Reference systems

Outline

- Concept of reference system
- The transformation in 1D and 2D: translations and rotations
- The transformation in 3D
- Definition of 3D reference systems
- The linearized transformation

The aim of positioning

To estimate positions of points by the available observations.

Needed constraints on the intrinsic degrees of freedom.

A reference system

Increasing complexity:

Case 1D, case 2D, case 3D.

Case 1D

To compute the position of P...

Case 1D

To compute the position of P (x_P) the unit of lengths and the origin are needed.

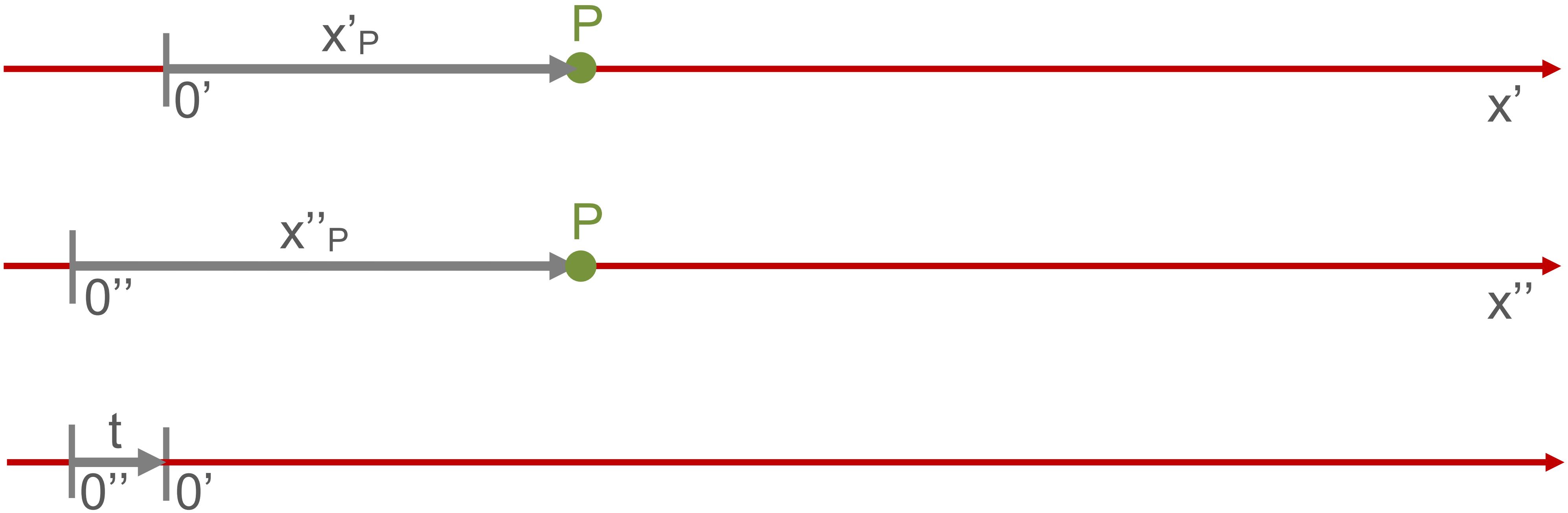
Case 1D

A different origin...

Case 1D

A different origin implies a different x''P

Case 1D



$$t = x''_P - x'_P = O' - O''$$

The lesson of the 1D case

Given a RS,

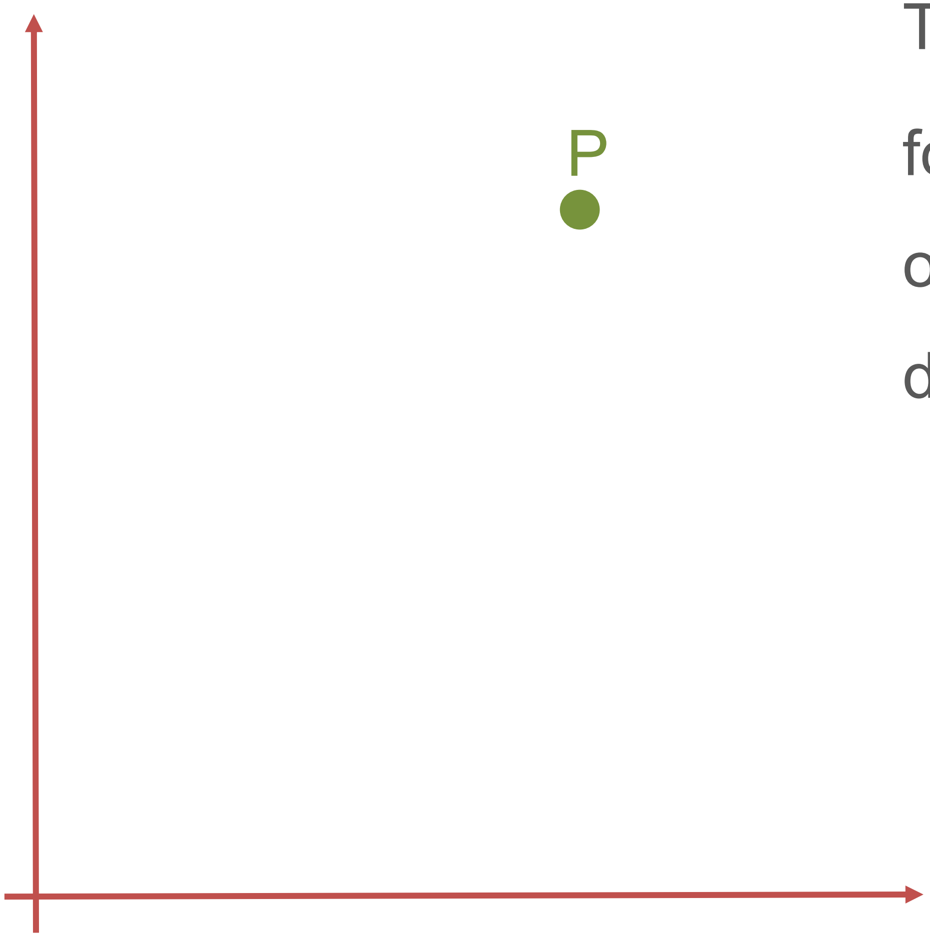
the position of a point could be its distance (oriented) from the origin.

The height RS

The origin height has to be imposed:

for example the mean sea level.

