Problem Set #5

Smooth and convex optimization

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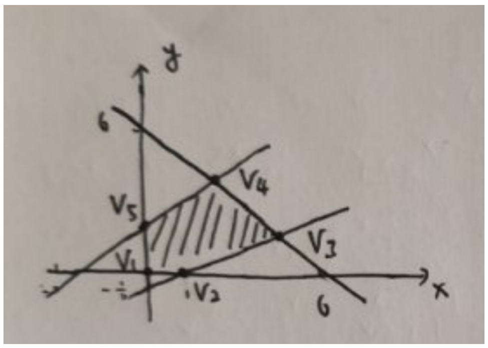
Exercise 8.1

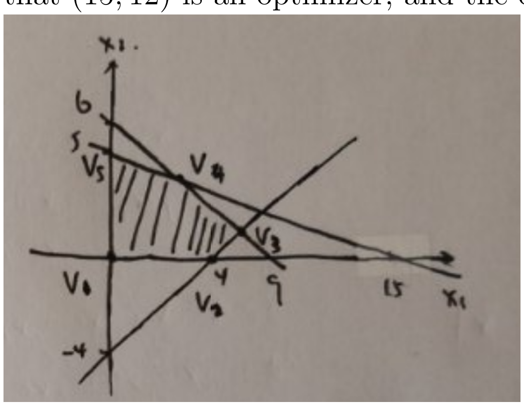
The feasible set is shown above. Observe that there are five vertices. $V_1 = (0,0), V_2 = (1,0), V_3 = (\frac{37}{7}, \frac{5}{7}), V_4 = (\frac{16}{5}, \frac{14}{5}), V_5 = (1,0)$. Easy calculation yields that $(\frac{37}{7}, \frac{5}{7})$ is an optimizer, and the optimized value is $\frac{165}{7}$.



Exercise 8.2

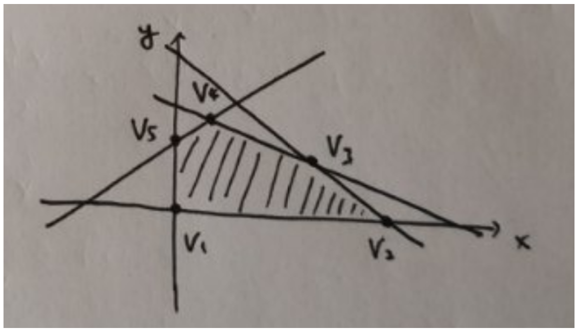
(1) The feasible set is shown above. Observe that there are five vertices. $V_1 = (0,0), V_2 = (4,0), V_3 = (6,2), V_4 = (3,4), V_5 = (5,0)$. Easy calculation yields that $(6,2)$ is an optimizer, and the optimized value is 20.

(2) The feasible set is shown above. Observe that there are five vertices. $V_1 = (0,0), V_2 = (27,0), V_3 = (15,12), V_4 = (5,16), V_5 = (11,0)$. Easy calculation yields that $(15,12)$ is an optimizer, and the optimized value is 132.



Exercise 8.3

$$\begin{aligned} &\max\{4x + 3y\} \\ \text{subject to } &15x + 10y \leq 1800 \\ &2x + 2y \leq 300 \\ &y \leq 200 \\ &x, y \geq 0. \end{aligned}$$



Exercise 8.4

$$\max\{2x_{AB} + 5x_{BC} + 2x_{CF} + 5x_{AD} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 4x_{DE} + 3x_{EF}\}$$

subject to $x_{AD} + x_{AB} = 10$

$$x_{BC} + x_{BE} - x_{AB} = 1$$
$$x_{CF} - x_{BC} = -2$$
$$x_{DE} - x_{AD} - x_{BD} = -3$$
$$x_{EF} - x_{BE} - x_{DE} = 4$$
$$-x_{CF} - x_{EF} - x_{BF} = -10$$
$$0 \le x_{AB} \le 6$$
$$0 \le x_{BC} \le 6$$
$$0 \le x_{CF} \le 6$$
$$0 \le x_{AD} \le 6$$
$$0 \le x_{BD} \le 6$$
$$0 \le x_{BE} \le 6$$
$$0 \le x_{BF} \le 6$$
$$0 \le x_{DE} \le 6$$
$$0 \le x_{EF} \le 6.$$

