Intermediate Machine Learning: Assignment 3

Deadline

Assignment 3 is due Wednesday, October 30 by 11:59pm. Late work will not be accepted as per the course policies (see the Syllabus and Course policies on Canvas).

Directly sharing answers is not okay, but discussing problems with the course staff or with other students is encouraged.

You should start early so that you have time to get help if you're stuck. The drop-in office hours schedule can be found on Canvas. You can also post questions or start discussions on Ed Discussion. The assignment may look long at first glance, but the problems are broken up into steps that should help you to make steady progress.

Submission

Submit your assignment as a pdf file on Gradescope, and as a notebook (.ipynb) on Canvas. You can access Gradescope through Canvas on the left-side of the class home page. The problems in each homework assignment are numbered. Note: When submitting on Gradescope, please select the correct pages of your pdf that correspond to each problem. This will allow graders to more easily find your complete solution to each problem.

To produce the .pdf, please do the following in order to preserve the cell structure of the notebook:

Go to "File" at the top-left of your Jupyter Notebook Under "Download as", select "HTML (.html)" After the .html has downloaded, open it and then select "File" and "Print" (note you will not actually be printing) From the print window, select the option to save as a .pdf

Topics

- Variational autoencoders
- Undirected graphs
- The graphical lasso

This assignment will also help to solidify your Python and Jupyter notebook skills.

Problem 1: Face time (35 points)

In this problem, we will implement a "shoestring" version of this amazing fake face generator, using a variational autoencoder (VAE). Building a generator like the one featured in the article can take a tremendous amount of computational resources, time, and parameter tuning. In this problem we will build a basic version to illustrate the main concepts, and help you to become more familiar with VAEs. Here is an outline of the process that we'll step you through:

Problem outline:

- Load data
- Create face groups based on attributes
- Construct the VAE
- Define the loss function and train the VAE (Problem 1.1)
- Encode and reconstruct faces (Problem 1.2)
- Visualize the latent space (Problem 1.3)
- Morph between faces (Problem 1.4)
- Shift attributes of faces (Problem 1.5)
- Generate new faces (Problem 1.6)
- Analyze the effect of the scaling factor in the loss function (Problem 1.7, optional)

In the next cell we load the packages that we'll need. If you don't have one or more of these, you can install them with `!pip install <package_name>` in the cell, or outside the notebook with `conda install -c conda-forge <package_name>`

In [1]:
```
import warnings
warnings.filterwarnings('ignore')

import matplotlib.pyplot as plt
from sklearn.decomposition import PCA
import os
import glob
import pandas as pd
import random
import numpy as np
import imageio
```

```
from tqdm import tqdm
from PIL import Image
from sklearn.model_selection import train_test_split
from scipy.stats import norm
import tensorflow
tensorflow.compat.v1.disable_eager_execution()
```

Loading the data

Labeled Faces in the Wild (LFW) is a database of face photographs. The images are placed in the folder lfw-deepfunneled. lfw_attributes.txt is a document including a set of attributes associated for each image, such as 'Male', 'Smile', 'Bold', etc. All the features are numerical and large positive values indicate that the keywords well describe the photo; large negative values indicate that the keywords don't fit the photo.

For this problem, we will keep only the middle parts of the photos to avoid complex backgrounds.

Download the data from the cloud at these URLs:

https://sds365.s3.amazonaws.com/lfw/lfw-deepfunneled.zip

https://sds365.s3.amazonaws.com/lfw/lfw_attributes.txt
Once you have the data, unzip it, and place it in a directory that we will call "YOUR_PATH" below.

Run all the cells in this section to load the data.

Note: Please write down the entire path instead of using something like '~/Desktop/datasets/' to avoid unnecessary compiling errors. Also, if you choose to use Colab to do your homework. We need to download the data into the same directory as your code. You may also need

"from google.colab import drive"

"drive.mount('/content/drive')"

to enable using paths in Google Drive before starting your code below. But personally, I would suggest using jupyter notebook to run the code locally instead of using Google Colab since the latter may take longer time.

In [2]:
```
# Change these path names to correspond with your directory
DATASET_PATH = r"C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfunneled"
ATTRIBUTES_PATH = r"C:/jofan/yale/sds_365/pset/p3/data/lfw_attributes.txt"
```

In [3]:
```
# # Make sure the above path is correct before running this cell
# dataset = []
# for path in glob.iglob(os.path.join(DATASET_PATH, "**", "*.jpg")):
#     person = path.split("/")[-2]
#     dataset.append({"person":person, "path": path})

# I change this code to work on Windows
dataset = []
for path in glob.iglob(os.path.join(DATASET_PATH, "**", "*.jpg"), recursive=True):
    person = os.path.basename(os.path.dirname(path))
    dataset.append({"person": person, "path": path})

dataset = pd.DataFrame(dataset)
dataset = dataset.groupby("person").filter(lambda x: len(x) < 100)
dataset.head(10)
```

Out[3]:

	person	path
0	Aaron_Eckhart	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
1	Aaron_Guiel	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
2	Aaron_Patterson	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
3	Aaron_Peirsol	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
4	Aaron_Peirsol	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
5	Aaron_Peirsol	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
6	Aaron_Peirsol	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
7	Aaron_Pena	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
8	Aaron_Sorkin	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...
9	Aaron_Sorkin	C:/jofan/yale/sds_365/pset/p3/data/lfw-deepfun...

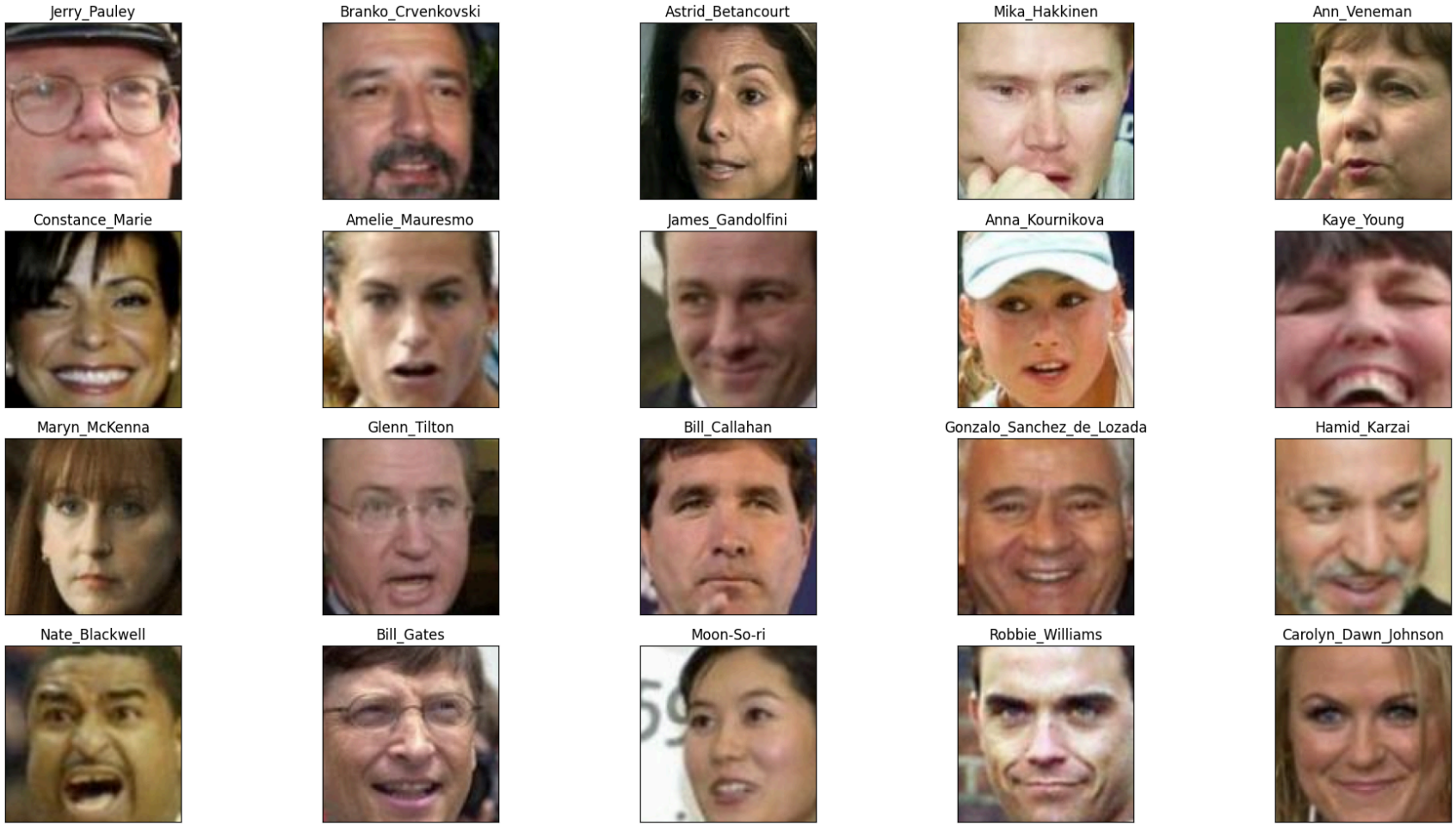
The following cell will display some sample images

In [4]:
```
sampled_id = []

plt.figure(figsize=(20,10))
for i in range(20):
    idx = random.randint(0, len(dataset))
    img = plt.imread(dataset.path.iloc[idx])
    plt.subplot(4, 5, i+1)
    plt.imshow(img)
    plt.title(dataset.person.iloc[idx])
    plt.xticks([])
    plt.yticks([])
    sampled_id.append(idx)
plt.tight_layout()
plt.show()
```

The following cell shows the images with some of the background removed.

In [5]:
```
dx=70
dy=70

plt.figure(figsize=(20,10))
for i in range(20):
    idx = sampled_id[i]
    img = plt.imread(dataset.path.iloc[idx])
    plt.subplot(4, 5, i+1)
    plt.imshow(img[dy:-dy,dx:-dx])
    plt.title(dataset.person.iloc[idx])
```

```
    plt.xticks([])
    plt.yticks([])
plt.tight_layout()
plt.show()
```



The following function crops the images to 45x45 pixels, which is what we will use in this problem.

In [6]:
```
def fetch_dataset(dx=70, dy=70, dimx=45, dimy=45):

    df_attrs = pd.read_csv(ATTRIBUTES_PATH, sep='\t', skiprows=1,)
    df_attrs = pd.DataFrame(df_attrs.iloc[:,:-1].values, columns = df_attrs.columns[1:])
```

```
    photo_ids = []
    for dirpath, dirnames, filenames in os.walk(DATASET_PATH):
        for fname in filenames:
            if fname.endswith(".jpg"):
                fpath = os.path.join(dirpath,fname)
                photo_id = fname[:-4].replace('_',' ').split()
                person_id = ' '.join(photo_id[:-1])
                photo_number = int(photo_id[-1])
                photo_ids.append({'person':person_id,'imagenum':photo_number,'photo_path':fpath})

    photo_ids = pd.DataFrame(photo_ids)
    df = pd.merge(df_attrs,photo_ids,on=('person','imagenum'))

    assert len(df)==len(df_attrs),"lost some data when merging dataframes"

    all_photos = df['photo_path'].apply(imageio.imread)\
                                 .apply(lambda img:img[dy:-dy,dx:-dx])\
                                 .apply(lambda img: np.array(Image.fromarray(img).resize([dimx,dimy])) )

    all_photos = np.stack(all_photos.values).astype('uint8')
    all_attrs = df.drop(["photo_path","person","imagenum"],axis=1)

    return all_photos,all_attrs
```

The variable `data` has all the face images and the variable `attrs` has all the attributes. The 8-bit RGB values are converted to values between 0 and 1 for modeling and plotting purposes.

In [7]:
```
data, attrs = fetch_dataset()
data = np.array(data / 255, dtype='float32')
```

Create Face Groups

We can now create groups of faces, by selecting the faces having the highest or lowest scores for each of the attributes. Run all the cells in this section to create and plot some face groups.

In [8]:
```
def plot_gallery(images, h, w, n_row=3, n_col=6, with_title=False, titles=[]):
    plt.figure(figsize=(1.75 * n_col, 2 * n_row))
    plt.subplots_adjust(bottom=0, left=.01, right=.99, top=.90, hspace=.35)
    for i in range(n_row * n_col):
```

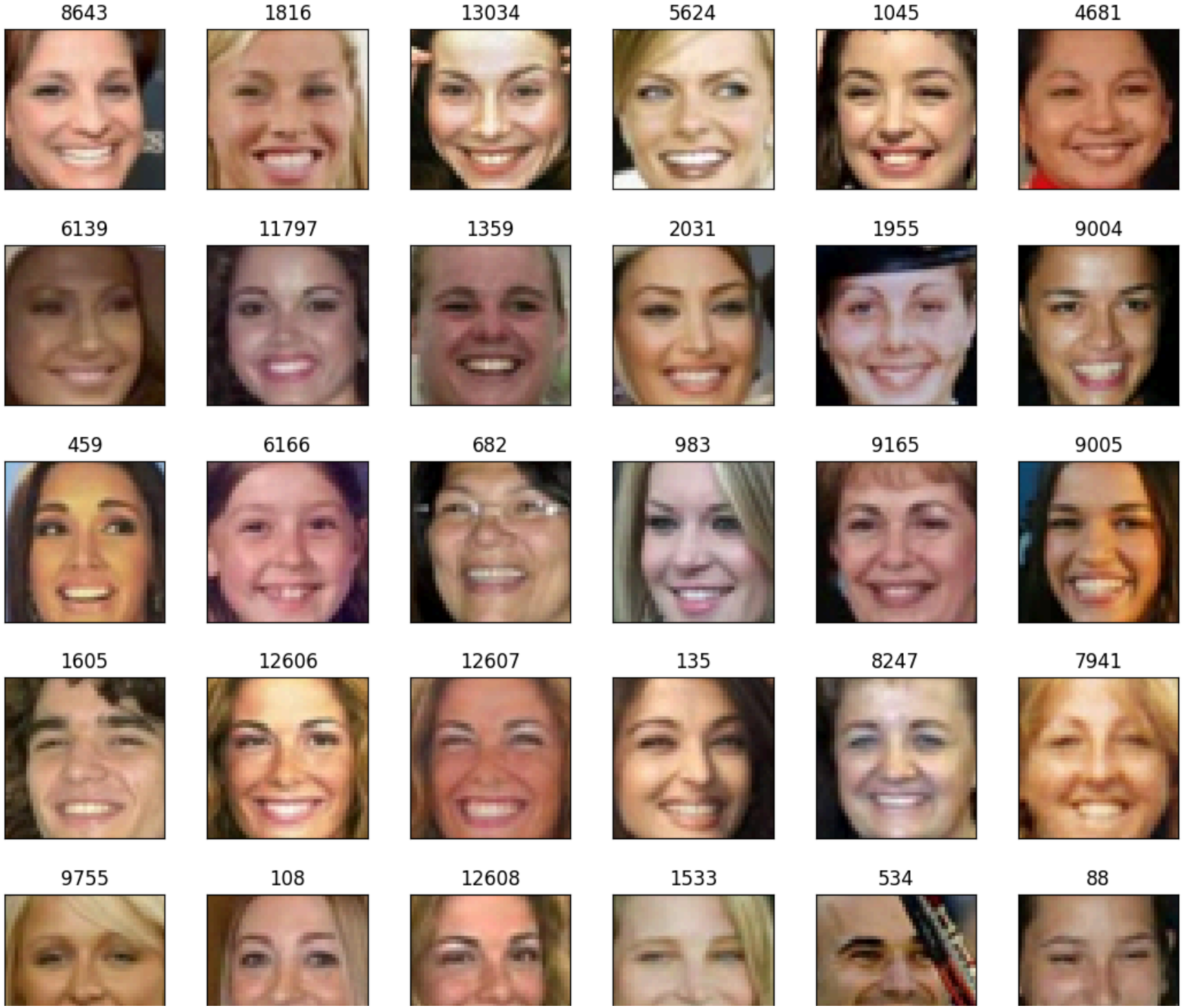
```
        plt.subplot(n_row, n_col, i + 1)
        try:
            plt.imshow(images[i].reshape((h, w, 3)), cmap=plt.cm.gray, vmin=-1, vmax=1, interpolation='nearest')
            if with_title:
                plt.title(titles[i])
            plt.xticks(())
            plt.yticks(())
        except:
            pass
```

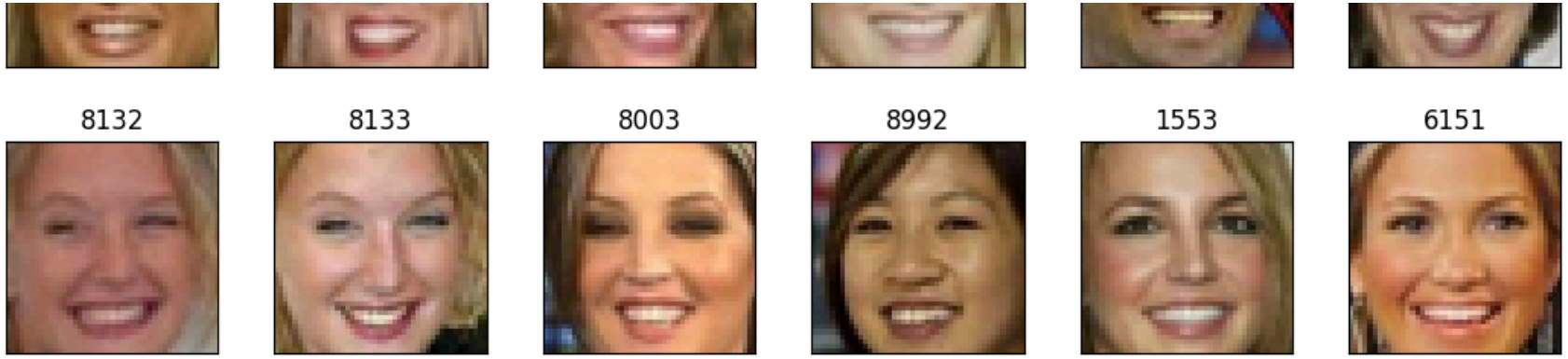
In [9]:
```
IMAGE_H = data.shape[1]
IMAGE_W = data.shape[2]
N_CHANNELS = 3
```

In [10]:
```
smile_ids = attrs['Smiling'].sort_values(ascending=False).head(36).index.values
smile_data = data[smile_ids]

no_smile_ids = attrs['Smiling'].sort_values(ascending=True).head(36).index.values
no_smile_data = data[no_smile_ids]

eyeglasses_ids = attrs['Eyeglasses'].sort_values(ascending=False).head(36).index.values
eyeglasses_data = data[eyeglasses_ids]

sunglasses_ids = attrs['Sunglasses'].sort_values(ascending=False).head(36).index.values
sunglasses_data = data[sunglasses_ids]

mustache_ids = attrs['Mustache'].sort_values(ascending=False).head(36).index.values
mustache_data = data[mustache_ids]

male_ids = attrs['Male'].sort_values(ascending=False).head(36).index.values
male_data = data[male_ids]

female_ids = attrs['Male'].sort_values(ascending=True).head(36).index.values
female_data = data[female_ids]

eyeclosed_ids = attrs['Eyes Open'].sort_values(ascending=True).head(36).index.values
eyeclosed_data = data[eyeclosed_ids]

mouthopen_ids = attrs['Mouth Wide Open'].sort_values(ascending=False).head(36).index.values
mouthopen_data = data[mouthopen_ids]
```

```
makeup_ids = attrs['Heavy Makeup'].sort_values(ascending=False).head(36).index.values
makeup_data = data[makeup_ids]
```

In [11]:
```
plot_gallery(smile_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=smile_ids)
```

