

Problem Set 3

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$$\mathbf{2} \quad D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\det(\lambda I - D) = 0$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 = 0$$

$$\Rightarrow \lambda = 0$$

\therefore Algebraic multiplicity is 3. The corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

\therefore Geometric multiplicity is 1.

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i) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^H = A$ implies that $b = \bar{c}$.

Recall that if $z \in \mathbb{C}$, then $z\bar{z} = |z|^2 \geq 0$

Now, $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$

$$\Delta = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4b\bar{b} = (a-d)^2 + 4|b|^2 \geq 0$$

\Rightarrow Real roots.

2 Suppose λ is an eigenvalue of A where $A^H = -A$. x is the corresponding eigenvector.

Then, $\langle Ax, x \rangle = \langle \lambda x, x \rangle = \lambda \langle x, x \rangle$.

Also, $\langle Ax, x \rangle = \langle x, A^H x \rangle = \langle x, -Ax \rangle = -\langle x, \lambda x \rangle = -\lambda \langle x, x \rangle$.

So we have $\bar{\lambda} = -\lambda$

$\Rightarrow \lambda$ is pure imaginary.

6 Suppose A is an upper triangular matrix.

$$A = \begin{bmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$$

Then the characteristic polynomial is

$$\det(zI - A) = \begin{vmatrix} z - a_1 & & & * \\ & z - a_2 & & \\ & & \ddots & \\ 0 & & & z - a_n \end{vmatrix} = \prod_{i=1}^n (z - a_i) = 0$$

Note that this polynomial has n zeros, which are a_1, a_2, \dots, a_n respectively.

The case of upper triangular matrix is the same.

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1 Since $V = \text{span}(s)$, it suffices to show that the four vectors are linearly independent.

Let $a \sin(x) + b \cos(x) + c \sin(2x) + d \cos(2x) = 0, \quad \forall x \in \mathbb{R}$.

Let $x = 0$: $b + d = 0$

Let $x = \frac{\pi}{2}$: $a - d = 0$

Let $x = \pi$: $-b + d = 0$

Let $x = \frac{\pi}{4}$: $a \sin(\frac{\pi}{4}) + b \cos(\frac{\pi}{4}) + c \sin(\frac{\pi}{2}) + d \cos(\frac{\pi}{2}) = 0$

From the above four conditions we can get $a = 0, b = 0, c = 0, d = 0$.

Since the only case that can let $a \sin(x) + b \cos(x) + c \sin(2x) + d \cos(2x) = 0, \quad \forall x \in \mathbb{R}$, is when $a = b = c = d = 0$,

\Rightarrow They are linearly independent.

$$2 \quad D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

3 $V_1 = \{\sin x, \cos x\}, V_2 = \{\sin 2x, \cos 2x\}$

13 To diagonalize A, we first need to find eigenvalues and eigenvectors.

$$p(\lambda) = \lambda^2 - 1.4\lambda + 0.4 = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = \frac{2}{5}$$

And $\Sigma_1 = \text{span}([2, 1]^T), \Sigma_2 = \text{span}([1, -1]^T)$

So A is semisimple.

$$\text{Let } p = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \text{ then } p^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}.$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

Then $D = p^{-1}Ap$.

15 Since $A \in M_n(\mathbb{F})$ is semisimple, we can diagonalize $A = pDp^{-1}$, where D is diagonalized, and $\{\lambda_i\}_1^n$ are the diagonal entries of D.

$$\begin{aligned}
\text{Now, } f(A) &= f(pDp^{-1}) \\
&= a_0I + a_1pDp^{-1} + \dots + a_npD^n p^{-1} \\
&= p[a_0I + a_1D + \dots + a_nD^n]p^{-1} \\
&= pf(D)p^{-1}
\end{aligned}$$

Observe that $f(A)$ and $f(D)$ are similar, so they have the same eigenvalues.

Also note that $f(D)$ is also diagonal, so each entry along the diagonal is $f(D)_{ii} = a_0 + a_1d_{ii} + \dots + a_nd_{ii}^n = f(d_{ii})$, where $D = [d_{ij}]_{ij}$

Hence, the eigenvalues of $f(D)$ are just its diagonals, which are $\{f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\}$

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$$\begin{aligned}
& \mathbf{1} \quad A = pD^n p^{-1} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \\
&\therefore \lim_{n \rightarrow \infty} A^n = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\
&\text{Let } B = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \text{ then it follows immediately from the definition of limit.}
\end{aligned}$$

2 The choice of norm does not affect the answer.

3 Let $f(x) = 3 + 5x + x^3$, then the eigenvalues of $f(A)$ are $f(\lambda_1) = f(1) = 9, f(\lambda_2) = f(0.4) = 5.064$.