Case 2D: the planimetric problem



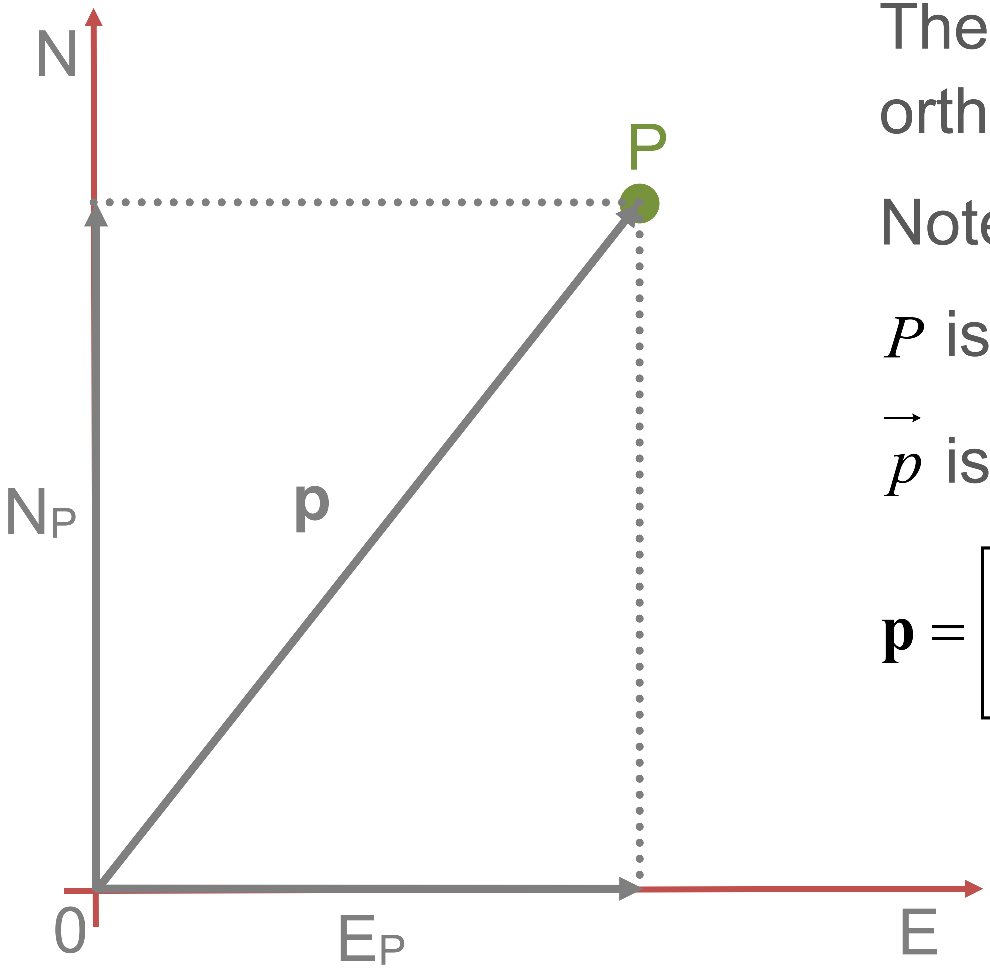
Two orthogonal axes,

for example $\vec{E}$ (East) e $\vec{N}$ (North),

one unit of lengths,

define the Reference System.

Case 2D: coordinates



The coordinates of a point are its orthogonal projections on the axes.

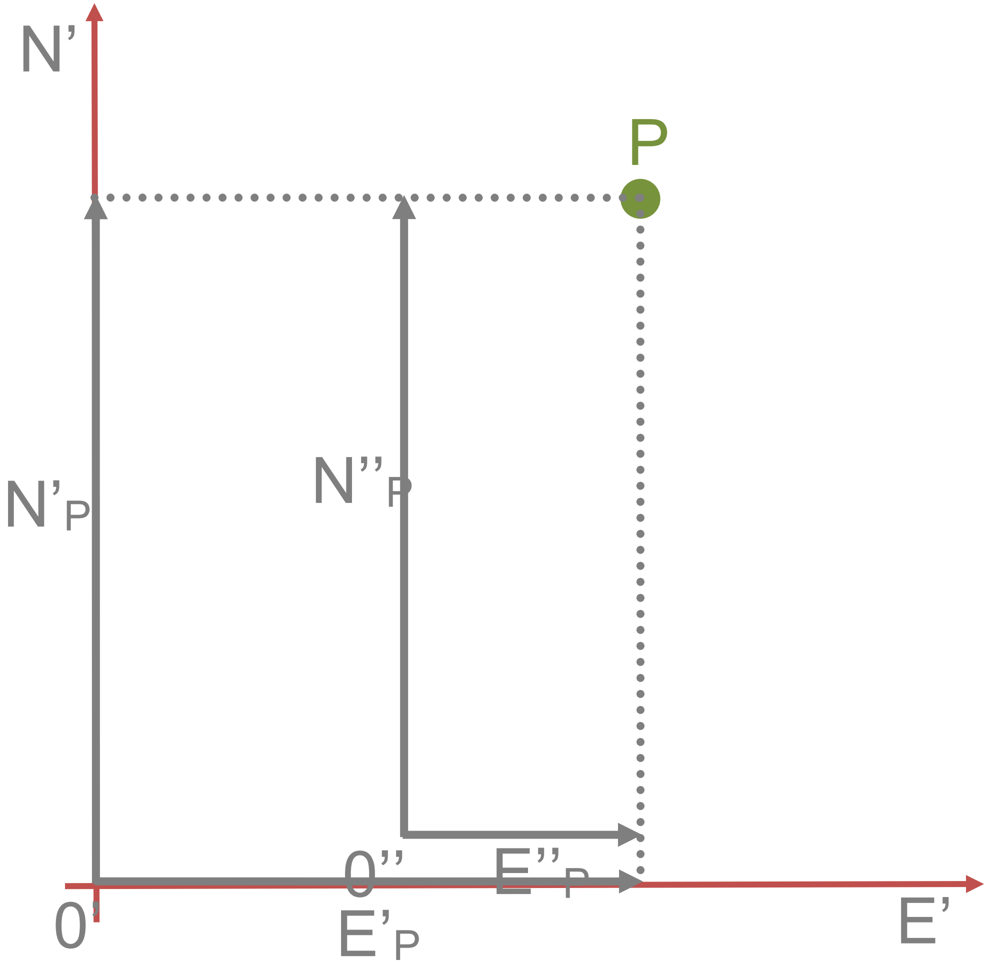
Note

P is the point,

$\vec{p}$ is the vector between the origin and P,

$\mathbf{p} = \begin{bmatrix} E_P \\ N_P \end{bmatrix}$ is P position wrt the axes.

Case 2D: the effect of an origin change



$\overrightarrow{E}', \overrightarrow{N}'$ e $\overrightarrow{E}'', \overrightarrow{N}''$

are 2 RS's

with parallel axes

but different origins.

