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initial conditions

(1) $\bar{P}_d \frac{1}{\bar{C}_n} = \beta \frac{A}{d_t} + \beta \frac{1}{C_n} \cdot \bar{P}_d (1-\delta)$

(2) $\bar{P}_k \cdot \cancel{\frac{1}{\bar{C}_n}} = \beta \cdot \cancel{\frac{1}{\bar{C}_n}} [\bar{r} + \bar{P}_k (1-\delta)]$

(3) $\bar{y} = \bar{C}_n + \bar{C}_d + \bar{i}$

(4) $\bar{C}_d = \bar{P}_d \cdot \bar{d}\, \delta$

(5) $\bar{i} = \bar{P}_k \cdot \bar{k} \cdot \delta$

(6) $\bar{r} = \alpha (\bar{k}_n)^{\alpha-1} (\bar{h}_n)^{1-\alpha}$

(7) $\bar{w} = (1-\alpha) (\bar{k}_n)^{\alpha} (\bar{h}_n)^{-\alpha}$

(8) $\bar{r} = \alpha \bar{P}_d (\bar{k}_d)^{\alpha-1} (\bar{h}_d)^{1-\alpha}$

(9) $\bar{w} = (1-\alpha) \bar{P}_d (\bar{k}_d)^{\alpha} (\bar{h}_d)^{-\alpha}$

(10) $\bar{r} = \alpha \bar{P}_k (\bar{k}_k)^{\alpha-1} (\bar{h}_k)^{1-\alpha}$

(11) $\bar{w} = (1-\alpha) \bar{P}_k (\bar{k}_k)^{\alpha} (\bar{h}_k)^{-\alpha}$

(12) $1 = \bar{h}_n + \bar{h}_d + \bar{h}_k$

(13) $\bar{k} = \bar{k}_n + \bar{k}_d + \bar{k}_k$

(14) $\delta \bar{k} = \bar{k}_k^{\alpha} \bar{h}_k^{1-\alpha}$

(15) $\delta \bar{d} = \bar{k}_d^{\alpha} \bar{h}_d^{1-\alpha}$

(16) $\bar{C}_n = \bar{k}_n^{\alpha} \bar{h}_n^{1-\alpha}$

(17) $\bar{z} = 0$

$\frac{\bar{r}}{\bar{w}} = \frac{\alpha}{1-\alpha} \frac{\bar{h}_n}{\bar{k}_n} = \frac{\alpha}{1-\alpha} \frac{\bar{h}_d}{\bar{k}_d} = \frac{\alpha}{1-\alpha} \frac{\bar{h}_k}{\bar{k}_k}$ ← 1°

$\frac{\bar{r}(1-\alpha)}{\bar{w} \cdot \alpha} = \frac{\bar{h}_n}{\bar{k}_n} = \frac{\bar{h}_d}{\bar{k}_d} = \frac{\bar{h}_k}{\bar{k}_k}$ ← 2°

From (2) $1 = \beta \frac{\bar{r}}{\bar{P}_k} + \beta (1-\delta)$

$\frac{\bar{r}}{\bar{P}_k} = \frac{[1-\beta(1-\delta)]}{\beta}$

From (10) $= \alpha \left(\frac{\bar{h}_k}{\bar{k}_k}\right)^{1-\alpha} = \alpha \left(\frac{\bar{h}_n}{\bar{k}_n}\right)^{1-\alpha}$ ← From 1°

From (6) $= \bar{r}$ ← 3°

$\Rightarrow \frac{\bar{r}}{\bar{P}_k} = \bar{r} \Rightarrow \boxed{\bar{P}_k = 1}$ ← 4°

$\Rightarrow \boxed{\bar{r} = \frac{1}{\beta} - (1-\delta)}$ Following the same logic, $\boxed{\bar{P}_d = 1}$

From 3° $\bar{r} = \alpha \cdot \left(\frac{\bar{h}_n}{\bar{k}_n}\right)^{1-\alpha}$

From 1° $\Rightarrow = \alpha \cdot \left(\frac{\bar{r}(1-\alpha)}{\bar{w}\, \alpha}\right)^{1-\alpha}$

