Smart SDFs

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Overview

Stochastic Discount Factor

- Market with N assets. Sequence random vectors of prices $\{p_t\}$ taking values on $\mathbb{R}^N$
- Formally $\{p_t\}$ is a sequence of random vectors defined on a Filtered Probability Space $(\Omega, \mathcal{I}, \{\mathcal{I}_t\}, \mathbb{P})$

SDF

A SDF between period t and $t+1$ is a random variable $m_{t,t+1} > 0$ a.s. such that:

$$p_t = \mathbb{E}_t[m_{t,t+1} p_{t+1}].$$

- Axioms of rational behaviour under uncertainty (completeness, transitivity, continuity, independence) $\implies$ agents maximize Expected Utility [Von Neumann, Morgentern (1947)].
- Agent maximizes expected life-time utility from consumption:

$$\max_{\{c_t\},\{\theta_t\}} \quad \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t U(c_t;\gamma)\right]$$
$$\text{s.t.} \qquad c_t + p_t^T \theta_t = p_t^T \theta_{t-1}$$

where $\{\theta_t\}$ is a sequence of portfolio weights.

- First-order condition wrt θ_t and Law Iterated Expectations $\implies$

$$p_t = \mathbb{E}_t\left[\beta \frac{U'(C_{t+1};\gamma)}{U'(C_t;\gamma)} p_{t+1}\right]$$

SDF - Financial justification

Assumptions

1. p_{t+1} are $\mathcal{I}_{t+1}$ measurable with finite conditional (on $\mathcal{I}_t$) second moment.
2. There exist a linear pricing functional $\pi_{t,t+1} : p_{t+1} \mapsto p_t$ conditioned on $\mathcal{I}_t$ and satisfies a conditional continuity restriction.
3. There is no arbitrage in the market: if $0 \neq p_{t+1} \geq 0$ a.s., then $p_t > 0$ a.s.

Fundamental theorem of Asset Pricing

Given the previous assumptions, the Riesz representation theorem implies

$$p_t = \pi_{t,t+1}(p_{t+1}) = \mathbb{E}_t[m_t p_{t+1}]$$

where $m_t > 0$ a.s. random variable.

SDF - Cox's justification

Cox

All models are wrong, but some of them are useful.

Useful?

- Any asset pricing model (CAPM, Fama-French 3-factor model, ecc.) is a particular specification of the SDF
- SDF incorporates time discounting, preferences, risk discounting, ecc. (Specify a SDF model to identify these features)
- SDF carries information on economic outlook. "High" SDF $\Longrightarrow$ "bad" economic outlook
- Trading strategy: sell financial portfolio replicating the SDF (see later)

Parametric SDFs

Economic specifications

$m_{t,t+1} = m(Y_{t+1};\alpha)$ where Y_{t+1} are relevant state variables and α is a vector of parameters.

- Time-separable preferences with CRRA utility

$$m_{t,t+1} = \beta(c_{t+1}/c_t)^{-\gamma}, \qquad \alpha = (\beta,\gamma)$$

- Time-nonseparable Epstein-Zin preferences

$$m_{t,t+1} = \beta^{\lambda}(c_{t+1}/c_t)^{-\gamma\lambda}(R^*_{t+1})^{\lambda-1}, \qquad \alpha = (\beta,\gamma,\lambda)$$

Parametric SDFs

Financial specifications

- Linear factor models:

$$m_{t,t+1} = \phi_0 + \phi_1^T F_{t+1}, \qquad \alpha = (\phi_0, \phi_1^T)$$

  - CAPM: $F_t = R_t^m$
  - Fama French 3-factor model: $F_t = (R_t^m, SMB_t, HML_t)^T$

- Exponentially affine factor models:

$$m_{t,t+1} = \exp(\phi_0 + \phi_1^T F_{t+1})$$

note that $m_{t,t+1} > 0$ a.s.

Parametric SDFs

Estimation

Two-step GMM estimator

$$\hat{\alpha}_T = \arg\min_{\alpha} \ \hat{g}_T(\alpha)^T \hat{V}_T^{-1} \hat{g}_T(\alpha)$$

where $\hat{g}_T(\alpha) = \frac{1}{T} \sum_{t=1}^{T-1} g(p_{t+1}, p_t, z_t; \alpha)$,

$$g(p_{t+1}, p_t, z_t; \alpha) = z_t \otimes (m_{t,t+1}(\alpha) p_{t+1} - p_t),$$

z_t is a vector of instruments and $\hat{V}_T$ is a consistent estimator of V_0

Nonparametric SDF - Hansen et al. 1990s

Assumptions

- $m, p' \in L_2$, $p \in L_1$
- *no arbitrage opportunities: if* $\theta^T p' \geq 0$ *and* $\theta^T p'$ *has positive probability, then* $\theta^T p > 0$.
- *no redundancies in the securities: if* $\theta^T p' = \theta^{*T} p'$ *and* $\theta^T p = \theta^{*T} p$, *then* $\theta = \theta *$.

