Problem Set 1

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Section 1

1.3)

i) G_1 is not an algebra, because its complement is closed and not included in the set. Since it's not algebra, it's not σ-algebra.

ii) G_2 is an algebra. It is not σ-algebra because $\bigcup_{i=1}^{\infty}(0,\frac{i-1}{i}] = (0,1)$, which is not included in the set.

iii) G_3 is both am algebra but also a σ-algebra, because the union of infinite subsets of all three forms is still included in the subset.

1.7 $\{\varnothing, X\}$ is the smallest because a σ-algebra has to contain the element and its complement, and also has to include $\varnothing$. $\wp(X)$ is the collection of all possible subsets of X, which satisfy all requirements of a σ-algebra.

1.10

$$\begin{aligned}
&\because X \in S_\alpha \quad \forall\alpha \\
&\therefore X \in \cap_\alpha S_\alpha \quad (1) \\
&\text{Let } E \in \cap_\alpha S_\alpha, \text{ then } X \setminus E \in S_\alpha \forall\alpha \\
&\therefore X \setminus E \in \cap_\alpha S_\alpha \quad (2) \\
&\text{Let } (E_n)_{n\in\mathbb{N}} \in \cap_\alpha S_\alpha, \text{ then } (E_n)_{n\in\mathbb{N}} \in S_\alpha \forall\alpha \\
&\therefore (E_n)_{n\in\mathbb{N}} \in \cap_\alpha S_\alpha \quad (3)
\end{aligned}$$

Based on 1, 2 and 3, $\cap_\alpha S_\alpha$ is a σ-algebra.

1.17

i.

$$\because A \subset B$$
$$\therefore B = A \cup (B \setminus A)$$
$$\therefore \mu(B) = \mu(A) + \mu(B \setminus A), \text{ where } \mu(B \setminus A) \geqslant 0 \text{ by assumption}$$
$$\mu A \leqslant \mu B$$

ii.

$$\text{Let } F_i = \bigcup_{i \in \mathbb{N}} A_i$$
$$\text{then } F_i \text{ is monotonically increasing, meaning } F_i \subseteq F_{i+1}$$
$$\begin{aligned}\mu\left(\bigcup_{i \in N} A_i\right) &= \lim_{i \to \infty} \mu(F_i) \\ &= \lim_{i \to \infty} (A_1 \cup A_2 \ldots \cup A_i) \\ &\leqslant \lim_{i \to \infty} \sum_{k=1}^{n} \mu(A_k) \\ &= \sum_{i=1}^{\infty} \mu(A_i)\end{aligned}$$

1.18

$$\text{Let } \mu_B(A) = \mu(A \cup B) \geqslant 0 \tag{4}$$
$$\begin{aligned}\mu_B(\bigcup_{n \in \mathbb{N}} A_n) &= \mu\left((\bigcup_{n \in \mathbb{N}} A_n) \cap B\right) \\ &= \mu\left(\bigcup_{n \in \mathbb{N}} (A_n \cap B)\right) \\ &= \sum_{n \in \mathbb{N}} \mu(A_n \cap B) \\ &= \sum_{n \in \mathbb{N}} \mu_B(A_n) \qquad (5)\end{aligned}$$
$$\text{Also, } \mu_B(\varnothing) = \mu(\varnothing \cap B) = 0 \tag{6}$$

Based on 4, 5,6, μ_B is a measure

1.20

$$A_{i+1} = A_{i+1} \cap A_i$$
$$\bigcap_{i=1}^{\infty} A_i = \lim_{i\to\infty} A_i$$
$$\text{Hence, } \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$$

Section 2

2.10 To prove the statement, we will shown that the $\geqslant$ can be replaced by $\leqslant$ in the Theorem 2.8. Since $B = (B \cap E) \cup (B \cap E^c)$ and μ^* is an outer-measure $\Rightarrow$ μ^* is countably sub-additive $\Rightarrow$ $\mu^*((B \cap E) \cup (B \cap E^c)) \leqslant \mu^*(B \cap E) + \mu^*(B \cap E^c) \Rightarrow \mu^*(B) \leqslant \mu^*(B \cap E) + \mu^*(B \cap E^c)$. From the Theorem 2.8, we obtain $\mu^*(B) \geqslant \mu^*(B \cap E) + \mu^*(B \cap E^c)$, hence we get the statement $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

