Problem Set 3

Fiona Fan

In collaboration with Zeshun, Shirley and Zunda

2 $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Let $\det(\lambda I - D) = 0$

$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 = 0$

$\Rightarrow \lambda = 0$

$\therefore$ Algebraic multiplicity is 3. The corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$Geometric multiplicity is 1.

4

i) Let A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^H = A$ implies that $b = \bar{c}$.

Recall that if $z \in \mathbb{C}$, then $z\bar{z} = |z|^2 \geq 0$

Now, $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$

$\Delta = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4b\bar{b} = (a-d)^2 + 4|b|^2 \geq 0$

$\Rightarrow$ Real roots.

2 Suppose λ is an eigenvalue of A where $A^H = -A$. x is the corresponding eigenvector.

Then, $\langle Ax, x \rangle = \langle \lambda x, x \rangle = lam\bar{b}da\langle x, x \rangle$.

Also, $\langle Ax, x \rangle = \langle x, A^H x \rangle = \langle x, -Ax \rangle = -\langle x, \lambda x \rangle = -\lambda \langle x, x \rangle$.

So we have $\bar{\lambda} = -\lambda$

$\Rightarrow \lambda$ is pure imagery.

6 Suppose A is an upper triangular matrix.

A$= \begin{bmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$

Then the characteristic polynomial is

$$\det(zI-A)=\begin{vmatrix} z-a_1 & & & * \\ & z-a_2 & & \\ & & \ddots & \\ 0 & & & z-a_n \end{vmatrix}=\Pi_{i=1}^{n}(z-a_i)=0$$

Note that this polynomial has n zeros, which are $a_1, a_2, \dots a_n$ respectively.
The case of upper triangular matrix is the same.

8

1 Since $V = span(s)$, it suffices to show that the four vectors are linearly independent.
Let $a\sin(x)+b\cos(x)+c\sin(2x)+d\cos(2x)=0, \quad \forall x \in \mathbb{R}$.
Let $x=0$: $b+d=0$
Let $x=\frac{\pi}{2}$: $a-d=0$
Let $x=\pi$: $-b+d=0$
Let $x=\frac{\pi}{4}$: $a\sin(\frac{\pi}{4})+b\cos(\frac{\pi}{4})+c\sin(\frac{\pi}{2})+d\cos(\frac{\pi}{2})=0$
From the above four conditions we can get $a=0, b=0, c=0, d=0$.
Since the only case that can let $a\sin(x)+b\cos(x)+c\sin(2x)+d\cos(2x)=0, \quad \forall x \in \mathbb{R}$. is when $a=b=c=d=0$,
$\Rightarrow$ They are linearly independent.

2 $D=\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

3 $V_1=\{\sin x, \cos x\}$, $V_2=\{\sin 2x, \cos 2x\}$

13 To diagonalize A, we first need to find eigenvalues and eigenvectors.
$p(\lambda)=\lambda^2-1.4\lambda+0.4=0$
$\Rightarrow \lambda_1=1, \lambda_2=\frac{2}{5}$
And $\Sigma_1=span([2,1]^T), \Sigma_2=span([1,-1]^T)$
So A is semisimple.
Let $p=\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, then p^{-1}=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$.
$D=\begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$
Then $D=p^{-1}Ap$.

15 Since $A \in M_n(\mathbb{F})$ is semisimple, we can diagonalize $A=pDp^{-1}$, where Dd is diagonalized, and $\{\lambda_i\}_1^n$ are the diagonal entries of D.

Now, $f(A) = f(pDp^{-1})$
$= a_0I + a_1pDp^{-1} + \cdots + a_npD^np^{-1}$
$= p[a_0I + a_1D + \cdots + a_nD^n]p^{-1}$
$= pf(D)p^{-1}$
Observe that f(A) and f(D) are similar, so they have the same eigenvalues.
Also note that $f(D)$ is also diagonal, so each entry along the diagonal is $f(D)_{ii} = a_0 + a_1d_{ii} + \cdots + a_nd_{ii}^n = f(d_{ii})$, where $D = [d_{ij}]_{ij}$
Hence, the eigenvalues of $f(D)$ are just its diagonals, which are $\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)\}$

16

1 $A = pD^np_{-1}$
$= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$
$\therefore lim_{n\to\infty}A^n = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$
$= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$
Let $B = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, then it follows immediately from the definition of limit.