Proposition

Consider

$$\inf_{m \in L_2,\ m \geq 0} \{\ \mathbb{E}[m^2/2]\ :\ p = \mathbb{E}[mp']\ \},$$

$$\max_{\theta \in \mathbb{R}^N} \{\ \theta^T p/2 - \mathbb{E}[(\theta^T p')^{+2}/2]\ \}.$$

Then

$$m^* = \theta^{*T} p'$$

Nonparametric SDF - Contribution

Assumptions

- *$p \in L_1$, $m \in L_a$ and $p'_i \in L_b$ for each security i, where a and b are Hölder conjugate*
- *there exits SDF $\bar{M} > 0$ a.s. such that $\|\mathbb{E}[mp'] - p\| < \lambda$*

Proposition

Consider

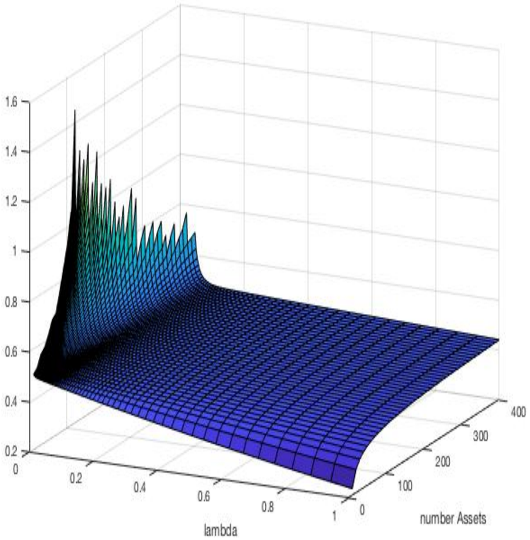
$$\inf_{m \in L_a,\ m \geq 0} \left\{ \, \mathbb{E}[\phi(m)] \ : \ \left\|\mathbb{E}[mp'] - p\right\| \leq \lambda \, \right\},$$

$$\max_{\theta \in \mathbb{R}^N} \left\{ \, \theta^T p - \phi_+^c(\theta^T p') - \lambda \, \|\theta\|^* \, \right\}.$$

