Ch 2 Conjugate Gaussian linear model

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Our model for the following problems is

$$(y \mid \beta, \sigma^{-2}) \sim N(X\beta, (\omega \Lambda)^{-1})$$

where $\omega = \frac{1}{\sigma^2}$ is the precision. The priors are

$$(\beta \mid \omega) \sim N(m, (\omega K)^{-1})$$
$$\omega \sim Gamma\left(\frac{d}{2}, \frac{\eta}{2}\right)$$

where K is a $p \times p$ precision matrix in the MVN prior for β which we assume to be known.

A) Derive the conditional posterior $p(\beta \mid y, \omega)$.

The joint posterior $p(\beta, \omega \mid y) \propto p(y \mid \beta, \omega)\, p(\beta \mid \omega)\, p(\omega)$

For a linear model, the likelihood is

$$p(y \mid \beta, \omega) = \frac{1}{|2\pi (\omega\Lambda)^{-1}|^{1/2}} \exp\left(-\frac{1}{2}(y - X\beta)^T \omega \Lambda (y - X\beta)\right)$$

The priors $p(\beta \mid \omega)$ and $p(\omega)$

$$p(\beta \mid \omega) = \frac{1}{|2\pi (\omega K)^{-1}|^{1/2}} \exp\left(-\frac{1}{2}(\beta - m)^T \omega K (\beta - m)\right)$$

$$p(\omega) = \frac{(\eta/2)^{d/2}}{\Gamma(\frac{d}{2})\Gamma(\frac{\eta}{2})} \omega^{\frac{d}{2}-1} \exp\left(-\frac{\eta\omega}{2}\right)$$

so $$p(\beta, \omega \mid y) \propto \frac{1}{|(\omega\Lambda)^{-1}|^{1/2} |(\omega K)^{-1}|^{1/2}} \exp\Big(\underbrace{-\frac{1}{2}(y - X\beta)^T \omega \Lambda (y - X\beta) - \frac{1}{2}(\beta - m)^T \omega K (\beta - m)}_{\text{complete the square in this term}}\Big) \cdot \omega^{\frac{d}{2}-1} \exp\left(-\frac{\eta\omega}{2}\right) \quad (1)$$

Completing the square:

$$-\frac{1}{2}(y - X\beta)^T \omega \Lambda (y - X\beta) - \frac{1}{2}(\beta - m)^T \omega K (\beta - m)$$

$$= -\frac{1}{2}\omega\left[(y - X\beta)^T \Lambda (y - X\beta) + (\beta - m)^T K (\beta - m)\right]$$

$$= -\frac{1}{2}\omega\left[(y^T\Lambda - \beta^T X^T \Lambda)(y - X\beta) + (\beta^T K - m^T K)(\beta - m)\right]$$

$$= -\frac{1}{2}\omega\left[y^T \Lambda y - 2y^T \Lambda X\beta + \beta^T X^T \Lambda X \beta + (\beta^T K \beta - 2m^T K \beta + m^T K m)\right]$$

$$= -\frac{\omega}{2}\Big[\beta^T \underbrace{(X^T \Lambda X + K)}_{\text{call this } A} \beta - 2\underbrace{(y^T \Lambda X + m^T K)}_{\text{call this } b}\beta + y^T \Lambda y + m^T K m\Big]$$

$$= -\frac{\omega}{2}\left[\beta^T A \beta - 2b\beta + b^T A^{-1} b - b^T A^{-1} b + y^T \Lambda y + m^T K m\right]$$

$$= -\frac{\omega}{2}\left[(\beta - A^{-1}b)^T A (\beta - A^{-1}b) - b^T A^{-1} b + y^T \Lambda y + m^T K m\right]$$

The parameters of interest are β and ω. Therefore, we can rearrange terms in the exponentials that do not contain β. Hence, we can plug our completed square back into (1), and reorganize to get

$$p(\beta, \omega \mid y) \propto \frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2} |(\omega K)^{-1}|^{1/2}} \exp\left(-\frac{\omega}{2}(\beta - A^{-1}b)^T A (\beta - A^{-1}b)\right) \exp\left(-\frac{\omega}{2}(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K m)\right)$$

To find the marginal posterior of β given ω, we can simply ignore all the parts of the joint posterior that do not include β.

$$p(\beta \mid \omega, y) \propto \exp\left(-\frac{\omega}{2}(\beta - A^{-1}b)^T A (\beta - A^{-1}b)\right)$$

$$\sim MVN\left((X^T X + K)^{-1}(y \Lambda X + m^T K), (\omega(X^T \Lambda X + K))^{-1}\right)$$ by plugging in the values for A and b.