2 The choice of norm does not affect the answer.

3 Let $f(x) = 3 + 5x + x^3$, then the eigenvalues of $f(A)$ are $f(\lambda_1) = f(1) = 9, f(\lambda_2) = f(0.4) = 5.064$.

18 Take $\vec{y}$ an eigenvector corresponding to λ. Then,
$x^TAy = x^T\lambda y = (\lambda x^Ty)$
$\Rightarrow x^TA = \lambda x^T$

20 Let $B = U^HAU$, then,
$B^H = U^HA^HU = U^HAU = B$, since $A^H = A$

24

1 $p(\vec{x}) = \frac{\langle x, Ax \rangle}{||x||^2}$
Observe that the denominator is always a real number. Hence to show that $p(\vec{x}) \in \mathbb{R}, itsufficestoshowthat \langle x, Ax \rangle \in \mathbb{R}$.
Now $\langle x, Ax \rangle = \langle A^H, x \rangle = -\langle Ax, x \rangle$
Since by definition, $\langle x, Ax \rangle = \langle A\bar{x}, x \rangle$, we have $\langle Ax, x \rangle = \langle A\bar{x}, x \rangle \in \mathbb{R}$
This implies

$$\langle x, Ax \rangle \in \mathbb{R}$$

$\Rightarrow p(x) \in \mathbb{R}$

2 If $A^H = -A$, then
$\langle x, Ax\rangle = \langle A^H, x\rangle = -\langle Ax, x\rangle$
Also, $\langle x, Ax\rangle = \langle A\bar{x}, x\rangle$
$\therefore \langle A\bar{x}, x\rangle = -\langle Ax, x\rangle$
This implies $\langle x, Ax\rangle = \langle A\bar{x}, x\rangle \in \mathbb{C} \setminus \mathbb{R} \cup \{0\}$
Hence $p(\vec{x}) = \frac{\langle x, Ax\rangle}{||x||^2}$ is pure imaginary number.

25

1 Since $A \in M_n(\mathbb{C})$ is a normal matrix, its eigenspace $\{x_1, x_2, \ldots, x_n\}$ spans $\mathbb{C}^n$.
Observe that $\forall j = 1, 2, \ldots, n$,
$(x_1x_1^H + x_2x_2^H + \cdots + x_nx_n^H)x_j = x_1x_1^Hx_j + x_2x_2^Hx_j + \cdots + x_nx_n^Hx_j = x_j$
This holds for any j.
Since $\{x_1, x_2, \ldots, x_n\}$ spans $\mathbb{C}^n$, $\forall \vec{v} = \mathbb{C}^n$, $\vec{v} = \sum a_i\vec{x}_i$
Let $B = x_1x_1^H + x_2x_2^H + \cdots + x_nx_n^H$, then $B\vec{v} = \sum a_iB\vec{x}_i = \sum a_i\vec{x}_i = \vec{v}$.
Let $\vec{v} = \vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$ respective, then we get
$Be_1 = e_1, Be_2 = e_2, \ldots, Be_n = e_n$
Hence $B = I$

2 Since A is a normal matrix and $\{x_1, x_2, \ldots, x_n\}$ forms an orthonormal eigenbasis, A admits a diagonalization.
$A = pDp{-}1 = pDp^H$, where

$$p = [x_1, x_2, \ldots, x_n] \quad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$p^{-1} = p^H = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$, since p is an orthonormal matrix.

Hence, $$A = [x_1, x_2, \ldots, x_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^H \\ x_2^H \\ \vdots \\ x_n^H \end{bmatrix} = \sum \lambda_i x_i x_i^H$$

27 Suppose $\begin{bmatrix} a_{11} & & \dots & a_{1n} \\ a_{21} & \lambda_2 & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & & & a_{nn} \end{bmatrix}$

By definition, $\forall x, \quad x^H Ax > 0$

Now, let $x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_i^{-1}$

then $e_1^H Ae_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = a_{11} > 0$

Similarly, let $x = e_2, e_3, \dots, e_n$

we have $a_{22} > 0, a_{33} > 0, \dots, a_{nn} > 0$

Here all diagonal elements are positive and real.