Exercise 8.5

(1)

$$\begin{aligned} \xi &= 3x_1 + x_2 \\ w_1 &= 15 - x_1 - 3x_2 \\ w_2 &= 18 - 2x_1 - 3x_2 \\ w_3 &= 4 - x_1 + x_2 \end{aligned}$$

Observe that we can increase x_1 to 9.

$$\begin{aligned} \xi &= 12 + 4x_2 - 3w_3 \\ w_1 &= 11 - 3x_2 - w_3 \\ w_2 &= 10 - 5x_2 + 2w_3 \\ x_1 &= 4 + x_2 - w_3 \end{aligned}$$

Observe that we can increase x_2 to 2.

$$\begin{aligned}
\xi &= 20 - 0.8w_2 - 1.4w_3 \\
w_1 &= 5 + 0.6w_2 - 2.2w_3 \\
x_2 &= 2 - 0.2w_2 + 0.4w_3 \\
x_1 &= 6 + 0.2w_2 - 0.6w_3
\end{aligned}$$

There is no more term to increase and we conclude that the optimal value is 20 when $x_1 = 6, x_2 = 2$.

(2)

$$\begin{aligned}
\xi &= 4x + 6y \\
w_1 &= 27 - x - y \\
w_2 &= 27 - x - y \\
w_3 &= 90 - 2x - 5y
\end{aligned}$$

Observe that we can increase x to 27.

$$\begin{aligned}
\xi &= 108 - 4w_2 + 2y \\
w_1 &= 38 - w_2 - 2y \\
x &= 27 - w_2 - y \\
w_3 &= 36 + 2w_2 - 3y
\end{aligned}$$

Observe that we can increase y to 12.

$$\begin{aligned}
\xi &= 132 - \frac{8}{3}w_2 + \frac{2}{3}w_3 \\
w_1 &= 38 - \frac{7}{3}w_2 - \frac{2}{3}w_3 \\
x &= 15 - \frac{5}{3}w_2 - \frac{1}{3}w_3 \\
y &= 12 + \frac{2}{3}w_2 + \frac{1}{3}w_3
\end{aligned}$$

There is no more term to increase and we conclude that the optimal value is 132 when $x = 15, y = 12$.

Exercise 8.6

Observe that our problem

$$\begin{aligned}
&\max\{4x + 3y\} \\
\text{subject to } &15x + 10y \le 1800 \\
&2x + 2y \le 300 \\
&y \le 200 \\
&x, y \ge 0
\end{aligned}$$

is equivalent to

$$\begin{aligned} &\max\{4x+3y\} \\ \text{subject to } &3x+2y \leq 360 \\ &x+y \leq 150 \\ &y \leq 200 \\ &x, y \geq 0. \end{aligned}$$

$$\begin{aligned} \xi &= 4x+3y \\ w_1 &= 360-3x-2y \\ w_2 &= 150-x-y \\ w_3 &= 200-7 \end{aligned}$$

Observe that we can increase y to 150.

$$\begin{aligned} \xi &= 450+x-3w_2 \\ w_1 &= 60-x+2w_2 \\ y &= 150-x-w_2 \\ w_3 &= 50+x+w_2 \end{aligned}$$

Observe that we can increase x to 60.

$$\begin{aligned} \xi &= 510-w_1-w_2 \\ x &= 60-w_1+2w_2 \\ y &= 90+w_1-3w_2 \\ w_3 &= 110-w_1+3w_2 \end{aligned}$$

There is no more term to increase and we conclude that the optimal value is 510 when $x = 60, y = 90$. So the company should produce 60 toy soldiers and 90 toy dolls.