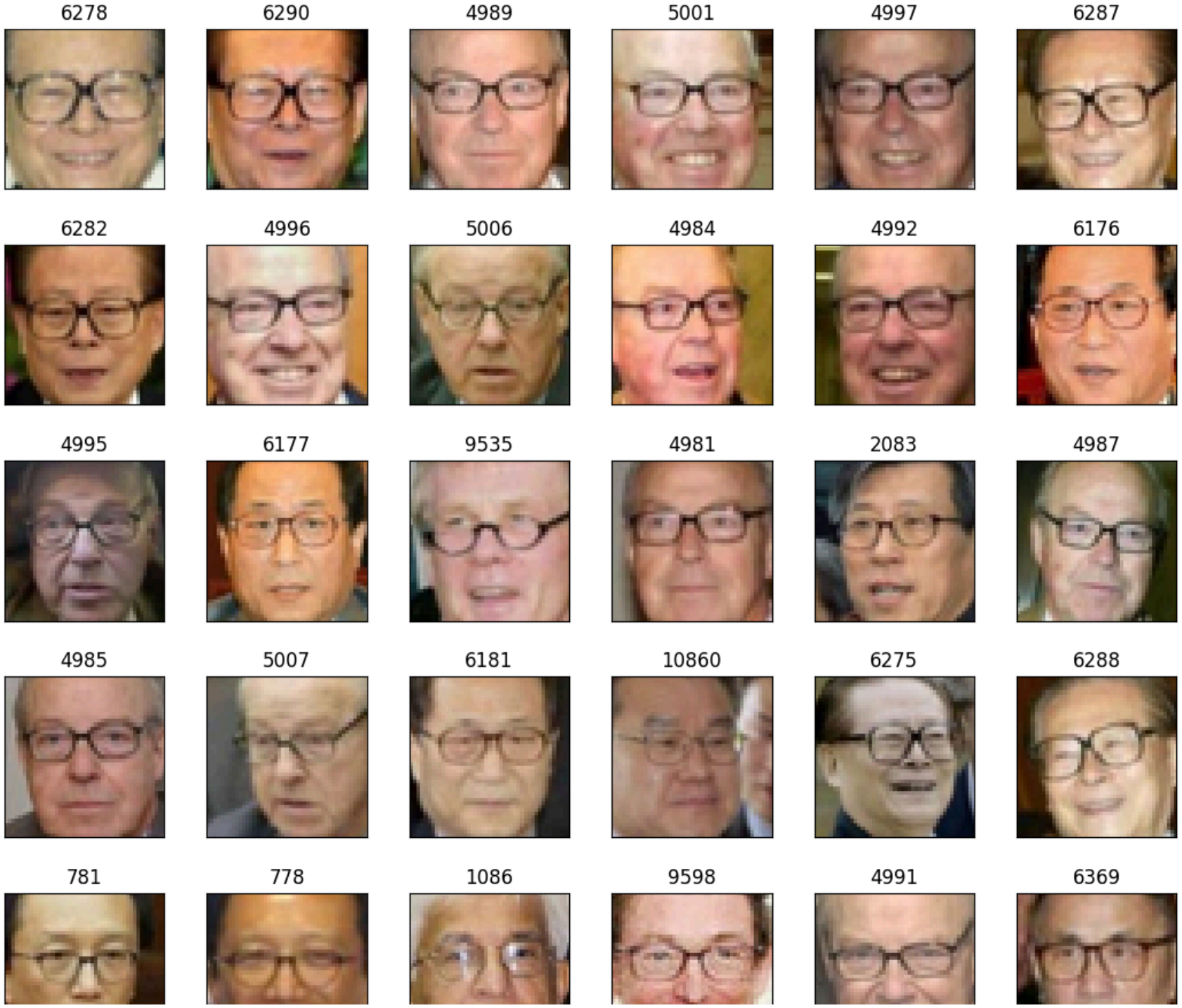

8643
1816
13034
5624
1045
4681
6139
11797
1359
2031
1955
9004
459
6166
682
983
9165
9005
1605
12606
12607
135
8247
7941
9755
108
12608
1533
534
88

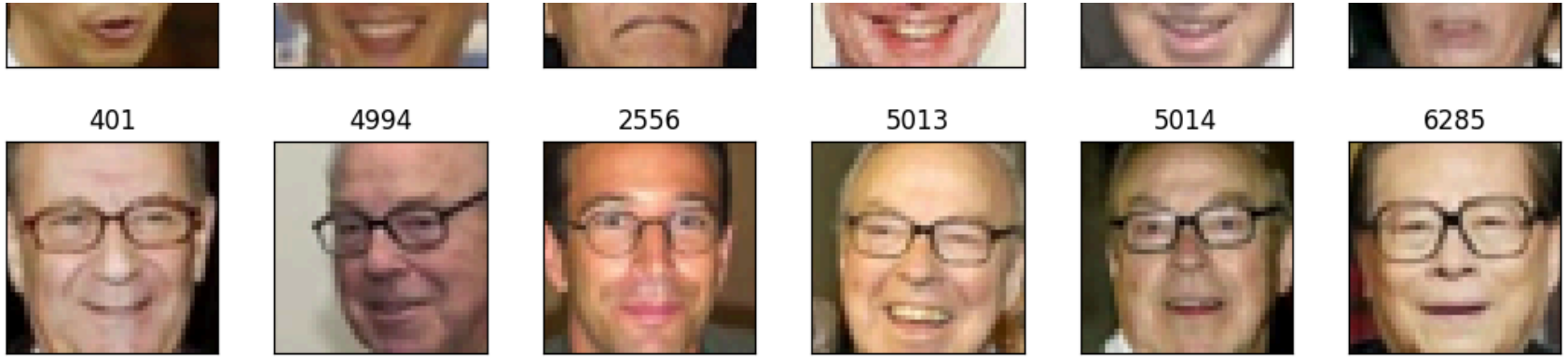


In [12]:
```
plot_gallery(no_smile_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=no_smile_ids)
```


11571
1234
7506
5458
7065
11912
1654
471
8410
8596
12688
1432
10097
10506
9114
6321
3798
8092
2390
6019
2529
12298
9201
4392
2000
12272
3935
10491
2647
12992

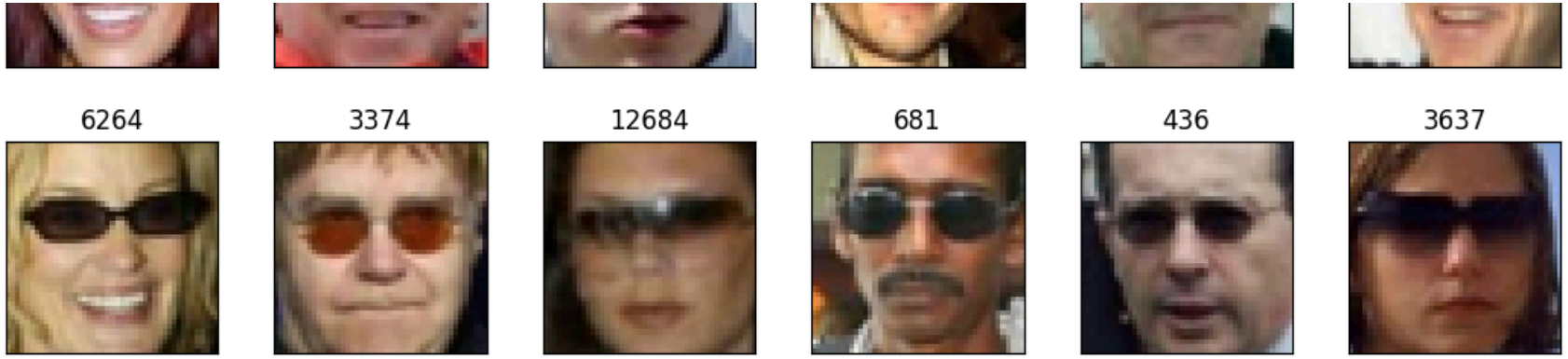
In [13]:
```
plot_gallery(eyeglasses_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=eyeglasses_ids)
```


6278
6290
4989
5001
4997
6287
6282
4996
5006
4984
4992
6176
4995
6177
9535
4981
2083
4987
4985
5007
6181
10860
6275
6288
781
778
1086
9598
4991
6369



In [14]:
```
plot_gallery(sunglasses_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=sunglasses_ids)
```

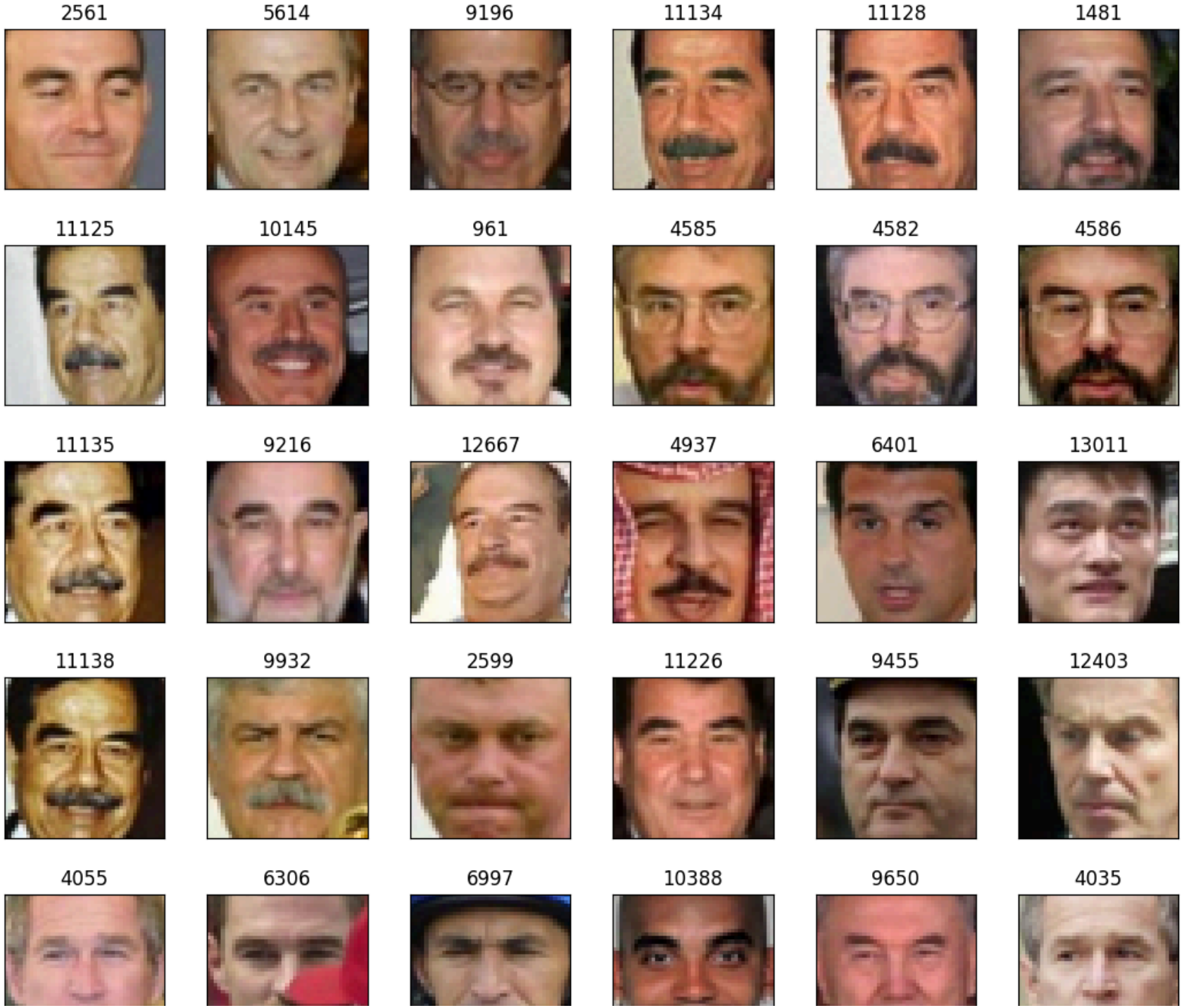

5791
5515
3602
8334
3732
1576
721
9163
10561
9021
7889
7357
1649
9780
9837
11790
9526
6477
3603
4608
7432
12683
1507
6342
9759
4195
7505
5082
4466
12110



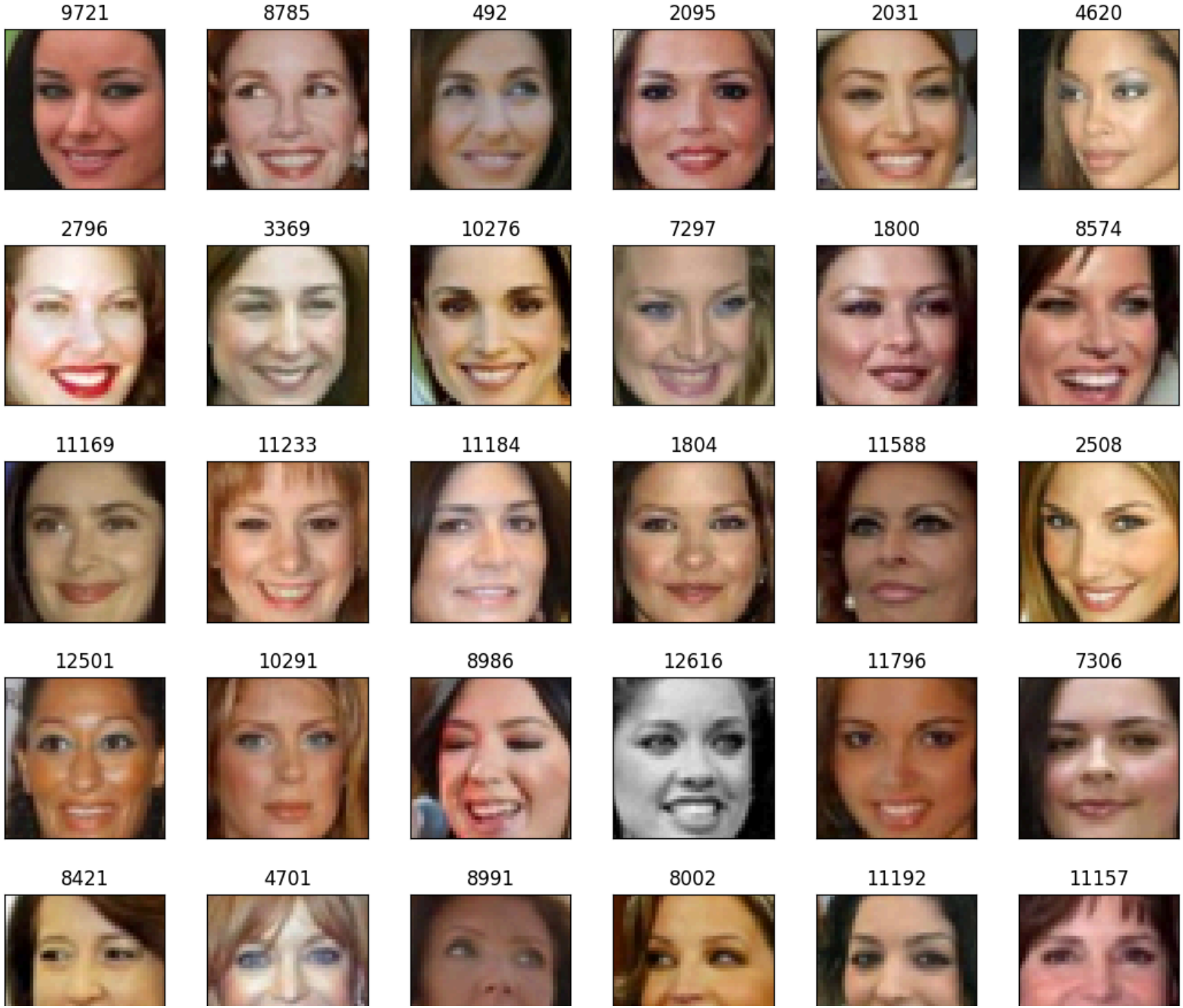
In [15]:
```
plot_gallery(mustache_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=mustache_ids)
```


961
12647
11850
5133
758
1481
9507
1482
8155
9329
9330
11840
4582
10284
960
2011
4585
6746
11839
2767
11843
4583
4579
43
3598
11153
11128
10
11846
13139

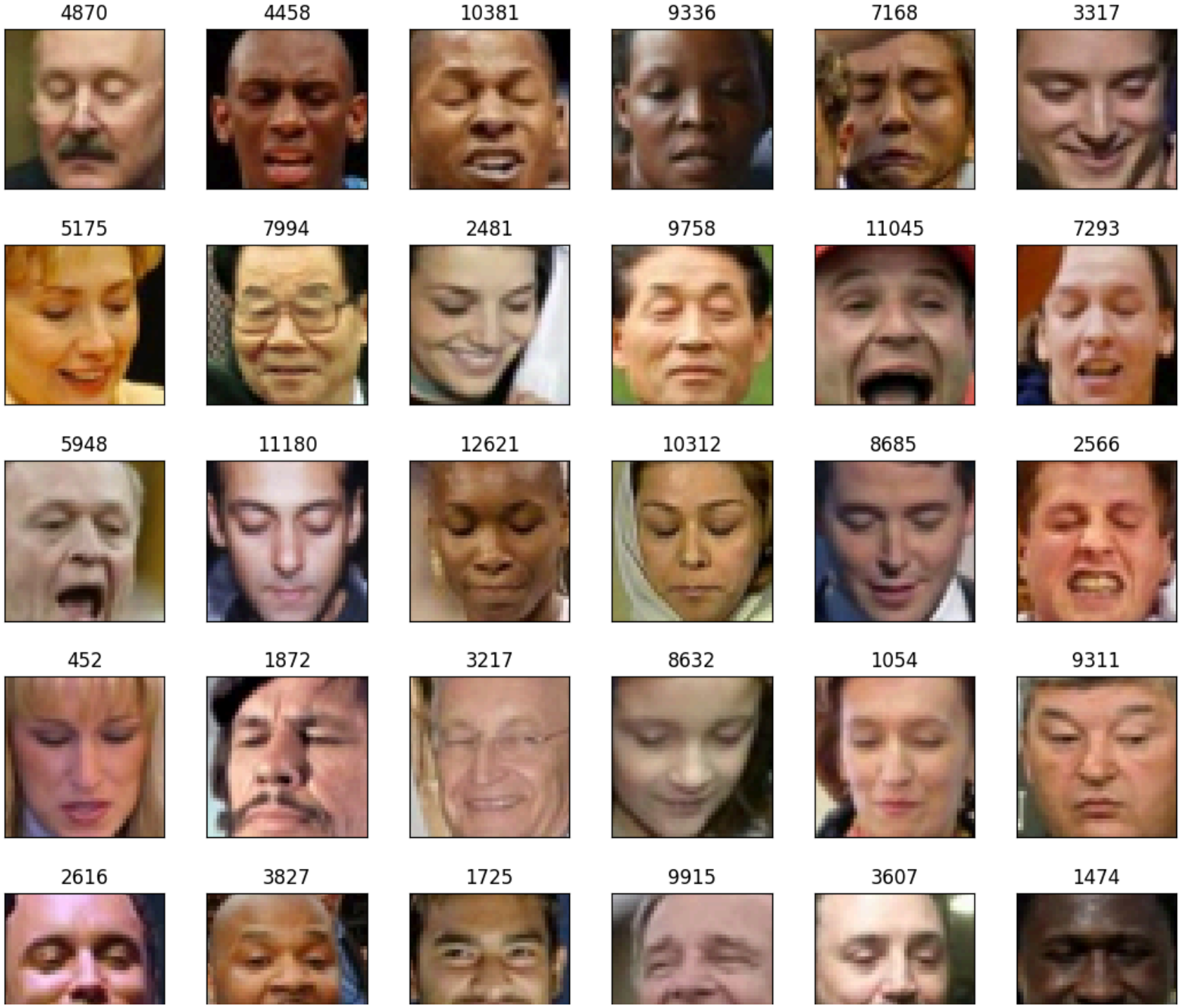
In [16]:
```
plot_gallery(male_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=male_ids)
```


2561
5614
9196
11134
11128
1481
11125
10145
961
4585
4582
4586
11135
9216
12667
4937
6401
13011
11138
9932
2599
11226
9455
12403
4055
6306
6997
10388
9650
4035

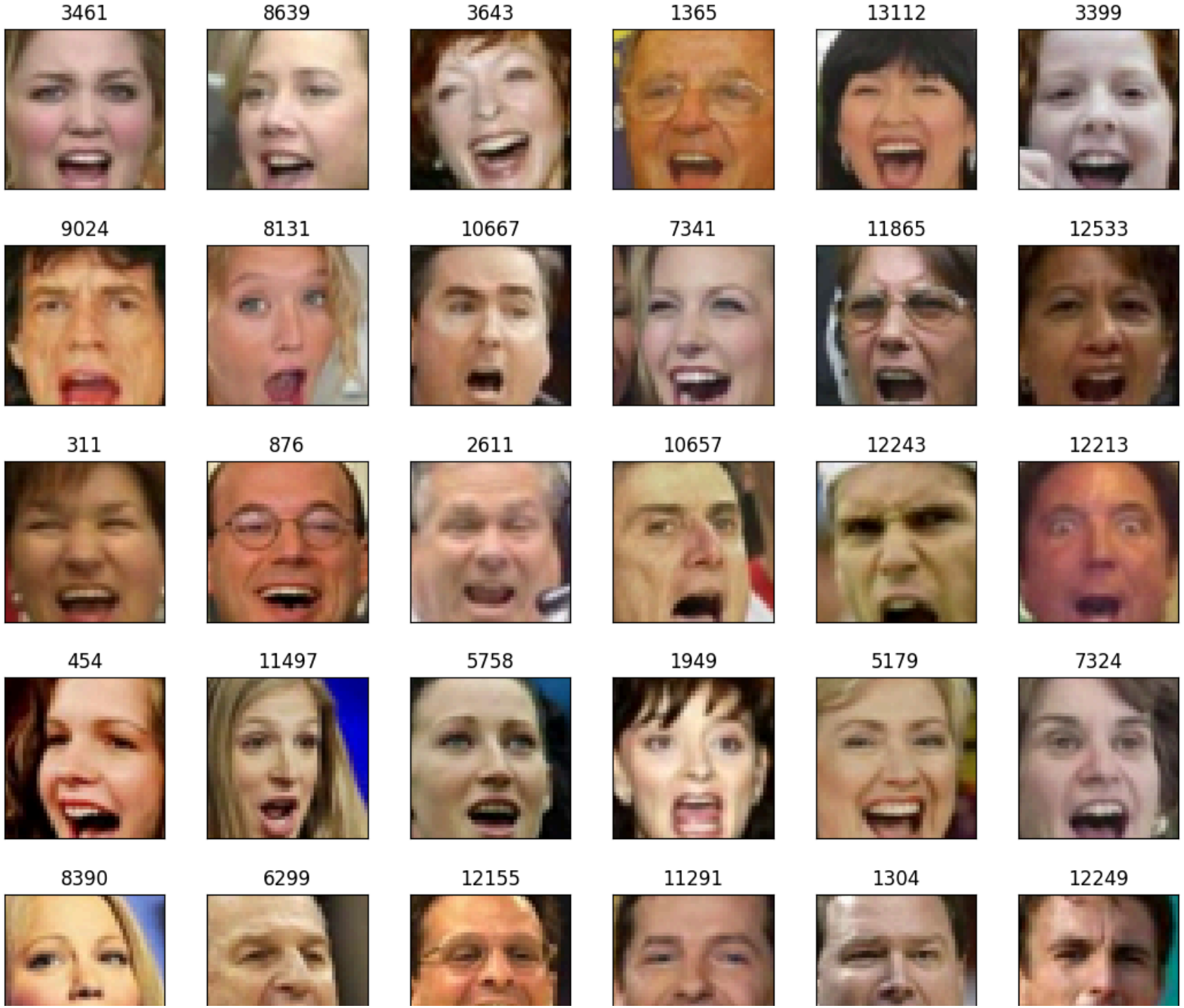
In [17]:
```
plot_gallery(female_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=female_ids)
```