28 Proof:

First we introduce the following lemmas used in the proof.

- Lemma 1: The diagonals of a positive semi-definite matrix are greater than or equal to zero. (Proof similar to exercise 4.27)
- Lemma 2: $tr(AB) = tr(BA)$ (Proof can be found in Problem Set 2)
- Lemma 3: If $A \in M_n(\mathbb{F})$ is a positive semi-definite matrix, $D \in M_n(\mathbb{F})$ is a diagonal matrix with non-negative diagonals, then $0 \le tr(AD) \le tr(A)tr(D)$.

Proof. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

,

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

,

then $tr(AD) = \sum_{i=1}^n a_{ii} d_i \ge 0$, since $a_{ii} \ge 0$ and $d_i \ge 0$ for $\forall i$

$tr(A)tr(D) = (\sum_{i=1}^n a_{ii})(\sum_{i=1}^n d_i) = \sum_{i=1}^n a_{ii} d_i + \sum_{i \neq j} a_{ii} d_j \ge \sum_{i=1}^n a_{ii} d_i$

$\Rightarrow tr(A)tr(D) \ge tr(AD) \ge 0$ $\square$

Now since B is a positive semi-definite matrix, it admits a diagonalization s.t. $B = PDP^{-1} = PDP^H$, where

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is an orthonormal eigenbasis,

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

is diagonal matrix with $d_i \geq 0 \quad \forall i$.

Then $tr(AB) = tr(APDP^H) = tr(P^H APD) \leq tr(P^H AP)tr(D)$
$= tr(APP^H)tr(D) = tr(A)tr(D) = tr(A)tr(B)$.

Meanwhile, $||AB||_F^2 = \text{ tr}(AA^H BB^H) \leq \text{ tr}(AA^H) \text{ tr}(BB^H) = ||A||_F ||B||_F^2$,

which makes $|| \cdot ||_F$ a matrix norm.

31

1 Suppose A has rank r, then $A^H A$ is positive definite and has r distinct eigenvalues.
Let $s = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be an orthonormal eigenspace of $A^H A$, and $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$ be the corresponding eigenvectors, where $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$.
Since s spans $\mathbb{F}^n$, $\forall \vec{x} \in \mathbb{F}^n$, we have
$\vec{x} = \sum_{i=1}^n c_i \vec{v_i}$, $c_i \in \mathbb{F}, \forall i$, and
$||x||_2 = \sqrt{(\sum c_i v_i^T)(\sum c_i v_i)} = \sqrt{(\sum c_i^2)}$
Hence if $||x||_2 = 1, then \sum_{i=1}^n c_i^2 = 1$
Now, observe that $\|Ax\|_2^2 = \langle Ax, Ax \rangle = (Ax)^H Ax = x^H A^H Ax$
$= (\sum_{i=1}^n c_i \vec{v_i}^H)(A^H A)(\sum_{i=1}^n c_i \vec{v_i})$
$= (\sum_{i=1}^n c_i \vec{v_i}^H)(\sum_{i=1}^n c_i A^H A \vec{v_i}^H)$
$= (\sum_{i=1}^n c_i \vec{v_i}^H)(\sum_{i=1}^n c_i \sigma^2 \vec{v_i}^H) = \sum c_i^2 \sigma^2$, where $s = \{v_1, v_2, \ldots, v_n\}$
Note that when $\sigma c_i^2 = 1$, and $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$,
$\sum c_i^2 \sigma_i^2 \leq \sigma_1^2$
Hence, $\|A\|_2^2 = \sup_{\|x\|_2 = 1} \|Ax\|_2^2 = \sigma_1^2$
$\Rightarrow \|A\|_2 = \sigma_1$