18 Take \vec{y} an eigenvector corresponding to λ . Then,

$$\begin{aligned}
x^T Ay &= x^T \lambda y = (\lambda x^T y) \\
&\Rightarrow x^T A = \lambda x^T
\end{aligned}$$

20 Let $B = U^H AU$, then,

$$B^H = U^H A^H U = U^H AU = B, \text{ since } A^H = A$$

24

$$\mathbf{1} \quad p(\vec{x}) = \frac{\langle x, Ax \rangle}{||x||^2}$$

Observe that the denominator is always a real number. Hence to show that $p(\vec{x}) \in \mathbb{R}$, it suffices to show that $\langle x, Ax \rangle \in \mathbb{R}$.

$$\text{Now } \langle x, Ax \rangle = \langle A^H, x \rangle = -\langle Ax, x \rangle$$

$$\text{Since by definition, } \langle x, Ax \rangle = \langle A\bar{x}, x \rangle, \text{ we have } \langle Ax, x \rangle = \langle A\bar{x}, x \rangle \in \mathbb{R}$$

This implies

$$\langle x, Ax \rangle \in \mathbb{R}$$

$$\Rightarrow p(x) \in \mathbb{R}$$

2 If $A^H = -A$, then

$$\langle x, Ax \rangle = \langle A^H, x \rangle = -\langle Ax, x \rangle$$

$$\text{Also, } \langle x, Ax \rangle = \langle A\bar{x}, x \rangle$$

$$\therefore \langle A\bar{x}, x \rangle = -\langle Ax, x \rangle$$

This implies $\langle x, Ax \rangle = \langle A\bar{x}, x \rangle \in \mathbb{C} \setminus \mathbb{R} \cup \{0\}$

Hence $p(\vec{x}) = \frac{\langle x, Ax \rangle}{\|x\|^2}$ is pure imaginary number.

25

1 Since $A \in M_n(\mathbb{C})$ is a normal matrix, its eigenspace $\{x_1, x_2, \dots, x_n\}$ spans \mathbb{C}^n .

Observe that $\forall j = 1, 2, \dots, n$,

$$(x_1x_1^H + x_2x_2^H + \dots + x_nx_n^H)x_j = x_1x_1^Hx_j + x_2x_2^Hx_j + \dots + x_nx_n^Hx_j = x_j$$

This holds for any j .

Since $\{x_1, x_2, \dots, x_n\}$ spans \mathbb{C}^n , $\forall \vec{v} = \mathbb{C}^n$, $\vec{v} = \sum a_i \vec{x}_i$

Let $B = x_1x_1^H + x_2x_2^H + \dots + x_nx_n^H$, then $B\vec{v} = \sum a_i B\vec{x}_i = \sum a_i \vec{x}_i = \vec{v}$.

Let $\vec{v} = \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ respective, then we get

$$Be_1 = e_1, Be_2 = e_2, \dots, Be_n = e_n$$

Hence $B = I$

2 Since A is a normal matrix and $\{x_1, x_2, \dots, x_n\}$ forms an orthonormal eigenbasis, A admits a diagonalization.

$A = pDp^{-1} = pDp^H$, where

$$p = [x_1, x_2, \dots, x_n] \quad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$p^{-1} = p^H = \begin{bmatrix} x_1^H \\ x_2^H \\ \vdots \\ x_n^H \end{bmatrix}, \text{ since } p \text{ is an orthonormal matrix.}$$

$$\text{Hence, } A = [x_1, x_2, \dots, x_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^H \\ x_2^H \\ \vdots \\ x_n^H \end{bmatrix} = \sum \lambda_i x_i x_i^H$$

27 Suppose
$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \lambda_2 & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

By definition, $\forall x, x^H A x > 0$

Now, let $x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_i^{-1}$

then $e_1^H A e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = a_{11} > 0$

Similarly, let $x = e_2, e_3, \dots, e_n$

we have $a_{22} > 0, a_{33} > 0, \dots, a_{nn} > 0$

Here all diagonal elements are positive and real.