Like in the univariate case, we can see that the precision adds.

B) Derive the marginal posterior $p(\omega \mid y)$.

To find the marginal posterior of ω, we need to integrate β out of the joint posterior:

$$p(\beta, \omega \mid y) \propto \frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2}\,|(\omega K)^{-1}|^{1/2}} \exp\left(-\frac{\omega}{2}(\beta - A^{-1}b)^T A(\beta - A^{-1}b)\right) \exp\left(-\frac{\omega}{2}(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K m)\right)$$

$$p(\omega \mid y) \propto \frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2}\,|(\omega K)^{-1}|^{1/2}} \exp\left(-\frac{\omega}{2}(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K)\right) \int_{-\infty}^{\infty} \underbrace{\exp\left(-\frac{\omega}{2}(\beta - A^{-1}b)^T A(\beta - A^{-1}b)\right) d\beta}_{\text{This is the kernel of a MVN}(A^{-1}b,\,(\omega A)^{-1})}$$

$$= \frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2}\,|(\omega K)^{-1}|^{1/2}} \exp\left(-\frac{\omega}{2}(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K)\right) \left|2\pi\left(\omega(X^T\Lambda X + K)\right)^{-1}\right|^{1/2}$$

This is some sort of gamma distribution. To find the α parameter, we need to use the determinant property $\det(cA) = c^n A$ where n is the dimension of A. We have

$$|(\omega\Lambda)^{-1}|^{1/2} = |\omega^{-1}\Lambda^{-1}|^{1/2} \propto \omega^{-\frac{n}{2}}$$

$$|(\omega K)^{-1}|^{1/2} = |\omega^{-1}K^{-1}|^{1/2} \propto \omega^{-\frac{p}{2}}$$

$$|(\omega(X^T\Lambda X + K))^{-1}|^{1/2} = |\omega^{-1}(X^T\Lambda X + K)^{-1}|^{1/2} \propto \omega^{-p/2}$$

Combining these with the $\omega^{\frac{d}{2}-1}$ from the prior gives us

$$\frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2}\,|(\omega K)^{-1}|^{1/2}} |(\omega(X^T\Lambda X + K))^{-1}|^{1/2}$$

$$\propto \omega^{\frac{d}{2}-1}(\omega^{\frac{n}{2}})(\omega^{\frac{p}{2}})(\omega^{-p/2})$$

$$= \omega^{\frac{d+n}{2}-1}$$

Thus, $p(\omega \mid y) \sim \text{Gamma}\left(\frac{d+n}{2}, \frac{1}{2}(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K)\right)$

where $A = (X^T X + K)$
$b = (y \Lambda X + m^T K)$

C) Putting these together, what is the marginal posterior of $p(\beta \mid y)$?

We can get $p(\beta \mid y) = \int p(\beta, \omega \mid y)\, d\omega$

$$= \int_0^\infty \frac{\omega^{\frac{d}{2}-1}}{|(\omega\Lambda)^{-1}|^{1/2}\,|(\omega K)^{-1}|^{1/2}} \exp\Big(-\frac{\omega}{2}\underbrace{(\beta - A^{-1}b)^T A(\beta - A^{-1}b)}_{\text{call this } B}\Big) \exp\Big(-\frac{\omega}{2}\underbrace{(\eta - b^T A^{-1} b + y^T \Lambda y + m^T K m)}_{\text{call this } \eta^*}\Big) d\omega$$

$$= \int_0^\infty \underbrace{\omega^{\left(\frac{d+n+p}{2}\right)-1} \exp\left(-\frac{\omega}{2}(B + \eta^*)\right)}_{\text{this is the Kernel of the gamma}\left(\frac{d+n+p}{2},\, \frac{B+\eta^*}{2}\right)} d\omega$$

