MATH 262: Homework #1

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1. *Determine which of the following statements are true for all sets A, B, C, and D. If double implication fails determine whether one or the other of the possible implication holds. If an equality fails determine whether the statement becomes true if the "equals" symbol is replace by one or the other of the inclusion symbols.*

   (a) $A \subset B$ *and* $A \subset C \Leftrightarrow A \subset B \cup C$
   The "only if" holds since $A \subset B \subset B \cup C$, but the converse does not, e.g., if B and C are disjoint, nonempty sets and A is a nonempty set contained entirely in B.

   (b) $A \subset B$ *or* $A \subset C \Leftrightarrow A \subset B \cup C$
   If $A \subset B$ then $A \subset B \subset B \cup C$, and if $A \subset C$ then $A \subset C \subset B \cup C$. The converse does not hold since, for example, B and C could be nonempty, disjoint sets and A could contain elements of both sets.

   (c) $A \subset B$ *and* $A \subset C \Leftrightarrow A \subset B \cap C$
   By hypothesis $a \in A \Rightarrow a \in B$ and $a \in C$, which is true is and only if $a \in B \cap C$.

   (d) $A \subset B$ *or* $A \subset C \Leftrightarrow A \subset B \cap C$
   From the previous part it follows that the "if" portion of the statement is true since $A \subset B$ and $A \subset C$ implies $A \subset B$ or $A \subset C$. The converse, however, is not true: let A, B, and C be nonempty sets with $B \cap C = \emptyset$ and $A \subset B$.

   (e) $A \setminus (A \setminus B) = B$
   Any sort of inclusion fails if A and B are disjoint, nonempty sets. Equality holds if $B \subset A$, and left inclusion holds if $B \cap A \neq \emptyset$ but one is not included in the other.

   (f) $A \setminus (B \setminus A) = A \setminus B$
   Let A and B be two sets in some universe X.

$$\begin{aligned} x \in A \setminus (B \setminus A) &\Leftrightarrow x \in A \cap (X \setminus (B \setminus A)) \\ &\Leftrightarrow x \in A \cap ((X \setminus B) \cup A) \\ &\Leftrightarrow x \in (A \setminus B) \cup A \end{aligned}$$

   Hence the right-hand side is included in the left, but not the opposite. For example, if $A \cap B \neq \emptyset$, then $x \in A \cap B$ is in the left-hand side but not the right-hand side.

   (g) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
   x is in the left-hand set iff $x \in A$ and $x \in B$ and $(x \notin A$ or $x \notin C)$. But $x \in A$, so this becomes $x \in A$ and $x \in B$ and $x \notin C$, i.e., $x \in A \cap (B \setminus C)$. So they are equal.

(h) $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$

Left inclusion does not hold, for example, if B and C are the empty set and A is nonempty.

(i) $(A \cap B) \cup (A \setminus B) = A$

This is true, since this is equivalent to the statement that $x \in A$ and $x \in B$, or $x \in A$ and $x \notin B$. Clearly this implies that $x \in A$. But if $x \in A$, then either $x \in B$ or $x \notin B$, and hence the statement is still true.

(j) $A \subset C$ *and* $B \subset D \Rightarrow A \times B \subset C \times D$

This is true since $A \times B \subset A \times D \subset C \times D$.

(k) *The converse of (j).*

If A is the empty set and B is nonempty, then this is not true since then $A \times B = \emptyset \subset C \times D$.

(l) *The converse of (j), assuming that A and B are nonempty.*

If they are nonempty then there exists $x \in (A \setminus C) \cup (B \setminus D)$ and $(x, y) \notin C \times D$ for any $y \in B$, hence the converse is true in this case.

(m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

Prof. Weinberger wins. I'm annoyed.

(n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(o) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

(p) $(A \setminus B) \times (C \setminus D) = (A \times C - B \times C) \setminus (A \times D)$

(q) $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$

2. *Let A and B set set of real numbers. Write the negation of each of the following statements:*

   (a) *For every $a \in A$, it is true that $a^2 \in B$.*

   $\exists a \in A$ such that $a^2 \notin B$

   (b) *For at least one $a \in A$, it is true that $a^2 \in B$.*

   $\forall a \in A, a^2 \notin B$

   (c) *For every $a \in A$, it is true that $a^2 \notin B$.*

   $\exists a \in A$ such that $a^2 \in B$

   (d) *For at least one $a \notin A$, it is true that $a^2 \in B$.*

   $\forall a \notin A, a^2 \notin B$

3. *Let $\mathcal{A}$ be a nomempty collection of sets. Determine the truth of each of the following statements and of their converses:*

   (a) $x \in \bigcup_{A \in \mathcal{A}} \Rightarrow x \in A$ *for at least one* $A \in \mathcal{A}$.

   Both this statement and its converse are true by the definition of an arbitrary union.

