Ch 2 Heavy tail error model

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The full model:

$$p(y|\beta, \omega, \Lambda) \sim N(X\beta, (\omega\Lambda)^{-1})$$

$$\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$$
$$\lambda_i \sim \mathrm{Gamma}\left(\frac{h}{2}, \frac{h}{2}\right) \quad \text{where } h \text{ is a fixed hyperparameter.}$$
$$(\beta|\omega) \sim N(m, (\omega K)^{-1})$$
$$\omega \sim \mathrm{Gamma}\left(\frac{d}{2}, \frac{\eta}{2}\right)$$

A) Under this model, what is the implied conditional distribution $p(y_i|X, \beta, \omega)$? Notice that λ_i has been marginalized out. This should look familiar.

The conditional distribution of y_i

$$P(y_i|X,\beta,\omega) = P(y_i|X,\beta,\omega,\lambda_i)P(\lambda_i)$$

$$\propto \int_0^\infty \sqrt{\omega\lambda_i}\exp\left(-\frac{\omega\lambda_i}{2}(y_i - X_i^T\beta)^2\right)\lambda_i^{h/2-1}\exp\left(-\frac{\lambda_i h}{2}\right)d\lambda_i$$

$$\propto \int_0^\infty \underbrace{\lambda_i^{\frac{h+1}{2}-1}\exp\left(-\frac{\lambda_i}{2}\left(\omega(y_i - X_i^T\beta)^2 + h\right)d\lambda_i\right.}_{\text{Kernel of } \mathrm{gamma}\left(\frac{h+1}{2}, \frac{1}{2}(\omega(y_i - X_i\beta)^2 + h)\right)}$$

$$\propto \left(\frac{1}{2}\left(\omega(y_i - X_i^T\beta)^2 + h\right)\right)^{-\frac{(h+1)}{2}}$$

$$\propto \left(1 + \frac{1}{h}\frac{(y_i - X_i^T\beta)^2}{(\sqrt{\omega^{-1}})^2}\right)^{-\frac{(h+1)}{2}}$$

$\sim T_h(X_i^T\beta, \sqrt{\omega^{-1}})$, a t dist'n with h degrees of freedom, centered at $X_i^T\beta$, with $\sqrt{\omega^{-1}}$ scale

B) What is the conditional posterior distribution $p(\lambda_i|y_i, \beta, \omega)$?

To find the conditional posterior, I will first find the full posterior. We have

$$p(\lambda_i, \beta, \omega|y_i) \propto p(y_i|\beta,\omega,\lambda_i)\,p(\beta|\omega)\,p(\omega)\,p(\lambda_i)$$

$$\propto \sqrt{\frac{\omega\lambda_i}{2\pi}}\exp\left(-\frac{\omega\lambda_i}{2}(y_i - x_i^T\beta)^2\right)\sqrt{\frac{\omega K}{2\pi}}\exp\left(-\frac{\omega K}{2}(\beta-m)^2\right)\omega^{\frac{d}{2}-1}\exp\left(-\frac{\omega\eta}{2}\right)\lambda_i^{\frac{h}{2}-1}\exp\left(-\frac{\lambda_i h}{2}\right) \quad (1)$$

To find $p(\lambda_i|y_i,\beta,\omega)$, we can set y_i, β, and ω to constants and only work with the pieces of the posterior that involve λ_i. Thus, (1) will simplify to

$$p(\lambda_i|y_i,\beta,\omega) \propto \sqrt{\frac{\omega\lambda_i}{2\pi}}\exp\left(-\frac{\omega\lambda_i}{2}(y_i - x_i^T\beta)^2\right)\lambda_i^{\frac{h}{2}-1}\exp\left(-\frac{\lambda_i h}{2}\right)$$

$$\propto \lambda^{\frac{h+1}{2}-1}\exp\left(-\lambda_i\left(\frac{1}{2}\left(\omega(y_i - x_i^T\beta)^2 + h\right)\right)\right)$$

$$\sim \mathrm{gamma}\left(\frac{h+1}{2}, \frac{1}{2}\left(\omega(y_i - x_i^T\beta)^2 + h\right)\right)$$

C) Code up a gibbs sampler that repeatedly cycles through

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$$p(\beta \mid y, \omega, \Lambda)$$

$$p(\omega \mid y, \Lambda)$$

$$p(\lambda_i \mid y, \beta, \omega)$$

The first two should follow identically from your previous results, except that we are explicitly conditioning on Λ, which is a random variable rather than a fixed hyperparameter.

$$p(\lambda_i, \beta, \omega \mid y_i) \propto \sqrt{\tfrac{\omega\lambda_i}{2\pi}} \exp\left(-\tfrac{\omega\lambda_i}{2}(y_i - x_i^T\beta)^2\right) \sqrt{\tfrac{\omega K}{2\pi}} \exp\left(-\tfrac{\omega K}{2}(\beta - m)^2\right) \omega^{\frac{d}{2}-1} \exp\left(-\tfrac{\omega \eta}{2}\right) \lambda_i^{\frac{h}{2}-1} \exp\left(-\tfrac{\lambda_i h}{2}\right) \quad 1$$