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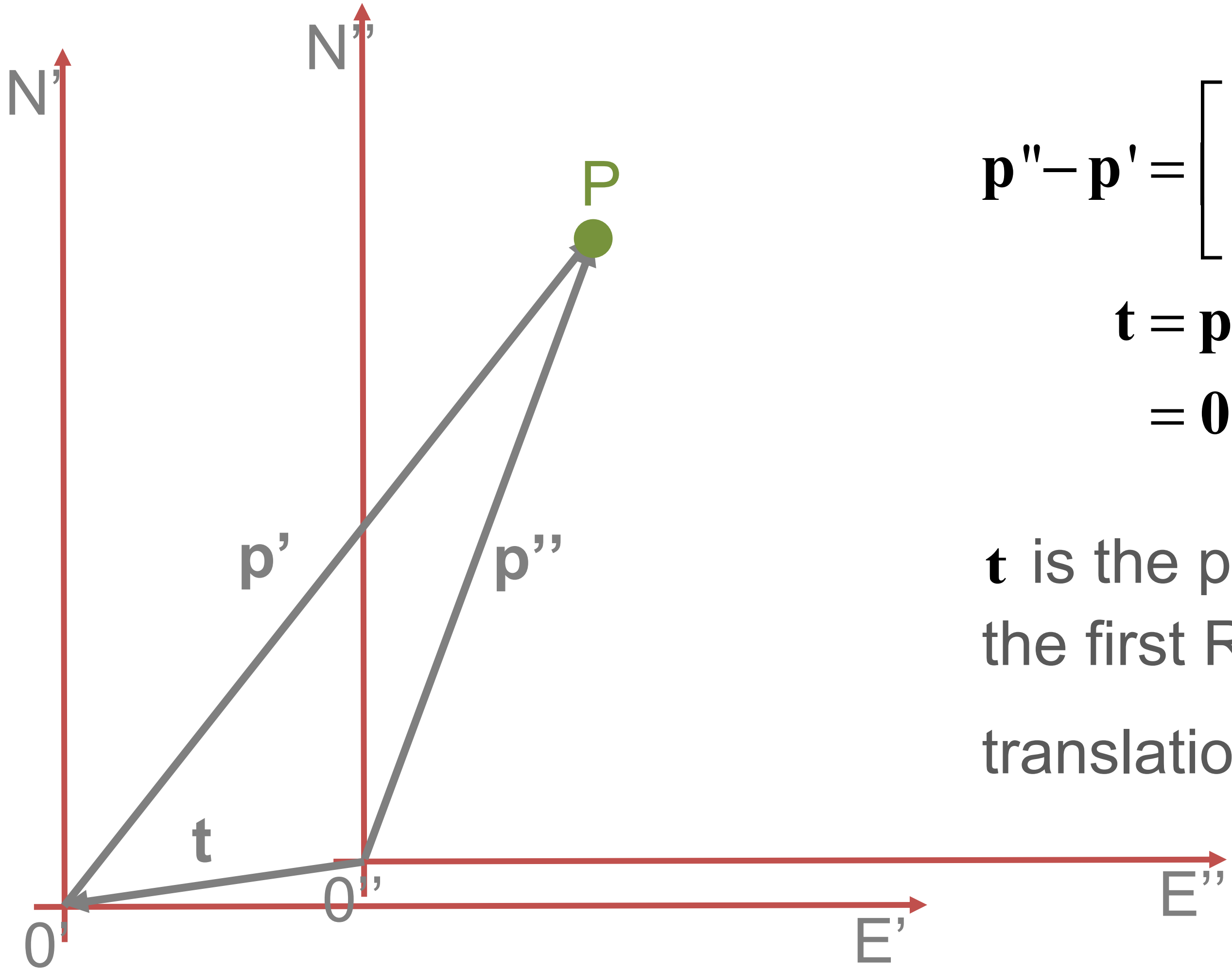
Case 2D: the effect of an origin change



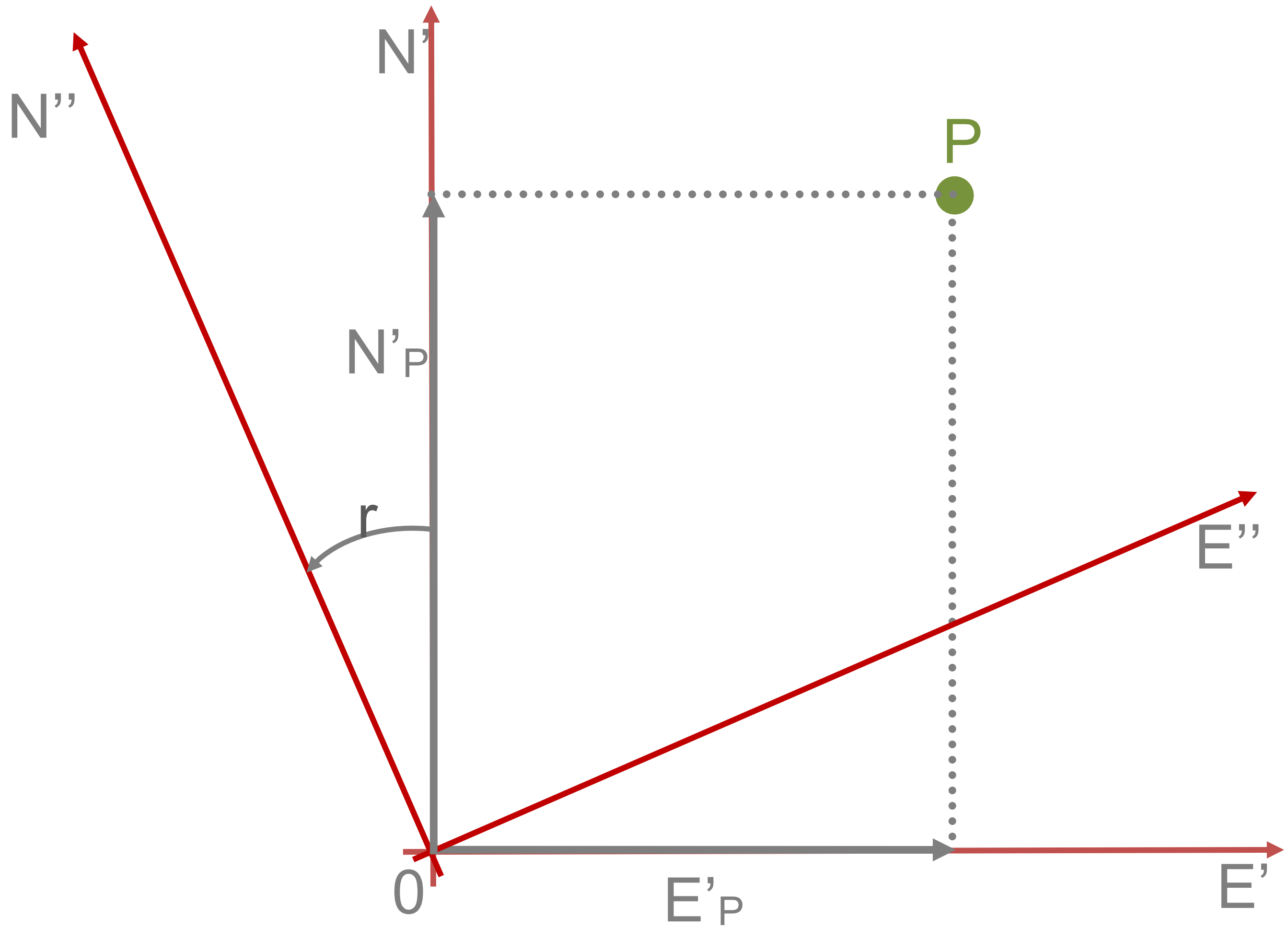
$$\mathbf{p}'' - \mathbf{p}' = \begin{bmatrix} E'' \\ N'' \end{bmatrix} - \begin{bmatrix} E' \\ N' \end{bmatrix} = \mathbf{t}$$

$$\begin{aligned} \mathbf{t} &= \mathbf{p}'' - \mathbf{p}' \\ &= \mathbf{0}' - \mathbf{0}'' \end{aligned}$$