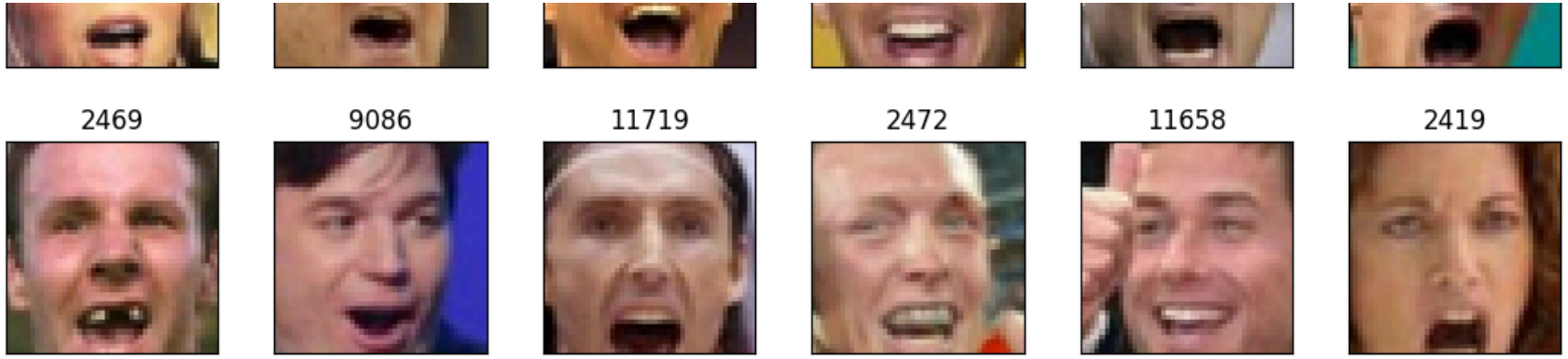

9721
8785
492
2095
2031
4620
2796
3369
10276
7297
1800
8574
11169
11233
11184
1804
11588
2508
12501
10291
8986
12616
11796
7306
8421
4701
8991
8002
11192
11157

In [18]:
```
plot_gallery(eyeclosed_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=eyeclosed_ids)
```

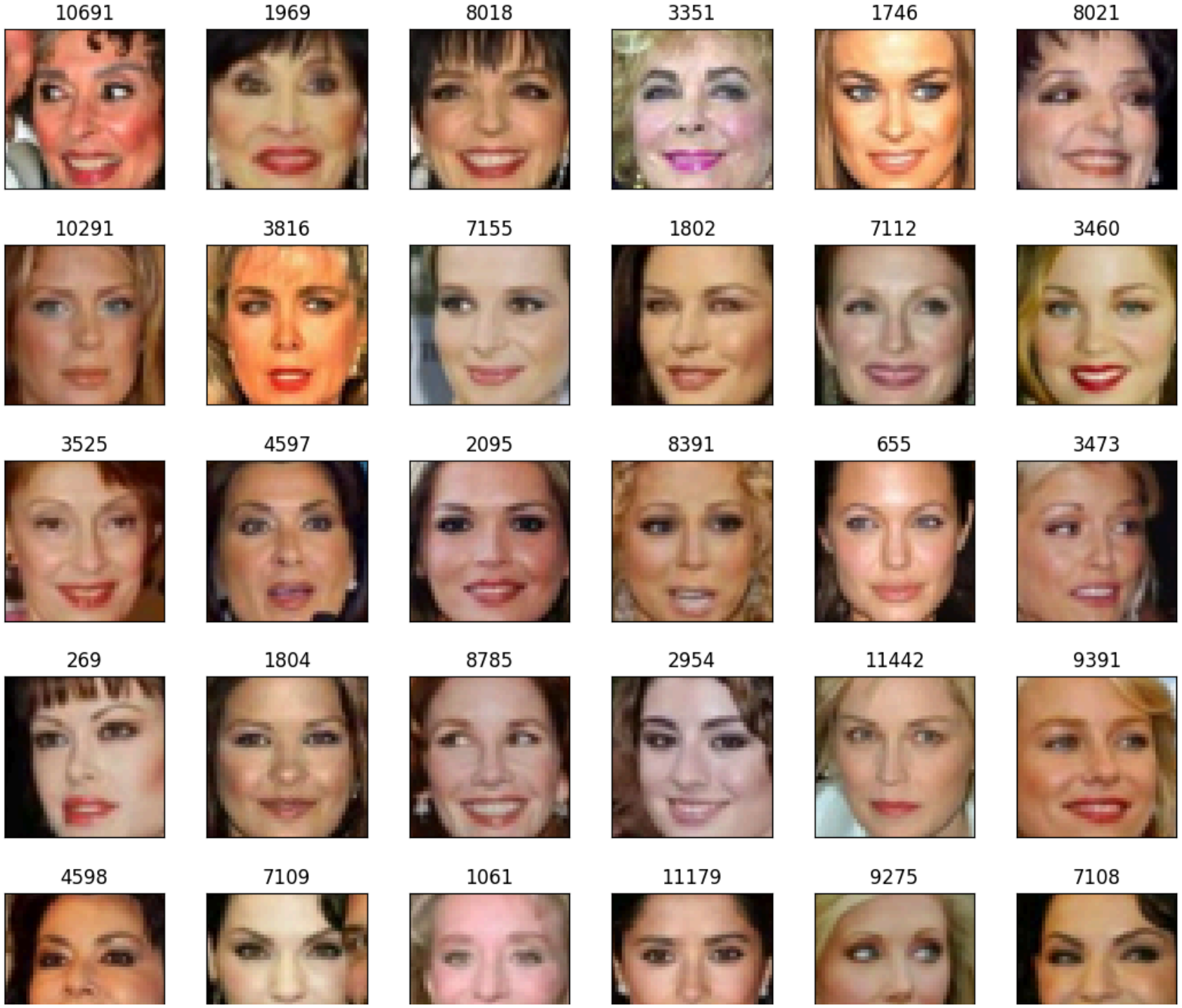

4870
4458
10381
9336
7168
3317
5175
7994
2481
9758
11045
7293
5948
11180
12621
10312
8685
2566
452
1872
3217
8632
1054
9311
2616
3827
1725
9915
3607
1474

In [19]:
```
plot_gallery(mouthopen_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=mouthopen_ids)
```


3461
8639
3643
1365
13112
3399
9024
8131
10667
7341
11865
12533
311
876
2611
10657
12243
12213
454
11497
5758
1949
5179
7324
8390
6299
12155
11291
1304
12249



In [20]:
```
plot_gallery(makeup_data, IMAGE_H, IMAGE_W, n_row=6, n_col=6, with_title=True, titles=makeup_ids)
```


10691
1969
8018
3351
1746
8021
10291
3816
7155
1802
7112
3460
3525
4597
2095
8391
655
3473
269
1804
8785
2954
11442
9391
4598
7109
1061
11179
9275
7108

Constructing the encoder

Recall that the encoder part of the VAE architecture maps a data point to a variational mean and (log) variance. The mean is a point in the latent space.

In [21]:
```
LATENT_SPACE_SIZE = 100
```

The "reparameterization trick" draws samples from the variational distribution that are parameterized by the variational mean and variance, so that the parameters of the encoder network can be trained.

In [22]:
```
def sample_latent_features(distribution):
    distribution_mean, distribution_variance = distribution
    batch_size = tensorflow.shape(distribution_variance)[0]
    random = tensorflow.keras.backend.random_normal(shape=(batch_size, tensorflow.shape(distribution_variance)[1]))
    return distribution_mean + tensorflow.exp(0.5 * distribution_variance) * random
```

In [23]:
```
input_data = tensorflow.keras.layers.Input(shape=(45, 45, 3))

encoder = tensorflow.keras.layers.Conv2D(64, (5,5), activation='relu')(input_data)
encoder = tensorflow.keras.layers.MaxPooling2D((2,2))(encoder)

encoder = tensorflow.keras.layers.Conv2D(64, (3,3), activation='relu')(encoder)
encoder = tensorflow.keras.layers.MaxPooling2D((2,2))(encoder)

encoder = tensorflow.keras.layers.Conv2D(32, (3,3), activation='relu')(encoder)
encoder = tensorflow.keras.layers.MaxPooling2D((2,2))(encoder)
```

```
encoder = tensorflow.keras.layers.Flatten()(encoder)

distribution_mean = tensorflow.keras.layers.Dense(LATENT_SPACE_SIZE, name='variational_mean')(encoder)
distribution_variance = tensorflow.keras.layers.Dense(LATENT_SPACE_SIZE, name='variational_log_variance')(encoder)
latent_encoding = tensorflow.keras.layers.Lambda(sample_latent_features)([distribution_mean, distribution_variance])

encoder_model = tensorflow.keras.Model(input_data, latent_encoding)
encoder_model.summary()
```

```
Model: "model"
__________________________________________________________________________________________________
 Layer (type)                   Output Shape         Param #     Connected to
==================================================================================================
 input_1 (InputLayer)           [(None, 45, 45, 3)]  0           []

 conv2d (Conv2D)                (None, 41, 41, 64)   4864        ['input_1[0][0]']

 max_pooling2d (MaxPooling2D)   (None, 20, 20, 64)   0           ['conv2d[0][0]']

 conv2d_1 (Conv2D)              (None, 18, 18, 64)   36928       ['max_pooling2d[0][0]']

 max_pooling2d_1 (MaxPooling2D)  (None, 9, 9, 64)    0           ['conv2d_1[0][0]']

 conv2d_2 (Conv2D)              (None, 7, 7, 32)     18464       ['max_pooling2d_1[0][0]']

 max_pooling2d_2 (MaxPooling2D)  (None, 3, 3, 32)    0           ['conv2d_2[0][0]']

 flatten (Flatten)              (None, 288)          0           ['max_pooling2d_2[0][0]']

 variational_mean (Dense)       (None, 100)          28900       ['flatten[0][0]']

 variational_log_variance (Dens  (None, 100)         28900       ['flatten[0][0]']
 e)

 lambda (Lambda)                (None, 100)          0           ['variational_mean[0][0]',
                                                                  'variational_log_variance[0][0]'
                                                                 ]

==================================================================================================
Total params: 118,056
Trainable params: 118,056
Non-trainable params: 0
__________________________________________________________________________________________________
```

Construct the decoder

The decoder network in the VAE architecture maps a latent vector to an image. This is done with a series of transposed convolutional layers, since it must map from low to high dimensions.

In [24]:
```
# decoder_input = tensorflow.keras.layers.Input(shape=LATENT_SPACE_SIZE)
decoder_input = tensorflow.keras.layers.Input(shape=(LATENT_SPACE_SIZE,))
decoder = tensorflow.keras.layers.Reshape((1, 1, 100))(decoder_input)
decoder = tensorflow.keras.layers.Conv2DTranspose(64, (3,3), activation='relu')(decoder)
decoder = tensorflow.keras.layers.UpSampling2D((2,2))(decoder)

decoder = tensorflow.keras.layers.Conv2DTranspose(32, (3,3), activation='relu')(decoder)
decoder = tensorflow.keras.layers.UpSampling2D((2,2))(decoder)

decoder = tensorflow.keras.layers.Conv2DTranspose(16, (5,5), activation='relu')(decoder)
decoder = tensorflow.keras.layers.UpSampling2D((2,2))(decoder)

decoder_output = tensorflow.keras.layers.Conv2DTranspose(3, (6,6), activation='relu')(decoder)

decoder_model = tensorflow.keras.Model(decoder_input, decoder_output)
decoder_model.summary()
```

```
Model: "model_1"
_________________________________________________________________
 Layer (type)                Output Shape              Param #
=================================================================
 input_2 (InputLayer)        [(None, 100)]             0

 reshape (Reshape)           (None, 1, 1, 100)         0

 conv2d_transpose (Conv2DTra  (None, 3, 3, 64)         57664
 nspose)

 up_sampling2d (UpSampling2D  (None, 6, 6, 64)         0
 )

 conv2d_transpose_1 (Conv2DT  (None, 8, 8, 32)         18464
 ranspose)

 up_sampling2d_1 (UpSampling  (None, 16, 16, 32)       0
 2D)

 conv2d_transpose_2 (Conv2DT  (None, 20, 20, 16)       12816
 ranspose)

 up_sampling2d_2 (UpSampling  (None, 40, 40, 16)       0
 2D)

 conv2d_transpose_3 (Conv2DT  (None, 45, 45, 3)        1731
 ranspose)

=================================================================
Total params: 90,675
Trainable params: 90,675
Non-trainable params: 0
_________________________________________________________________
```

In [25]:
```
encoded = encoder_model(input_data)
decoded = decoder_model(encoded)
autoencoder = tensorflow.keras.models.Model(input_data, decoded)
```

In [26]:
```
def get_loss(distribution_mean, distribution_variance, factor, batch_size):

    def get_reconstruction_loss(y_true, y_pred, factor, batch_size):
```

```
        reconstruction_loss = tensorflow.math.squared_difference(y_true, y_pred)
        reconstruction_loss_batch = tensorflow.reduce_sum(reconstruction_loss)/batch_size
        return 0.5*reconstruction_loss_batch*factor

    def get_kl_loss(distribution_mean, distribution_variance, batch_size):
        kl_loss = LATENT_SPACE_SIZE + distribution_variance - tensorflow.square(distribution_mean) - tensorflow.exp(
        kl_loss_batch = tensorflow.reduce_sum(kl_loss)/batch_size
        return kl_loss_batch*(-0.5)

    def total_loss(y_true, y_pred):
        reconstruction_loss_batch = get_reconstruction_loss(y_true, y_pred, factor, batch_size)
        kl_loss_batch = get_kl_loss(distribution_mean, distribution_variance, batch_size)
        return reconstruction_loss_batch + kl_loss_batch

    return total_loss
```

1.1 Deriving the loss function (5 points)

Derive the loss function defined in the cell above from the probability model perspective. You can ignore the scalar `factor` in your derivation. Show your work using either LaTeX or a picture of your written solution.

Hint: Think about how how the total loss is related to the ELBO.

Answer 1.1

Proof

Denote Z as the latent random variable and X as the data random variable. We are given the following assumptions about the model:

$$Z \sim N(0, I_L)$$

$$X|Z \sim N(G(Z), I_N)$$

where $G(Z)$ is the decoder.

Denote Latent Space Size as L and Batch Size as N.

We wish to maximize $p(x)$. This is equivalent to minimizing $-\log p(x)$. However, this is hard to do given the nonlinearity of $G(Z)$.

Thus, we utilize Variational Inference and assume a "nice" function $q(z|x) \sim N(\hat{\mu_e}, \hat{\sigma^2}_e I_L)$

From class derivation we know that $-\log p(x) \leq -E_{q(z|x)}[\log(p(x|z))] + D_{KL}(q(z|x)||p(z|x))$. Our goal is to to minimize the upper bound.

Hence, we know that $L_R = -E_{q(z|x)}[\log(p(x|z))]$ and $L_{KL} = D_{KL}(q(z)||p(z))$.

For notation C denotes any arbitrary constant that vanishes whenever we take the derivative.

[1] Let us first examine $L_R = -E_{q(z|x)}[\log(p(x|z))]$

$$p(x|z) = \frac{1}{(2\pi)^{N/2}} \exp[-\frac{1}{2}(X - G(Z))^T(X - G(Z))]$$

$$log(p(x|z)) = -\frac{1}{2}(X - G(Z))^T(X - G(Z)) + C$$

$$-E_{q(z|x)}[\log(p(x|z))] \approx \frac{1}{N}\sum_{i=1}^{N} \log(p(x|z_i)) \quad \text{(by WLLN)} \tag{1}$$

$$\approx \frac{1}{2N}\sum_{i=1}^{N} (x_i - \hat{x}_i)^2 \, \square \tag{2}$$

[2] Let us next examine $L_{KL} = D_{KL}(q(z)||p(z))$

The KL divergence between two multivariate Gaussian distributions $\mathcal{N}(\mu_q, \Sigma_q)$ and $\mathcal{N}(\mu_p, \Sigma_p)$ is given by:

$$D_{KL}(q \parallel p) = \frac{1}{2}\left(\operatorname{tr}\left(\Sigma_p^{-1}\Sigma_q\right) + (\mu_p - \mu_q)^T \Sigma_p^{-1}(\mu_p - \mu_q) - k + \ln\left(\frac{\det \Sigma_p}{\det \Sigma_q}\right)\right)$$

Where k is the dimension of the latent space

For our case:

- $\mu_p = 0$
- $\Sigma_p = I_L$
- $\mu_q = \hat{\mu_e}$

- $\Sigma_q = \hat{\sigma^2}_e I_L$
- $k = L$

Plugging these into the KL divergence formula:

$$D_{KL}(q(z|x) \parallel p(z)) = \frac{1}{2}\left(\text{tr}\left(I_L^{-1} \cdot \hat{\sigma^2}_e I_L\right) + (\hat{\mu_e} - 0)^T I_L^{-1}(\hat{\mu_e} - 0) - L + \ln\left(\frac{\det I_L}{\det(\hat{\sigma^2}_e I_L)}\right)\right) \tag{3}$$

$$= \frac{1}{2}\left(\hat{\sigma^2}_e L + \|\hat{\mu_e}\|^2 - L - \sum_{i=1}^{L} \ln \hat{\sigma^2}_e\right) \tag{4}$$

$$\approx \frac{1}{2N}\sum_{i=1}^{L}\left(e^v + \hat{\mu_{ei}}^2 - 1 - v\right) \square \tag{5}$$

where the encoder estimates distribution variance as $v = \ln(\hat{\sigma^2}_e)$ and we normalize by batch size.

The following three cells train the model. You can just run them. It may take a while to run on your laptop.

In [27]:
```
X_train, X_val = train_test_split(data, test_size=0.2, random_state=365)
```

In [28]:
```
batch_size = 64
autoencoder.compile(loss=get_loss(distribution_mean, distribution_variance, factor = 100,
                                  batch_size = batch_size), optimizer='adam')
autoencoder.summary()
```

```
Model: "model_2"
_________________________________________________________________
 Layer (type)                Output Shape              Param #
=================================================================
 input_1 (InputLayer)        [(None, 45, 45, 3)]       0

 model (Functional)          (None, 100)               118056

 model_1 (Functional)        (None, 45, 45, 3)         90675

=================================================================
Total params: 208,731
Trainable params: 208,731
Non-trainable params: 0
_________________________________________________________________
```

In [ ]:
```
autoencoder.fit(X_train, X_train, epochs=50, batch_size=64, validation_data=(X_val, X_val))
```

In [ ]:
```
# Save your trained model (for later use)

# autoencoder.save("My_Trained_VAE")
# encoder_model.save("My_Trained_encoder")
# decoder_model.save("My_Trained_decoder")
```

If you experience issues while training the models above or if the process is taking too long, you may also use our pre-trained versions of the VAE, encoder, and decoder, which are available on Canvas.

No points will be taken off if you decide to use the pre-trained models, but we do recommend trying out the training process above. In case you encounter any issues when loading in the models, please post your questions on Ed discussion.