$$= \frac{\Gamma\left(\frac{d+n+p}{2}\right)}{\left(\frac{B+\eta^*}{2}\right)^{\frac{d+n+p}{2}}}$$

$$\propto \left(\frac{\eta^*}{2} + \frac{1}{2}(\beta - A^{-1}b)^T A(\beta - A^{-1}b)\right)^{-\frac{(d+n+p)}{2}}$$

$$\propto \left(\eta^* + (\beta - A^{-1}b)^T A(\beta - A^{-1}b)\right)^{-\frac{(d+n+p)}{2}}$$

$$= \left(1 + (\beta - A^{-1}b)^T \frac{A}{\eta^*}(\beta - A^{-1}b)\right)^{-\frac{(d+n+p)}{2}}$$

$$= \left(1 + \frac{1}{\quad} (\beta - A^{-1}b)^T \frac{A(d+n)}{\quad} (\beta - A^{-1}b)\right)^{-\frac{(d+n+p)}{2}}$$

$$\left(1 + (\beta - A^{-1}b)^T \frac{1}{\eta^*}(\beta - A^{-1}b)\right)$$

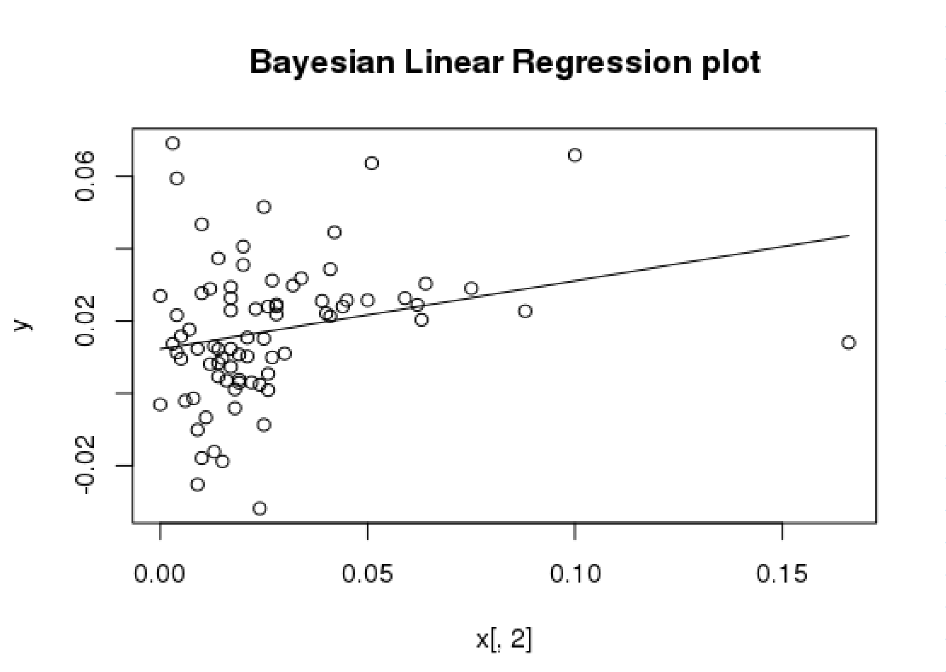
$$= \left(1 + \frac{1}{d+n}(\beta - A^{-1}b)^T \frac{A(d+n)}{\eta^*}(\beta - A^{-1}b)\right)^{-\frac{(d+n+p)}{2}}$$

$$\sim \text{multivariate } T\left(A^{-1}b, \frac{A(d+n)}{\eta^*}\right)$$

where

$$A = (X^T X + K)$$
$$b = (Y \Lambda X + m^T K)$$
$$\eta^* = \eta - b^T A^{-1} b + Y^T \Lambda Y + m^T K m$$

D) Fit the Bayesian linear model of GR6096 vs DEF60 from the "gdpgrowth.csv" file. Use $\Lambda = I$ and something diagonal and vague for the prior precision matrix. Are you happy with the fit of the line?



The figure above was generated using $\Lambda = I$, $m = (0.01, 0.2)$, $K = \text{diag}(0.01, 0.01)$. I am not very satisfied with the fit of this line since it is heavily influenced by the outlying points and does not represent the data well. It would make more sense for the slope to be larger.