$, so F is also bounded.

(2) $\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\left(Y_{t-1}(\rho-\hat{\rho}) + \varepsilon_t\right)^2 = \frac{1}{T}\sum_{t=1}^{T}\left(Y_{t-1}^2(\rho-\hat{\rho})^2 + \varepsilon_t^2 + 2Y_{t-1}(\rho-\hat{\rho})\varepsilon_t\right)$

~~First do~~

$\cancel{\rho - \hat{\rho}}$ $\hat{\rho} = \dfrac{\sum_{t=1}^{T} Y_t Y_{t-1}}{\sum_{t=1}^{T} Y_{t-1}^2} = \dfrac{\rho\sum_{t=1}^{T} Y_{t-1}^2}{\sum_{t=1}^{T} Y_{t-1}^2} + \dfrac{\sum_{t=1}^{T}\varepsilon_t Y_{t-1}}{\sum_{t=1}^{T} Y_{t-1}^2} = \rho + \dfrac{\sum_{t=1}^{T}\varepsilon_t Y_{t-1}}{\sum_{t=1}^{T} Y_{t-1}^2}$

$$\hat{\rho} - \rho = \frac{\frac{1}{T^2}\sum_{t=1}^{T}\varepsilon_t Y_{t-1}}{\frac{1}{T^2}\sum_{t=1}^{T} Y_{t-1}^2}$$

$$\frac{1}{T}\left(\sum_{t=1}^{T} Y_{t-1}\varepsilon_t\right)(\rho-\hat{\rho}) = \frac{1}{T}\,\frac{\frac{1}{T^2}\left(\sum_{t=1}^{T}\varepsilon_t Y_{t-1}\right)^2}{\frac{1}{T^2}\sum_{t=1}^{T} Y_{t-1}^2}$$

① $\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t Y_{t-1} \xrightarrow{d} \frac{\sigma^2}{2}(W^2(1)-1)$ by FCLT and LLN

$\Rightarrow$ $\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t Y_{t-1}\right)^2 \xrightarrow{d} \left(\frac{\sigma^2}{2}W^2(1)-1\right)^2$ by FCLT

② $\frac{1}{T^2}\sum_{t=1}^{T} Y_{t-1}^2 \xrightarrow{d} \sigma^2\int_0^1 W(z)^2\,dz$ by FCLT.

③ $\frac{1}{T} \xrightarrow[T\to\infty]{} 0$.

$\rightarrow$ ~~$\frac{1}{T}$~~ by ~~Short~~ slutsky theorem:

$$\frac{1}{T}\left(\sum_{t=1}^{T} Y_{t-1}\varepsilon_t\right)(\rho-\hat{\rho}) = \frac{1}{T}\,\frac{\frac{1}{T^2}\left(\sum_{t=1}^{T}\varepsilon_t Y_{t-1}\right)^2}{\frac{1}{T^2}\sum_{t=1}^{T} Y_{t-1}^2} \xrightarrow[T\to\infty]{} 0\cdot\frac{\left(\frac{\sigma^2}{2}W^2(1)-1\right)^2}{\sigma^2\int_0^1 W(z)^2\,dz} = 0.$$

$$\left(\frac{1}{T}\sum_{t=1}^{T} Y_{t-1}^2\right)(\rho-\hat{\rho})^2 = \frac{1}{T}\,\frac{\frac{1}{T^2}\left(\sum_{t=1}^{T}\varepsilon_t Y_{t-1}\right)^2}{\frac{1}{T^2}\sum_{t=1}^{T} Y_{t-1}^2} \xrightarrow[T\to\infty]{} 0\cdot\frac{\left(\frac{\sigma^2}{2}W^2(1)-1\right)^2}{\sigma^2\int_0^1 W^2(z)\,dz} = 0.$$

(also by slutsky theorem)

$\varepsilon_t^2 \sim \text{iid}(0,\sigma^2)$, by LLN. $\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2 \xrightarrow{p} E(\varepsilon_t^2) = \sigma^2$

so $\hat{\sigma} \xrightarrow{p} 0+0+\sigma^2$ by algebric sum of convergence.