In [30]:
```
# Load Pre-trained Model

batch_size = 64
autoencoder = tensorflow.keras.models.load_model("C:\jofan\yale\sds_365\pset\p3\model\Trained_VAE",compile=False)
autoencoder.compile(loss=get_loss(distribution_mean, distribution_variance, factor = 100, batch_size = batch_size), c

encoder_model = tensorflow.keras.models.load_model("C:\jofan\yale\sds_365\pset\p3\model\Trained_encoder",compile=Fals
encoder_model.compile(loss=get_loss(distribution_mean, distribution_variance, factor = 100, batch_size = batch_size),
```

```
decoder_model = tensorflow.keras.models.load_model("C:\jofan\yale\sds_365\pset\p3\model\Trained_decoder",compile=Fals
decoder_model.compile(loss=get_loss(distribution_mean, distribution_variance, factor = 100, batch_size = batch_size).
```

```
WARNING:tensorflow:SavedModel saved prior to TF 2.5 detected when loading Keras model. Please ensure that you are sav
ing the model with model.save() or tf.keras.models.save_model(), *NOT* tf.saved_model.save(). To confirm, there shoul
d be a file named "keras_metadata.pb" in the SavedModel directory.
WARNING:tensorflow:SavedModel saved prior to TF 2.5 detected when loading Keras model. Please ensure that you are sav
ing the model with model.save() or tf.keras.models.save_model(), *NOT* tf.saved_model.save(). To confirm, there shoul
d be a file named "keras_metadata.pb" in the SavedModel directory.
WARNING:tensorflow:SavedModel saved prior to TF 2.5 detected when loading Keras model. Please ensure that you are sav
ing the model with model.save() or tf.keras.models.save_model(), *NOT* tf.saved_model.save(). To confirm, there shoul
d be a file named "keras_metadata.pb" in the SavedModel directory.
```

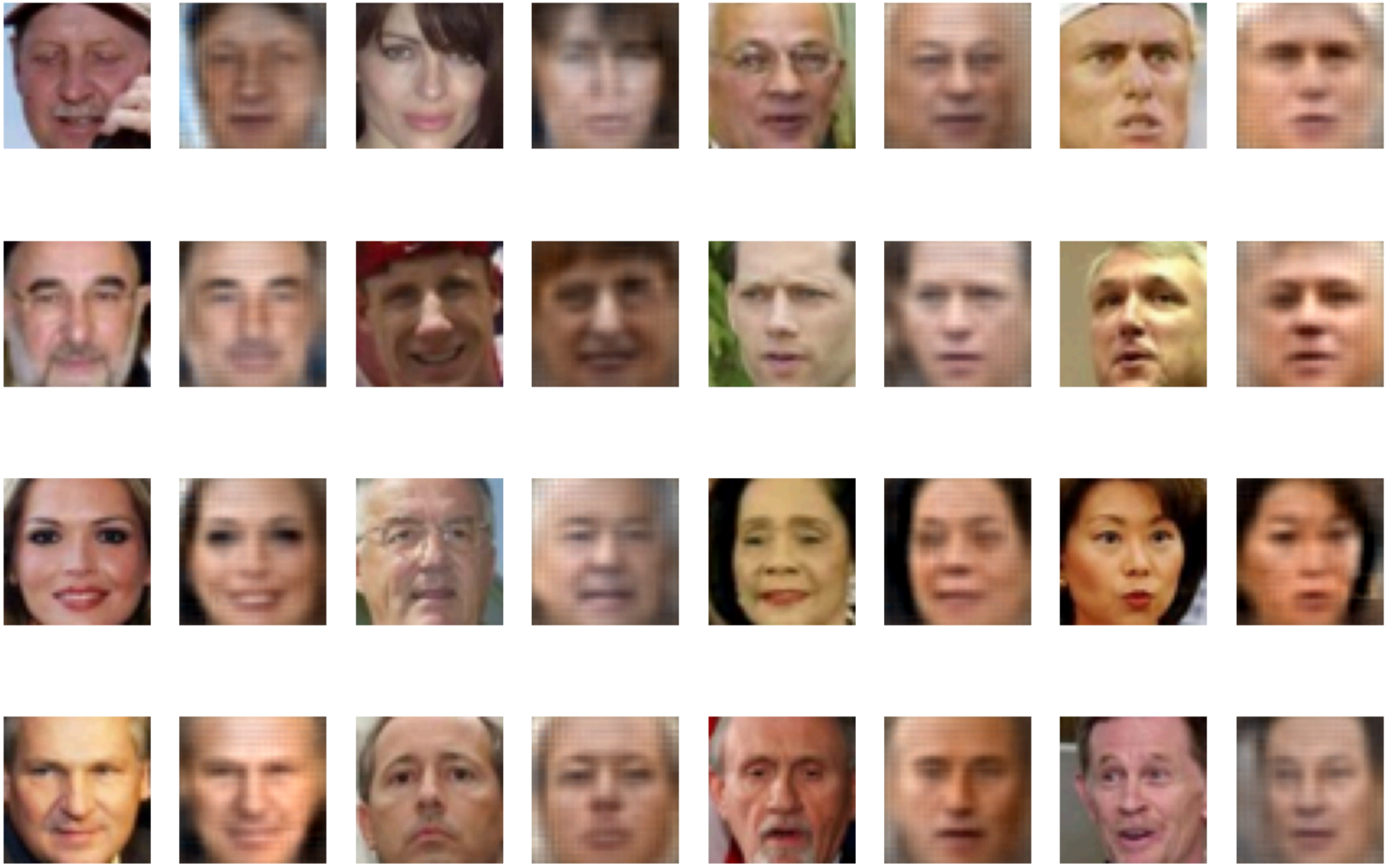
1.2 Reconstructing faces (3 points)

The following cell encodes and reconstructs 16 random faces from the validation set with the trained VAE. Run the cell and comment on the reconstructed faces. (3 points)

- Do the reconstructed faces resemble the original images? How are they similar/different?
- Are there any faces that are reconstructed better or worse than the others? Can you think of why?
- Comment on any other aspects of your findings that are interesting to you.

In [34]:

```
sample_index = random.sample(range(1, len(X_val)), 16)

fig, axs = plt.subplots(4, 8)
fig.set_figheight(10)
fig.set_figwidth(15)

for i in range(4):
    for j in range(4):
        axs[i, 2*j].imshow(X_val[sample_index[4*i+j], :, :, :])
        axs[i, 2*j].axis('off')
        axs[i, 2*j+1].imshow(np.clip(autoencoder.predict(np.array([X_val[sample_index[4*i+j], :, :, :]]))[0],0,1))
        axs[i, 2*j+1].axis('off')
```



Answer 1.2

The reconstructed images closely resemble the original test images in terms of capturing overall facial structure and prominent features, but they appear somewhat "blurred" compared to the originals. For example, fine details like wrinkles are smoothed out, and the eyes lack distinct details, appearing as solid patches of color without clear separation between the pupil and the sclera.

Interestingly, the model performs better on faces with distinct or prominent features, such as a sharply defined jawline or prominent cheekbones. This aligns with the typical behavior of CNNs, where the initial layers capture broad and structural features,

like facial shape and contours, before finer details. This layering likely explains why the model excels at identifying and reproducing bold, easily recognizable facial characteristics while smoothing out smaller details.

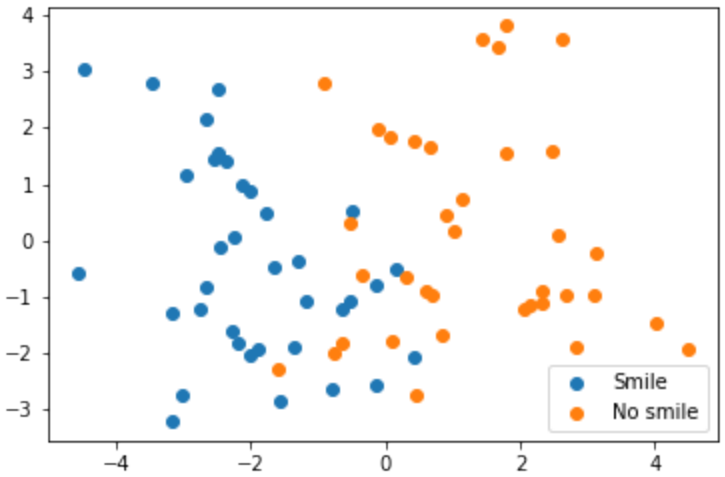
Additionally, the model tends to perform better at reconstructing smiles compared to faces with subtle expressions, such as a frown or a neutral expression. This may be due to a higher representation of smiling faces in the training data, making it easier for the model to generalize this feature. Moreover, smiles often have more pronounced and recognizable contours—like lifted cheeks and the shape of the mouth—which are easier for the model to capture than the more nuanced distinctions of a frown or neutral expression.

1.3 Visualizing the latent space (10 points)

In `vae_demo` from class, the MNIST digits were generated from a two-dimensional latent space. In the current model, the latent space has more than two dimensions, so to visualize it we need to use a dimensionality reduction technique. (If you are not familiar with PCA, please refer to the material for Week 7 of iML.)

In this problem, you will first implement the function `LatentSpace_2D`. (6 points)

1. Calculate the latent space encodings for two sets of faces that are different in one attribute, e.g. smile vs. no smile.
2. Use PCA to reduce the dimension of the latent space codes to two.
3. Visualize the latent space after dimensionality reduction with a scatter plot. Clearly color-code and label the two different groups.



Here is an example using smile_data and no_smile_data.

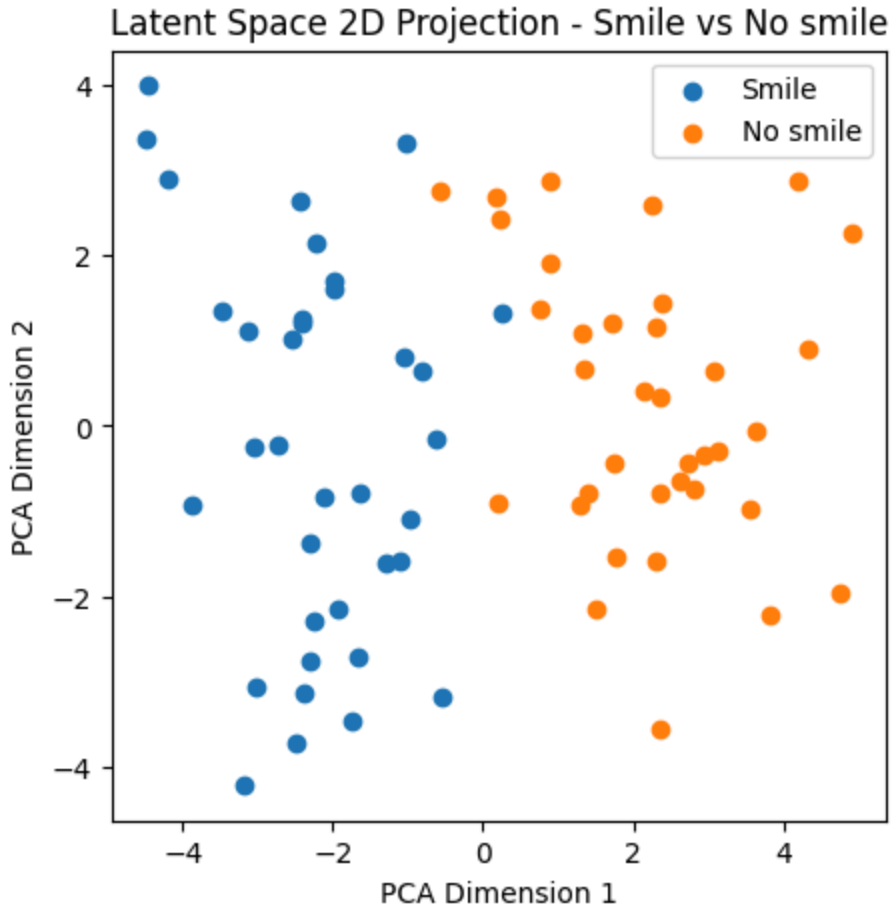
Visualize the latent space for at least three pairs of face groups including smile vs. no smile. Comment on how the scatter plots look.

- Are the two groups separable in the two-dimensional latent space? Is this what you expected? Why or why not? (2 points)
- How do the plots for the three different attributes differ from each other? (2 points)

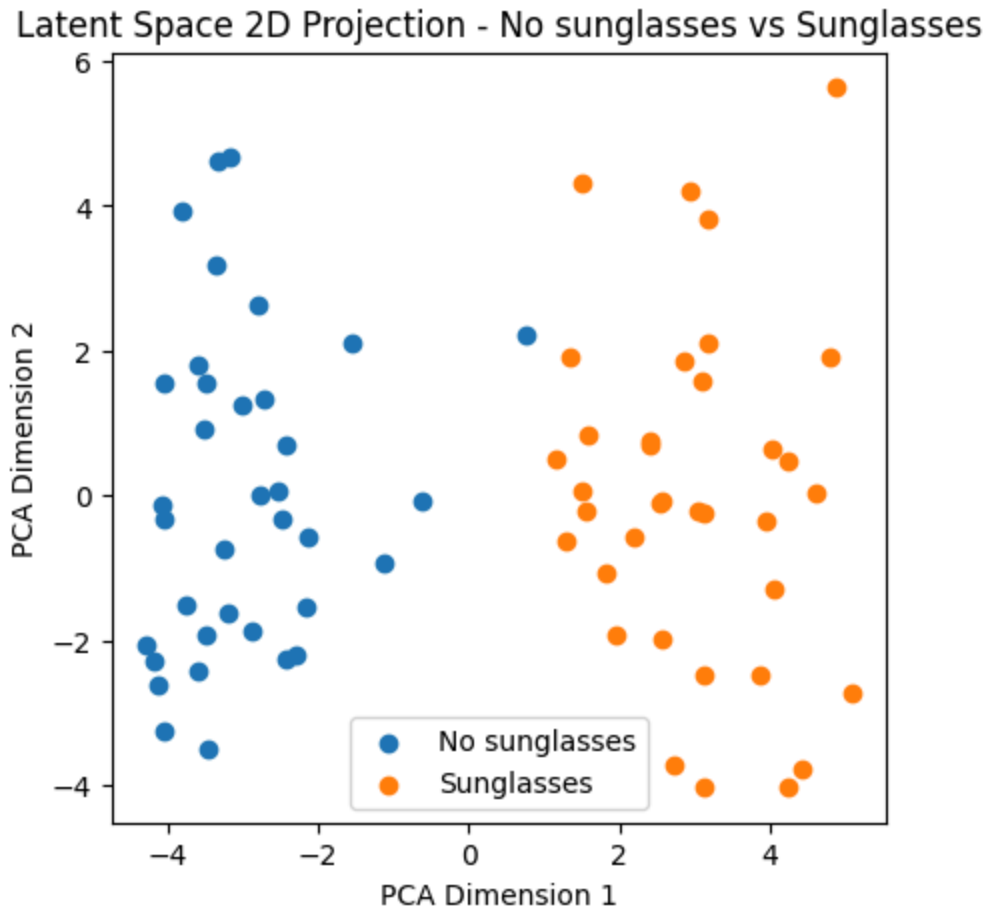
In [62]:
```
def LatentSpace_2D(encoder_model, data1, label1, data2, label2):
    # Get latent space
    l_data1 = encoder_model.predict(data1)
    l_data2 = encoder_model.predict(data2)

    # Run pca
    combined_data = np.concatenate([l_data1, l_data2])
    pca = PCA(n_components=2)
    r_data = pca.fit_transform(combined_data)

    # Split data
    r_data1 = r_data[:len(data1)]
    r_data2 = r_data[len(data1):]

    # Plot
    plt.figure(figsize=(5, 5))
```

```
    plt.scatter(r_data1[:, 0], r_data1[:, 1], label=label1, alpha=1)
    plt.scatter(r_data2[:, 0], r_data2[:, 1], label=label2, alpha=1)
    plt.xlabel("PCA Dimension 1")
    plt.ylabel("PCA Dimension 2")
    plt.legend()
    plt.title(f"Latent Space 2D Projection - {label1} vs {label2}")
    plt.show()
```

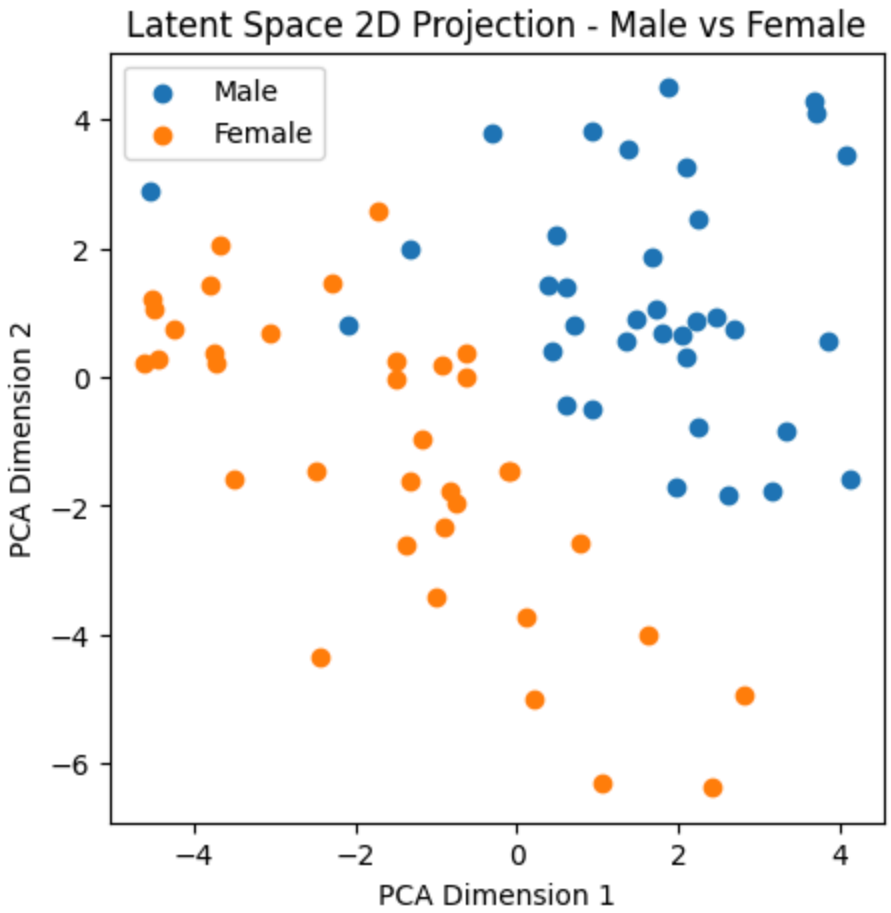
In [63]:
```
LatentSpace_2D(encoder_model, smile_data, 'Smile', no_smile_data, 'No smile')
```



In [83]:
```
LatentSpace_2D(encoder_model,smile_data,'No sunglasses',sunglasses_data,'Sunglasses')
```



In [49]:
```
LatentSpace_2D(encoder_model,male_data,'Male',female_data,'Female')
```



Answer 1.3

All PCA plots show a clear separation within the data, with distinct groupings that could be divided by a line. Interestingly, for features like Sunglasses and Smile, the data can be split with a vertical line, while Gender displays a separation along a diagonal line, roughly at a -45° angle. This separability aligns with expectations, as the model likely identifies more prominent, high-contrast features first. For example, the presence or absence of sunglasses creates a stark difference, as the eyes are covered in one case and not in the other, making it easier for CNN feature maps to capture. In contrast, less noticeable features, such as subtle variations in ear shape, may be more challenging for the model to distinguish and would require deeper layers to capture these finer details. Also, most variation is captured within a compact 4x4 square in the PCA plots, suggesting that a small subset of principal components effectively represents the key distinguishing features in the data.

1.4 Morphing between faces (4 points)

Morph at least 5 pairs of faces with the function `morphBetweenImages` and comment on what you observe.

- Briefly explain how the morphing works. (2 points)
- Do the generated faces look like what you expected? Does any of the pairs work better than the others? If so, what kind of image pairs work better? (2 points)

In [60]:
```
# Don't change the function
def morphBetweenImages(img1, img2, num_of_morphs):
    alpha = np.linspace(0,1,num_of_morphs)
    z1 = encoder_model.predict(np.array([img1]))
    z2 = encoder_model.predict(np.array([img2]))
    fig = plt.figure(figsize=(30,5))

    ax = fig.add_subplot(1, num_of_morphs+2, 1)
    ax.imshow(img1)
    ax.axis('off')
    ax.set_title(loc='center', label='original image 1',fontsize=10)

    for i in range(num_of_morphs):
        z = z1*(1-alpha[i]) + z2*alpha[i]
        new_img = decoder_model.predict(z)

        ax = fig.add_subplot(1, num_of_morphs+2, i+2)
        ax.imshow(np.clip(new_img.squeeze(),0,1))
        ax.axis('off')
        ax.set_title(loc='center', label='alpha={:.2f}'.format(alpha[i]))

    ax = fig.add_subplot(1, num_of_morphs+2, num_of_morphs+2)
    ax.imshow(img2)
    ax.axis('off')
    ax.set_title(loc='center', label='original image 2',fontsize=10)
    return
```

In [61]:
```
sample_index = random.sample(range(1, len(data)), 2)
morphBetweenImages(data[sample_index[0]],data[sample_index[1]],10)
```

Answer 1.4

The morphing process works by first encoding both images into their latent space representations. Then, an alpha parameter is introduced to control the blend between the two images' latent spaces. A higher alpha means a greater influence from the second image's latent space. This linear combination is then decoded to generate a new image. As alpha transitions from 0 to 1, we observe a smooth progression from the first image to the second.

Images with similar latent space representations (measurable via Cosine Similarity) tend to produce smoother, more coherent morphs. For example, morphing two male faces typically yields better results than morphing a female face with a male face, as they share more similar features.

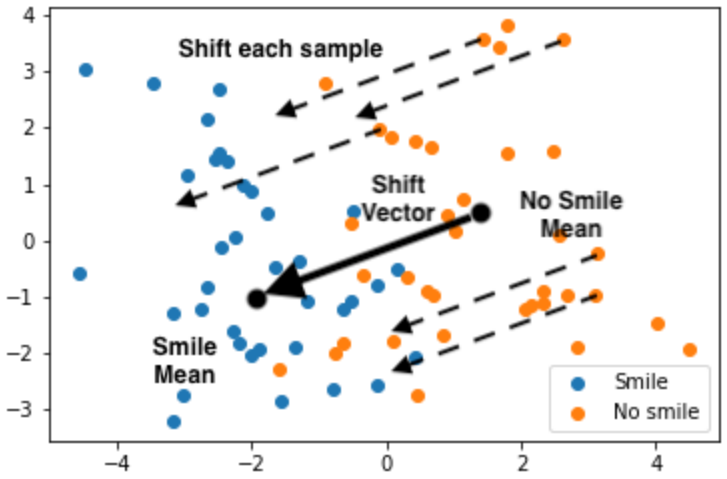
1.5 Attribute shift (10 points)

In 1.3, we've seen that faces with the same attributes form clusters in the latent space. In this problem, you will implement a function `AttributeShift` to change one attribute of the faces.

First implement the function `AttributeShift`. (5 points)

1. Calculate the latent space codes for two sets of faces that are different in one attribute, e.g. smile vs. no smile.
2. Calculate the mean latent space code for each group.
3. Get the attribute shifting vector by taking the difference between the two codes.
4. Perform attribute shift by adding the attribute shifting vector to the latent space code of the faces you want to modify.
5. Generate the image using the new latent space codes.

Here is a diagram demonstrating the shift in the latent space. Please note that the two-dimensional latent space is just for demonstration purpose. You should *not* use PCA in this problem. Instead, use the original latent space.