2 Since $A = U\Sigma V^H$
$A^{-1} = (U\Sigma V^H)^{-1} = (V^H)^{-1} \Sigma^{-1} (U)^{-1} = V \Sigma^{-1} U^H$
$\Rightarrow$ This is still an SVD of A^{-1}
And $\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$
i.e. The singular values of $A^{-1} are$
$\frac{1}{\sigma_1} \leq \cdots \leq \frac{1}{\sigma_n}$
By(1), $\left\|A^{-1}\right\|_2$ is the largest singular value of A^{-1}, i.e. $\frac{1}{\sigma_n}$

3 Since $A = U\Sigma V^H$
$A^H = (V^H)^H \Sigma^H U^H = V\Sigma^H U = V\Sigma U$
$\Rightarrow A^H$ and A has the same singular values.
So $\left\|A^H\right\|_2^2 = \|A\|_2^2 = \sigma_1^2$
(A^T) is just A^H restricted on $\mathbb{R}$.
So $\left\|A^T\right\|_2^2 = \left\|A^H\right\|_2^2$
By the previous argument, we know that $A^H A$ has an orthonormal eigenbasis $\{v_1, v_2, \ldots, v_n\}$, and $\forall \|x\|_2 = 1$, $\left\|A^H Ax\right\|_2 = \left\|A^H A \sum c_i v_i\right\|_2 = \sqrt{(\sum c_i \sigma_i^2 v_i^T)(\sum c_i \sigma_i^2 v_i)} = \sqrt{\sum c_i \sigma_i^4} \leq \sigma_1^2$
Hence $\left\|A^H A\right\|_2 = \sup_{\|x\|=1} \left\|A^H Ax\right\| = \sigma_1^2$
It follows that $\left\|A^H A\right\|_2 = \|A\|_2^2 = \left\|A^H\right\|_2^2 = \left\|A^T\right\|_2^2 = \sigma_1^2$

4 Lemma: Let Q be an orthonormal matrix, then $||AQ||_2 = ||A||_2$.

Proof. Let $S_1 = \{\ ||AQ\vec{x}||, ||x||_2 = 1\ \}$, $S_2 = \{\ ||Ax||, ||x||_2 = 1$

Proof. Since Q is orthonormal, so Q is also invertible.
$\forall s_1 \in S_1, \exists x, \|x\| = 1, \quad s.t. \|AQx\|_2 = s_1$
Now, let $y = Qx$, it follows that $\|Qx\| = \|y\|_2 = 1$
Since orthonormal matrix preserves length, $\|Ay\|_2 = \|AQx\|_2 = s_1 \in S_2$
i.e. $S_1 \subset S_2$
$\forall s_2 \in S_2, \exists x, \|x\|_2 = 1 \quad s.t. \|Ax\|_2 = s_2$
Now, let $y = Q^{-1}x$, then $\|y\|_2 = \left\|Q^{-1}x\right\|_2 = 1$
Hence $\|AQy\|_2 = \left\|AQQ^{-1}x\right\|_2 = \|Ax\|_2 = s_2 \in S_1$
i.e. $S_2 \subset S_1$
$\therefore \|AQ\|_2 = \sup S_1 = \sup S_2 = \|A\|_2$

□

Now $\|UAV\|_2 = \|UA\|_2$ by lemma since V is an orthonormal matrix.
$\|UA\|_2 = \sup_{\|x\|_2=1} \sqrt{(UAx)^H(UAx)} = \sup_{\|x\|_2=1} \sqrt{x^H A^H U^H U A x}$
$= \sup_{\|x\|_2=1} \sqrt{\langle Ax, Ax\rangle} = \|A\|_2$
Hence, $\|UAV\|_2 = \|UA\|_2 = \|A\|_2$ □

32

1 We need the following lemmas:

- lemma 1: if $A, B \in M_n(\mathbb{F}), then tr(AB) = tr(BA)$
- lemma 2: $||A||_p^2 = tr(A^T A)$

Proof. let $A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{1m} & \ldots & a_{mn} \end{bmatrix}$

Then $A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$
Observe that $(A^TA)_{nm} = a_{1n}^2 + a_{2n}^2 + \cdots + a_{mn}^2$
$\Rightarrow tr(A^TA) = ||A||_p^2$ $\square$