   (b) $x \in \bigcup_{A \in \mathcal{A}} \Rightarrow x \in A$ *for every* $A \in \mathcal{A}$.

   This statement is false since x need only be in one A, but the converse is true.

   (c) $x \in \bigcap_{A \in \mathcal{A}} \Rightarrow x \in A$ *for at least one* $A \in \mathcal{A}$.

   This statement is true since by definition x is in every $A \in \mathcal{A}$. The converse is false, however, since x must be in every $A \in \mathcal{A}$.

(d) $x \in \bigcap_{A \in \mathcal{A}} \Rightarrow x \in A$ *for every* $A \in \mathcal{A}$.

Both this statement and its converse are true by the definition of an arbitrary intersection.

4. *If a set A has two elements, show that $\mathcal{P}(A)$ has four elements. How many elements does the power set have if A has one element? Three elements? No elements? Why is $\mathcal{P}(A)$ called the power set of A?*

$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ has four elements. If A has one element then the power set has two elements, and if A has three elements then $\mathcal{P}(A)$ has eight elements. Presumably it is called the power set because if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

5. *Let $f : A \to B$, $A_0 \subset A$, and $B_0 \subset B$.*

(a) *Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.*

Recall that $f^{-1}(f(A_0)) = \{x \in A \mid f(x) \in f(A_0)\}$. Since, for every $x \in A_0$, $f(x) \in f(A_0)$ by definition, the inclusion is obvious. The only possible situation where the opposite inclusion wouldn't hold is if there is some $x \in A \setminus A_0$ that is also in $f^{-1}(f(A_0))$. But if f is injective then no such x can exist since there is precisely one $x \in A$ for which $f(x) = a$ for $a \in f(A_0)$, and there is at least one such x in A_0.

(b) *Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.*

Let $y \in f(f^{-1}(B_0))$. Then there is some $x \in f^{-1}(B_0)$ such that $f(x) = y$. But this implies $f(x) \in B_0$, so that $y = f(x) \in B_0$. Let $y \in B_0$ and assume f is surjective. Then there is some $x \in A$ such that $f(x) = y$. But then $x \in f^{-1}(y) \subset f^{-1}(B_0)$, so that $y = f(x) \in f(f^{-1}(B_0))$.

6. *Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$. Show the following:*

(a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$

Let $x \in f^{-1}(B_0)$ be arbitrary, and $B_0 \subset B_1$, then

$$\begin{aligned} x \in f^{-1}(B_0) &\Rightarrow f(x) \in B_0 \\ &\Rightarrow f(x) \in B_1 \\ &\Rightarrow x \in f^{-1}(B_1) \end{aligned}$$

(b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

If $x \in f^{-1}(B_0 \cup B_1)$ then $f(x) \in B_0 \cup B_1$, so that $f(x)$ belongs to at least one of B_0 or B_1. But then x belongs to at least one of the $f^{-1}(B_0)$ or $f^{-1}(B_1)$, i.e., $x \in f^{-1}(B_0) \cup f^{-1}(B_1)$. Conversely, if $x \in f^{-1}(B_0) \cup f^{-1}(B_1)$ then x belongs to at least one of $f^{-1}(B_0)$ or $f^{-1}(B_1)$ so that $f(x)$ belongs to at least one of B_0 or B_1. But then $f(x) \in B_0 \cup B_1$, and hence $x \in f^{-1}(B_0 \cup B_1)$.

(c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

This proof follows *mutatis mutandis* from above by replacing every instance of "or" with "and," but here it goes anyhow:

If $x \in f^{-1}(B_0 \cap B_1)$ then $f(x) \in B_0 \cap B_1$, so that $f(x)$ belongs to both B_0 and B_1. But then x belongs to both $f^{-1}(B_0)$ and $f^{-1}(B_1)$, i.e., $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$.

Conversely, if $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$ then x belongs to both $f^{-1}(B_0)$ and $f^{-1}(B_1)$ so that $f(x)$ belongs to both B_0 and B_1. But then $f(x) \in B_0 \cap B_1$, and hence $x \in f^{-1}(B_0 \cap B_1)$.

(d) $f^{-1}(B_0 \setminus B_1) = f^{-1}(B_0) \setminus f^{-1}(B_1)$

Let B_0 and B_1 be in some universe X. Then $x \in f^{-1}(B_0 \setminus B_1)$ iff $f(x) \in B_0 \setminus B_1$, but this is equivalent to $f(x) \in B_0$ and $f(x) \in X \setminus B_1$, i.e., $x \in f^{-1}(B_0)$ and $x \in X \setminus f^{-1}(B_1)$. This is identical to $x \in f^{-1}(B_0) \setminus f^{-1}(B_1)$.