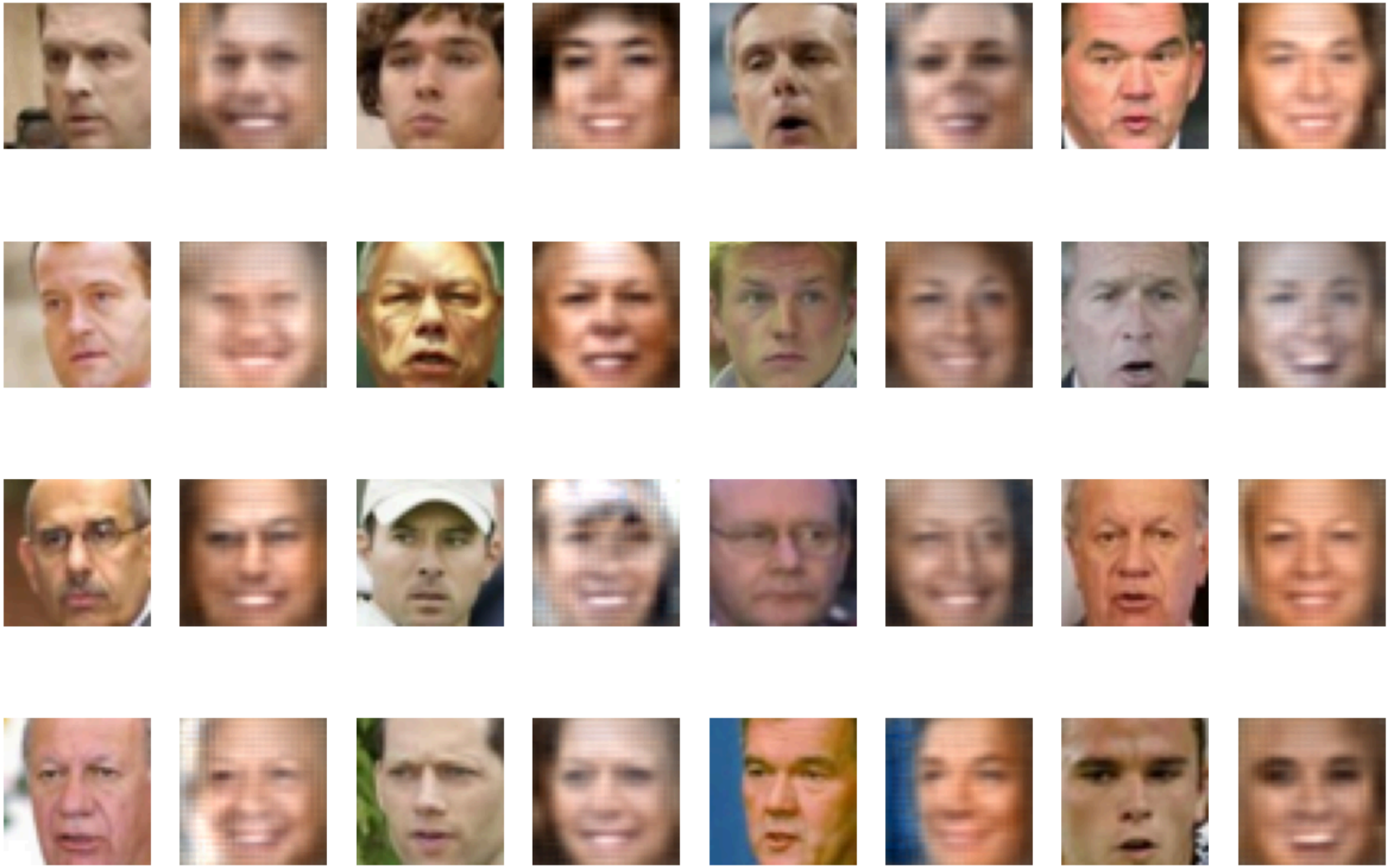
Perform attribute shift on at three attributes including smile. Comment on the faces with shifted attributes. (5 points)

- Do the generated faces look like what you expected? If not, can you think of some possible reasons?
- Do the faces with new attributes resemble the original faces? If not, can you think of some possible reasons?
- Which of the attribute shift is more successful? What are some possible reasons?
- Comment on any other aspects of your findings that are interesting to you.

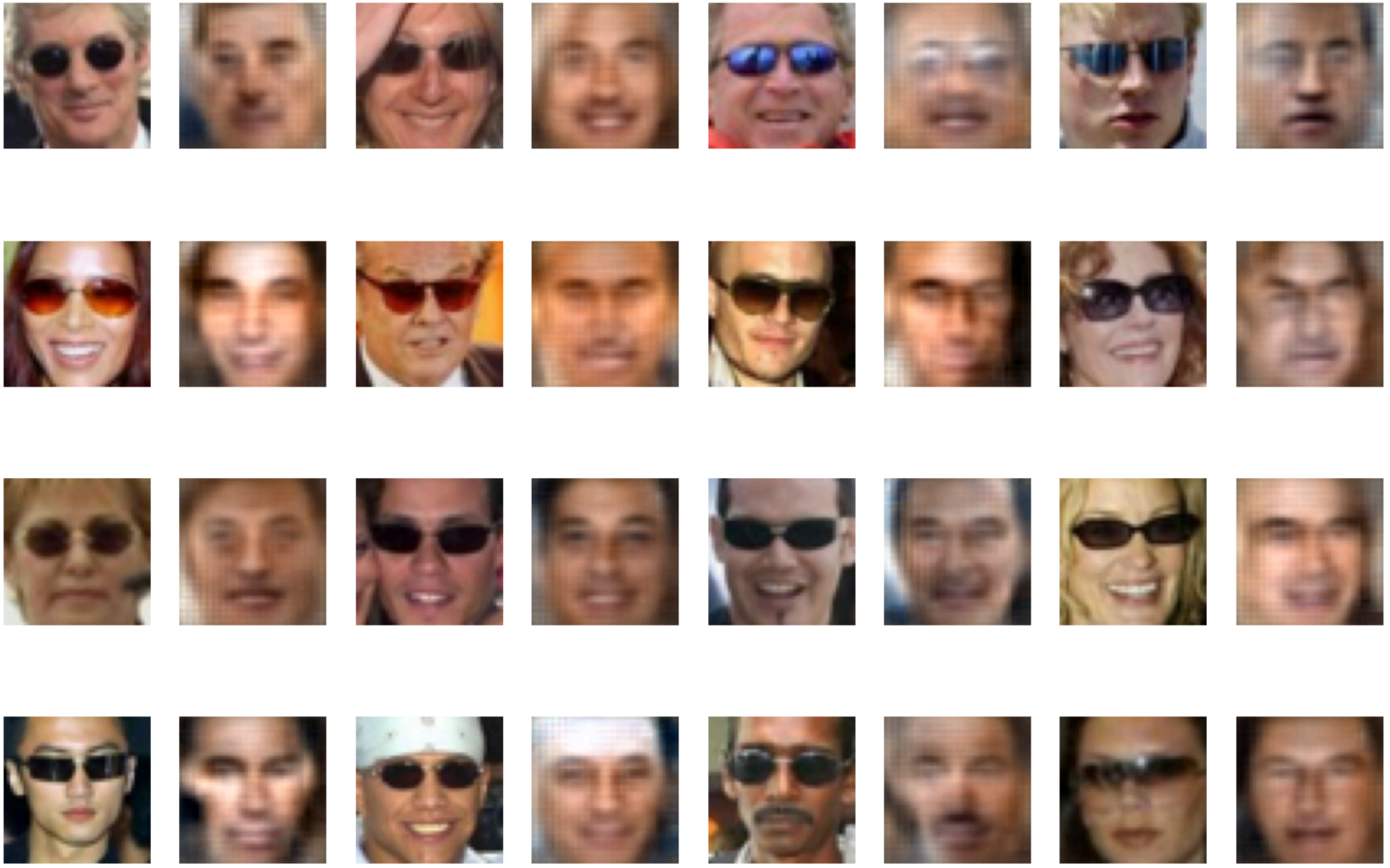
In [67]:
```
# Don't change this helper function!
def PlotAttributeShift(data2,pic_output):
    sample_index = random.sample(range(1, len(data2)), 16)

    fig, axs = plt.subplots(4, 8)
    fig.set_figheight(10)
    fig.set_figwidth(15)

    for i in range(4):
        for j in range(4):
            axs[i, 2*j].imshow(data2[sample_index[4*i+j], :, :, :])
            axs[i, 2*j].axis('off')
            axs[i, 2*j+1].imshow(np.clip(pic_output[sample_index[4*i+j]],0,1))
            axs[i, 2*j+1].axis('off')
```

In [77]:
```
def AttributeShift(encoder_model,decoder_model,data1,data2):
    # Latence space
    l_data1 = encoder_model.predict(data1)
    l_data2 = encoder_model.predict(data2)

    # Mean
    m_code1 = np.mean(l_data1, axis=0)
    m_code2 = np.mean(l_data2, axis=0)

    # Diff
    shift_vector = m_code2 - m_code1

    # Shift
    shift_l_data2 = l_data2 - shift_vector

    # Gen
    pic_output = decoder_model.predict(shift_l_data2)
    return pic_output
```

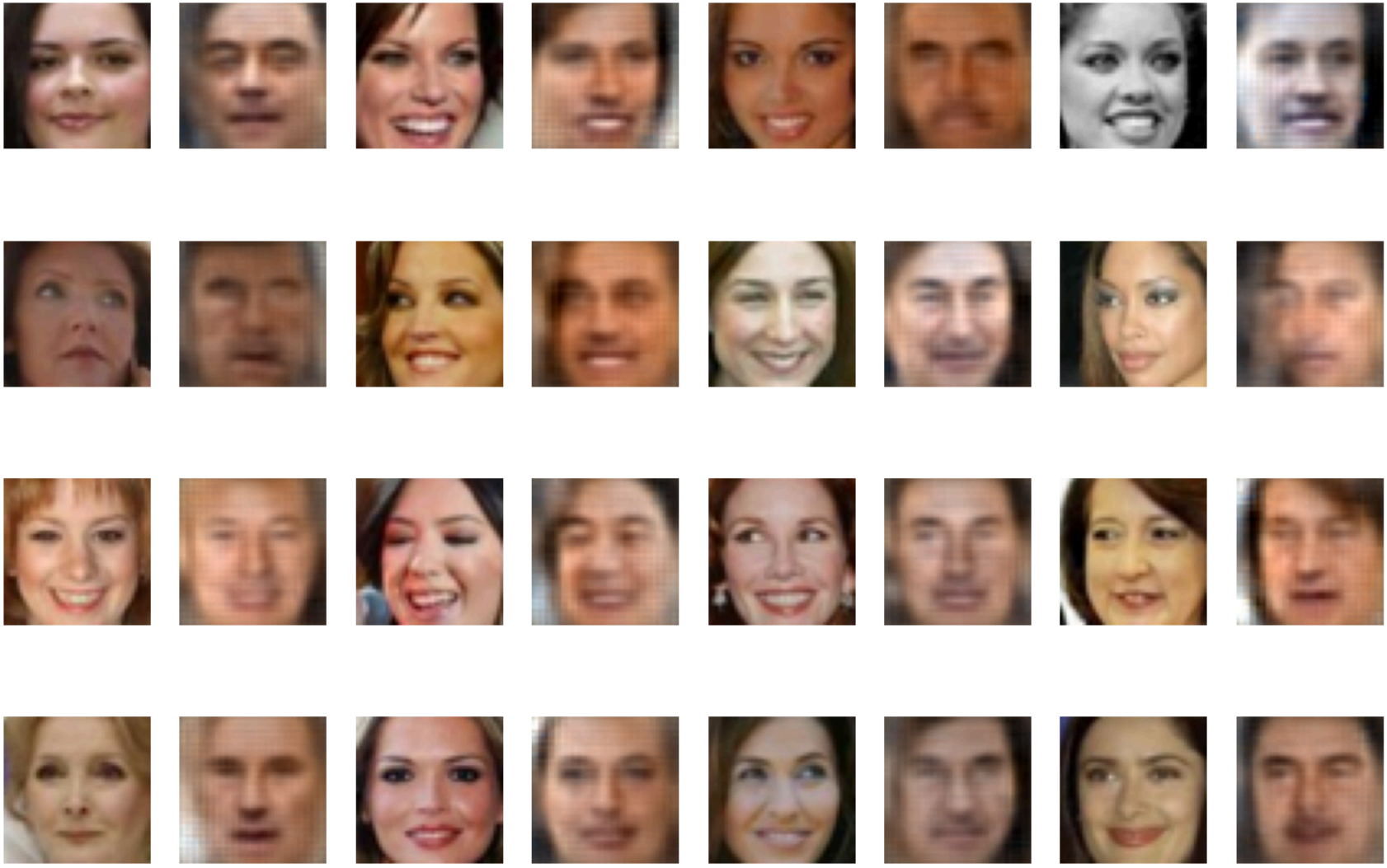
In [78]:
```
pic_output = AttributeShift(encoder_model,decoder_model,smile_data,no_smile_data)
PlotAttributeShift(no_smile_data,pic_output)
```



In [84]:
```
pic_output = AttributeShift(encoder_model,decoder_model,male_data,sunglasses_data)
PlotAttributeShift(sunglasses_data,pic_output)
```



In [81]:
```
pic_output = AttributeShift(encoder_model,decoder_model,male_data,female_data)
PlotAttributeShift(female_data,pic_output)
```



Answer 1.5

When we apply a shift in the latent space, we're essentially moving from one region to another—such as transitioning from the "No Smile" region to the "Smile" region. This is analogous to shifting from one cluster to another in a PCA plot (e.g., moving from the orange to the blue cluster). With this shift, the decoder outputs the opposite attribute, such as transforming "sunglasses" to "no sunglasses." Among all attributes, the "smile" and "no smile" transformations produce the most consistent results. This aligns with the PCA plot, where the model effectively captures and separates the variance between "Smile" and "No Smile" data. This clear

separation suggests that the model has learned distinct latent representations for these expressions, making it easier to transition between them.

Interestingly, the model's generated male version of a female image is quite accurate. It effectively captures prominent features—like face shape and hair color—and subtly tweaks them to resemble a male. This accuracy likely stems from the model's strong understanding of fundamental human features (e.g., eyes, mouth, cheeks) and its ability to differentiate subtle traits between male and female faces.

Lastly, the model performs slightly worse on the "sunglasses" attribute. The shifted output sometimes displays "eye bags" around the eyes rather than a clear transformation. This suggests that a simple mean shift may be insufficient; we might need to amplify the shift vector to achieve a more definitive transition. This observation implies that the latent spaces for "sunglasses" and "no sunglasses" are further apart geometrically, requiring a stronger push to fully cross over between these regions.

1.6 Generating new faces (3 points)

Variational autoencoders can be used to generate new data; this is why they are generative models. We can sample new data points from the distribution in latent space and reconstruct new, fake faces based on them.

To draw a sample close to an existing sample in the latent space, we can add a scaled random sample from the normal distribution to the latent space code of an existing sample. The scalar, which we call the `noise_level` is a parameter that we can tune.

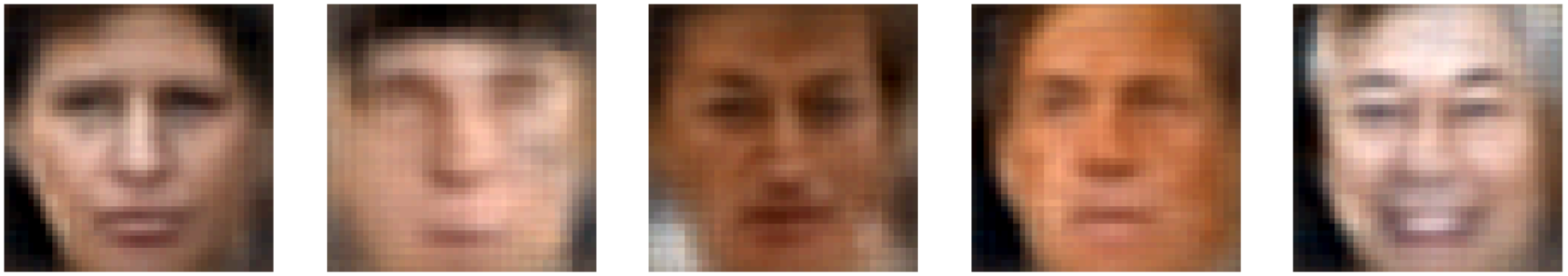
Run `GenerateFaces` with three different values of `noise_level` and comment on the generated faces. (3 points)

- Do the generated images look like faces?
- What happens when the new samples diverge more from the existing samples? What is a possible reason?

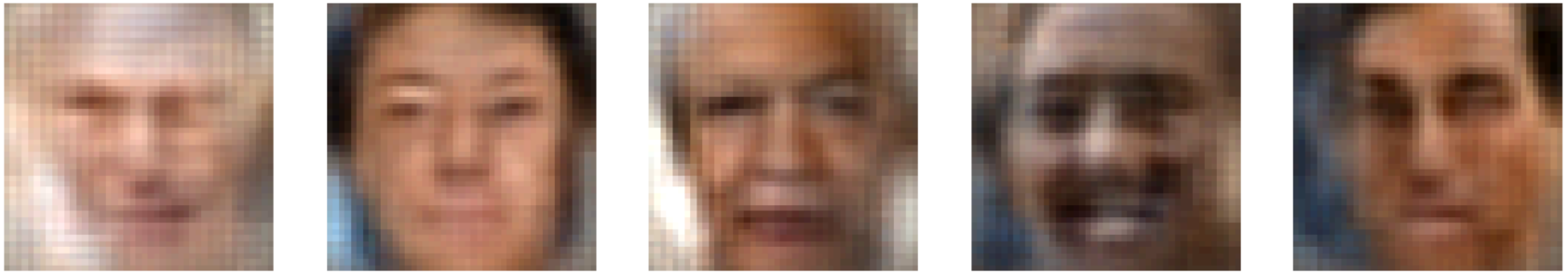
In [88]:
```
def GenerateFaces(data, LATENT_SPACE_SIZE, noise_level):
    sample_index = random.sample(range(1, len(data)), 15)
    latent_space = noise_level*np.random.normal(size=(15,LATENT_SPACE_SIZE))+encoder_model.predict(data[sample_index]
    generated_image = decoder_model.predict(latent_space)
    fig = plt.figure(figsize=(15,10))
    for i in range(number_of_images):
        ax = fig.add_subplot(3, 5, i+1)
        ax.imshow(np.clip(generated_image[i, :,:,:],0,1))
        ax.axis('off')
```

In [92]:
```
number_of_images = 5
```

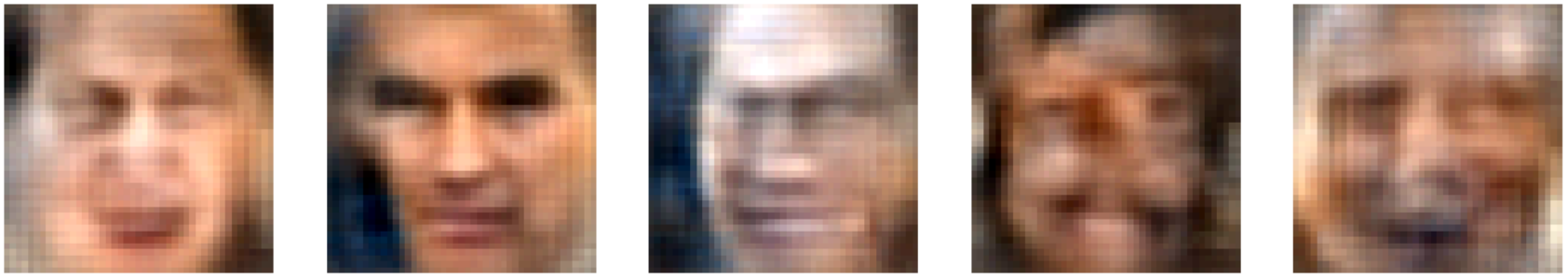
In [93]:
```
GenerateFaces(data,LATENT_SPACE_SIZE,0.1)
```



In [94]:
```
GenerateFaces(data,LATENT_SPACE_SIZE,0.5)
```



In [95]:
```
GenerateFaces(data,LATENT_SPACE_SIZE,1)
```



Answer 1.6

Images generated with lower noise resemble realistic faces, while those with higher noise levels appear distorted, almost resembling a "jump scare." Adding noise means the model is generating from a part of the latent space it hasn't encountered during training. If the model had excellent generalization capabilities, it would produce coherent faces from any latent vector, making noise addition unnecessary. However, at a noise level of 1, the model struggles to generate recognizable images, indicating that it lacks experience with such unfamiliar latent vectors and thus cannot effectively generalize in these extreme cases.

Optional: Changing the loss function (2 points extra credit)

The variable `factor` is a parameter of the loss function. In this optional problem, you will play with it and think about its effect on the performance of VAE.

Retrain the model with a smaller factor. Repeat 1.2, 1.6 (using `noise_level` = 1) and latent space visualization. Comment on how the reconstruction results, generated new faces and latent space distributions change. (2 points)

- Which model reconstructs the faces better?
- Do the generated faces look different?
- How do the latent variable distributions differ?
- Do the differences make sense? Can you explain what you observed?

In [ ]:
```
# Your code here
```

[Your markdown here]

Problem 2: Are you Schur? (10 points)

The graphical lasso is based on conditional independence properties of Gaussian distributions. This problem asks you to reason about the graphs underlying a multivariate Gaussian.