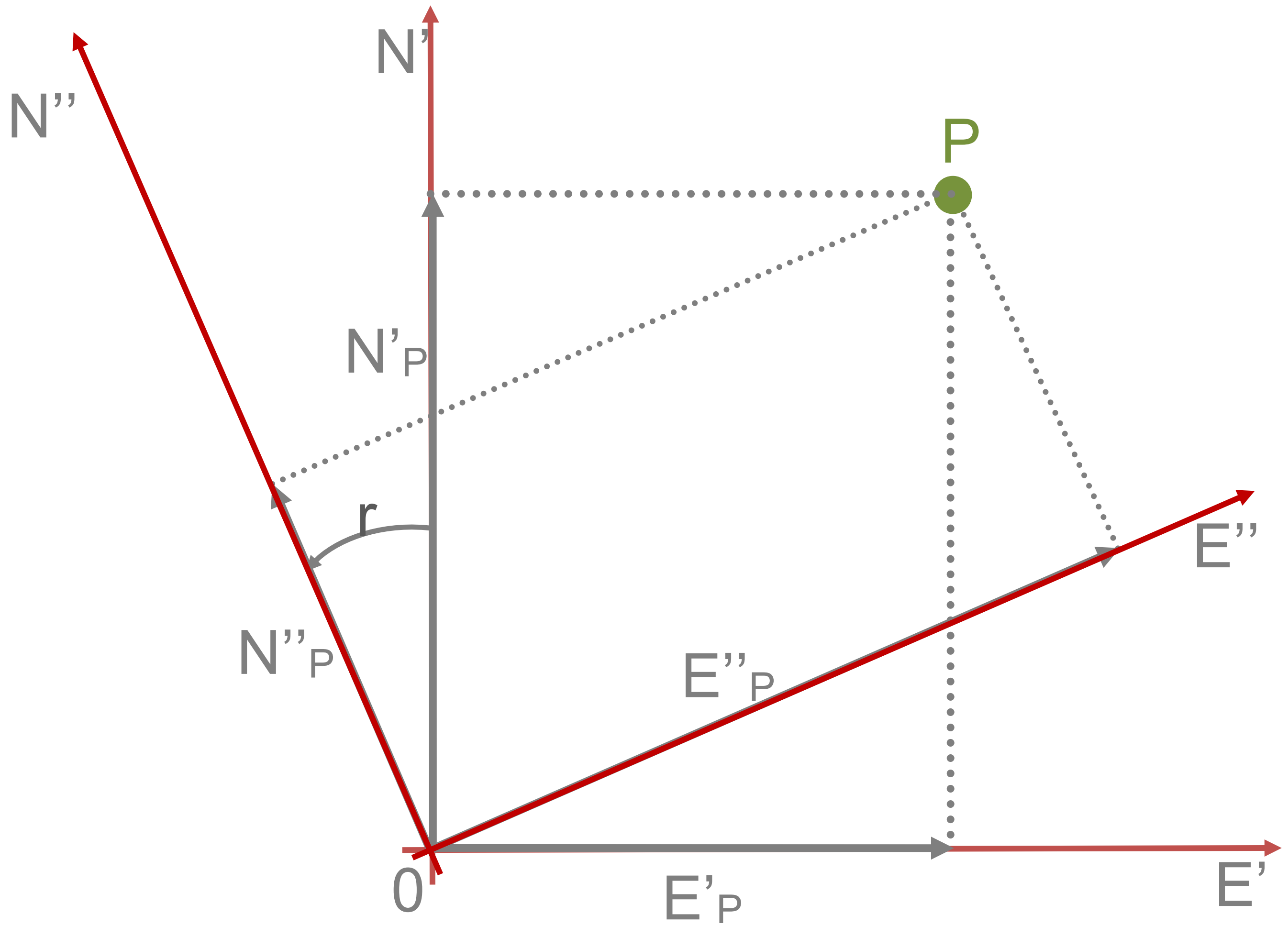
$\mathbf{t}$ is the position of the origin of the first RS wrt the second:

translation

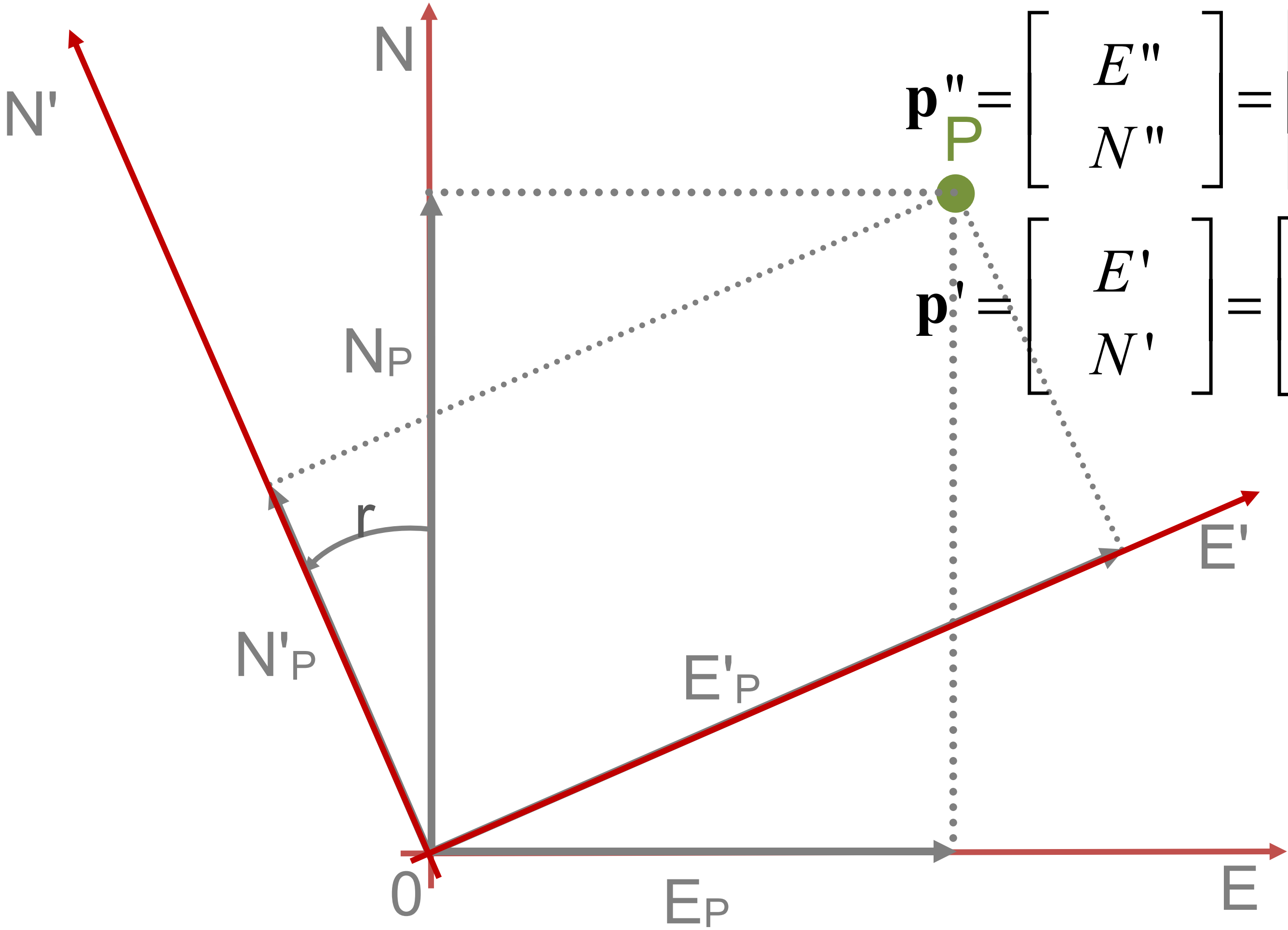
Case 2D: effect of a rotation of the axes