(3) (a)

$E(u_t^2) = \sigma^2$

$\sigma^2 = E(u_t^2) = E(\sigma_t^2) = w + \alpha E(u_{t-1}^2) + \beta E(\sigma_{t-1}^2) = w + \alpha \sigma^2 + \beta \sigma^2$

$\cancel{\sigma^2 = \frac{1-w}{\alpha+\beta}}$ $\quad \sigma^2 = \frac{w}{1-\alpha-\beta} \quad 0<\alpha+\beta<1$

Let $v_t = u_t^2 - E(u_t^2 | u_{t-1}, \dots) = u_t^2 - \sigma_t^2$, $\boxed{v_t \text{ is known when } u_t, u_{t-1}, \dots \text{ is given}}$

① $E(v_t) = E(E(v_t | u_{t-1} \dots)) = E(\sigma_t^2 - \sigma_t^2) = 0$

② ~~for t>s~~ for $\forall t > s$

$E(v_t v_s) = E(E(v_t v_s | u_{t-1}, \dots)) = \cancel{E\ v_s\ E\ v_t | u_{t-1} \dots\ 0}$

$= E(v_s E(v_t | u_{t-1} \dots))$

$= E(v_s \cdot 0) = 0$

$\cancel{v_t \sim WN(\delta^2)}$ $\quad v_t$ is WN.

$u_t^2 - v_t = w + \alpha u_{t-1}^2 + \beta (u_{t-1}^2 - v_{t-1})$

$u_t^2 = w + (\alpha+\beta) u_{t-1}^2 + v_t - \beta v_{t-1}$

$\cancel{E_{T} \ E_T u_{T+1}^2 = E(u_{T+1}^2 | u_T, u_{T-1} \dots)}$

$E_T u_{T+1}^2 = E(u_{T+1}^2 | Y_T, Y_{T-1}, \dots) = E(u_{T+1}^2 | u_T, u_{T-1} \dots)$

$\cancel{= w + (\alpha+\beta) u_{T-1}^2}$

$= w + (\alpha+\beta) u_T^2 + E(v_{T+1} - \beta v_T | u_T, u_{T-1} \dots)$

$\cancel{= w + (\alpha+\beta) u_T^2 + E(v_{T+1} | u_T \dots u_{T-1})}$

$= w + (\alpha+\beta) u_T^2 + E(v_{T+1} | u_T, u_{T-1} \dots) - \beta E(v_T | u_T, u_{T-1} \dots)$

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$$= w + (\alpha+\beta) u_T^2 + 0 - \beta \left(u_T^2 - E(u_T^2 \mid u_{T-1} \cdots)\right)$$

$$= w + (\alpha+\beta) u_T^2 - \beta (u_T^2 - b_T^2) = \cancel{w + (\alpha+\beta)}\ w + \alpha u_T^2 + \beta b_T^2$$

$$E_T u_{T+2}^2 = \cancel{E\, u_{T+2}^2 \mid u_T, u_{T-1}\ E\, u_{T+2}^2 \mid u_{T+1}, u_{T}, \ldots)} = E(u_{T+2}^2 \mid Y_T, Y_{T-1} \cdots)$$

$$= E(u_{T+2}^2 \mid u_T, u_{T-1} \cdots)$$

$$\cancel{= w + (\alpha+\beta)(u_{T+1}^2 \mid u_T \cdots)}$$

$$= w + (\alpha+\beta) E(u_{T+1}^2 \mid u_T, u_{T-1} \cdots) + E(v_{T+2} - \beta v_{T+1} \mid u_T, u_{T-1} \cdots)$$

$$= w + (\alpha+\beta)(w + \alpha u_T^2 + \beta b_T^2) + E(v_{T+2} \mid u_T, u_{T-1} \cdots) - \beta E(v_{T+1} \mid u_T, u_{T-1} \cdots)$$

$$E(v_{T+1} \mid u_T, u_{T-1} \cdots) = 0$$

$$E(v_{T+2} \mid u_T, u_{T-1} \cdots) = E\left(E(v_{T+2} \mid u_{T+1}, u_T, u_{T-1} \cdots)\right) = E(0) = 0$$

$\cdot (0) =$

$$E_T u_{T+2}^2 = w + (\alpha+\beta)(w + \alpha u_T^2 + \beta b_T^2)$$

$$E_T(u_{T+3}^2) = E(u_{T+3}^2 \mid u_T, u_{T-1} \cdots) = w + (\alpha+\beta) E(u_{T+2}^2 \mid u_T, \cdots) + E(v_{T+3} - \beta v_{T+2} \mid u_T \cdots)$$

$$E(v_{T+2} \mid u_T \cdots) = 0. \quad E(v_{T+3} \mid u_T \cdots) = E\left(E(v_{T+3} \mid u_{T+2}, u_{T+1}, u_T \cdots)\right) = E(0) = 0$$

$$E_T(u_{T+3}^2) = w + (\alpha+\beta)\left(w + (\alpha+\beta)(w + \alpha u_T^2 + \beta b_T^2)\right)$$

$$= w + (\alpha+\beta) w + (\alpha+\beta)^2 w + (\alpha+\beta)^2 (\alpha u_T^2 + \beta b_T^2)$$

$$= \frac{1-(\alpha+\beta)^3}{1-(\alpha+\beta)} w + (\alpha+\beta)^2 (\alpha u_T^2 + \beta b_T^2)$$