Let $X = (Y, Z) \in \mathbb{R}^6 \sim N(0, \Sigma)$ be a random Gaussian vector where $Y = (Y_1, Y_2) \in \mathbb{R}^2$ and $Z = (Z_1, Z_2, Z_3, Z_4) \in \mathbb{R}^4$, with $\Sigma^{-1} = \Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \end{pmatrix} \qquad C = \begin{pmatrix} 2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 2 & 0 & 0 \\ 0 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 2 \end{pmatrix}.$$

Problem 2.1

Draw the undirected graph of X, arranging the vertices in a hexagon. Explain your answer.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \end{pmatrix} \qquad C = \begin{pmatrix} 2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 2 & 0 & 0 \\ 0 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 2 \end{pmatrix}.$$

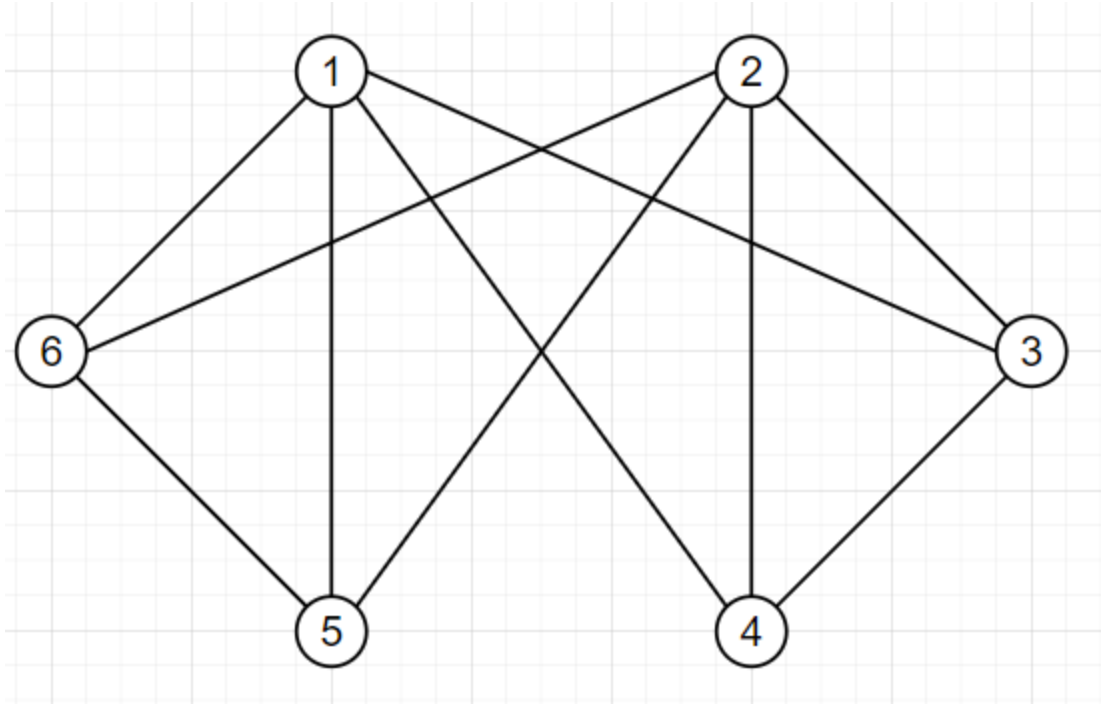
Answer 2.1

Combining A, B, C, we can construct:

$$\Omega = \begin{pmatrix} 2 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 2 & -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \\ 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 2 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 2 \end{pmatrix}$$

There are 6 diagnoal elements. Let $v_i \in V = \{i \,|\, \Omega_{ii} \,\text{exists}\}$ denote each vertex in the hexagon graph.

We draw an edge connecting vertex $v_i \in V$ and $v_j \in V$ if $\Omega_{ij} = \Omega_{ji} \neq 0$ by proposition introduced in class. Thus, we have:



Problem 2.2

Draw the undirected graph of Z, arranging the vertices in a square. Explain your answer. Hint: The Schur complement $S = C - B^T A^{-1} B$ has the property that

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}B^TA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}B^TA^{-1} & S^{-1} \end{pmatrix}$$

Answer 2.2

Lemma 1: $Z \in \mathbb{R}^4 \sim N(0,\ S^{-1})$, where $S = C - B^T A^{-1} B$ and $\Sigma = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}^{-1}$

Proof: Denote the projection row vector $A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$. Then $Z = A \cdot X$.

Thus, $E[Z] = E[A \cdot X] = A \cdot E[X] = 0 \in \mathbb{R}^4$ and $Cov[Z] = Cov[A \cdot X] = A \cdot \Sigma \cdot A^T = S^{-1}$.

Since A is a constant vector, this is a linear transformation and henze Z is normally distributed □

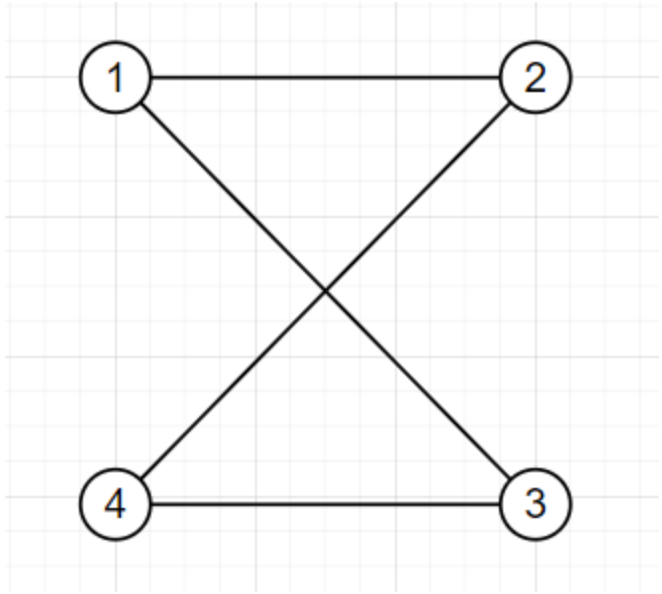
Thus, $\Omega_Z = \Sigma_Z^{-1} = (S^{-1})^{-1} = S$.

$$S = \begin{pmatrix} 2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 2 & 0 & 0 \\ 0 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

In [7]:
```
import numpy as np
C = np.array([
    [2, 0.5, 0, 0],
    [0.5, 2, 0, 0],
    [0, 0, 2, 0.5],
    [0, 0, 0.5, 2]
])
B = np.array([
    [1, -1],
    [0.5, 0.5],
    [1/3, -1/3],
    [0.25, 0.25]
])
A = np.array([
    [0.5, 0],
    [0, 0.5]
])
S = C- B @ A @ B.T
print(S)
```

```
[[ 1.          0.5        -0.33333333  0.        ]
 [ 0.5         1.75        0.         -0.125     ]
 [-0.33333333  0.          1.88888889  0.5       ]
 [ 0.         -0.125       0.5         1.9375    ]]
```

Following same procedure as Answer 2.1, we can draw:



Problem 2.3

Write down the set of all conditional independence relations on the undirected graph of X.

Answer 2.3

The set contains all conditional independence where $\Omega_{ij} = 0$:

$$X_1 \perp X_2 \mid \{X_3, X_4, X_5, X_6\}$$
$$X_3 \perp X_5 \mid \{X_1, X_2, X_4, X_6\}$$
$$X_3 \perp X_6 \mid \{X_1, X_2, X_4, X_5\}$$
$$X_4 \perp X_5 \mid \{X_1, X_2, X_3, X_6\}$$
$$X_4 \perp X_6 \mid \{X_1, X_2, X_3, X_5\}$$

Problem 3: Taking stock (15 points)

A joint distribution of data has a natural graph associated with it. When the distribution is multivariate normal, this graph is encoded in the pattern of zeros and non-zeros in the inverse of the covariance matrix, also known as the "precision matrix."

In class we demonstrated the graphical lasso for estimating the graph on ETF data. In this problem you will construct two different "portolios" of stocks, and run the graphical lasso to estimate a graph, commenting on your results.

All of the code you might need for this is contained in the demo.

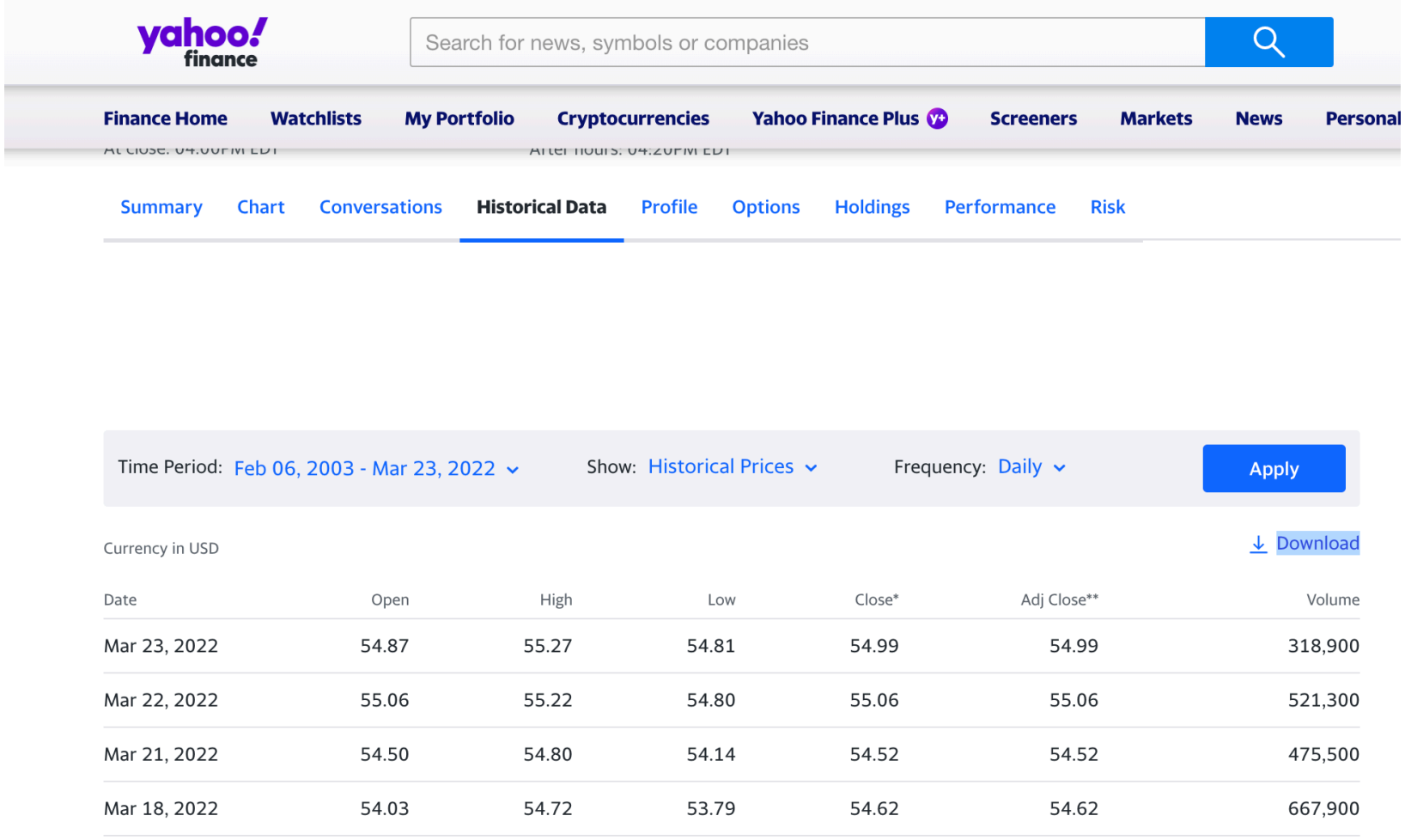
Downloading data

As demonstrated in class, you will run on equities data downloaded from Yahoo finance. Your job is to construct two "portfolios" of stocks, each of which has some kind of organization to it. For example, in one portfolio you might have 5 energy stocks, 5 tech stocks, 5 consumer staples stocks, and 5 ETF stocks. Each portfolio should have at least 20 stocks.

The page https://en.wikipedia.org/wiki/List_of_S%26P_500_companies has GICS sectors, which you may find useful for the glasso problem.

To download the data, follow the procedure outlined below (and discussed briefly in class):

- Search on a ticker symbol, like EZA, using https://finance.yahoo.com/quote/EZA/history?p=EZA
- Select the range of the query, the frequency (daily, weekly, or monthly) and then issue the query. This will give you results like this:



Date	Open	High	Low	Close*	Adj Close**	Volume
Mar 23, 2022	54.87	55.27	54.81	54.99	54.99	318,900
Mar 22, 2022	55.06	55.22	54.80	55.06	55.06	521,300
Mar 21, 2022	54.50	54.80	54.14	54.52	54.52	475,500
Mar 18, 2022	54.03	54.72	53.79	54.62	54.62	667,900

- Next, hover over the Download link, and grab the URL. In this case it gives

https://query1.finance.yahoo.com/v7/finance/download/EZA?period1=1044576000&period2=1648080000&interval=1d&events=history&includeAdjustedClose=true

- Then, you can use this same URL, but swap in different ticker symbols, to get the corresponding data for range of companies or funds.

Analyzing your portfolios

Your task is to analyze each porfolio using the graphical lasso, and comment on your findings. Here are the types of questions you should address:

- How did you choose the portolio? How did you choose the date range and frequency (daily, weekly, etc.)? Remember, each of the portfolios must contain at least 20 stocks, and be organized in some reasonable way.
- Display the graph obtained with the graphical lasso, using networkx. How did you choose the regularization level? Does the structure of the graph make sense? Is it sensitive to the choice of regularization level? Is this the structure you expected to see when you designed the portfolio? Why or why not?
- What are some of the conditional independence assumptions implied by the graph? Are some parts of the graph more densely connected than others? Why?

Answer 3

In [416…

```
import warnings
warnings.filterwarnings('ignore')

import numpy as np
import pandas as pd
import seaborn as sns
import networkx as nx
import matplotlib.pyplot as plt

from tqdm.notebook import tqdm
from scipy.stats import mstats
from joblib import Parallel, delayed
from sklearn.model_selection import KFold
from sklearn.covariance import GraphicalLasso
```

Data Logistics

1. price_d_sp500_2023.parquet.brotli is price data for all tickers in SP500 2023 ticker list.
2. wrds_link.parquet.brotli is linking data compiled from Wharton Research Data Service (WRDS) that links tickers with other identifies (i.e., permno, gvkey, etc.)

3. wrds_ind.parquet.brotli is industry data compiled from Wharton Research Data Service (WRDS) that contains industry information for the stocks.

Note: All of this data was compiled before by me for ongoing Finance Research at Yale SOM. This data is verified and tested in our research.

In [158…
```
price_d = pd.read_parquet('price_d_sp500_2023.parquet.brotli')
wrds_link = pd.read_parquet('wrds_link.parquet.brotli')
wrds_ind = pd.read_parquet('wrds_ind.parquet.brotli')
```

In [159…
```
price_d
```

Out[159…

ticker	date	open	high	low	close	adj_close	volume	unadj_volume	change	change_percent	vwap	label
A	2000-01-03	53.06	53.18	45.39	48.51	43.60	4962189.0	4962189.0	-4.55	-8.58000	50.0350	January 03, 00
	2000-01-04	45.90	46.40	43.63	44.17	39.70	5058514.0	5058514.0	-1.73	-3.77000	45.0250	January 04, 00
	2000-01-05	44.64	44.64	40.64	42.03	37.77	6113793.0	6113793.0	-2.61	-5.85000	42.9875	January 05, 00
	2000-01-06	41.52	41.77	39.16	39.75	35.73	2690591.0	2690591.0	-1.77	-4.26000	40.5500	January 06, 00
	2000-01-07	39.79	44.43	39.75	43.79	39.36	2993521.0	2993521.0	4.00	10.05000	41.9400	January 07, 00
…	…	…	…	…	…	…	…	…	…	…	…	…
ZTS	2023-12-22	195.32	195.91	192.74	194.98	193.99	1548412.0	1548412.0	-0.34	-0.17407	194.7375	December 22, 23
	2023-12-26	194.88	196.34	194.09	195.50	194.50	814641.0	814641.0	0.62	0.31814	195.2025	December 26, 23
	2023-12-27	195.41	197.01	194.74	196.90	195.90	766411.0	766411.0	1.49	0.76250	196.0150	December 27, 23
	2023-12-28	197.62	198.60	196.53	197.16	196.15	880094.0	880094.0	-0.46	-0.23277	197.4775	December 28, 23
	2023-12-29	196.68	198.01	196.25	197.37	196.36	1007235.0	1007235.0	0.69	0.35082	197.0775	December 29, 23

2737829 rows × 13 columns

In [160…

```
print(price_d.index.get_level_values('ticker').unique().tolist())
```

```
['A', 'AAL', 'AAPL', 'ABBV', 'ABNB', 'ABT', 'ACGL', 'ACN', 'ADBE', 'ADI', 'ADM', 'ADP', 'ADSK', 'AEE', 'AEP', 'AES',
'AFL', 'AIG', 'AIZ', 'AJG', 'AKAM', 'ALB', 'ALGN', 'ALL', 'ALLE', 'AMAT', 'AMCR', 'AMD', 'AME', 'AMGN', 'AMP', 'AMT',
'AMZN', 'ANET', 'ANSS', 'AON', 'AOS', 'APA', 'APD', 'APH', 'APTV', 'ARE', 'ATO', 'AVB', 'AVGO', 'AVY', 'AWK', 'AXON',
'AXP', 'AZO', 'BA', 'BAC', 'BALL', 'BAX', 'BBWI', 'BBY', 'BDX', 'BEN', 'BF-B', 'BG', 'BIIB', 'BIO', 'BK', 'BKNG', 'BK
R', 'BLDR', 'BLK', 'BMY', 'BR', 'BRK-B', 'BRO', 'BSX', 'BWA', 'BX', 'BXP', 'C', 'CAG', 'CAH', 'CARR', 'CAT', 'CB', 'C
BOE', 'CBRE', 'CCI', 'CCL', 'CDAY', 'CDNS', 'CDW', 'CE', 'CEG', 'CF', 'CFG', 'CHD', 'CHRW', 'CHTR', 'CI', 'CINF', 'C
L', 'CLX', 'CMA', 'CMCSA', 'CME', 'CMG', 'CMI', 'CMS', 'CNC', 'CNP', 'COF', 'COG', 'COO', 'COP', 'COR', 'COST', 'CPA
Y', 'CPB', 'CPRT', 'CPT', 'CRL', 'CRM', 'CSCO', 'CSGP', 'CSX', 'CTAS', 'CTLT', 'CTRA', 'CTSH', 'CTVA', 'CVS', 'CVX',
'CZR', 'D', 'DAL', 'DAY', 'DD', 'DE', 'DFS', 'DG', 'DGX', 'DHI', 'DHR', 'DIS', 'DISCK', 'DLPH', 'DLR', 'DLTR', 'DOC',
'DOV', 'DOW', 'DPZ', 'DRI', 'DTE', 'DUK', 'DVA', 'DVN', 'DXCM', 'EA', 'EBAY', 'ECL', 'ED', 'EFX', 'EIX', 'EL', 'ELV',
'EMN', 'EMR', 'ENPH', 'EOG', 'EPAM', 'EQIX', 'EQR', 'EQT', 'ES', 'ESS', 'ETN', 'ETR', 'ETSY', 'EVRG', 'EW', 'EXC', 'E
XPD', 'EXPE', 'EXR', 'F', 'FANG', 'FAST', 'FB', 'FCX', 'FDS', 'FDX', 'FE', 'FFIV', 'FI', 'FICO', 'FIS', 'FITB', 'FL
T', 'FMC', 'FOX', 'FOXA', 'FRT', 'FSLR', 'FSR', 'FTNT', 'FTV', 'GD', 'GE', 'GEHC', 'GEN', 'GILD', 'GIS', 'GL', 'GLW',
'GM', 'GNRC', 'GOOG', 'GOOGL', 'GPC', 'GPN', 'GRMN', 'GS', 'GWW', 'HAL', 'HAS', 'HBAN', 'HCA', 'HD', 'HES', 'HIG', 'H
II', 'HLT', 'HOLX', 'HON', 'HPE', 'HPQ', 'HRL', 'HSIC', 'HST', 'HSY', 'HUBB', 'HUM', 'HWM', 'IBM', 'ICE', 'IDXX', 'IE
X', 'IFF', 'ILMN', 'INCY', 'INTC', 'INTU', 'INVH', 'IP', 'IPG', 'IQV', 'IR', 'IRM', 'ISRG', 'IT', 'ITW', 'IVZ', 'J',
'JBHT', 'JBL', 'JCI', 'JEC', 'JKHY', 'JNJ', 'JNPR', 'JPM', 'K', 'KDP', 'KEY', 'KEYS', 'KHC', 'KIM', 'KLAC', 'KMB', 'K
MI', 'KMX', 'KO', 'KR', 'KVUE', 'L', 'LDOS', 'LEN', 'LH', 'LHX', 'LIN', 'LKQ', 'LLY', 'LMT', 'LNT', 'LOW', 'LRCX', 'L
ULU', 'LUV', 'LVS', 'LW', 'LYB', 'LYV', 'MA', 'MAA', 'MAR', 'MAS', 'MCD', 'MCHP', 'MCK', 'MCO', 'MDLZ', 'MDT', 'MET',
'META', 'MGM', 'MHK', 'MKC', 'MKTX', 'MLM', 'MMC', 'MMM', 'MNST', 'MO', 'MOH', 'MOS', 'MPC', 'MPWR', 'MRK', 'MRNA',
'MRO', 'MS', 'MSCI', 'MSFT', 'MSI', 'MTB', 'MTCH', 'MTD', 'MU', 'NCLH', 'NDAQ', 'NDSN', 'NEE', 'NEM', 'NFLX', 'NI',
'NKE', 'NOC', 'NOW', 'NRG', 'NSC', 'NTAP', 'NTRS', 'NUE', 'NVDA', 'NVR', 'NWS', 'NWSA', 'NXPI', 'O', 'ODFL', 'OKE',
'OMC', 'ON', 'ORCL', 'ORLY', 'OTIS', 'OXY', 'PANW', 'PARA', 'PAYC', 'PAYX', 'PCAR', 'PCG', 'PEAK', 'PEG', 'PEP', 'PF
E', 'PFG', 'PG', 'PGR', 'PH', 'PHM', 'PKG', 'PLD', 'PM', 'PNC', 'PNR', 'PNW', 'PODD', 'POOL', 'PPG', 'PPL', 'PRU', 'P
SA', 'PSX', 'PTC', 'PWR', 'PXD', 'PYPL', 'Q', 'QCOM', 'QRVO', 'RCL', 'RE', 'REG', 'REGN', 'RF', 'RHI', 'RJF', 'RL',
'RMD', 'ROK', 'ROL', 'ROP', 'ROST', 'RSG', 'RTX', 'RVTY', 'SBAC', 'SBUX', 'SCHW', 'SHW', 'SJM', 'SLB', 'SNA', 'SNPS',
'SO', 'SPG', 'SPGI', 'SRE', 'STE', 'STLD', 'STT', 'STX', 'STZ', 'SWK', 'SWKS', 'SYF', 'SYK', 'SYY', 'TAP', 'TDG', 'TD
Y', 'TECH', 'TEL', 'TER', 'TFC', 'TFX', 'TGT', 'TJX', 'TMO', 'TMUS', 'TPR', 'TRGP', 'TRMB', 'TROW', 'TRV', 'TSCO', 'T
SLA', 'TSN', 'TT', 'TTWO', 'TXN', 'TXT', 'TYL', 'UAL', 'UBER', 'UDR', 'UHS', 'ULTA', 'UNH', 'UNP', 'UPS', 'URI', 'US
B', 'V', 'VFC', 'VICI', 'VLO', 'VLTO', 'VMC', 'VRSK', 'VRSN', 'VRTX', 'VTR', 'VTRS', 'VZ', 'WAB', 'WAT', 'WBA', 'WB
D', 'WDC', 'WEC', 'WELL', 'WFC', 'WHR', 'WM', 'WMB', 'WMT', 'WRB', 'WRK', 'WST', 'WTW', 'WY', 'WYNN', 'XEL', 'XOM',
'XRAY', 'XYL', 'YUM', 'ZBH', 'ZBRA', 'ZION', 'ZTS']
```