2.14 Define $\mathcal{O} = \{A : A \text{ is open}, A \subset \mathbb{R}\}$, ν is a premeasure on $\mathbb{R}$, denote μ^* as the outer measure generated by ν. Let $\sigma(\mathcal{O})$ be the σ-algebra generated by $\mathcal{O}$ and $\mathcal{M}$ denote the σ-algebra from the Caratheodory construction. By Theorem 2.12, we obtain $\sigma(\mathcal{O}) \subset \mathcal{M}$, since $\sigma(\mathcal{O})$ is the σ-algebra generated by $\mathcal{O}$, which is the Borel-algebra. Hence, we have shown $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$.

Section 3

3.1

$$\text{Let } a \in \mathbb{R} \text{ , then } a \subset [a-\epsilon, a+\epsilon] \quad \forall \epsilon > 0.$$
$$\text{Also, } \lambda^*(a) \leqslant \lambda^*([a-\epsilon, a+\epsilon]) = 2\epsilon \quad \forall \epsilon > 0$$
$$\text{Therefore, } \lambda^*(a) = 0 \quad \forall a \in \mathbb{R}$$
$$\text{Let } A = a_1, a_2, \ldots = \bigcup_{n=1}^{\infty} \{a_n\} \text{ be a countable set,}$$
$$\text{then, } \lambda^*(A) \leqslant \sum_{n=1}^{\infty} \lambda^*(a_n) = 0$$

3.4 1). First, let set $\{x \in X : f(x) < a\}, \quad \forall a$ be A
$\because$ $\mathbb{M}$ is σ-algebra, $\therefore A^C = \{x \in X : f(x) \geqslant a\} \in \mathbb{M}, \quad \forall a$, and the definition still holds.
2). Then, we show that set $\{x \in X : f(x) > a\} \in \mathbb{M}$. Define $\{a_n = a + \frac{1}{n}\}_{n \in \mathbb{N}}$ in $\mathbb{R}$, then $\lim_{n\to\infty} a_n = a$.
By the proof above, we know that $A_n = x \in X : f(x) \geqslant a_n \in \mathbb{M} \quad \forall a_n \in \mathbb{R}$.
Thus $\bigcup_{n=1}^{\infty} A_n \in \mathbb{M}$
$\Rightarrow \bigcup_{n=1}^{\infty}\{x \in X : f(x) \geqslant a\} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) > \lim_{n\to\infty} a_n\}$
$\Rightarrow \{x \in X, f(x) > a\} \in \mathbb{M}$ 3). Thus, by the same logic in 1), $\{x \in X, f(x) > a\}$ $\{x \in X, f(x) \leqslant a\}$. Thus we have the sets composed of all four operators belonging to $\mathbb{M}$.

3.7 1). For case of $f+g$: Let $F(x,y)=x+y$, then $f+g=F(f,g)$ and $f+g$ is a continuous function. $\Rightarrow$ By property 4, we show that $f+g$ is measurable.
2). For case of $f\cdot g$: Let $F(x,y)=xy$, then $f\cdot g=F(f,g)$ and $f\cdot g$ is a continuous function. $\Rightarrow$ By property 4, we show that $f\cdot g$ is measurable.
3). Let $f=\sup_{n\in\mathbb{N}} f_n(x), \quad g=\sup_{n\in\mathbb{N}} g_n(x)$. Also, let $\{K_n \mid_{n\in\mathbb{N}}\}=\{\{f_n\}_{n\in\mathbb{N}},\{g_n\}_{n\in\mathbb{N}}\}$. $sup_{n\in\mathbb{N}}K_n(x)=\max(sup_{n\in\mathbb{N}}f_n(x), sup_{n\in\mathbb{N}}g_n(x))=\max(f,g)$
$\Rightarrow \forall n,\ K_n(x)$ is measurable. $\Rightarrow \{K_n(x)\}_{n\in\mathbb{N}}$ is measurable
$\Rightarrow$ By property (2) we show that $\max(f,g)$ is measurable.
4). Similar to the above proof, change sup to inf, then $\inf_{n\in\mathbb{N}} K_n(x)=min(f,g)$
$\Rightarrow min(f,g)$ is measurable.
5). $|f|=\max(f,-f)$, by proof aboce $\Rightarrow$ we know that $|f|$ is measurable.