By Induction.

$$E_T(u_{T+h}^2) = \frac{1-(\alpha+\beta)^h}{1-(\alpha+\beta)} w + (\alpha+\beta)^{h-1} (\alpha u_T^2 + \beta b_T^2)$$

Proof: Suppose $E_T(u_{T+h-1}) = \dfrac{1-(\alpha+\beta)^{h-1}}{1-(\alpha+\beta)} + (\alpha+\beta)^{h-2} (\alpha\, u_T^2 + \beta b_T^2) \quad h \geq 3$

$$E_T(u^2_{T+h}) = E(u_{T+h} \mid u_T, u_{T-1} \cdots)$$

$$= w + (\alpha+\beta) E(u_{T+h-1} \mid u_T, u_{T-1} \cdots) + E(V_{T+h} - \beta V_{T+h-1} \mid u_T \cdots)$$

$$E(V_{T+h} \mid u_T \cdots) = \cancel{E_{\{u_{T+h-1} \cdots u_{T+1}\}} E(V_{T+h} \mid u_{T+h-1}, u_{T+h-2} \cdots u_{T+1}, u_T \cdots)}$$

$$= E_{\{u_{T+h-1} \cdots u_{T+1}\}}(E(V_{T+h} \mid u_{T+h-1} \cdots u_{T+1}, u_T \cdots)) \cancel{= E(0) = 0}$$

$$= E_{\{u_{T+h-1}, \cdots u_{T+1}\}}(0) = 0$$

$$E(V_{T+h-1} \mid u_T \cdots) = E_{\{u_{T+h-2}, \cdots u_{T+1}\}} E(V_{T+h-1} \mid u_{T+h-2}, \cdots u_{T+1}, u_T \cdots)$$

$$= E_{\{u_{T+h-2}, \cdots u_{T+1}\}}(0) = 0$$

$$E_T(u^2_{T+h}) = w + (\alpha+\beta) E(u_{T+h-1} \mid u_T \cdots) = \frac{1-(\alpha+\beta)^h}{1-(\alpha+\beta)} + (\alpha+\beta)^{h-1}(\alpha u_T^2 + \beta \sigma_T^2)$$

$$E_T(\sigma^2_{T+h}) = E_T u^2_{T+h} - E_T V_{T+h} = \frac{1-(\alpha+\beta)^h}{1-(\alpha+\beta)} + (\alpha+\beta)^{h-1}(\alpha u_T^2 + \beta \sigma_T^2)$$ ~~because~~ because $E_T V_{T+h} = 0$

~~(a)~~

(b) (i) $E_T Y_{T+1} = \cancel{E_T Y_T} \; \rho Y_T + E(u_{T+1} \mid u_T \cdots) = \rho Y_T$

$$E_T Y_{T+2} = \rho E_T Y_{T+1} + E_T u_{T+2}$$

$E_T u_{T+2} = \cancel{E\, u_{T+2} | u_{T}} \; E(u_{T+2} | u_T, u_{T-1} \cdots)$

$= E_{\{u_{T+1}\}} E(u_{T+2} | u_{T+1}, u_T \cdots) = \cancel{E_{u_{T+1}}} \; E_{\{u_{T+1}\}} \, 0 = 0$

$E_T Y_{T+2} = \rho^2 Y_T$.

~~Eq~~. By Induction

Suppose $E_T Y_{T+h-1} = \rho^{h-1} Y_T$, $\forall h \geq 2$ then,

$E_T Y_{T+h} = \rho E_T Y_{T+h-1} + E_T u_{T+h}$.