In [161…

```
wrds_link
```

Out[161…

	gvkey	conm	ticker	cusip	cik	sic	naics	linkprim	linktype	liid	permno	lpermco	t
0	001000	A & E PLASTIK PAK INC	AE.2	000032102	None	3089	None	P	LU	01	25881	23369.0	
1	001001	A & M FOOD SERVICES INC	AMFD.	000165100	0000723576	5812	722	P	LU	01	10015	6398.0	
2	001002	AAI CORP	AAIC.1	000352104	0001306124	3825	None	C	LC	01	10023	22159.0	
3	001003	A.A. IMPORTING CO INC	ANTQ	000354100	0000730052	5712	442110	C	LU	01	10031	6672.0	
4	001004	AAR CORP	AIR	000361105	0000001750	5080	423860	P	LU	01	54594	20000.0	
…	…	…	…	…	…	…	…	…	…	…	…	…	
31533	349994	CLEARMIND MEDICINE INC	CMND	185053402	0001892500	2834	325412	P	LC	01	23514	59438.0	
31534	350681	GETNET ADQUIRENCIA E	GET	37428A103	0001867325	7374	518210	P	LC	90	22205	58855.0	
31535	351038	QUOIN PHARMACEUTICALS LTD	QNRX	74907L300	0001671502	2834	325412	P	LC	90	16161	55612.0	
31536	352262	COOL COMPANY LTD	CLCO	G2415A113	0001944057	4400	4831	P	LC	01	23773	59507.0	
31537	353444	HALEON PLC	HLN	405552100	0001900304	2834	325412	P	LC	90	23209	59330.0	

31538 rows × 14 columns

In [162…

```
wrds_ind
```

Out[162…

	ticker	ind	sub_ind
51	AAL	203020	20302010
83	PNW	551010	55101010
165	AMD	453010	45301020
213	APD	151010	15101040
338	SWKS	453010	45301020
...	...	...	...
45560	EPAM	451020	45102010
45676	XYL	201060	20106020
46149	CBRE	602010	60201040
46413	LYB	151010	15101010
46462	ALLE	201020	20102010

417 rows × 3 columns

Format Data

I choose to use weekly return data to reduce number of observations and computation time for GLasso, but also retain a sufficient amount of data for accurate estimatin of Ω. I selected the date range from 2012 to 2018, as this period exhibits relatively stable "random" returns, unaffected by significant events such as the 2008 Financial Crisis or the 2020 COVID-19 pandemic.

In [185…

```
# Create return data (use adj_close instead of close because this is yields more accurate backtest)
price_w = price_d.reset_index('ticker').groupby('ticker').resample('W').last()
price_w = price_w.drop('ticker', axis=1)
price_w['ret'] = price_w.groupby('ticker')['adj_close'].pct_change(1)

# Compute log returns
price_w['log_ret'] = np.log(1 + price_w['ret'])
```

```
# Winsorize log returns at 1% and 99%
price_w['log_ret'] = mstats.winsorize(price_w['log_ret'], limits=[0.01, 0.01])
```

In [217…

```
# Add industry
price_ind = price_w.reset_index('date').join(wrds_ind.set_index('ticker')).reset_index().set_index(['ticker', 'date']

# Set timeframe
price = price_ind.loc[(price_ind.index.get_level_values('date') >= '2012-01-01') & (price_ind.index.get_level_values(

# Filter out any tickers that do not have same length of data from 2012 to 2018
price_unstack = price['log_ret'].unstack('ticker')
price_unstack = price_unstack.dropna(axis=1)
price = price.loc[price.index.get_level_values('ticker').isin(price_unstack.columns.get_level_values('ticker').unique
price = price.dropna(subset='ind')
```

In [218…

```
# Display unique industrys
price.ind.unique()
```

Out[218…

```
array([352030., 203020., 452020., 403010., 451020., 451030., 453010.,
       202020., 551030., 551050., 151010., 351010., 201040., 352010.,
       402030., 601080., 255030., 201020., 101020., 452030., 251010.,
       601050., 551020., 601060., 151030., 551040., 201010., 402020.,
       255040., 302020., 253010., 101010., 601040., 351020., 602010.,
       303010., 203010., 502010., 401010., 301010., 402010., 202010.,
       452010., 203040., 252010., 502020., 201060., 551010., 303020.,
       201070., 601070., 251020., 502030., 255010., 252020., 601030.,
       352020., 302010., 252030., 151020., 151040., 601025., 302030.,
       201030., 501010.])
```

Create Portfolio

To enhance diversification and reduce risk, I construct two distinct portfolios. Each portfolio is built by randomly selecting 10 industries and, within each selected industry, randomly choosing up to 5 tickers. This approach ensures a broad spread across sectors while maintaining exposure within each industry.

In [219…

```
# Create portfolio
def create_portfolio(price):
    # Randomly sample 10 industries
    unique_inds = price.ind.unique()
    sampled_inds = np.random.choice(unique_inds, size=10, replace=False)
```

```
    # Randomly sample maximum 5 tickers per industry
    sampled_tic = {}
    for ind in sampled_inds:
        tic_in_ind = price.loc[price.ind == ind].index.get_level_values('ticker').unique()
        ss = min(5, len(tic_in_ind))
        sampled_tic[ind] = np.random.choice(tic_in_ind, size=ss, replace=False).tolist()
    return sampled_tic
```

In [220…
```
# Portfolio 1
port_1 = create_portfolio(price)
port_1_tic = list(set(ticker for tickers in port_1.values() for ticker in tickers))
port_1
```

Out[220…
```
{601060.0: ['ESS', 'CPT', 'EQR', 'UDR', 'AVB'],
 252030.0: ['LULU', 'TPR', 'VFC', 'RL', 'NKE'],
 302020.0: ['HRL', 'MKC', 'CAG', 'BG', 'SJM'],
 203010.0: ['FDX', 'CHRW', 'EXPD', 'UPS'],
 201060.0: ['XYL', 'IEX', 'SWK', 'PCAR', 'WAB'],
 403010.0: ['PGR', 'WTW', 'BRO', 'AIZ', 'AJG'],
 502010.0: ['IPG', 'CMCSA', 'PARA', 'CHTR', 'OMC'],
 501010.0: ['VZ'],
 151020.0: ['MLM', 'VMC'],
 301010.0: ['WBA', 'DG', 'SYY', 'TGT', 'COST']}
```

In [221…
```
# Portfolio 2
port_2 = create_portfolio(price)
port_2_tic = list(set(ticker for tickers in port_2.values() for ticker in tickers))
port_2
```

Out[221…
```
{201030.0: ['PWR'],
 551010.0: ['LNT', 'EVRG', 'ES', 'PNW', 'NEE'],
 301010.0: ['SYY', 'DG', 'DLTR', 'TGT', 'WBA'],
 101010.0: ['BKR'],
 601050.0: ['VTR', 'ARE', 'WELL', 'DOC'],
 351020.0: ['CNC', 'COR', 'HUM', 'UNH', 'UHS'],
 402010.0: ['FIS', 'FI', 'V', 'CPAY', 'GPN'],
 601040.0: ['BXP'],
 502030.0: ['MTCH', 'GOOGL'],
 352030.0: ['WST', 'CRL', 'RVTY', 'ILMN', 'TMO']}
```

Estimate Ω

I define a parameter grid and perform a 5-fold cross-validation for each parameter setting. For each fitted model, I compute the average likelihood score to assess risk. Finally, I refit the model using the parameter that achieved the lowest risk.

In [243…
```
# Unstack price
log_ret = price[['log_ret']].unstack('ticker')
log_ret.columns = log_ret.columns.get_level_values('ticker')
```

In [320…
```
# Compute cross-validation score for a given alpha
def compute_alpha_score(alpha, X, kf):
    cv_score = 0
    for train_index, test_index in kf.split(X):
        X_train, X_test = X[train_index], X[test_index]
        glasso = GraphicalLasso(alpha=alpha).fit(X_train)
        cv_score += -glasso.score(X_test)

    avg_score = cv_score / kf.get_n_splits()
    return alpha, avg_score

# 5 fold cross-validation over parameter space (parallelized per batch)
def glasso_cv_5(X, alpha_grid=np.linspace(0.1, 1.0, 100), batch_size=16):
    # Standardize
    X = np.array(X)
    p = X.shape[1]
    X = (X - np.mean(X, axis=0)) / np.std(X, axis=0)

    # 5-Fold Split
    kf = KFold(n_splits=5, shuffle=True, random_state=0)

    # Parallelize batch processing (i.e., run all alphas in batch at the same time)
    results = []
    num_batches = len(alpha_grid) // batch_size + (1 if len(alpha_grid) % batch_size != 0 else 0)

    for i in range(0, len(alpha_grid), batch_size):
        batch_alphas = alpha_grid[i:i + batch_size]
        batch_results = Parallel(n_jobs=-1)(delayed(compute_alpha_score)(alpha, X, kf) for alpha in batch_alphas)
        results.extend(batch_results)
        print(f"Progress: {(i // batch_size) + 1}/{num_batches}")
```

```
    # Format results
    results_df = pd.DataFrame(results, columns=['alpha', 'score'])

    # Plot alpha vs. score
    plt.plot(results_df['alpha'], results_df['score'], marker='o')
    plt.xlabel('Alpha')
    plt.ylabel('Score')
    plt.title('Cross-Validation Score vs. Alpha')
    plt.show()

    # Get best alpha
    best_alpha = results_df.loc[results_df['score'].idxmin(), 'alpha']
    print(f"Best alpha: {best_alpha}\n")

    # Refit model using best alpha
    glasso_best = GraphicalLasso(alpha=best_alpha).fit(X)
    Omega_hat = glasso_best.precision_

    # Set diagonal elements to 0 for visualization
    for j in np.arange(p):
            Omega_hat[j,j] = 0

    # Plot heatmap of the precision matrix
    sns.heatmap(Omega_hat, cmap='jet')
    plt.show()

    # Return the fitted precision matrix and results DataFrame
    return Omega_hat
```

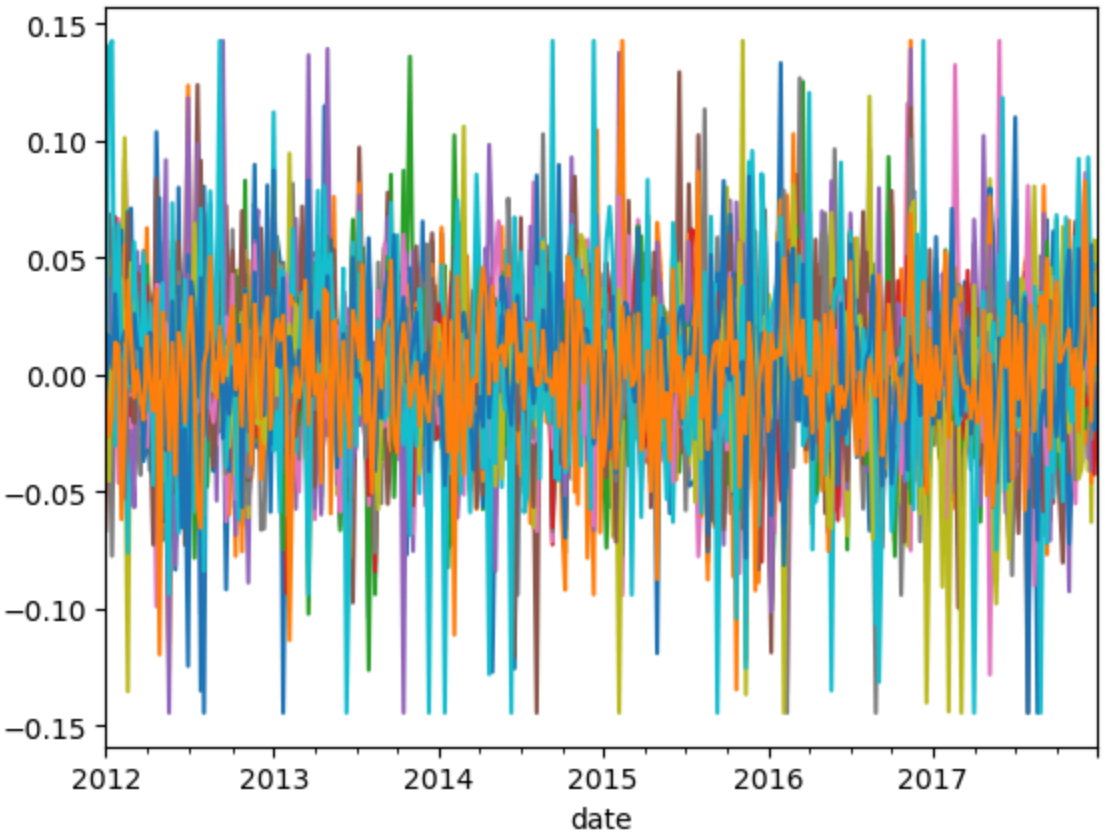
In [321…

```
# Params
port_1_X = log_ret[port_1_tic]
port_2_X = log_ret[port_2_tic]
alpha_grid = np.linspace(0, 1, 100)
batch_size = 16
```

In [322…

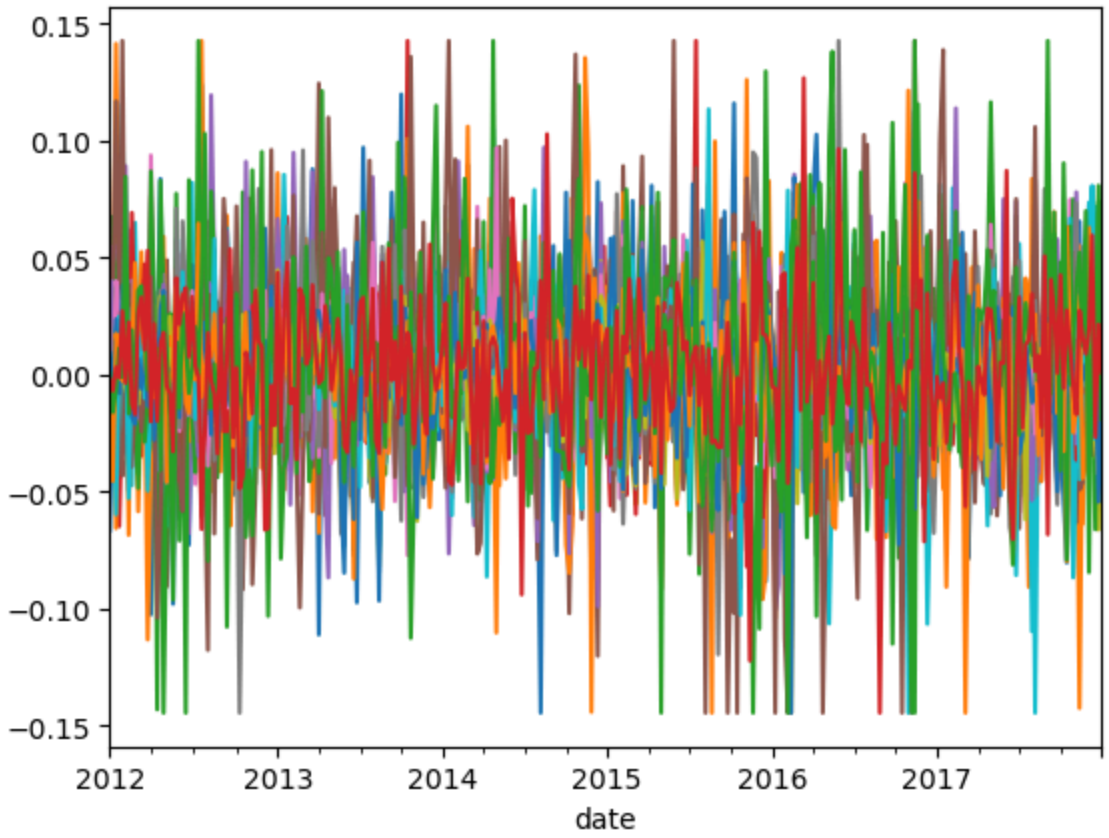
```
# Port 1 (I looks normal!)
port_1_X.plot(legend=False)
```