$\Rightarrow \bar{w}^{1-\alpha} = \alpha \cdot \bar{r}^{-\alpha} \cdot \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}$

$\boxed{\bar{w} = \frac{1-\alpha}{\alpha} \cdot \sqrt[1-\alpha]{\alpha \cdot \bar{r}^{-\alpha}}}$ ← 5°

From (14)+(16)

$\bar{C}_n = \left(\frac{\bar{h}_n}{\bar{k}_n}\right)^{1-\alpha} \bar{k}_n = \left(\frac{\bar{r}}{\bar{w}} \cdot \frac{1-\alpha}{\alpha}\right)^{1-\alpha} \cdot \bar{k}_n$ 6°

$\bar{C}_d = \left(\frac{\bar{h}_d}{\bar{k}_d}\right)^{1-\alpha} \cdot \bar{k}_d = \left(\frac{\bar{r}}{\bar{w}} \cdot \frac{1-\alpha}{\alpha}\right)^{1-\alpha} \cdot \bar{k}_d$ 7°

$\bar{i} = \left(\frac{\bar{h}_k}{\bar{k}_k}\right)^{1-\alpha} \cdot \bar{k}_k = \left(\frac{\bar{r}}{\bar{w}} \cdot \frac{1-\alpha}{\alpha}\right)^{1-\alpha} \cdot \bar{k}_k$ 8°

$\bar{C}_n = \left(\frac{\bar{h}_n}{\bar{k}_n}\right)^{-\alpha} \cdot \bar{h}_n$

$\bar{C}_d = \delta \bar{d} = \left(\frac{\bar{h}_d}{\bar{k}_d}\right)^{-\alpha} \cdot \bar{h}_d$

$\bar{i} = \delta \bar{k} = \left(\frac{\bar{h}_k}{\bar{k}_k}\right)^{-\alpha} \cdot \bar{h}_k$

From data, $\bar{C}_n : \bar{C}_d : \bar{i} = 3.91 : 0.61 : 1 = \bar{h}_n : \bar{h}_d : \bar{h}_k$

$\Rightarrow \because \bar{h}_n + \bar{h}_d + \bar{h}_k = 1$

$\bar{h}_k = 0.18 \quad \bar{h}_d = 0.11 \quad \bar{h}_n = 0.71$

⇒ then we know $\bar{C}_n, \bar{C}_d, \bar{i}$ ⇒ then we know $\bar{k}_n, \bar{k}_d, \bar{k}_k$

From (1), $\frac{\bar{P}_d \cdot \bar{d}}{\bar{C}_n} = \beta A + \beta(1-\delta) \frac{\bar{P}_d \cdot \bar{d}}{\bar{C}_n}$ $(\bar{C}_d = \bar{P}_d \cdot \bar{d} \cdot \delta)$

$\frac{\bar{C}_d}{\bar{C}_n} = \delta \cdot \beta A + \beta(1-\delta) \frac{\bar{C}_d}{\bar{C}_n}$

$\Rightarrow A = \frac{[1-\beta(1-\delta)]}{\delta \beta \frac{\bar{C}_n}{\bar{C}_d}}$

1) $$\frac{\bar{P}_d}{\bar{C}_n}\cdot e^{\tilde{P}_{d,t}-\tilde{C}_{n,t}} = \beta\frac{A}{\bar{d}}\cdot e^{-\tilde{d}_t} + \beta\,\mathbb{E}\left\{\frac{\bar{P}_d}{\bar{C}_n}\cdot e^{\tilde{P}_{d,t+1}-\tilde{C}_{n,t+1}}(1-\delta)\right\}$$

$$0=\mathbb{E}\left\{\beta\frac{\bar{P}_d}{\bar{C}_n}(1-\delta)\cdot e^{\tilde{P}_{d,t+1}-\tilde{C}_{n,t+1}} + \beta\frac{A}{\bar{d}}\cdot e^{-\tilde{d}_t} - \frac{\bar{P}_d}{\bar{C}_n}e^{\tilde{P}_{d,t}-\tilde{C}_{n,t}}\right\}$$