$E_T u_{T+h} = E(u_{T+h} | u_T \cdots) = E_{\{u_{T+h-1}, \ldots u_{T+1}\}} E(u_{T+h} | u_{T+h-1} \cdots u_{T+1}, u_T \cdots)$

$= E_{\{u_{T+h-1}, \ldots u_{T+1}\}} (0) = 0$

$E_T Y_{T+h} = \rho E_T Y_{T+h-1} = \rho^h Y_T$

(2)

$E_T Y_{T+h}^2 = E_T(\rho^2 Y_{T+h-1}^2 + u_{T+h}^2 + 2\rho u_{T+h} Y_{T+h-1})$

$E_T(u_{T+1} Y_T) = Y_T \cdot E_T(u_{T+1}) = 0$

~~$E_T(u_{T+h} Y_{T+h-1}$~~

$E_T(u_{T+h} Y_{T+h-1}) = E(u_{T+h} Y_{T+h-1} | u_T, u_{T-1} \cdots)$

$= E_{\{u_{T+h-1}, \cdots u_{T+1}\}} E(u_{T+h} Y_{T+h-1} | u_{T+h-1}, \cdots u_{T+1}, u_T \cdots)$

$= E_{\{u_{T+h-1} \cdots u_{T+1}\}} (0) = 0$

$$E_T Y_{T+h}^2 = E_T(\rho^2 Y_{T+h-1}^2) + E_T u_{T+h}^2$$

$$= \rho^2 E_T Y_{T+h-1}^2 + E_T u_{T+h}^2$$

$$E_T u_{T+h} = E_{\{u_{T+h-1},\dots u_{T+1}\}} E(u_{T+h} \mid u_{T+h-1} \dots u_{T+1}, u_T \dots)$$

$$= E_{\{u_{T+h-1},\dots u_{T+1}\}}(0) = 0$$

$$E_T Y_{T+1}^2 = \rho^2 E_T Y_T^2 + E_T u_{T+1}^2 + 2\rho E_T Y_T u_{T+1}$$

$$= \rho^2 Y_T^2 + \frac{1-(2+\beta)^h}{1-(2+\beta)} + (2+\beta)^{h-1}(2u_T^2 + \rho b_T^2)$$

$$\cancel{E_T Y_{T+2}^2 = \rho^2 E_T Y_{T+1}^2 + E_T u_{T+2} = \rho^2 E_T Y_{T+1}^2 = \rho^4 Y_T^2}$$

~~Suppose~~ $\cancel{E_T Y_{T+h-1}^2 = \rho^{2h-2} Y_T^2}$

~~Then~~ $\cancel{E_T Y_{T+h} = \rho^2 E_T Y_{T+h-1}^2}$

$$E_T Y_{T+2}^2 = \rho^2 E_T Y_{T+1}^2 + E_T u_{T+2}^2 + 2\rho E_T Y_{T+1} u_{T+2}$$

$$= \rho^2 E_T Y_{T+1}^2 + E_T u_{T+2}^2$$

$$= \rho^4 E_T Y_T^2 + \rho^2 E_T u_{T+1}^2 + E_T u_{T+2}^2$$

suppose $E_T Y_{T+h-1}^2 = \rho^{2(h-1)} Y_T^2 + \sum_{i=0}^{h-2} \rho^{2i} E_T u_{T+h-1-i}$ $\quad h \geq 3$

Then.

$$E_T Y_{T+h}^2 = \rho^2 E_T Y_{T+h-1}^2 + E_T u_{T+h}^2$$

$$= \rho^{2h} Y_T^2 + \sum_{i=0}^{h-2} \rho^{2i+2} E_T u_{T+h-1-i}^2 + E_T u_{T+h}^2$$

$$= \rho^{2h} Y_T^2 + \sum_{i=0}^{h-1} \rho^{2i} E_T u_{T+h-i}^2 + E_T u_{T+h}^2$$

$$= \rho^{2h} Y_T^2 + \sum_{i=0}^{h-1} \rho^{2i} E_T u_{T+h-i}^2$$

$$\sum_{i=0}^{h-1} \rho^{2i}\left(\frac{1-(\alpha+\beta)^{h-i}}{1-(\alpha+\beta)}w + (\alpha+\beta)^{h-i-1}(\alpha u_T^2+\beta b_T^2)\right)$$

$$= \sum_{i=0}^{h-1}\rho^{2i}\frac{1-(\alpha+\beta)^{h-i}}{1-(\alpha+\beta)}w + \sum_{i=0}^{h-1}\frac{(\rho^2)^i}{(\alpha+\beta)^i}(\alpha+\beta)^{h-1}(\alpha u_T^2+\beta b_T^2)$$

$$= \frac{1-(\rho^2)^h}{1-\rho^2}\frac{w}{1-\alpha-\beta} - \frac{(\alpha+\beta)^h}{1-(\alpha+\beta)}\sum_{i=0}^{h-1}\left(\frac{\beta^2}{\alpha+\beta}\right)^i w + (\alpha+\beta)^{h-1}(\alpha u_T^2+\beta b_T^2)\sum_{i=0}^{h-1}\left(\frac{\rho^2}{\alpha+\beta}\right)^i$$