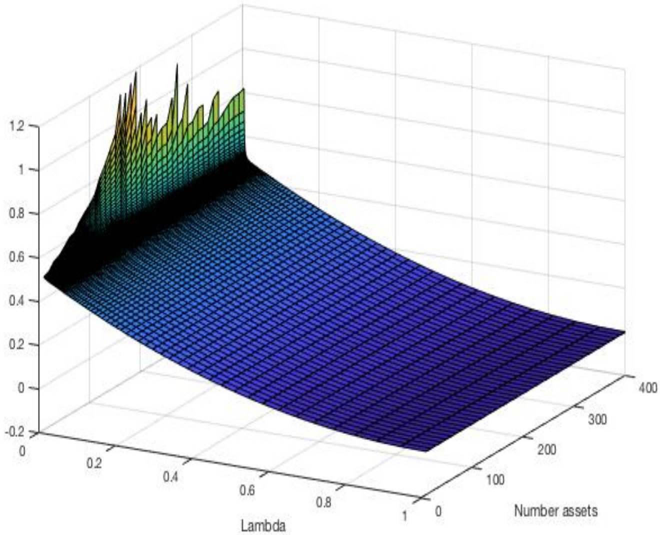
Then

$$m^* = (\phi_+^c)'({\theta^*}^T p')$$

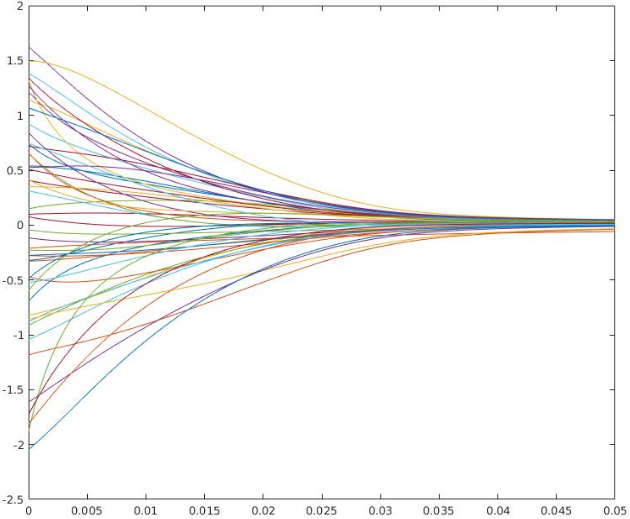
Optimal value l_2



Optimal value l_1

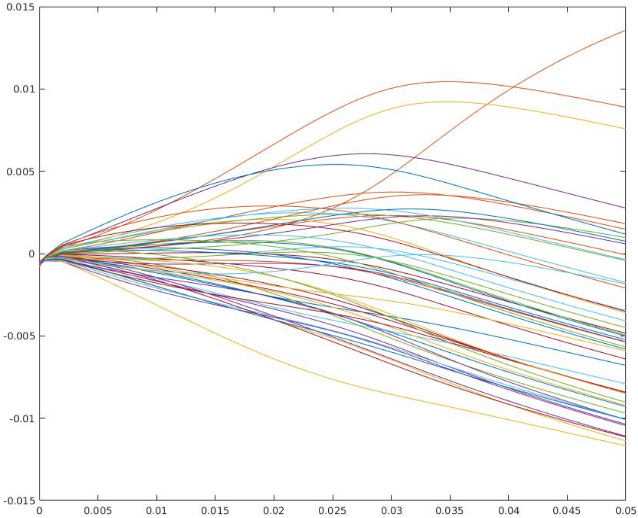


Weights l_2

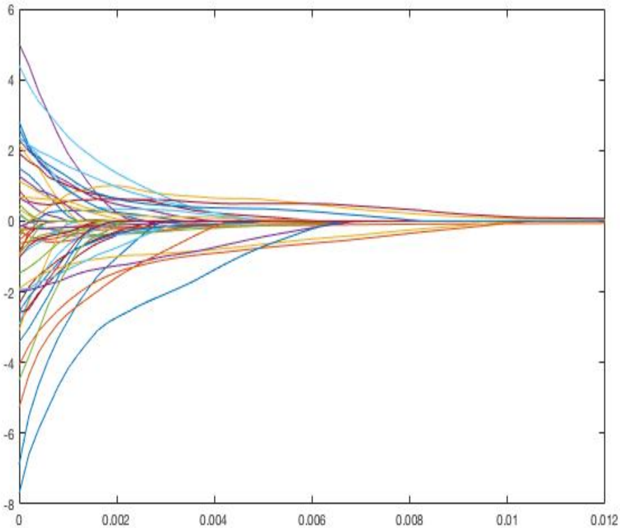


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05

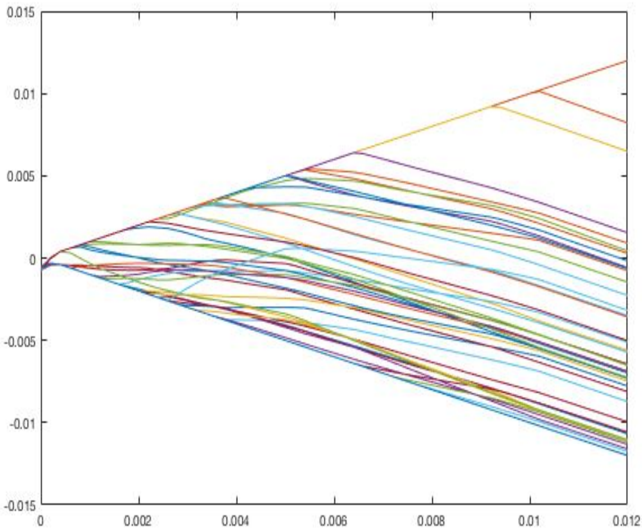
Pricing errors l_2



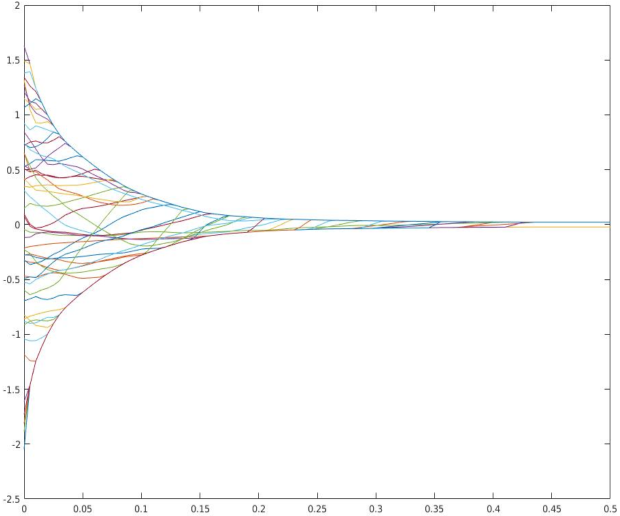
Weights l_1



Pricing errors l_1



Weights l_∞



Pricing errors l_∞