Quiz #1: Suggested Solutions

ECON 302

Wellesley College

1. Consider a neoclassical growth model in which the representative consumer has preference given by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

and faces the budget constraint

$$c_t + k_{t+1} + \phi(k_{t+1} - k_t)^2 = (1-\delta)k_t + r_t k_t + \pi_t$$

at each date $t \geq 0$, where $\phi > 0$ is a capital adjustment cost — the more the consumer wants to adjust their capital stock (either up or down), the more costly it will be — and π represents the profits (net of taxes) earned by the representative firm — the representative consumer owns all shares of the firm, and therefore is entitled to the profit stream that it generates. The production function of the representative firm in this model economy is

$$f(k_t) = k_t^{\alpha}$$

for some $\alpha \in (0,1)$. There is a government that finances its exogenously given spending g_t with a proportional profits tax τ_t levied on the representative firm. The profits, net of taxes, earned by the representative firm are thus

$$\pi_t = (1-\tau_t)[f(k_t) - r_t k_t]$$

and the government's budget constraint is given by

$$g_t = \tau_t[f(k_t) - r_t k_t]$$

In this economy, there is no population growth or growth in productivity – so all variables are already expressed in per capital terms.

(a) *Characterize* the general solution to this version of the neoclassical growth model.

Solution: The problem of the representative consumer is given by

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$c_t + k_{t+1} + \phi(k_{t+1} - k_t)^2 = (1-\delta)k_t + r_t k_t + \pi_t \tag{1}$$

The associated Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \lambda_t \Big[(1-\delta)k_t + r_t k_t + \pi_t - c_t - k_{t+1} - \phi(k_{t+1} - k_t)^2\Big]$$

Now to the first order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t \Big[1 + 2\phi(k_{t+1} - k_t)\Big] + \lambda_{t+1}\Big[r_{t+1} + 2\phi(k_{t+2} - k_{t+1}) + (1-\delta)\Big] = 0$$

Combining these first order conditions yields the Euler equation:

$$\frac{c_{t+1}}{c_t} = \beta \left[\frac{r_{t+1} + 2\phi(k_{t+2} - k_{t+1}) + (1-\delta)}{1 + 2\phi(k_{t+1} - k_t)}\right] \tag{2}$$

The problem of the representative firm is given by

$$\max_{k_t} (1-\tau_t)[k_t^{\alpha} - r_t k_t]$$

The first order condition is

$$(1-\tau_t)[\alpha k_t^{\alpha-1} - r_t] = 0$$

which can be solved for the rental rate:

$$r_t = \alpha k_t^{\alpha-1} \tag{3}$$

Finally, recall the government's budget constraint:

$$g_t = \tau_t[f(k_t) - r_t k_t] \tag{4}$$

Together, equations (1) – (4) fully characterize the competitive equilibrium of this model economy.

(b) What is the steady state level of the capital stock k^*?

Solution: Substituting (3) into above and setting $\Delta c_t = 0$, we have:

$$1 = \beta \left[\frac{\alpha k_{t+1}^{\alpha-1} + 2\phi(k_{t+2} - k_{t+1}) + (1-\delta)}{1 + 2\phi(k_{t+1} - k_t)} \right]$$

Setting $k_t = k_{t+1} = k_{t+2} = k^*$ yields:

$$1 = \beta[\alpha(k^*)^{\alpha-1} + (1-\delta)]$$

which implies that

$$k^* = \left[\frac{\alpha}{\beta^{-1} - (1-\delta)} \right]^{\frac{1}{1-\alpha}}$$

(c) For what values of c_t is $\Delta k_t = 0$?

Solution: First, we must derive an expression for Δk_t. Actual investment is simply

$$i_t = f(k_t) - c_t - g_t$$

while required investment is given by

$$\tilde{i}_t = \delta k_t$$

since $\gamma = 0$ and $n = 0$. The change in the capital stock is the difference between actual and required investment:

$$\Delta k_t = i_t - \tilde{i}_t = k_t^\alpha - c_t - g_t - \delta k_t$$

So $\Delta k_t = 0$ when

$$c_t = k_t^\alpha - g_t - \delta k_t$$

(d) The social planner's problem for this model economy is:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$k_t^\alpha = c_t + \underbrace{k_{t+1} - (1-\delta)k_t + \phi(k_{t+1} - k_t)^2}_{i_t} + g_t$$

Is the competitive equilibrium of this model economy efficient?

Solution: The Lagrangian for the planner's problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[k_t^\alpha - c_t - k_{t+1} + (1-\delta)k_t - \phi(k_{t+1} - k_t)^2 - g_t\right]$$

Now to the first order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t \Big[1 + 2\phi(k_{t+1} - k_t)\Big] + \lambda_{t+1}\Big[\alpha k_{t+1}^{\alpha-1} + 2\phi(k_{t+2} - k_{t+1}) + (1-\delta)\Big] = 0$$

Combining these first order conditions yields the Euler equation:

$$\frac{c_{t+1}}{c_t} = \beta \left[\frac{\alpha k_{t+1}^{\alpha-1} + 2\phi(k_{t+2} - k_{t+1}) + (1-\delta)}{1 + 2\phi(k_{t+1} - k_t)}\right]$$

So the Pareto optimal (PO) allocation is characterized by this Euler equation and the economy-wide resource constraint above. The competitive equilibrium (CE) is efficient if it is characterized by the same equations. Equations (1) – (4) from part (a) reduce to precisely this same system of equations! Hence, the CE is PO.

(e) **Take–home Assignment:** Construct the second order difference equation $h(k_t, k_{t+1}, k_{t+2}) = 0$ for this model economy. Then use the shooting algorithm to approximate the equilibrium growth path $\{k_t\}_{t=0}^{T}$ given $k_0 = 0.5k^*$, $T = 100$ simulate periods (years), and the parameter values $\beta = 0.96$, $\delta = 0.06$, $\alpha = 0.36$ for the cases in which $\phi = 0, 0.1, 0.2$. Plot all three equilibrium trajectories on the same graph (k_t versus t). Provide some intuition for your results.