Now, $||UAV||_1^2 = tr((UAV)^T(UAV)) = tr(V^TA^TU^TUAV) = tr(V^TA^TAV) = tr(VV^TA^A) = tr(A^TA) = ||A||_1^2$
$\Rightarrow ||UAV||_2 = ||A||_2$

2 Observe that $A = U\Sigma V^T$, with U and V^T orthonormal and
$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$ Now, $\|A\|_p^2 = tr(A^TA) = tr((U\Sigma V)^T(U\Sigma V)) = tr(U\Sigma^TU^TU\Sigma V^T) = tr(U\Sigma^2V^T)$
$= tr(\Sigma^2) = \sum_{i=1}^r \sigma_i^2$
Hence $\|A\|_p = \sqrt{\sum_{i=1}^r \sigma_i^2}$

33 Note that the Y^HAx will be a field element. Consider it as a linear map $Y^HAx : \mathbb{F} \to \mathbb{F}$, then the spectral norm of this map is:
$||Y^HAx||_2 = \sup_{f\in\mathbb{F}} \frac{||(Y^HAx)f||_2}{||f||_2} = |Y^HAx|$
, where the first norm is spectral norm and the norm in fraction is the standard 2-norm.

36 One example can be $A = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$
then $A^TA = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$
$\det(A^TA - \lambda I) = \lambda^2 - 50\lambda + 400 = 0$
$\lambda_1 = 40, \lambda_2 = 10$
Thus its singular value are $s_1 = \sqrt{40}, s_2 = \sqrt{10}$
To calculate its eigenvalues,
$\det(A - \lambda I) = \lambda^2 - 9\lambda + 20 = 0$
Thus its eigenvalues are $\lambda_1 = 4, \lambda_2 = 5$, which are different from its singular values.

38 (i) Suppose $U\Sigma V^H$ is an SVD of A, then $A^\dagger = V\Sigma^{-1}U^H$
$AA^\dagger A = (U\Sigma V^H)(V\Sigma^{-1}U^H)(U\Sigma V^H) = U\Sigma V^H = A$
(ii)
$A^\dagger AA^\dagger = (V\Sigma^{-1}U^H)(U\Sigma V^H)(V\Sigma^{-1}U^H) = V\Sigma^{-1}U^H = A^\dagger$
(iii)
$(AA^\dagger)^H = ((U\Sigma V^H)(V\Sigma^{-1}U^H))^H = U\Sigma^{-1}V^HV\Sigma U^H = UU^H = AA^\dagger$
(iv)
$(A^\dagger A)^H = ((V\Sigma^{-1}U^H)(U\Sigma V^H))^H = V\Sigma U^HU\Sigma^{-1}V^H = VV^H = A^\dagger A$

(v)

By prop (iii) $\Rightarrow AA^\dagger$ is hermitian.

Also by prop (i), $AA^\dagger AA^\dagger = AA^\dagger \Rightarrow AA^\dagger$ is idempotent.

Next we will check whether $\mathcal{R}(AA^\dagger) = \mathcal{R}(A)$.

It is trivially $\mathcal{R}(AA^\dagger) \subset \mathcal{R}(A)$, and by prop(i) $\Rightarrow \mathcal{R}(A) \subset \mathcal{R}(AA^\dagger)$

$\Rightarrow \mathcal{R}(AA^\dagger) = \mathcal{R}(A)$

(vi)

By prop (iv) $A^\dagger A$ is hermitian.

Also by prop (ii) $A^\dagger AA^\dagger A = A^\dagger A \Rightarrow A^\dagger A$ is idempotent

Next we will check whether $\mathcal{R}(A^\dagger A) = \mathcal{R}(A^H)$

By prop (iv), $AA^\dagger = (AA^\dagger)^H = A^H(A^\dagger)^H \implies \mathcal{R}(A^\dagger A) \subset \mathcal{R}(A^H)$

Then we take the hermitian of both sides of prop (i),

we have $(A^\dagger A)^H A^H = (A^\dagger A)^H A^H A^H \Rightarrow \mathcal{R}(A^H) \subset \mathcal{R}(A^\dagger A)$

$\Rightarrow \mathcal{R}(A^\dagger A) = \mathcal{R}(A^H)$