Exercise 8.7

(1) Observe that the origin is not in the feasible set, so we need an auxillary problem to find an inital vertex, which is

$$\begin{aligned} &\max\{-x_0\} \\ \text{subject to } &-4x_1-2x_2-x_0 \leq -8 \\ &-2x_1+3x_2-x_0 \leq 6 \\ &x_1+x_0 \leq 3 \\ &x_0, x_1, x_2 \geq 0 \end{aligned}$$

We use simplex method to solve it:

$$
\begin{aligned}
\xi &= -x_0 \\
w_1 &= -8 + 4x_1 + 2x_2 + x_0 \\
w_2 &= 6 + 2x_1 - 3x_2 + x_0 \\
w_3 &= 3 - x_1 - x_0
\end{aligned}
$$

We start with $x_0 = 8$.

$$
\begin{aligned}
\xi &= -8 - w_1 + 4x_1 + 2x_2 \\
x_0 &= 8 + w_1 - 4x_1 - 2x_2 \\
w_2 &= 14 + w_1 - 2x_1 - 5x_2 \\
w_3 &= -5 - w_1 + 3x_1 + 2x_2
\end{aligned}
$$

Observe that we can increase x_1 to 2.

$$
\begin{aligned}
\xi &= -x_0 \\
x_1 &= 2 + 0.25w_1 - 0.5x_2 - 0.25x_0 \\
w_2 &= 10 + 0.5w_1 + x_2 + 0.5x_0 \\
w_3 &= 1 - 0.25w_1 + 0.5x_2 - 0.75x_0
\end{aligned}
$$

Hence we can see that a feasible vertex is $x_1 = 2, x_2 = 0$. We start from the original problem and let x_1 be 2.

$$
\begin{aligned}
\xi &= 2 + 1.5x_2 + 0.5w_1 \\
x_1 &= 2 - 0.5x_2 + 0.5w_1 \\
w_2 &= 10 + w_1 - 4x_2 \\
w_3 &= 1 + 0.5x_2 - 0.5w_1
\end{aligned}
$$

Observe that we can incrase w_1 to 2.

$$
\begin{aligned}
\xi &= 3 + 2x_2 - w_3 \\
x_1 &= 3 - w_3 \\
w_2 &= 12 - 3x_2 - 2w_3 \\
w_1 &= 2 + x_2 - 2w_3
\end{aligned}
$$

Observe that we can incrase x_2 to 4.

$$
\begin{aligned}
\xi &= 11 - \frac{2}{3}w_2 - \frac{7}{3}w_3 \\
x_1 &= 3 - w_3 \\
x_2 &= 4 - \frac{1}{3}w_2 - \frac{2}{3}w_3 \\
w_1 &= 6 - \frac{1}{3}w_2 - \frac{8}{3}w_3
\end{aligned}
$$

There is no more term to increase. So the optimal value is 11 when $x_1 = 3, x_2 = 4$.
(2)Write down the auxillary problem and we can see that x_0 can never be 0. So this problem is infeasible.
(3)We write down the table.

$$\begin{aligned}
\xi &= -3x_1 + x_2 \\
w_1 &= 4 - x_2 \\
w_2 &= 6 + 2x_1 - 3x_2
\end{aligned}$$

Observe that we can increase x_2 to 2.

$$\begin{aligned}
\xi &= 2 - \frac{7}{3}x_1 - \frac{1}{3}w_2 \\
w_1 &= 2 - \frac{2}{3}x_1 + \frac{1}{3}w_2 \\
x_2 &= 2 + \frac{2}{3}x_1 \frac{1}{3}w_2
\end{aligned}$$

There is no more term to increase. So the optimal value is 2 when $x_1 = 0, x_2 = 2$.

Exercise 8.8

$$\max -x_1 - x_2 - x_3 \text{ s.t. } x_1, x_2, x_3 \geq 0.$$

Exercise 8.9

$$\max x_1 + x_2 + x_3 \text{ s.t. } x_1, x_2, x_3 \geq 0.$$

Exercise 8.10

$$\max x_1 + x_2 + x_3 \text{ s.t. } x_1, x_2 \geq 0, x_3 \geq 3 \text{ and } x_3 \leq 2.$$

Exercise 8.11

$$\max x_1 + x_2 + x_3 \text{ s.t. } , x_1 + x_2 + x_3 \geq 1, \text{ and } 0 \leq x_1, x_2, x_3 \leq 2.$$

The auxillary problem is

$$\max -x_0 \text{ s.t. } - x_1 - x_2 - x_3 - x_0 \leq -1 \text{ and } x_0, x_1, x_2, x_3 \geq 0.$$

Exercise 8.12
By Bland's rule, among possible leaving and entering variables, we should always choose the one with the smallest index.