$$= \begin{cases} \frac{1-\rho^{2h}}{(1-\rho^2)(1-(\alpha+\beta))}w - \frac{(\alpha+\beta)^h}{1-(\alpha+\beta)}\frac{1-\left(\frac{\rho^2}{\alpha+\beta}\right)^h}{1-\left(\frac{\rho^2}{\alpha+\beta}\right)}w + (\alpha+\beta)^{h-1}(\alpha u_T^2+\beta b_T^2)\frac{1-\left(\frac{\rho^2}{\alpha+\beta}\right)^h}{1-\left(\frac{\rho^2}{\alpha+\beta}\right)} & \rho^2\neq\alpha+\beta \\ \frac{1-\rho^{2h}}{(1-\rho^2)(1-\alpha-\beta)}w - \frac{(\alpha+\beta)^h h}{1-(\alpha+\beta)}w + (\alpha+\beta)^{h-1}(\alpha u_T^2+\beta b_T^2)h & \rho^2=\alpha+\beta \end{cases}$$

$$V_T = E_T Y_{T+h}^2 - (E_T Y_{T+h})^2 = E_T Y_{T+h}^2 - \rho^{2h}Y_T^2$$

$$= \sum_{i=0}^{h-1}\rho^{2i}E_T u_{T+h-i}^2$$

$$= \begin{cases} \frac{1-\rho^{2h}}{(1-\rho^2)(1-\alpha-\beta)}w - \frac{(\alpha+\beta)^h}{1-(\alpha+\beta)}\frac{1-\left(\frac{\rho^2}{\alpha+\beta}\right)^h}{1-\frac{\rho^2}{\alpha+\beta}}w + (\alpha+\beta)^{h-1}(\alpha u_T^2+\beta b_T^2)\frac{1-\left(\frac{\rho^2}{\alpha+\beta}\right)^h}{1-\frac{\rho^2}{\alpha+\beta}} & \rho^2\neq\alpha+\beta \\ \frac{1-\rho^{2h}}{(1-\rho^2)(1-\alpha-\beta)}w - \frac{(\alpha+\beta)^h h}{1-(\alpha+\beta)}w + (\alpha+\beta)^{h-1}(\alpha u_T^2+\beta b_T^2)h & \rho^2=\alpha+\beta \end{cases}$$

We don't need to consider $\alpha+\beta\geq 1$ because $\alpha+\beta<1$ and $w>0$ is necessary for $b_t^2>0$ for $\forall t$

(c) when $\rho = 1$ $\quad Y_t = Y_{t-1} + u_t$.

$$\sigma^2 = E(u_t^2) = E(\sigma_t^2) = w + \alpha E(u_{t-1}^2) + \beta E(\sigma_{t-1}^2) = w + \alpha\sigma^2 + \beta\sigma^2$$

$$\alpha + \beta < 1 \Rightarrow \sigma^2 = \frac{w}{1-(\alpha+\beta)}$$

~~(a) is not~~

(a) doesn't change. Ior (b)

$$E_T Y_{T+h} = E_T Y_{T+h-1} + E_T u_{T+h}.$$

$$E_T u_{T+h} = E(u_{T+h} \mid u_T, u_{T-1} \cdots) = E_{\{u_{T+h-1}, \cdots u_{T+1}\}} E(u_{T+h} \mid u_{T+h-1} \cdots u_{T+1}, u_T \cdots)$$

$$= E_{\{u_{T+h-1}, \cdots u_{T+1}\}}(0) = 0$$

$$E_T Y_{T+h} = E_T Y_{T+h-1}$$

$$E_T Y_{T+1} = E_T Y_T = Y_T$$

$$E_T Y_{T+2} = E_T Y_{T+1} = Y_T$$

By induction, suppose $E_T Y_{T+h-1} = Y_T \quad h \geq 3$

$$E_T Y_{T+h} = E_T Y_{T+h-1} = Y_T$$

For: $\quad E_T Y_{T+h}^2 = E_T Y_{T+h-1}^2 + E_T u_{T+h}^2 + 2E_T Y_{T+h-1} u_{T+h}$

$$E_T u_{T+h} Y_{T+h-1} = E(u_{T+h} Y_{T+h-1} \mid u_T, u_{T-1} \cdots)$$

$$= E_{\{u_{T+h-1}, \cdots u_{T+1}\}}\left(E(u_{T+h} Y_{T+h-1} \mid u_{T+h-1}, \cdots u_{T+1}, u_T \cdots)\right)$$