Case 2D: effect of a rotation of the axes



Case 2D: effect of a rotation of the axes



$$\mathbf{p}'' = \begin{bmatrix} E'' \\ N'' \end{bmatrix} = \begin{bmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{bmatrix} \begin{bmatrix} E' \\ N' \end{bmatrix}$$

$$\mathbf{p}' = \begin{bmatrix} E' \\ N' \end{bmatrix} = \begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix} \mathbf{p}''$$

$$\mathbf{p}'' = \mathbf{R}\mathbf{p}', \mathbf{p}' = \mathbf{R}^T \mathbf{p}'' \Rightarrow \mathbf{R}^T \mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$$

Case 2D

combination of translation and rotation

$$\mathbf{p}'' = \mathbf{t} + \mathbf{R}\mathbf{p}', \ \mathbf{p}' = \mathbf{R}^T(\mathbf{p}'' - \mathbf{t}) = \mathbf{R}^T\mathbf{p}'' - \mathbf{R}^T\mathbf{t}$$

Moreover a scale factor should be introduced,

i.e. a ratio between the units of lengths of the 2 RS's.

$$\mathbf{p}'' = \mathbf{t} + \lambda\mathbf{R}\mathbf{p}', \ \mathbf{p}' = \lambda^{-1}\mathbf{R}^T(\mathbf{p}'' - \mathbf{t}) = \lambda^{-1}\mathbf{R}^T\mathbf{p}'' + \tau$$

The lesson of the 2D case

Given the unit of lengths,

one origin (point) and

one geographical direction (North) are imposed.

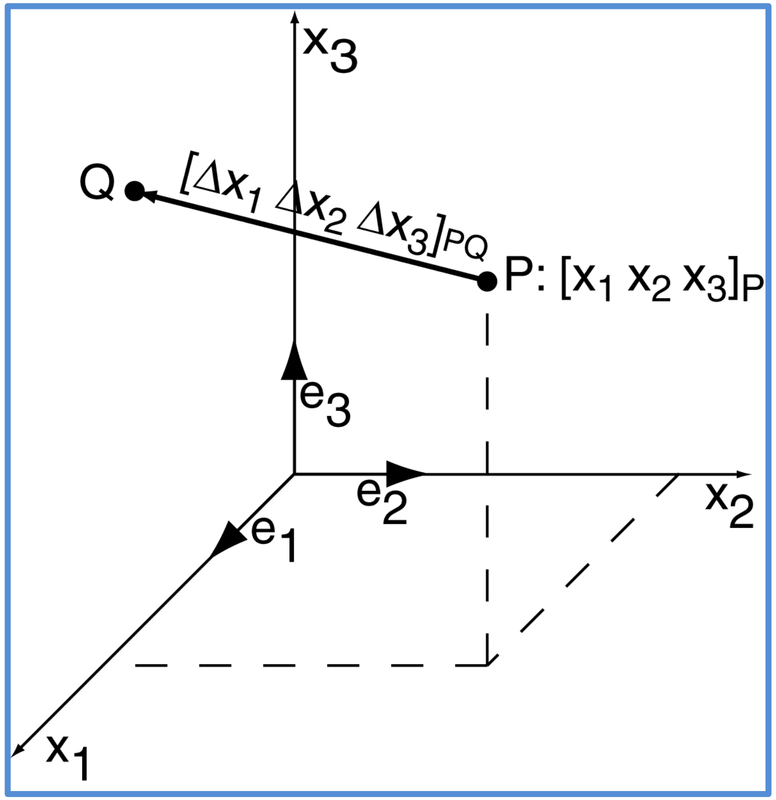
One alternative approach

One point is materialized, it is the origin,

another point defines the X axis.

By properly combined observations

the positions of other points can be estimated.



3D Reference Systems

Three axes $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$ with common origin, reciprocally orthogonal, and the unit of lengths (unitary vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$).

The coordinates of a point are the lengths of its orthogonal projections on the three axes. The vector between two points is the oriented difference between their coordinates.

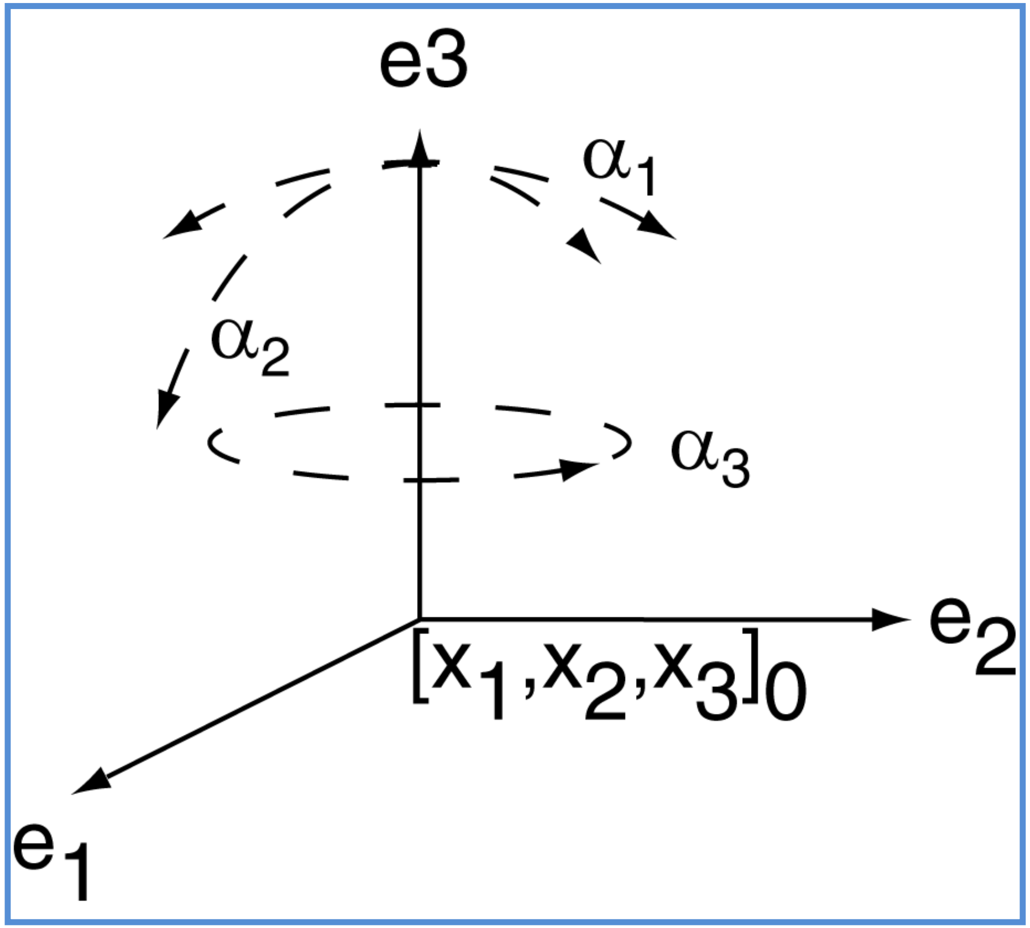
Position of a point: cartesian coordinates

$$\mathbf{x}_P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_P, \quad \|\mathbf{x}\| = \sqrt{x_{1_P}^2 + x_{2_P}^2 + x_{3_P}^2}$$

Vector between two points: cartesian components

$$\Delta\mathbf{x}_{PQ} = \begin{bmatrix} x_{1_Q} - x_{1_P} \\ x_{2_Q} - x_{2_P} \\ x_{3_Q} - x_{3_P} \end{bmatrix}, \Delta\mathbf{x}_{PQ} = -\Delta\mathbf{x}_{QP},$$

$$\|\Delta\mathbf{x}_{PQ}\| = \sqrt{(x_{1_Q} - x_{1_P})^2 + (x_{2_Q} - x_{2_P})^2 + (x_{3_Q} - x_{3_P})^2}$$



The degrees of freedom of a 3D RS

Given the unit of lengths:

one translation of the origin:

3 DOF's.

Two orthogonal direction angles for one axis: α_1, α_2 for $\vec{x}_3$,

one rotation angle around the same axis: α_3.