From here, we can see that if we were to find the conditional posterior of β and ω, we can simply disregard the prior for λ_i since both the $p(\beta \mid \omega)$ and $p(\omega)$ are independent of λ_i. Thus, the conditional posterior $p(\beta, \omega \mid y, \Lambda)$ is:

$$p(\beta, \omega \mid y, \Lambda) \propto \frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2} \, |(\omega K)^{-1}|^{1/2}} \exp\left(-\tfrac{\omega}{2}(\beta - A^{-1}b)^T A(\beta - A^{-1}b)\right) \exp\left(-\tfrac{\omega}{2}(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K m)\right)$$

where $A = (X^T X + K)$
$b = (y \Lambda x + m^T K)$

Thus, $p(\beta \mid y, \omega, \Lambda) \propto \exp\left(-\tfrac{\omega}{2}(\beta - A^{-1}b)^T A(\beta - A^{-1}b)\right)$

$$\sim MVN(A^{-1}b, (\omega A)^{-1})$$

$$p(\omega \mid y, \beta, \Lambda) \propto \omega^{\frac{d+n+p}{2}-1} \exp\left[-\tfrac{\omega}{2}\left((\beta - A^{-1}b)^T A(\beta - A^{-1}b) + (\eta + b^T A^{-1} b + y^T \Lambda y + m^T K m)\right)\right]$$

$$\sim Gam\left(\tfrac{d+n+p}{2}, \tfrac{1}{2}\left((\beta - A^{-1}b)^T A(\beta - A^{-1}b) + (\eta + b^T A^{-1} b + y^T \Lambda y + m^T K m)\right)\right)$$

$$p(\lambda_i \mid y, \beta, \omega) \propto \lambda^{\frac{h+1}{2}-1} \exp\left(-\lambda_i \left(\tfrac{1}{2}\left(\omega(y_i - x_i^T\beta)^2 + h\right)\right)\right)$$

$$\sim gamma\left(\tfrac{h+1}{2}, \tfrac{1}{2}\left(\omega(y_i - x_i^T\beta)^2 + h\right)\right)$$

The Gibbs sampler algorithm is:

1.) Start with some $(\beta, \omega, \Lambda)^{(0)}$

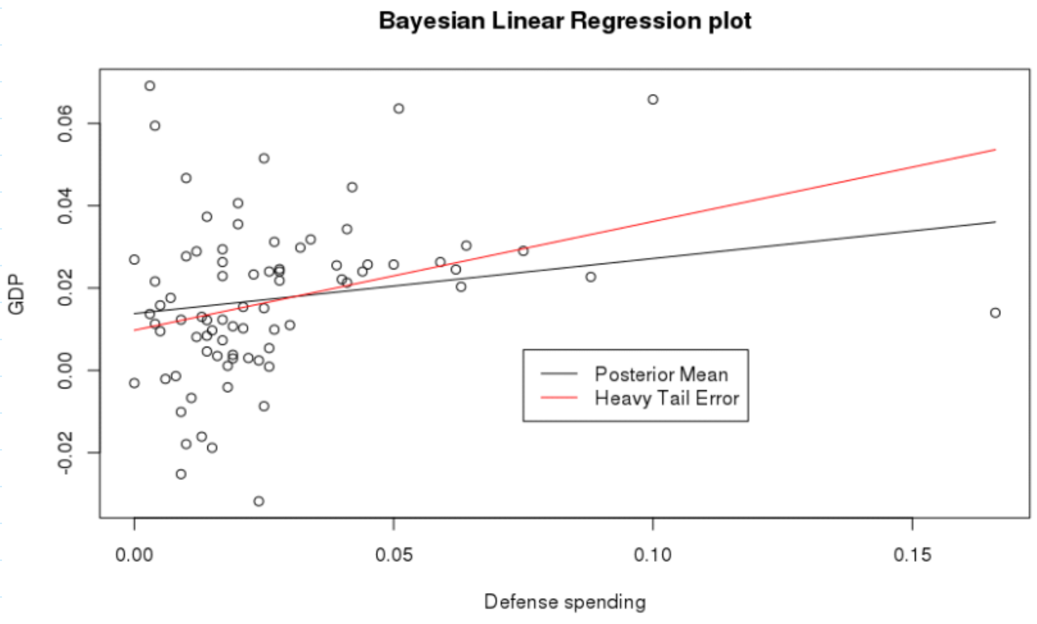
2) At each iteration t, for each $j = 1, \ldots, p$, sample $(\beta^{(t)}, \omega^{(t)}, \Lambda^{(t)})$ from

$$p(\beta^{(t)} \mid \omega^{(t-1)}, \Lambda^{(t-1)})$$

$$p(\omega^{t} \mid \beta^{(t)}, \Lambda^{(t-1)})$$

$$p(\lambda_i^{(t)} \mid \beta^{(t)}, \omega^{(t)}, \lambda_1^{(t)}, \ldots, \lambda_{i-1}^{(t)}, \lambda_{i+1}^{(t-1)}, \ldots, \lambda_p^{(t-1)})$$

We can discard the first thousand draws as the burn in and draw an additional 3000 samples.



The hyperparameters I used for this simulation were:

$$d = 10$$
$$\eta = 10$$
$$h = 10$$
$$m = (0.4, 0.4)$$
$$K = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

After burning 1000 samples, I thinned the remaining 3000 samples by a factor of 3. To compute the red fitted line, I took the average of my sampled betas. As you can see from the figure above, the red line has a slightly higher slope than the original Bayesian linear model. The increased slope is due to the fact that the heavy tail error allows the model to assign large variances to the outliers, rendering them less trustworthy. The influence of the outlying points is therefore diminished. In particular, the data point on the furthest right is given less weight and the fitted line is better able to represent the remaining data points.