$$0=\mathbb{E}\left\{\beta\frac{\bar{P}_d}{\bar{C}_n}(1-\delta)(1+\tilde{P}_{d,t+1}-\tilde{C}_{n,t+1}) + \beta\frac{A}{\bar{d}}\cdot(1-\tilde{d}_t) - \frac{\bar{P}_d}{\bar{C}_n}(1+\tilde{P}_{d,t}-\tilde{C}_{n,t})\right\}$$

$$0=\underbrace{\beta\frac{\bar{P}_d}{\bar{C}_n}(1-\delta)+\beta\frac{A}{\bar{d}}-\frac{\bar{P}_d}{\bar{C}_n}}_{=0} + \mathbb{E}\left\{\beta\frac{\bar{P}_d}{\bar{C}_n}(1-\delta)\cdot\tilde{P}_{d,t+1} - \beta\frac{\bar{P}_d}{\bar{C}_n}(1-\delta)\tilde{C}_{n,t+1} - \beta\frac{A}{\bar{d}}\cdot\tilde{d}_t - \frac{\bar{P}_d}{\bar{C}_n}\cdot\tilde{P}_{d,t} + \frac{\bar{P}_d}{\bar{C}_n}\tilde{C}_{n,t}\right\}$$

2). $$\frac{\bar{P}_k}{\bar{C}_n}\cdot e^{\tilde{P}_{k,t}-\tilde{C}_{n,t}} = \beta\,\mathbb{E}\left\{\frac{\bar{r}}{\bar{C}_n}\cdot e^{\tilde{r}_{t+1}-\tilde{C}_{n,t+1}} + \frac{\bar{P}_k(1-\delta)}{\bar{C}_n}\cdot e^{\tilde{P}_{k,t+1}-\tilde{C}_{n,t+1}}\right\}$$

$$\cancel{\frac{\bar{P}_k}{\bar{C}_n}=\bar{C}_n}\quad \bar{P}_k=\beta\left[\bar{r}+\bar{P}_k(1-\delta)\right]$$

$$0=\mathbb{E}\left\{\frac{\beta\bar{r}}{\bar{C}_n}\cdot e^{\tilde{r}_{t+1}-\tilde{C}_{n,t+1}} + \frac{\beta\bar{P}_k(1-\delta)}{\bar{C}_n}\cdot e^{\tilde{P}_{k,t+1}-\tilde{C}_{n,t+1}} - \frac{\bar{P}_k}{\bar{C}_n}\cdot e^{\tilde{P}_{k,t}-\tilde{C}_{n,t}}\right\}$$

$$0=\beta\bar{r}+\beta\bar{P}_k(1-\delta)-\bar{P}_k+\mathbb{E}\left\{\beta\bar{r}(\tilde{r}_{t+1}-\tilde{C}_{n,t+1})+\beta\bar{P}_k(1-\delta)(\tilde{P}_{k,t+1}-\tilde{C}_{n,t+1})-\bar{P}_k(\tilde{P}_{k,t}-\tilde{C}_{n,t})\right\}$$

$$0=\mathbb{E}\left\{\beta\bar{r}\,\tilde{r}_{t+1}-\left[\beta\bar{r}+\beta\bar{P}_k(1-\delta)\right]\tilde{C}_{n,t+1}+\beta\bar{P}_k(1-\delta)\,\tilde{P}_{k,t+1}-\bar{P}_k\,\tilde{P}_{k,t}+\bar{P}_k\tilde{C}_{n,t}\right\}.$$

3). $\bar{y}\cdot e^{\hat{y}_t} = \bar{c}_n \cdot e^{\tilde{c}_{n,t}} + \bar{c}_d \cdot e^{\hat{c}_{d,t}} + \bar{i} e^{\tilde{i}_t}$

$\bar{y}(1+\tilde{y}_t) = \bar{c}_n \cdot (1+\tilde{c}_{n,t}) + \bar{c}_d(\tilde{c}_{d,t}+1) + \bar{i}(1+\tilde{i}_t)$.