28 Proof:

First we introduce the following lemmas used in the proof.

- Lemma 1: The diagonals of a positive semi-definite matrix are greater than or equal to zero. (Proof similar to exercise 4.27)
- Lemma 2: $tr(AB) = tr(BA)$ (Proof can be found in Problem Set 2)
- Lemma 3: If $A \in M_n(\mathbb{F})$ is a positive semi-definite matrix, $D \in M_n(\mathbb{F})$ is a diagonal matrix with non-negative diagonals, then $0 \leq tr(AD) \leq tr(A)tr(D)$.

Proof. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

,

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

,

then $tr(AD) = \sum_{i=1}^n a_{ii}d_i \geq 0$, since $a_{ii} \geq 0$ and $d_i \geq 0$ for $\forall i$

$$tr(A)tr(D) = (\sum_{i=1}^n a_{ii})(\sum_{i=1}^n d_i) = \sum_{i=1}^n a_{ii}d_i + \sum_{i \neq j} a_{ii}d_j \geq \sum_{i=1}^n a_{ii}d_i$$

$$\Rightarrow tr(A)tr(D) \geq tr(AD) \geq 0$$

□

Now since B is a positive semi-definite matrix, it admits a diagonalization s.t. $B = PDP^{-1} = PDP^H$, where

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is an orthonormal eigenbasis,

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

is diagonal matrix with $d_i \geq 0 \quad \forall i$.

$$\begin{aligned} \text{Then } tr(AB) &= tr(APDP^H) = tr(P^H APD) \leq tr(P^H AP)tr(D) \\ &= tr(APP^H)tr(D) = tr(A)tr(D) = tr(A)tr(B). \end{aligned}$$

Meanwhile, $\|AB\|_F^2 = tr(AA^H BB^H) \leq tr(AA^H) tr(BB^H) = \|A\|_F \|B\|_F^2$, which makes $\|\cdot\|_F$ a matrix norm.

31

1 Suppose A has rank r , then $A^H A$ is positive definite and has r distinct eigenvalues.

Let $s = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be an orthonormal eigenspace of $A^H A$, and $\{\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2\}$ be the corresponding eigenvalues, where $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2$.

Since s spans \mathbb{F}^n , $\forall \vec{x} \in \mathbb{F}^n$, we have

$$\vec{x} = \sum_{i=1}^n c_i \vec{v}_i, c_i \in \mathbb{F}, \forall i, \text{ and}$$

$$\|\vec{x}\|_2 = \sqrt{(\sum c_i v_i^T)(\sum c_i v_i)} = \sqrt{(\sum c_i^2)}$$

Hence if $\|\vec{x}\|_2 = 1$, then $\sum_{i=1}^n c_i^2 = 1$

$$\text{Now, observe that } \|Ax\|_2^2 = \langle Ax, Ax \rangle = (Ax)^H Ax = x^H A^H Ax$$

$$= (\sum_{i=1}^n c_i \vec{v}_i^H)(A^H A)(\sum_{i=1}^n c_i \vec{v}_i)$$

$$= (\sum_{i=1}^n c_i \vec{v}_i^H)(\sum_{i=1}^n c_i A^H A \vec{v}_i)$$

$$= (\sum_{i=1}^n c_i \vec{v}_i^H)(\sum_{i=1}^n c_i \sigma_i^2 \vec{v}_i) = \sum c_i^2 \sigma_i^2, \text{ where } s = \{v_1, v_2, \dots, v_n\}$$

Note that when $\sigma_i^2 = 1$, and $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2$,

$$\sum c_i^2 \sigma_i^2 \leq \sigma_1^2$$

$$\text{Hence, } \|A\|_2^2 = \sup_{\|\vec{x}\|_2=1} \|Ax\|_2^2 = \sigma_1^2$$

$$\Rightarrow \|A\|_2 = \sigma_1$$

2 Since $A = U\Sigma V^H$

$$A^{-1} = (U\Sigma V^H)^{-1} = (V^H)^{-1}\Sigma^{-1}(U)^{-1} = V\Sigma^{-1}U^H$$