$$= E_{\{u_{T+h-1}, \cdots u_{T+1}\}}(0) = 0.$$

$$E_T Y_{T+h}^2 = E_T Y_{T+h-1}^2 + E_T u_{T+h}^2$$

$$E_T Y^2_{T+1} = E_T Y_T^2 + E_T u^2_{T+1} = Y_T^2 + E_T u^2_{T+1}$$

$$E_T Y^2_{T+2} = E_T Y^2_{T+1} + E_T u^2_{T+2} = Y_T^2 + E_T u^2_{T+1} + E_T u^2_{T+2}$$

Suppose: $E_T Y^2_{T+h-1} = Y_T^2 + \sum_{i=0}^{h-2} E_T u^2_{T+h-1-i}$

$$E_T Y^2_{T+h} = E_T Y^2_{T+h-1} + E_T u^2_{T+h}$$

$$= Y_T^2 + \sum_{i=0}^{h-2} E_T u^2_{T+h-1-i} + E_T u^2_{T+h}$$

$$= Y_T^2 + \sum_{i=1}^{h-1} E_T u^2_{T+h-i} + E_T u^2_{T+h} = Y_T^2 + \sum_{i=0}^{h-1} E_T u^2_{T+h-i}$$

$$V_T Y_{T+h} = E_T Y^2_{T+h} - (E_T Y_{T+h})^2$$

$$= Y_T^2 + \sum_{i=0}^{h-1} E_T u^2_{T+h-i} - Y_T^2 = \sum_{i=0}^{h-1} E_T u^2_{T+h-i}$$

$$= \sum_{i=0}^{h-1} \left(\frac{1-(\alpha+\beta)^{h-i}}{1-(\alpha+\beta)} w + (\alpha+\beta)^{h-i-1} (\alpha u_T^2 + \beta \sigma_T^2) \right)$$

$$= \frac{wh}{1-(\alpha+\beta)} - \frac{w(\alpha+\beta)^h}{1-(\alpha+\beta)} \frac{1-\left(\frac{1}{\alpha+\beta}\right)^h}{1-\frac{1}{\alpha+\beta}} + (\alpha+\beta)^{h-1} (\alpha u_T^2 + \beta \sigma_T^2) \frac{1-\left(\frac{1}{\alpha+\beta}\right)^h}{1-\frac{1}{\alpha+\beta}}$$

$$= \frac{wh}{1-(\alpha+\beta)} - \frac{w\left((\alpha+\beta)^h - 1\right)}{(1-(\alpha+\beta))\left(1-\frac{1}{\alpha+\beta}\right)} + \frac{\alpha u_T^2 + \beta \sigma_T^2}{\alpha+\beta-1} \left((\alpha+\beta)^h - 1\right)$$

$\alpha+\beta<1$

(4)

$E(u_t^2) = \sigma^2$

$E(u_t^2) = E(b_t^2) = w + \alpha E(u_{t-1}^2) + \beta E(b_{t-1}^2) = w + \alpha \sigma^2 + \beta E(b_{t-1}^2)$

$\sigma^2 = w + \alpha \sigma^2 + \beta \sigma^2 \qquad \alpha+\beta<1 \Rightarrow \sigma^2 = \frac{w}{1-(\alpha+\beta)}$

$\gamma(h) = Cov(u_{t+h}^2, u_t^2) = E(u_{t+h}^2 u_t^2) - E(u_{t+h}^2)E(u_t^2)$

$\gamma(0) = Cov(u_t^2, u_t^2) = E(u_t^4) - (E(u_t^2))^2 = E(u_t^4) - \sigma^4$

Let $v_t = u_t^2 - E_{t-1}u_t^2 = u_t^2 - E(u_t^2 \mid u_{t-1}, \dots) = u_t^2 - b_t^2$

$E(v_t) = E(E(v_t \mid u_{t-1} \dots)) = E(0) = 0.$

$\forall\, t>s$

$E(v_t v_s) = E(E(v_t v_s \mid u_{t-1} \dots)) = E(v_s E(v_t \mid u_{t-1} \dots)) = E(0) = 0$

So $v_t \sim WN$.