$\bar{c}_n \cdot \tilde{c}_{n,t} + \bar{c}_d \cdot \tilde{c}_{d,t} + \bar{i} \cdot \tilde{i}_t - \bar{y} \cdot \tilde{y}_t = 0$

4). $\bar{c}_d \cdot e^{\tilde{c}_{d,t}} = \bar{p}_d \cdot e^{\tilde{p}_{d,t}} [\bar{d} \cdot e^{\tilde{d}_t} - \bar{d}(1-\delta) e^{\tilde{d}_{t-1}}]$

$\bar{c}_d(1+\tilde{c}_{d,t}) = \bar{p}_d \cdot \bar{d}(1+\tilde{p}_{d,t}+\tilde{d}_t) - \bar{p}_d \cdot \bar{d}(1-\delta)(1+\tilde{p}_{d,t}+\tilde{d}_{t-1})$.

$\bar{c}_d \cdot \tilde{c}_{d,t} = \bar{p}_d \cdot \bar{d} \cdot \delta\, \tilde{p}_{d,t} + \bar{p}_d \cdot \bar{d}\, \tilde{d}_t - \bar{p}_d \cdot \bar{d}(1-\delta)\tilde{d}_{t-1}$

5) $\bar{i} e^{\tilde{i}_t} = \bar{p}_k \cdot e^{\tilde{p}_{k,t}} \bar{k} [e^{\tilde{k}_t} - (1-\delta) e^{\tilde{k}_{t-1}}]$

$\bar{i}(1+\tilde{i}_t) = \bar{p}_k \bar{k}(1+\tilde{p}_{k,t}+\tilde{k}_t) - \bar{p}_k \bar{k}(1-\delta)(1+\tilde{p}_{k,t}+\tilde{k}_{t-1})$.

$\bar{i} \cdot \tilde{i}_t = \bar{p}_k \cdot \bar{k} \cdot \delta\, \tilde{p}_{k,t} + \bar{p}_k \cdot \bar{k} \cdot \tilde{k}_t - \bar{p}_k \bar{k}(1-\delta)\tilde{k}_{t-1}$

6). $\bar{r} e^{\tilde{r}_t} = \alpha \cdot e^{z_t} (\bar{k}_n)^{\alpha-1} (\bar{h}_n)^{1-\alpha} \cdot e^{(\alpha-1)\cdot \tilde{k}_{n,t} + (1-\alpha)\tilde{h}_{n,t}}$

$\bar{r}(1+\tilde{r}_t) = \alpha(\bar{k}_n)^{\alpha-1}(\bar{h}_n)^{1-\alpha}[1+\tilde{k}_{n,t}(\alpha-1)+\tilde{h}_{n,t}(1-\alpha)+z_t]$

$\bar{r}\tilde{r}_t = \alpha\left(\frac{\bar{h}_n}{\bar{k}_n}\right)^{1-\alpha}[\tilde{k}_{n,t}(\alpha-1)+\tilde{h}_{n,t}(1-\alpha)+z_t]$

$0 = \tilde{k}_{n,t}(\alpha-1)+\hat{h}_{n,t}(1-\alpha)+z_t-\bar{r}\tilde{r}_t$

7). $0 = \tilde{k}_{n,t}\alpha - \alpha \cdot \hat{h}_{n,t} - \hat{w}_t + z_t \Rightarrow$

8) $\bar{r} e^{\tilde{r}_t} = \alpha \bar{p}_d \cdot e^{\tilde{p}_{d,t}+z_t+(\alpha-1)\tilde{k}_{d,t}+(1-\alpha)\hat{h}_{d,t}} \cdot \bar{k}_d^{\alpha-1}\bar{h}_d^{1-\alpha}$

$\bar{r}(1+\tilde{r}_t) = \alpha \bar{p}_d \cdot \bar{k}_d^{\alpha-1}\bar{h}_d^{1-\alpha} \cdot (1+\tilde{p}_{d,t}+z_t+(\alpha-1)\tilde{k}_{d,t}+(1-\alpha)\tilde{h}_{d,t})$.