\Rightarrow This is still an SVD of A^{-1}

$$\text{And } \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$$

i.e. The singular values of A^{-1} are

$$\frac{1}{\sigma_1} \leq \dots \leq \frac{1}{\sigma_n}$$

By (1), $\|A^{-1}\|_2$ is the largest singular value of A^{-1} , i.e. $\frac{1}{\sigma_n}$

3 Since $A = U\Sigma V^H$

$$A^H = (V^H)^H \Sigma^H U^H = V \Sigma^H U = V \Sigma U$$

$\Rightarrow A^H$ and A has the same singular values.

$$\text{So } \|A^H\|_2^2 = \|A\|_2^2 = \sigma_1^2$$

(A^T) is just A^H restricted on \mathbb{R} .

$$\text{So } \|A^T\|_2^2 = \|A^H\|_2^2$$

By the previous argument, we know that $A^H A$ has an orthonormal eigenbasis $\{v_1, v_2, \dots, v_n\}$,

$$\text{and } \forall \|x\|_2 = 1, \|A^H A x\|_2 = \|A^H A \sum c_i v_i\|_2 = \sqrt{(\sum c_i \sigma_i^2 v_i^T)(\sum c_i \sigma_i^2 v_i)} = \sqrt{\sum c_i \sigma_i^4} \leq \sigma_1^2$$

$$\text{Hence } \|A^H A\|_2 = \sup_{\|x\|=1} \|A^H A x\|_2 = \sigma_1^2$$

$$\text{It follows that } \|A^H A\|_2 = \|A\|_2^2 = \|A^H\|_2^2 = \|A^T\|_2^2 = \sigma_1^2$$

4 Lemma: Let Q be an orthonormal matrix, then $\|AQ\|_2 = \|A\|_2$.

Proof. Let $S_1 = \{ \|AQx\|, \|x\|_2 = 1 \}$, $S_2 = \{ \|Ax\|, \|x\|_2 = 1 \}$

Proof. Since Q is orthonormal, so Q is also invertible.

$$\forall s_1 \in S_1, \exists x, \|x\|_2 = 1, \text{ s.t. } \|AQx\|_2 = s_1$$

$$\text{Now, let } y = Qx, \text{ it follows that } \|Qx\|_2 = \|y\|_2 = 1$$

$$\text{Since orthonormal matrix preserves length, } \|Ay\|_2 = \|AQx\|_2 = s_1 \in S_2$$

$$\text{i.e. } S_1 \subset S_2$$

$$\forall s_2 \in S_2, \exists x, \|x\|_2 = 1 \text{ s.t. } \|Ax\|_2 = s_2$$

$$\text{Now, let } y = Q^{-1}x, \text{ then } \|y\|_2 = \|Q^{-1}x\|_2 = 1$$

$$\text{Hence } \|AQy\|_2 = \|AQQ^{-1}x\|_2 = \|Ax\|_2 = s_2 \in S_1$$

$$\text{i.e. } S_2 \subset S_1$$

$$\therefore \|AQ\|_2 = \sup S_1 = \sup S_2 = \|A\|_2$$

□

Now $\|UAV\|_2 = \|UA\|_2$ by lemma since V is an orthonormal matrix.

$$\|UA\|_2 = \sup_{\|x\|_2=1} \sqrt{(UAx)^H (UAx)} = \sup_{\|x\|_2=1} \sqrt{x^H A^H U^H U A x}$$

$$= \sup_{\|x\|_2=1} \sqrt{\langle Ax, Ax \rangle} = \|A\|_2$$

$$\text{Hence, } \|UAV\|_2 = \|UA\|_2 = \|A\|_2$$

□

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1 We need the following lemmas:

• lemma 1: if $A, B \in M_n(\mathbb{F})$, then $\text{tr}(AB) = \text{tr}(BA)$

• lemma 2: $\|A\|_F^2 = \text{tr}(A^T A)$

$$\text{Proof. let } A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{1m} & \dots & a_{mn} \end{bmatrix}$$

$$\text{Then } A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

$$\text{Observe that } (A^T A)_{nm} = a_{1n}^2 + a_{2n}^2 + \dots + a_{mn}^2 \\ \Rightarrow \text{tr}(A^T A) = \|A\|_p^2 \quad \square$$

$$\text{Now, } \|UAV\|_1^2 = \text{tr}((UAV)^T(UAV)) = \text{tr}(V^T A^T U^T U AV) = \text{tr}(V^T A^T AV) = \text{tr}(VV^T A^T A) = \text{tr}(A^T A) = \|A\|_1^2 \\ \Rightarrow \|UAV\|_2 = \|A\|_2$$