$u_t^2 - v_t = w + \alpha u_{t-1}^2 + \beta(u_{t-1}^2 - v_{t-1})$

~~$u_t^2 = w + \alpha u_t$~~

$u_t^2 = w + (\alpha+\beta)u_{t-1}^2 + v_t - \beta v_{t-1}$

$u_t^2 - \sigma^2 = (\alpha+\beta)(u_{t-1}^2 - \sigma^2) + v_t - \beta v_{t-1}$. ($\varepsilon_t = v_t - \beta v_{t-1}$ is not WN)

$= (\alpha+\beta)(u_{t-1}^2 - \sigma^2) + \varepsilon_t \qquad (\varepsilon_t = v_t - \beta v_{t-1})$

~~$(1-(\alpha+\beta)L)^{-1}(u_t^2 - \sigma^2) = \varepsilon_t$~~

$(1-(\alpha+\beta)L)(u_t^2 - \sigma^2) = \varepsilon_t$

$u_t^2 - \sigma^2 = \left(\sum_{i=0}^{\infty}((\alpha+\beta)L)^i\right)\varepsilon_t$

$= \sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t-i}$

(4)

$$Cov(U_{t+h}^2, U_t^2) = Cov\left(\sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t-i}, \sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t+h-i}\right)$$

$$= Cov\left(\sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t-i}, \sum_{i=-h}^{\infty}(\alpha+\beta)^{i+h}\varepsilon_{t-i}\right)$$

$$Cov(\varepsilon_t, \varepsilon_t) = E(\varepsilon_t^2) - E(\varepsilon_t)^2 = E(V_t - \beta V_{t-1})^2 = E(V_t^2) + \beta^2 E(V_{t-1}^2)$$

$$Cov(\varepsilon_t, \varepsilon_{t+1}) = E(\varepsilon_t \varepsilon_{t+1}) - E(\varepsilon_t)E(\varepsilon_{t+1})$$

$$= E(V_t - \beta V_{t-1})(V_{t+1} - \beta V_t) - E(V_t - \beta V_{t-1}) \cdot E(V_{t+1} - \beta V_t)$$

$$= E(V_t V_{t+1}) - \beta E(V_{t-1} V_{t+1}) + \beta^2 E(V_{t-1} V_t) - \beta E V_t^2 - 0$$

$$= -\beta E(V_t^2)$$

for $h \geq 2$

$$Cov(\varepsilon_t, \varepsilon_{t+h}) = \cancel{E(V_{t+h} - \beta V_{t+h-1})(V_{t+h-1} - \beta}$$

$$E(V_{t+h} - \beta V_{t+h-1})(V_t - \beta V_{t-1}) - E(\varepsilon_t)E(\varepsilon_{t+h}) = 0$$

$$Cov(U_{t+h}^2, U_t^2) = Cov\left(\sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t-i}, \sum_{i=-1}^{\infty}(\alpha+\beta)^{i+h}\varepsilon_{t-i}\right)$$

$$= \sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h}\left(E(V_{t-i}^2) + \beta^2 E(V_{t-i-1}^2)\right) +$$

$$\sum_{i=0}^{\infty}(\alpha+\beta)^i(\alpha+\beta)^{i+h-1}E(\varepsilon_{t-i}\varepsilon_{t-i+1}) +$$

$$\sum_{i=0}^{\infty}(\alpha+\beta)^i(\alpha+\beta)^{i+1+h}E(\varepsilon_{t-i}\varepsilon_{t-i-1})$$

$$= \sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h}\left(E(V_{t-i}^2) + \beta^2 E(V_{t-i-1}^2)\right) +$$

$$\sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h}\left(\frac{-\beta}{\alpha+\beta}E(V_{t-i}^2)\right) +$$

$$\sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h}(\alpha+\beta)(-\beta)E(V_{t-i-1}^2)$$

$$= \sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h}\left(\frac{\alpha}{\alpha+\beta}E(V_{t-i}^2) - \alpha\beta E(V_{t-i-1}^2)\right) \cancel{+ \sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h}E(V_{t-i}^2) + \sum_{i=0}^{\infty}(\alpha+\beta)^{2i+h-2}EV_{t-i}^2}$$

$$= Var(U_t) - Var(b_t)$$

$$= \sum_{i=0}^{\infty} (\alpha+\beta)^{2i+h-1} \alpha E(V_{t-i}^2) - \sum_{i=0}^{\infty} \alpha (\alpha+\beta)^{2i+h} \beta E(V_{t-i-1}^2)$$

$$= \sum_{i=0}^{\infty} (\alpha+\beta)^{2i+h-1} \alpha E(V_{t-i}^2) - \sum_{i=1}^{\infty} \alpha (\alpha+\beta)^{2i+h-2} \beta E(V_{t-i}^2) + \cancel{\sum_{i=0}^{\infty}((\alpha+\beta)^2+\beta^2)(\alpha+\beta)^{2i+h-2} E V_{t-i}^2}$$