3.14 $\forall\epsilon>0$, we constrict intervals and simple function as the proof in note.
$\exists N_1\in\mathbb{N}$, s.t. $\frac{1}{2^{N_1}}<\epsilon$
$\exists N_2\in\mathbb{N}$, s.t. $f(x)<N_2$
Let $N=\max\{N_1,N_2\}$
for $n>N$, $\forall x\in X, x\in E_i^n$ for $0\leq i\leq N, i\in\mathbb{N}$
$\Rightarrow f(x)\in[\frac{i-1}{2^n},\frac{i}{2^n})$ and $s_n(x)=\frac{i-1}{2^n}$
$\Rightarrow |f(x)-s_n(x)|<\frac{1}{2^n}<\frac{1}{2^N}<\epsilon$
$\Rightarrow$ the convergence in (1) is uniform.

Section 4

4.13 Since $0\leqslant ||f||<M$ on $E\in\mathbb{M}$, and $\mu(E)<\infty$
by proposition 4.5, $0\leqslant\int_E ||f||d\mu\leqslant M\mu(E)<\infty$
$\Rightarrow f\in\mathbb{L}^1(\mu,E)$

4.14 We will prove by contradiction.
Without loss of generality, we need to show that $f=\infty$
Suppose $\exists A\subset E$ with positive measure μ s.t. $f=\infty$ somewhere on A.
Then, $\infty=\int_A fd\mu\leqslant\int_E fd\mu\leqslant\int_E ||f||d\mu$
$\Rightarrow f\notin\mathbb{L}^1(\mu,E)$, which contradicts with $f\in\mathbb{L}^1(\mu,E)$

4.15 Let $S(f)=\{s: 0\leqslant s\leqslant f$, s measurable and simple$\}$.
$f<g\Rightarrow f^+<g^+$ and $f^->g^-$
$\Rightarrow S(f^+)\subset S(g^+) \quad \Rightarrow \int_E f^+d\mu\leqslant\int_E g^+d\mu$
Similarly, $\Rightarrow S(g^-)\subset S(f^-) \quad \Rightarrow \int_E g^-d\mu\leqslant\int_E f^-d\mu$
$\Rightarrow \int_E fd\mu=\int_E f^+d\mu-\int_E f^-d\mu\leqslant\int_E g^+d\mu-\int_E g^-d\mu=\int_E gd\mu$
Hence $\int_E fd\mu\leqslant\int_E gd\mu$

4.16 Take an arbitrary simple function $s(x)=\sum_1^N c_i\chi_i E_i$, where E_i is measurable.
Then since $A\subset E\Rightarrow A\cap E_i\subset E\cap E_i \quad \forall i$

$\Rightarrow \mu(A \cap E_i) \leq \mu(E \cap E_i) \quad \forall i$
$\Rightarrow \int_A s d\mu = \sum_{i=1}^{N} c_i \mu(A \cap E_i) \leq \sum_{i=1}^{N} c_i \mu(E \cap E_i) = \int_E s d\mu$
$\Rightarrow \int_A ||f|| d\mu \leq \int_E ||f|| d\mu < \infty$
$\Rightarrow f \in \mathbb{L}^1(\mu, A)$

4.21 Let $\lambda(\cdot)$ be a measure on μ
$A = (A \setminus B) \cup (A \cap B)$
$\because B \subset A, \Rightarrow A = (A \setminus B) \cup B$
$\Rightarrow \lambda(A) = \lambda((A \setminus B) \cup B) = \lambda(A \setminus B) + \lambda(B)$
$\int_A f d\mu = \int_{A \setminus B} f d\mu + \int_B f d\mu$
$\because \int_{A \setminus B} f d\mu = 0$
$\Rightarrow \int_A f d\mu = \int_B f d\mu$
$\Rightarrow \int_A f d\mu \leqslant \int_B f d\mu$