2 Observe that $A = U\Sigma V^T$, with U and V^T orthonormal and

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix} \quad \text{Now, } \|A\|_p^2 = \text{tr}(A^T A) = \text{tr}((U\Sigma V)^T(U\Sigma V)) = \text{tr}(U\Sigma^T U^T U\Sigma V^T) = \text{tr}(U\Sigma^2 V^T) \\ = \text{tr}(\Sigma^2) = \sum_{i=1}^r \sigma_i^2 \\ \text{Hence } \|A\|_p = \sqrt{\sum_{i=1}^r \sigma_i^2}$$

33 Note that the $Y^H Ax$ will be a field element. Consider it as a linear map $Y^H Ax : \mathbb{F} \rightarrow \mathbb{F}$, then the spectral norm of this map is:

$$\|Y^H Ax\|_2 = \sup_{f \in \mathbb{F}} \frac{\|(Y^H Ax)f\|_2}{\|f\|_2} = |Y^H Ax|$$

, where the first norm is spectral norm and the norm in fraction is the standard 2-norm.

36 One example can be $A = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$

$$\text{then } A^T A = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = \lambda^2 - 50\lambda + 400 = 0$$

$$\lambda_1 = 40, \lambda_2 = 10$$

$$\text{Thus its singular value are } s_1 = \sqrt{40}, s_2 = \sqrt{10}$$

To calculate its eigenvalues,

$$\det(A - \lambda I) = \lambda^2 - 9\lambda + 20 = 0$$

Thus its eigenvalues are $\lambda_1 = 4, \lambda_2 = 5$, which are different from its singular values.

38 (i) Suppose $U\Sigma V^H$ is an SVD of A , then $A^\dagger = V\Sigma^{-1}U^H$

$$AA^\dagger A = (U\Sigma V^H)(V\Sigma^{-1}U^H)(U\Sigma V^H) = U\Sigma V^H = A$$

(ii)

$$A^\dagger AA^\dagger = (V\Sigma^{-1}U^H)(U\Sigma V^H)(V\Sigma^{-1}U^H) = V\Sigma^{-1}U^H = A^\dagger$$

(iii)

$$(AA^\dagger)^H = ((U\Sigma V^H)(V\Sigma^{-1}U^H))^H = U\Sigma^{-1}V^H V\Sigma U^H = UU^H = AA^\dagger$$

(iv)

$$(A^\dagger A)^H = ((V\Sigma^{-1}U^H)(U\Sigma V^H))^H = V\Sigma U^H U\Sigma^{-1}V^H = VV^H = A^\dagger A$$

(v)

By prop (iii) $\Rightarrow AA^\dagger$ is hermitian.

Also by prop (i), $AA^\dagger AA^\dagger = AA^\dagger \Rightarrow AA^\dagger$ is idempotent.

Next we will check whether $\mathcal{R}(AA^\dagger) = \mathcal{R}(A)$.

It is trivially $\mathcal{R}(AA^\dagger) \subset \mathcal{R}(A)$, and by prop(i) $\Rightarrow \mathcal{R}(A) \subset \mathcal{R}(AA^\dagger)$
 $\Rightarrow \mathcal{R}(AA^\dagger) = \mathcal{R}(A)$

(vi)

By prop (iv) $A^\dagger A$ is hermitian.

Also by prop (ii) $A^\dagger AA^\dagger A = A^\dagger A \Rightarrow A^\dagger A$ is idempotent

Next we will check whether $\mathcal{R}(A^\dagger A) = \mathcal{R}(A^H)$

By prop (iv), $AA^\dagger = (AA^\dagger)^H = A^H(A^\dagger)^H \Rightarrow \mathcal{R}(A^\dagger A) \subset \mathcal{R}(A^H)$

Then we take the hermitian of both sides of prop (i),

we have $(A^\dagger A)^H A^H = (A^\dagger A)^H A^H A^H \Rightarrow \mathcal{R}(A^H) \subset \mathcal{R}(A^\dagger A)$
 $\Rightarrow \mathcal{R}(A^\dagger A) = \mathcal{R}(A^H)$