The transformation between RS's

Two RS's: I and II,

with a different origin, different axes orientation and scale factor λ

$\mathbf{t} = [t_1 \quad t_2 \quad t_3]^T$: coordinates of I origin wrt II,

$\mathbf{R}$: rotation to bring I axes parallel to II axes.

Given a point P, whose coordinates in I are $\mathbf{x}_{P_I} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{P,I}$.

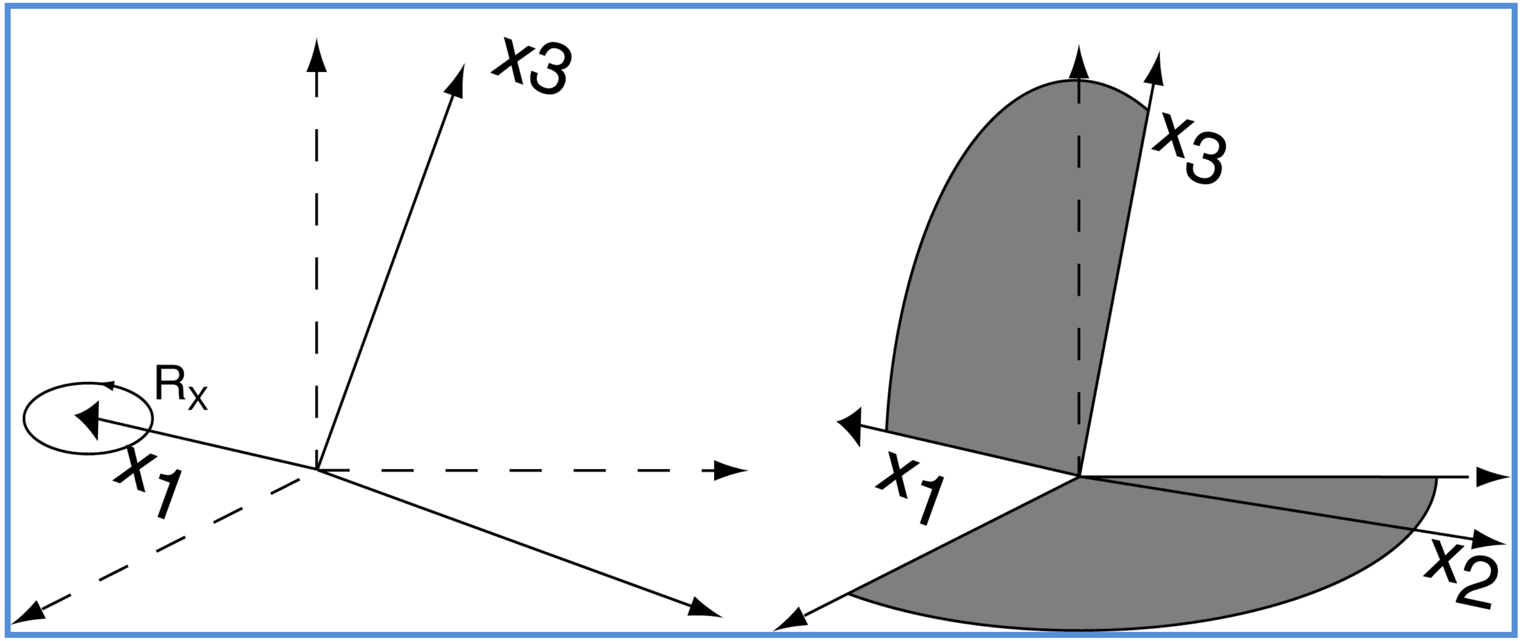
Its coordinates in II are given by the

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{P,II} = \mathbf{x}_{P_{II}} = \mathbf{t} + \lambda \mathbf{R} \mathbf{x}_{P_I}$$

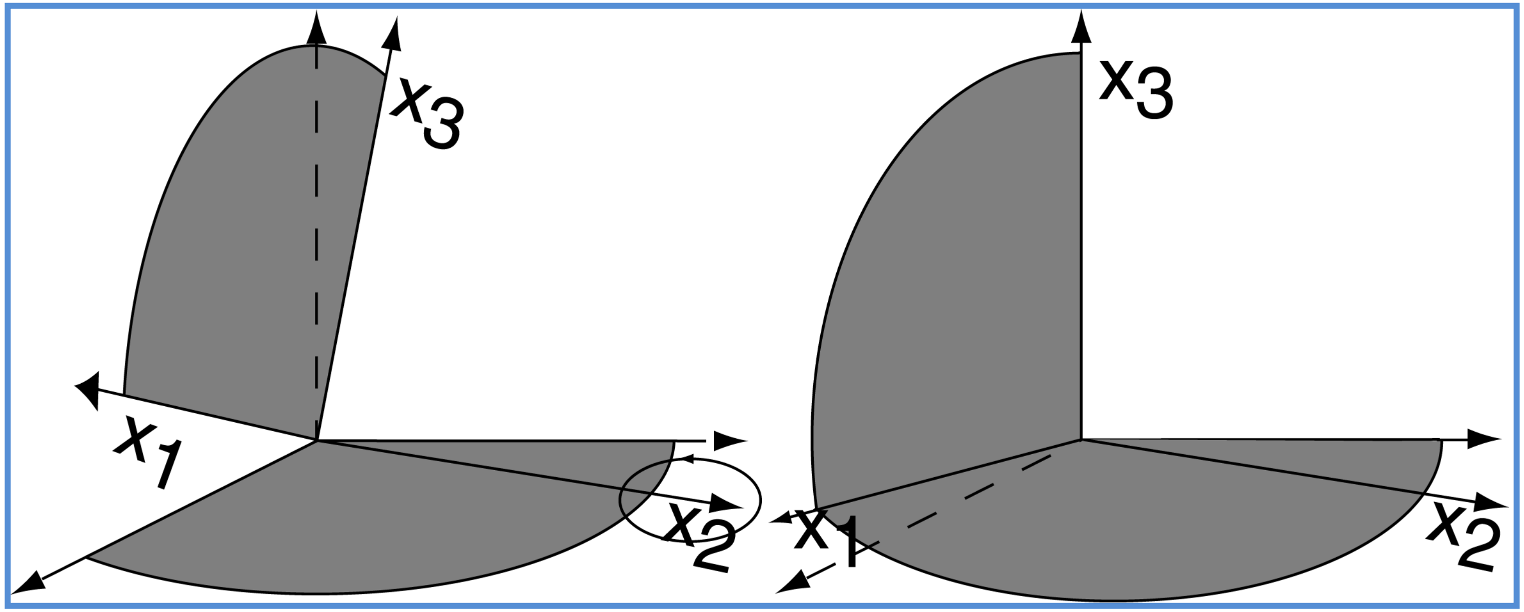
3D rotation: a matrix $[3 \times 3]$, with only 3 independent angles!