$$= (\alpha+\beta)^{h-1} \alpha E(V_t^2) + \sum_{i=1}^{\infty} \alpha^2 (\alpha+\beta)^{2i+h-2} E(V_{t-i}^2) \; \cancel{= (\alpha+\beta)^{h-2}(\dots) E V_t^2 + \sum_{i=1}^{\infty}(\dots)(\alpha+\beta)^{2i+h} E V_{t-i}^2}$$

$$Cov(u_t^2, u_t^2) = Cov\left(\sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t-i}, \sum_{i=0}^{\infty}(\alpha+\beta)^i \varepsilon_{t-i}\right)$$

$$= \sum_{i=0}^{\infty}(\alpha+\beta)^{2i}\left(E(V_{t-i}^2) + \beta^2 E(V_{t-i-1}^2)\right) + 2\sum_{i=0}^{\infty}(\alpha+\beta)^{2i+1} Cov(\varepsilon_{t-i}, \varepsilon_{t-i-1})$$

$$= \sum_{i=0}^{\infty}(\alpha+\beta)^{2i}\left(E(V_{t-i}^2) + \beta^2 E(V_{t-i-1}^2)\right) + 2\sum_{i=0}^{\infty}(\alpha+\beta)^{2i+1}\left(-\beta E(V_{t-i-1}^2)\right)$$

$$= \sum_{i=0}^{\infty}(\alpha+\beta)^{2i} E V_{t-i}^2 + \sum_{i=1}^{\infty}\beta^2(\alpha+\beta)^{2i-2} E V_{t-i}^2 - 2\beta\sum_{i=1}^{\infty}(\alpha+\beta)^{2i-1} E V_{t-i}^2$$

$$\cancel{= \left(1 + \frac{\beta^2}{(\alpha+\beta)^2}\right) E V_t^2}$$

$$\cancel{= \sum_{i=0} E V_t^2 + \quad -\beta^2(\alpha+\beta)}$$

$$= E V_t^2 + \sum_{i=1}^{\infty}\left((\alpha+\beta)^2 - \beta^2 - 2\alpha\beta\right)(\alpha+\beta)^{2i-2} E V_{t-i}^2$$

$$= E V_t^2 + \sum_{i=1}^{\infty}\alpha^2(\alpha+\beta)^{2i-2} E V_{t-i}^2$$

One important thing to notice is that $E(b_t^4)$ doesn't change with time. may or may not hold.

~~Proof:~~ $E(u_t^4 \mid u_{t-1}\dots) - E(u_t^2 \mid u_{t-1}\dots)^2 = Var(u_t^2 \mid u_{t-1}\dots)$

$\cancel{E(u_t^4) =}$ $E(E(u_t^4 \mid u_{t-1}\dots)) - E(b_t^4) = E(Var(u_t^2 \mid u_{t-1}\dots))$

$$= Var(u_t^2) - Var(E(u_t^2 \mid u_{t-1}\dots))$$

$$E(u_t^4) - E(b_t^4) = Var(u_t^2) - Var(E(u_t^2 \mid u_{t-1}\dots))$$

$= Var(u_t^2) - Var(b_t^2)$ so not sure $E(b_t^4)$ is time-invariant

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{(\alpha+\beta)^{h-1}\alpha E V_t^2 + \sum_{i=1}^{\infty} \alpha^2 (\alpha+\beta)^{2i+h-2} E V_{t-i}^2}{E V_t^2 + \sum_{i=1}^{\infty} \alpha^2 (\alpha+\beta)^{2i-2} E V_{t-i}^2}$$

When $E V_t^2$ doesn't depend on t.

$$\cancel{\rho(h) = \alpha(\alpha+\beta)^{h-1} + \alpha^2(\alpha+\beta)^{h-2} \; 1-(\alpha+\beta)}$$

$$\rho(h) = \frac{\alpha(\alpha+\beta)^{h-1} + \alpha^2(\alpha+\beta)^{h-2}\frac{1}{1-(\alpha+\beta)^2}(\alpha+\beta)^2}{1 + \alpha^2 \frac{1}{1-(\alpha+\beta)^2}}$$

$$= (\alpha+\beta)^h \frac{\frac{\alpha}{\alpha+\beta} + \frac{\alpha^2}{1-(\alpha+\beta)^2}}{1 + \frac{\alpha^2}{1-(\alpha+\beta)^2}}, \quad h \geq 1$$

$$\rho(0) = 1$$