Out[322… <Axes: xlabel='date'>



In [323…
```
# Port 2 (I looks normal!)
port_2_X.plot(legend=False)
```

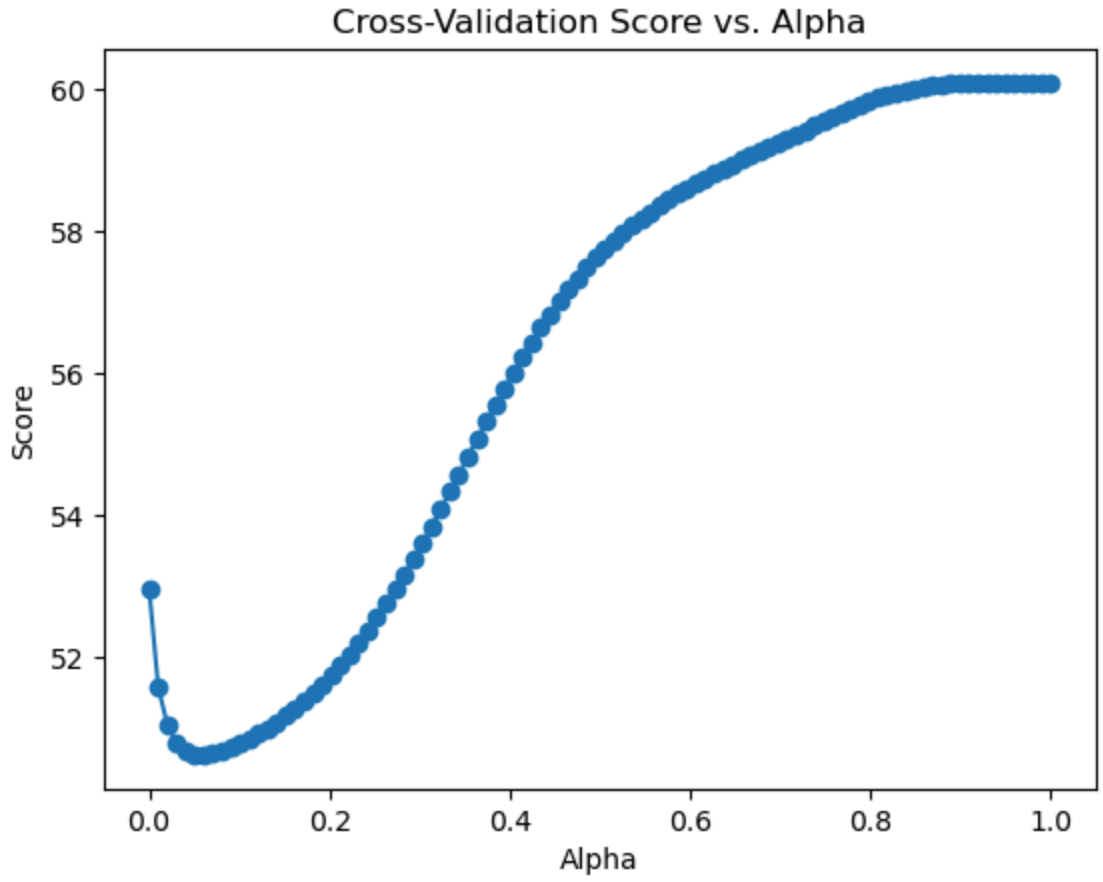
Out[323… <Axes: xlabel='date'>



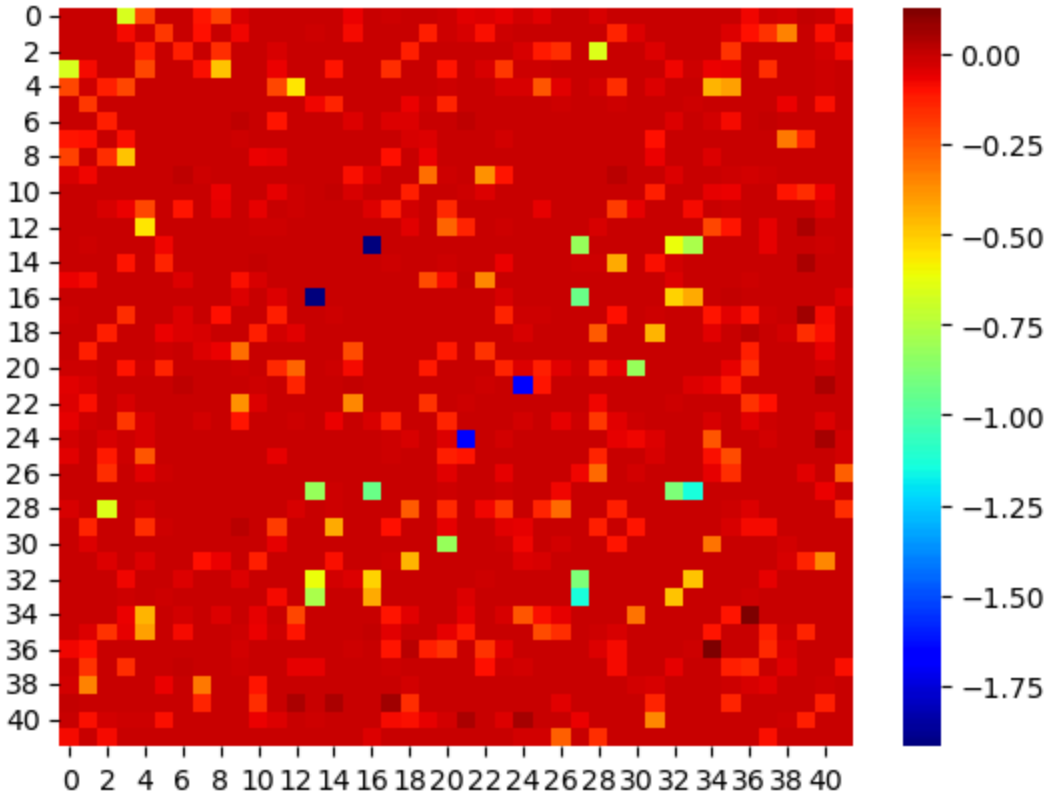
In [324…

```
# Run Port 1
Omega_port_1 = glasso_cv_5(port_1_X, alpha_grid, batch_size)
```

```
Progress: 1/7
Progress: 2/7
Progress: 3/7
Progress: 4/7
Progress: 5/7
Progress: 6/7
Progress: 7/7
```



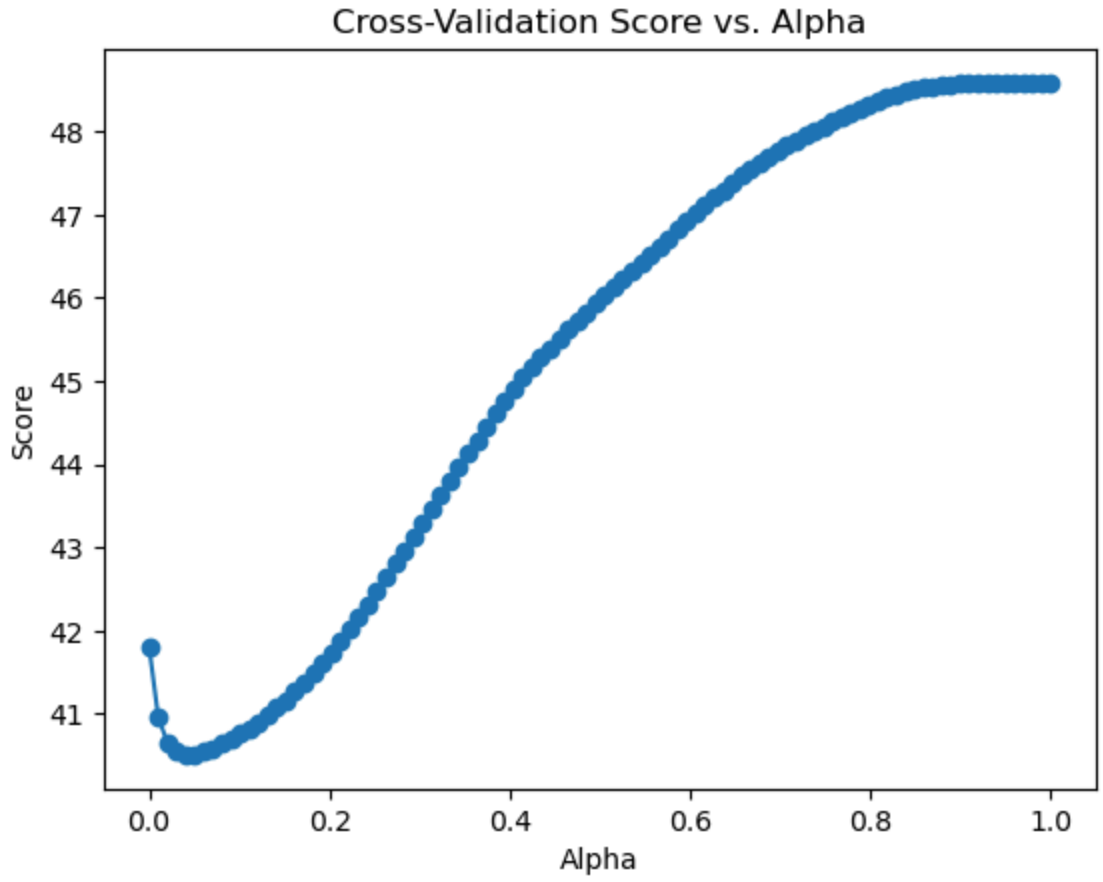

```
Best alpha: 0.06060606060606061
```



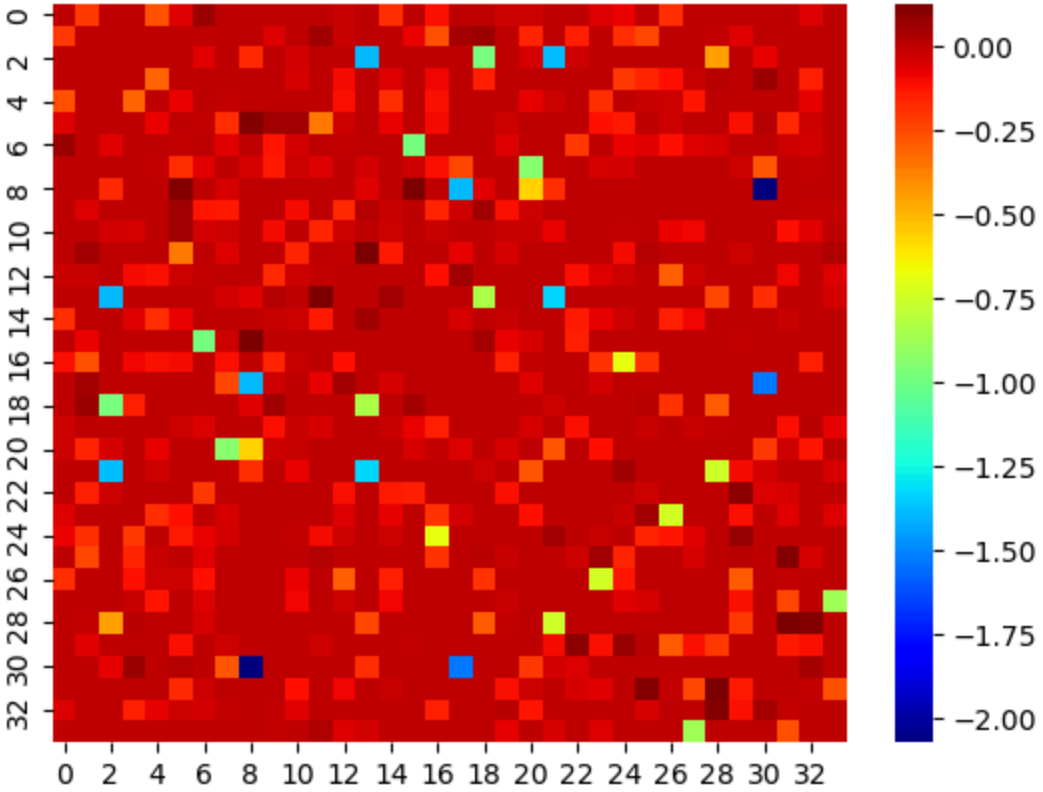
In [325…

```
# Run Port 2
Omega_port_2 = glasso_cv_5(port_2_X, alpha_grid, batch_size)
```

```
Progress: 1/7
Progress: 2/7
Progress: 3/7
Progress: 4/7
Progress: 5/7
Progress: 6/7
Progress: 7/7
```




```
Best alpha: 0.04040404040404041
```



Plot Graph

In [342…

```python
def plot_precision_graph(Omega, tickers, threshold=0.1, figsize=(8, 8)):
    # Convert precision matrix
    precdf = pd.DataFrame(Omega, columns=tickers, index=tickers)
    links = precdf.stack().reset_index()
    links.columns = ['var1', 'var2', 'value']
    links = links[(abs(links['value']) > threshold) & (links['var1'] != links['var2'])]

    # Build the graph
    G = nx.from_pandas_edgelist(links, 'var1', 'var2', create_using=nx.Graph())
    pos = nx.spring_layout(G, k=1.7/np.sqrt(len(G.nodes())), iterations=20)

    # Draw the graph
    plt.figure(figsize=figsize)
    nx.draw(G, pos=pos, node_size=50, edge_color="gray", node_color="skyblue", with_labels=False)
```

```
    nx.draw_networkx_labels(G, pos=pos, font_size=8)
    plt.show()
```

In [343…

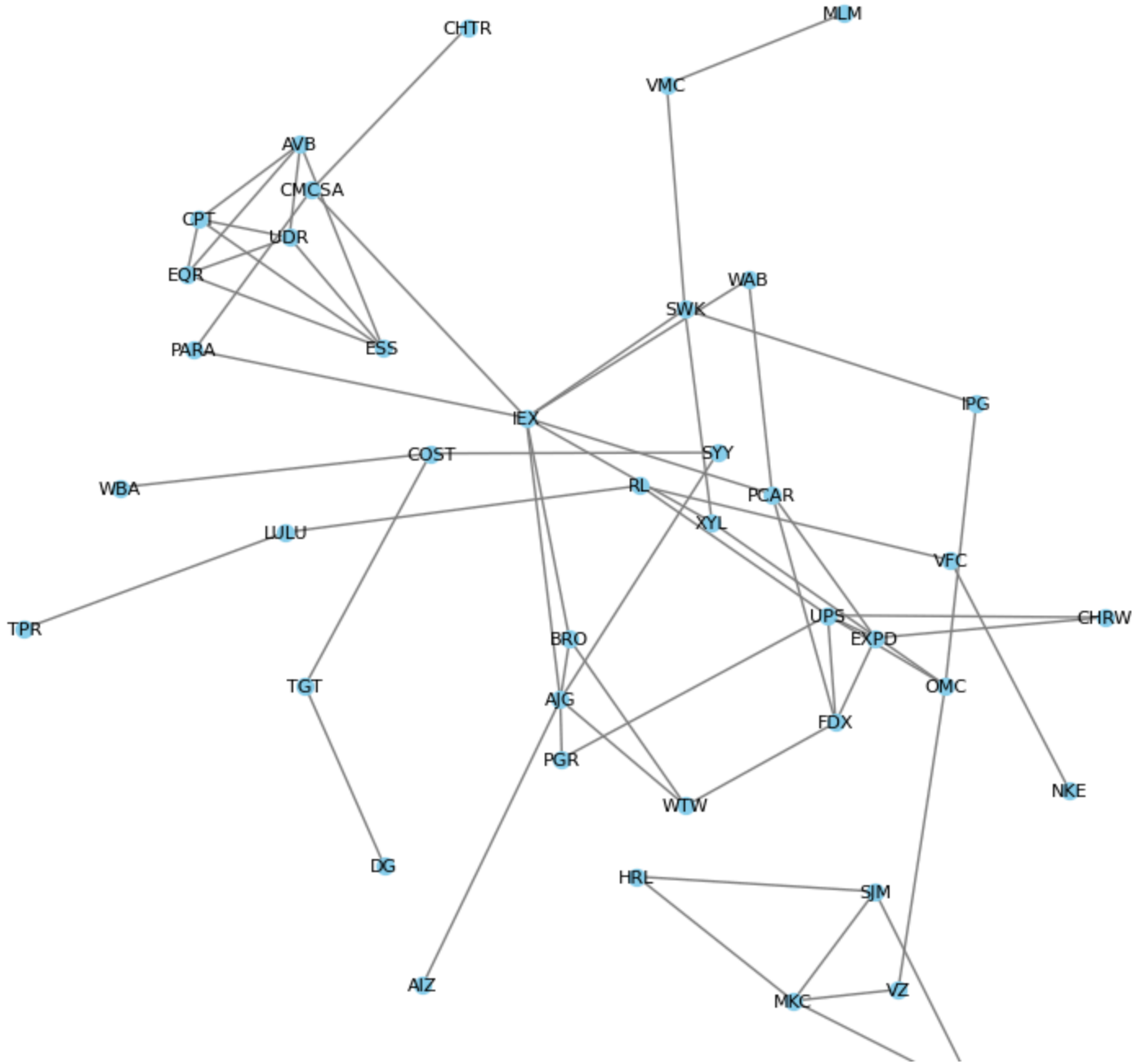
```
port_1
```

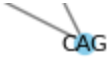
Out[343…

```
{601060.0: ['ESS', 'CPT', 'EQR', 'UDR', 'AVB'],
 252030.0: ['LULU', 'TPR', 'VFC', 'RL', 'NKE'],
 302020.0: ['HRL', 'MKC', 'CAG', 'BG', 'SJM'],
 203010.0: ['FDX', 'CHRW', 'EXPD', 'UPS'],
 201060.0: ['XYL', 'IEX', 'SWK', 'PCAR', 'WAB'],
 403010.0: ['PGR', 'WTW', 'BRO', 'AIZ', 'AJG'],
 502010.0: ['IPG', 'CMCSA', 'PARA', 'CHTR', 'OMC'],
 501010.0: ['VZ'],
 151020.0: ['MLM', 'VMC'],
 301010.0: ['WBA', 'DG', 'SYY', 'TGT', 'COST']}
```

In [348…

```
plot_precision_graph(Omega_port_1, port_1_tic, threshold=0.15)
```

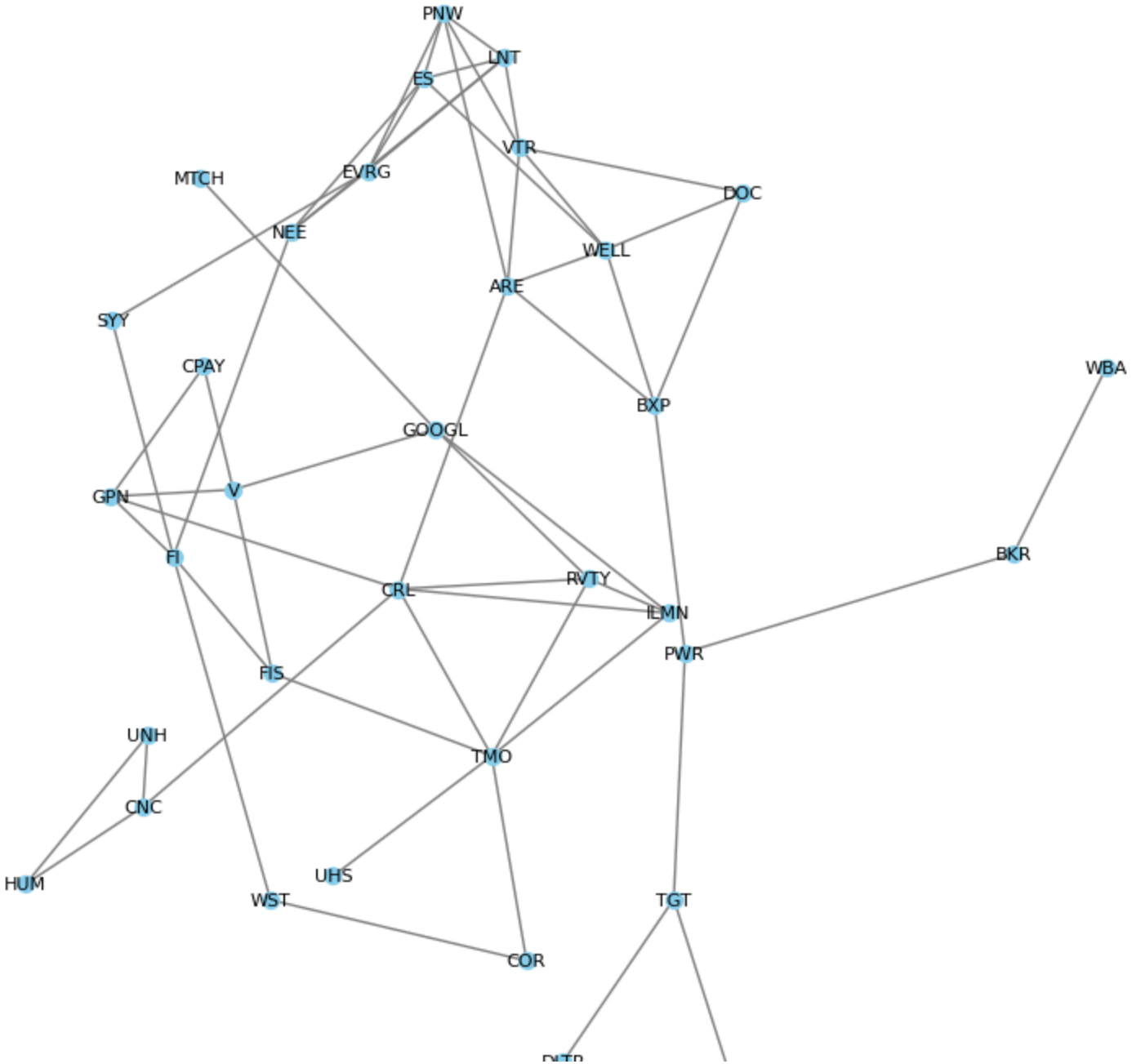

MLM
CHTR
VMC
AVB
CMCSA
CPT
UDR
EQR
WAB
SWK
PARA
ESS
IPG
IEX
COST
SYY
WBA
RL
PCAR
XYL
LULU
VFC
TPR
UPS
CHRW
BRO
EXPD
TGT
OMC
AJG
FDX
PGR
NKE
WTW
DG
HRL
SJM
AIZ
MKC
VZ



In [340…
```
port_2
```

Out[340…
```
{201030.0: ['PWR'],
 551010.0: ['LNT', 'EVRG', 'ES', 'PNW', 'NEE'],
 301010.0: ['SYY', 'DG', 'DLTR', 'TGT', 'WBA'],
 101010.0: ['BKR'],
 601050.0: ['VTR', 'ARE', 'WELL', 'DOC'],
 351020.0: ['CNC', 'COR', 'HUM', 'UNH', 'UHS'],
 402010.0: ['FIS', 'FI', 'V', 'CPAY', 'GPN'],
 601040.0: ['BXP'],
 502030.0: ['MTCH', 'GOOGL'],
 352030.0: ['WST', 'CRL', 'RVTY', 'ILMN', 'TMO']}
```

In [349…
```
plot_precision_graph(Omega_port_2, port_2_tic, threshold=0.15)
```


PNW
LNT
ES
VTR
MTCH
EVRG
DOC
NEE
WELL
ARE
SYY
CPAY
WBA
BXP
GOOGL
GPN
V
BKR
FI
RVTY
CRL
ILMN
PWR
FIS
UNH
TMO
CNC
HUM
UHS
WST
TGT
COR

I selected a regularization parameter of 0.15, as it provided the best alignment with the industry-to-ticker mapping. For example, in the plot, GOOGL is connected to MTCH, which makes sense since both belong to the same industry of Tech. When the regularization threshold is set too low, the plot becomes noisy, with excessive edges connecting stocks across different industries, making it harder to interpret. Too high regularization yields no points and connections. With the chosen regularization, we observe clear clusters of stocks, indicating connections both within and across industries. For instance, GOOGL is connected to CRL, linking the tech and pharma sectors, which reflects the real-world connection between these industries. However, it's crucial to highlight that GOOGL is conditionally independent of WBA (Walgreens) given the other return data, even though WBA is also in the pharma sector. This insight suggests that the threshold could be adjusted ad hoc based on this domain knowledge.

Overall, the graph aligns with expectations based on the portfolio construction method of randomly selecting 10 distinct industries and stocks within each industry. It’s reasonable to anticipate conditional independence between sectors like Tech and contrasting ones, which is reflected in pairs such as GOOGL (Tech) and HMA (Insurance). This makes sense, as a new GOOGL search engine upgrade is unlikely to impact the Insurance sector, which relies more on factors like human trust.