In geodetic framework, $\mathbf{R}$ is given by the composition of three planar rotations around the 3 axes (Euler angles and rotations).

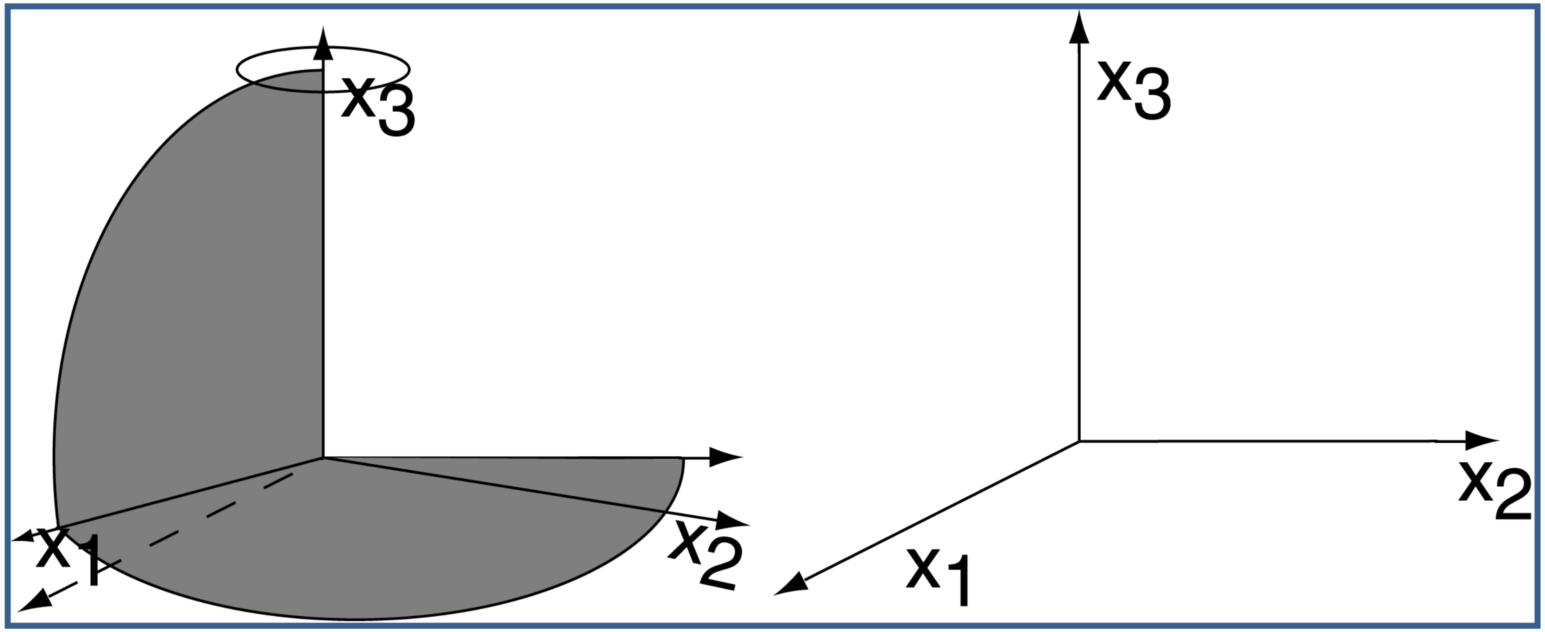
$$\mathbf{R}_1(r_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos r_1 & \sin r_1 \\ 0 & -\sin r_1 & \cos r_1 \end{bmatrix}$$



$$\mathbf{R}_2(r_2) = \begin{bmatrix} \cos r_2 & 0 & -\sin r_2 \\ 0 & 1 & 0 \\ \sin r_2 & 0 & \cos r_2 \end{bmatrix}$$



$$
\mathbf{R}_3(r_3) = \begin{bmatrix} \cos r_3 & \sin r_3 & 0 \\ -\sin r_3 & \cos r_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$



The order of the three rotations is important:

we adopt the sequence

$\mathbf{R}_1 \Rightarrow \mathbf{R}_2 \Rightarrow \mathbf{R}_3$, i.e. $\mathbf{R}(r_1, r_2, r_3) = \mathbf{R}_3(r_3)\mathbf{R}_2(r_2)\mathbf{R}_1(r_1)$,

$\mathbf{R}$ depends on r_1, r_2, r_3:

the whole transformation depends on 7 parameters.

Helmert or similarity transformation.

The final rotation matrix

$$\mathbf{R} = \begin{bmatrix} \mathrm{c}r_2\,\mathrm{c}r_3 & \mathrm{c}r_2\,\mathrm{s}r_3 + \mathrm{s}r_1\,\mathrm{s}r_2\,\mathrm{c}r_3 & \mathrm{s}r_1\,\mathrm{s}r_3 - \mathrm{c}r_1\,\mathrm{s}r_2\,\mathrm{c}r_3 \\ -\mathrm{c}r_2\,\mathrm{s}r_3 & \mathrm{c}r_1\,\mathrm{c}r_3 - \mathrm{s}r_1\,\mathrm{s}r_2\,\mathrm{s}r_3 & \mathrm{s}r_1\,\mathrm{c}r_3 + \mathrm{c}r_1\,\mathrm{s}r_2\,\mathrm{s}r_3 \\ \mathrm{s}r_2 & -\mathrm{s}r_1\,\mathrm{c}r_2 & \mathrm{c}r_1\,\mathrm{c}r_2 \end{bmatrix}$$

$$cr_j = \cos(r_j), \quad sr_j = \sin(r_j)$$

The definition of a 3D Reference System

Given the unit of lengths,

the translation and 3 direction angles must be assigned.

The global reference system

International Terrestrial Reference System:

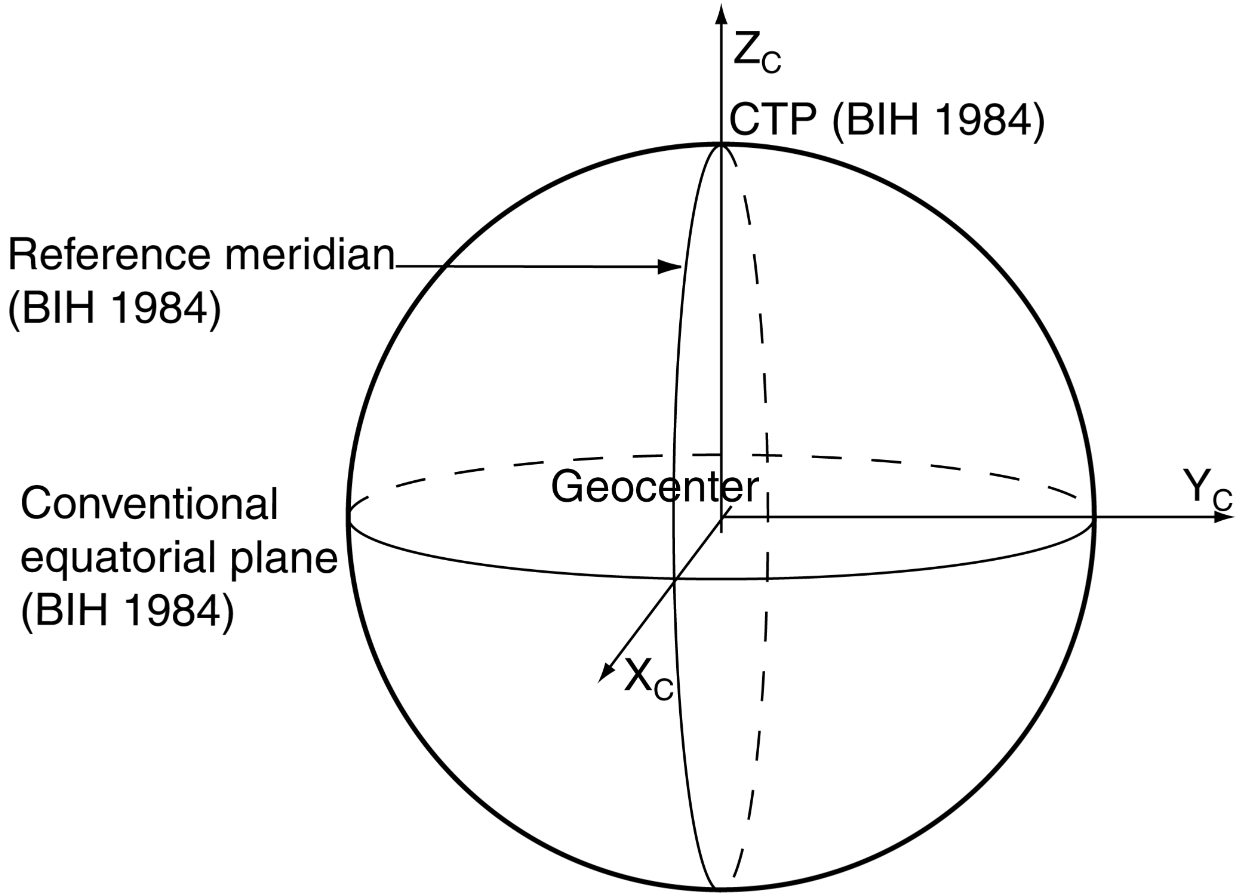
Standard Unit of lengths,

origin in the mass center of the Earth,

$\overrightarrow{x_3}(Z)$ axis toward the conventional Earth rotation axis,

$\overrightarrow{x_1}(X)$ axis orthogonal to Z (conventional equator),

toward fundamental (Greenwich) meridian.


Z_C
CTP (BIH 1984)
Reference meridian
(BIH 1984)
Geocenter
Y_C
Conventional
equatorial plane
(BIH 1984)
X_C

Example of a local reference system for indoor positioning:

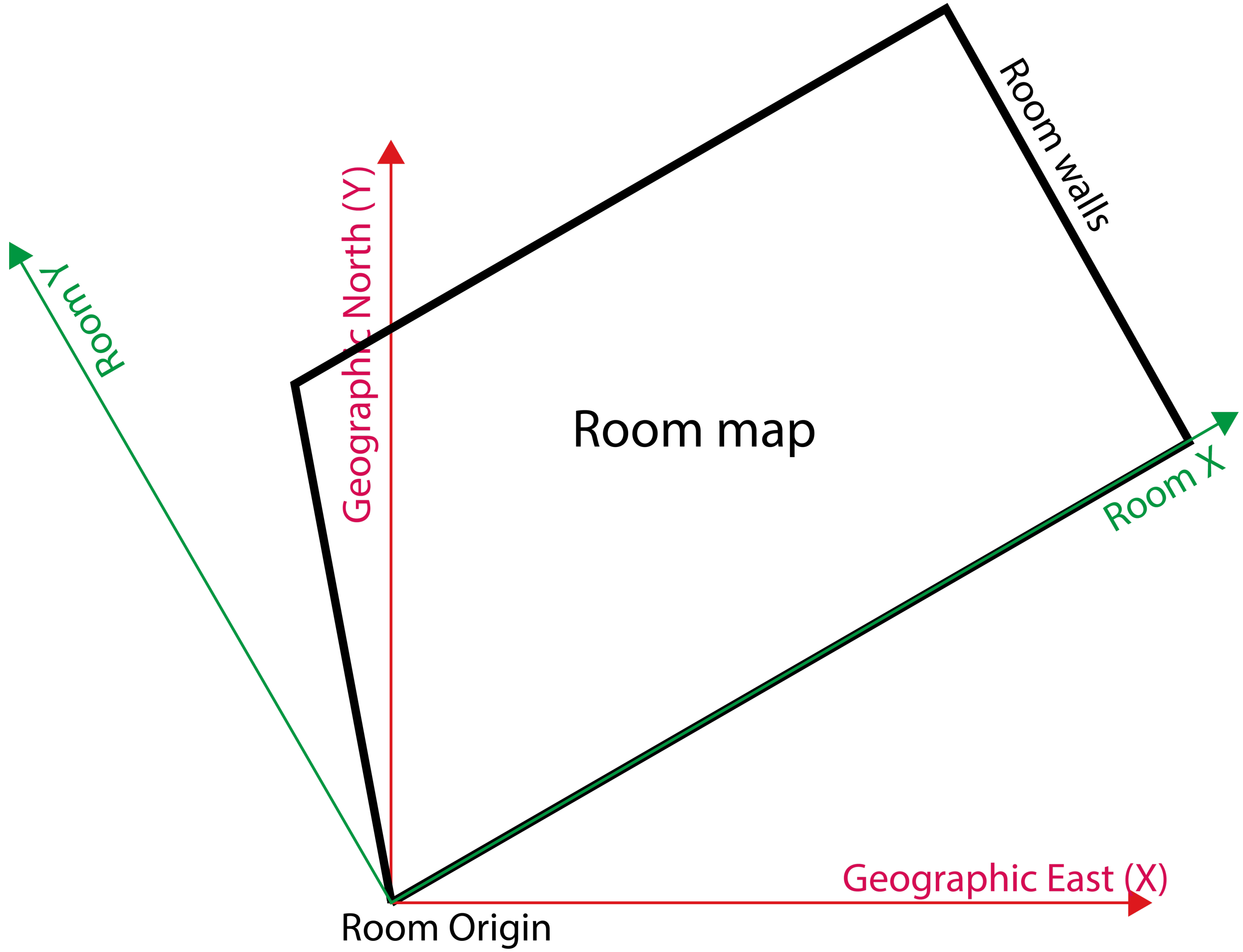
room in a building

Z axis toward vertical direction

X and Y in the orthogonal plane to Z

X and Y reciprocally orthogonal

At least two possible definitions of them (see notes in 2D RS)



Linearized transformation between RS's

In the case of small rotations (≅ 1'') and scale factor near to unity, the similarity transformation can be linearized.

$$\cos r_i \cong 1,\ \sin r_i \cong r_i,\ r_i r_j \cong 0,\ \lambda = 1 + \mu,\ \mu r_j \cong 0$$

$$\mathbf{R} \cong \begin{bmatrix} 1 & r_3 & -r_2 \\ -r_3 & 1 & r_1 \\ r_2 & -r_1 & 1 \end{bmatrix} = \mathbf{I} + \delta\mathbf{R},\ \delta\mathbf{R} = \begin{bmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{bmatrix} = -[\mathbf{r}\times]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{SR\ II} \cong \mathbf{t} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{SR\ I} + \delta\mathbf{R} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{SR\ I} + \mu \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{SR\ I}$$

Counter clockwise rotations!