$$\begin{aligned}
\xi &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_6 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_7 &= 1 - x_1
\end{aligned}$$

We choose to increase x_1 to zero.

$$\begin{aligned}
\xi &= -27x_2 + x_3 - 44x_4 - 20x_5 \\
x_1 &= 3x_2 + x_3 - 2x_4 - 2x_5 \\
x_6 &= 4x_2 + 2x_3 - 8x_4 + x_5 \\
x_7 &= 1 - 3x_2 - x_3 + 2x_4 + 2x_5
\end{aligned}$$

We choose to increase x_3 to 1.

$$\begin{aligned}
\xi &= 1 - 30x_2 - 42x_4 - 18x_5 - x_7 \\
x_1 &= 1 - x_7 \\
x_6 &= 2 - 2x_2 - 4x_4 + 5x_5 - 2x_7 \\
x_3 &= 1 - 3x_2 + 2x_4 + 2x_5 - x_7
\end{aligned}$$

Now we conclude that the maximum is 1. This is obtained when $\mathbf{x} = (1, 0, 1, 0)$.

Exercise 8.15

Proof.

$$\begin{aligned}
\mathbf{c}^T\mathbf{x} &= \mathbf{x}^T\mathbf{c} \\
&\leq \mathbf{x}^T(A^T\mathbf{y}) \\
&= (A\mathbf{x})^T\mathbf{y} \\
&\leq \mathbf{b}^T\mathbf{y}
\end{aligned}$$

□

Exercise 8.17

Proof. Our primal problem is

$$\max \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} \preceq \mathbf{b} \text{ and } \mathbf{x} \succeq \mathbf{0}.$$

The corresponding dual problem is

$$\min \mathbf{b}^T\mathbf{y} \text{ s.t. } A^T\mathbf{y} \succeq \mathbf{c} \text{ and } \mathbf{y} \succeq \mathbf{0},$$

which is equivalent to

$$\max -\mathbf{b}^T\mathbf{y} \text{ s.t. } -A^T\mathbf{y} \preceq -\mathbf{c} \text{ and } \mathbf{y} \succeq \mathbf{0}.$$

Now take this as the primal problem, and observe that the dual problem is

$$\min -\mathbf{c}^T\mathbf{z} \text{ s.t. } -A\mathbf{z} \succeq -\mathbf{b} \text{ and } \mathbf{z} \succeq \mathbf{0},$$

which is equivalent to

$$\max \mathbf{c}^T\mathbf{z} \text{ s.t. } A\mathbf{z} \preceq \mathbf{b} \text{ and } \mathbf{z} \succeq \mathbf{0}.$$

This is the same as the original primal problem. □

Exercise 8.18
We first solve for the primal problem.

$$\begin{aligned}
\xi &= x_1 + x_2 \\
w_1 &= 3 - 2x_1 - x_2 \\
w_2 &= 5 - x_1 - 3x_2 \\
w_3 &= 4 - 2x_1 - 3x_2
\end{aligned}$$

Obseve that we can increase x_1 to 1.5.

$$\begin{aligned}
\xi &= 1.5 + 0.5x_2 - 0.5x_1 \\
x_1 &= 1.5 - 0.5x_2 - 0.5w_1 \\
w_2 &= 3.5 - 2.5x_2 + 0.5w_1 \\
w_3 &= 1 - 2x_2 + w_1
\end{aligned}$$

Obseve that we can increase x_2 to 0.5.

$$\begin{aligned}
\xi &= 1.75 - 0.25w_1 - 0.25w_3 \\
x_1 &= 1.25 - 0.75w_1 - 0.25w_3 \\
w_2 &= 2.25 - 0.75w_1 + 1.75w_3 \\
x_2 &= 0.5 + 0.5w_1 - 0.5w_3
\end{aligned}$$

We conclude that the maximum value is 1.75. Now, the dual problem is

$$\min 3y_1 + 5y_2 + 4y_3 \text{ s.t. } 2y_1 + y_2 + 2y_3 \geq 1, y_1 + 3y_2 + 3y_3 \geq 1, \text{ and } y_1, y_2, y_3 \geq 0.$$

Using the same technique we can see that the minimum value is attained at $(0.25, 0, 0.25, 0)$, and the minimum is 1.75.