$\cancel{\bar{r}}\ \tilde{r}_t = \cancel{\alpha \bar{p}_d \bar{k}_d^{\alpha-1}\bar{h}_d^{1-\alpha}} \cdot (\tilde{p}_{d,t}+z_t+(\alpha-1)\tilde{k}_{d,t}+(1-\alpha)\tilde{h}_{d,t})$

$0 = \tilde{p}_{d,t}+z_t+(\alpha-1)\hat{k}_{d,t}+(1-\alpha)\tilde{h}_{d,t}-\tilde{r}_t$

9). $0 = \tilde{p}_{d,t}+z_t+\alpha\tilde{k}_{d,t}-\alpha\hat{h}_{d,t}-\hat{w}_t$

(10). $0 = \tilde{p}_{k,t}+z_t+(\alpha-1)\tilde{k}_{k,t}+(1-\alpha)\hat{h}_{k,t}-\tilde{r}_t$,

(11). $0 = \tilde{p}_{k,t}+z_t+\alpha\tilde{k}_{k,t}-\alpha\hat{h}_{k,t}-\hat{w}_t$.

(16) $\bar{c}_n e^{\tilde{c}_{n,t}} = \bar{k}_n^{\alpha}\bar{h}_n^{1-\alpha} e^{z_t+\tilde{k}_{n,t}\cdot\alpha+}$

$\because \bar{c}_n = \bar{k}_n^{\alpha}\bar{h}_n^{1-\alpha}$

$\cancel{1}+\tilde{c}_{n,t} = \cancel{1}+z_t+\tilde{k}_{n,t}\cdot\alpha+(1-\alpha)\tilde{h}$

(12). $\bar{h}_n\tilde{h}_{n,t}+\bar{h}_d\cdot\tilde{h}_{d,t}-\bar{h}_k\cdot\tilde{h}_{k,t}=0$.

(13) $\bar{k}_n\tilde{k}_{n,t}+\bar{k}_d\tilde{k}_{d,t}+\tilde{k}_{k,t}\cdot\bar{k}_k-\bar{k}\tilde{k}_{t-1}=0$.

(14). $\bar{k}[1+\tilde{k}_t-(1-\delta)(1+\tilde{k}_{t-1})] = \bar{k}_k^{\alpha}\bar{h}_k^{1-\alpha}.[1+z_t+\alpha\tilde{k}_{k,t}+(1-\alpha)\tilde{h}_{k,t}]$

$\because \frac{\bar{k}_k^{\alpha}\bar{h}_k^{1-\alpha}}{\bar{k}}=\delta \Rightarrow \cancel{\delta}+\tilde{k}_t-(1-\delta)\tilde{k}_{t-1} = \cancel{\delta}+z_t+\alpha\tilde{k}_{k,t}+(1-\alpha)\tilde{h}_{k,t}$

(15) $\tilde{d}_t-(1-\delta)\tilde{d}_{t-1} = z_t+\alpha\tilde{k}_{d,t}+(1-\alpha)\tilde{h}_{d,t}$

(16.) $\tilde{c}_{n,t} = z_t + \tilde{k}_{n,t} \cdot \alpha + (1-\alpha)\, \tilde{h}_{n,t}$

(18) $c_{d,t} = [\, d_t - d_{t-1}(1-\delta)\,] \times P_{d,t} \quad \Leftarrow \quad \bar{c}_d = \delta \bar{d} \cdot \bar{P}_d$

$\bar{c}_d \cdot e^{\tilde{c}_{d,t}} = \bar{d}\,[\, e^{\tilde{d}_t} - e^{\tilde{d}_{t-1}}(1-\delta)\,] \times \bar{P}_d \cdot e^{\tilde{P}_{d,t}} \qquad \frac{\bar{c}_d}{\bar{P}_d \cdot \bar{d}} = \delta$

$\delta \cdot (1+\tilde{c}_{d,t}) = (1 + \tilde{d}_t + \tilde{P}_{d,t}) - (1-\delta)(1 + \tilde{d}_{t-1} + \tilde{P}_{d,t}).$

$\cancel{\delta} + \delta \tilde{c}_{d,t} = \cancel{\delta} + \tilde{d}_t + \delta \tilde{P}_{d,t} - (1-\delta)\, \tilde{d}_{t-1}$

$\tilde{d}_t - (1-\delta)\tilde{d}_{t-1} + \delta \tilde{P}_{d,t} - \delta \tilde{c}_{d,t} = 0$

(19) $\tilde{i} = [\, k_t - k_{t-1}(1-\delta)\,] \times P_{k,t}$

$\tilde{k}_t - (1-\delta)\, \tilde{k}_{t-1} + \delta \tilde{P}_{k,t} - \delta \tilde{i} = 0$

(20). $c_{n,t} + P_{d,t}\,[\, d_t - d_{t-1}(1-\delta)\,] + P_{k,t}\,[\, k_t - k_{t-1}(1-\delta)\,] = r_t\, k_{t-1} + w_t$

$\bar{c}_n + \bar{P}_d \cdot \delta \bar{d} + \bar{P}_k \cdot \delta \bar{k} = \bar{r} \bar{k} + \bar{w}$

$\bar{c}_n e^{\tilde{c}_{n,t}} + \bar{P}_d \cdot \bar{d} \cdot e^{\tilde{P}_{d,t}} [\, e^{\tilde{d}_t} - e^{\tilde{d}_{t-1}}(1-\delta)\,] + \bar{P}_k \cdot \bar{k}\,[\, e^{\tilde{k}_t} - e^{\tilde{k}_{t-1}}(1-\delta)\,] \cdot e^{\tilde{P}_k}$

$= \bar{r}\bar{k} \cdot e^{\tilde{r}_t + \tilde{k}_{t-1}} + \bar{w}\, e^{\tilde{w}_t}$

$\bar{c}_n(1+\tilde{c}_{n,t}) + \bar{P}_d \cdot \bar{d}\,(1+\tilde{P}_{d,t} + \tilde{d}_t) - \bar{P}_d \cdot \bar{d}\,(1-\delta)(1+\tilde{P}_{d,t} + \tilde{d}_{t-1})$

$+ \bar{P}_k \cdot \bar{k}\,(1 + \tilde{P}_{k,t} + \tilde{k}_t) - \bar{P}_k \cdot \bar{k} \cdot (1-\delta)\,(1 + \tilde{P}_{k,t} + \tilde{k}_{t-1}) = \bar{r}\bar{k}(\cancel{1}+\tilde{r}_t + \tilde{k}_{t-1}) + \bar{w}(1+\tilde{w}_t$

$\bar{c}_n \tilde{c}_{n,t} + \bar{P}_d \cdot \bar{d} \cdot \delta \tilde{P}_{d,t} + \bar{P}_d \cdot \bar{d} \cdot \tilde{d}_t - \bar{P}_d \cdot \bar{d}\,(1-\delta)\, \tilde{d}_{t-1}$

$+ \bar{P}_k \cdot \bar{k} \cdot \delta\, \tilde{P}_{k,t} + \bar{P}_k \cdot \bar{k}\, \tilde{k}_t - [\bar{P}_k \cdot \bar{k}\,(1-\delta) + \bar{r}\bar{k}] \cdot \tilde{k}_{t-1} - \bar{r}\bar{k} \cdot \tilde{r}_t - \bar{w}\, \tilde{w}_t = 0$

model correation matrx	cn	cd	i	y	
cn	1.00	-0.22	0.23	1.00	
cd	-0.22	1.00	-1.00	-0.20	
i	0.23	-1.00	1.00	0.22	
y	1.00	-0.20	0.22	1.00	
real data correlation	cn	cd	i	y	
cn	1.00	0.76	0.60	0.83	
cd	0.76	1.00	0.47	0.55	
i	0.60	0.47	1.00	0.64	
y	0.83	0.55	0.64	1.00	
	cn_y	cd_y	i_y	y	cd_cn
model relative volatility	0.53	2.93	4.58	1.00	5.53
real relative volatility	1.40	5.01	2.96	1.00	3.57
pct captured	2.65	1.71	0.65		0.65