(e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$

This one is obvious, but, since $A_0 \subset A_1$, we have

$$f(A_0) = \{f(x) \mid x \in A_0\} \subset \{f(x) \mid x \in A_1\} = f(A_1)$$

(f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$

If $y \in f(A_0 \cup A_1)$ then $y = f(x)$ where x is in at least one of A_0 or A_1. Therefore y is in at least one of $f(A_0)$ or $f(A_1)$. Conversely, if y is in $f(A_0)$ or $f(A_1)$ then $y = f(x)$ where x is in at least one of A_0 or A_1, and hence $y = f(x) \in f(A_0 \cup A_1)$.

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$, *and that equality holds if f is injective.*

If $y \in f(A_0 \cap A_1)$ then there exists an $x \in A_1 \cap A_0$ such that $f(x) = y$, but then $f(x) \in f(A_1)$ and $f(x) \in f(A_0)$, so that $y \in f(A_0) \cap f(A_1)$. We know that there is at least one $x \in A_0 \cap A_1$ such that $f(x) = y$ for any $y \in f(A_0) \cap f(A_1)$. If f is injective then that is the only one, so $f^{-1}(y) \in A_0 \cap A_1$ and $y \in f(A_0 \cap A_1)$.

(h) $f(A_0 \setminus A_1) \supset f(A_0) \setminus f(A_1)$, *and that equality holds if f is surjective.*

Let $y \in f(A_0) \setminus f(A_1)$, then there is some $x \in A_0 \setminus A_1$ such that $f(x) = y$ and therefore $f(x) \in f(A_0 \setminus A_1)$. If f is surjective then for every $y \in f(A_0 \setminus A_1)$ there is some $x \in A_0 \setminus A_1$ such that $f(x) = y$. But then $f(x) \in A_0 \setminus A_1$.

7. *Let $f : A \to B$ and $g : B \to C$.*

(a) *If $C_0 \subset C$ show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.*

$$\begin{aligned}
f^{-1}(g^{-1}(C_0)) &= \{x \in A \mid f(x) \in g^{-1}(C_0)\} \\
&= \{x \in A \mid f(x) \in \{y \in B \mid g(y) \in C_0\}\} \\
&= \{x \in A \mid g(f(x)) \in C_0\} \\
&= (g \circ f)^{-1}(C_0)
\end{aligned}$$

(b) *If f and g are injective, show that $g \circ f$ is injective.*

Let $(g \circ f)(x) = (g \circ f)(x')$, then the injectivity of g implies that $f(x) = f(x')$ and the injectivity of f implies that $x = x'$.

(c) *If $g \circ f$ is injective, what can you say about the injectivity of f and g?*

(d) *If f and g are surjective, show that $g \circ f$ is surjective.*

Let $z \in C$, then there exist $x \in A$ and $y \in B$ such that $f(x) = y$ and $g(y) = z$ by the surjectivity of f amd g. But then $g(f(x)) = z$, and so $g \circ f$ is surjective.

(e) *If $g \circ f$ is surjective, what can you say about the surjectivity of f and g.*

One of the functions must be surjective, since if neither were, f would map A to a proper subset of B and g would take that proper subset to yet another proper subset of C, and hence $g \circ f$ would not be surjective.

(f) *Summarize these results in a theorem.*

The composition of two bijections is itself a bijection.

8. *Prove that* $(x_0, y_0) \sim (x_1, y_1)$ *if and only if* $y_0 - x_0^2 = y_1 - x_1^2$ *is an equivalence relation on* $\mathbb{R}^2$*. Describe the equivalence classes.*

   That this is an equivalence relation follows directly from the fact that equality is an equivalence relation. The equivalence class of a point $(a, b) \in \mathbb{R}^2$ is the parabola given by $y = x^2 + a - b^2$.

9. *Let* $f : A \to B$ *be a surjective function. Let us define a relation on* A *by setting* $a_0 \sim a_1$ *if* $f(a_0) = f(a_1)$*.*

   (a) *Show that* $\sim$ *is an equivalence relation.*

   That this is an equivalence relation follows directly, as above, from the fact that equality is an equivalence relation on any set.

   (b) *Let* $\mathcal{A}$ *be the set of equivalence classes induced by* $\sim$*. Show there is a bijective correspondence of* $\mathcal{A}$ *with* B*.*

   From the definition of $\sim$ it is clear that the equivalence classes are precisely the preimages of all the points in $f(A)$, but since f is surjective we have $f(A) = B$. The function $b \mapsto f^{-1}(b)$, where $f^{-1}(b) = \{a \in A \mid f(a) = b\}$, is a bijection between B and $\mathcal{A}$. That this is injective follows from the fact that if two points have the same preimage then they must be the same point, otherwise f would not be a function. It is surjective because f is also surjective: every point in A corresponds to at least one point in B.

10. *Prove that* $(x_0, y_0) < (x_1, y_1)$ *if either* $y_0 - x_0^2 < y_1 - x_1^2$*, or* $y_0 - x_0^2 = y_1 - x_1^2$ *and* $x_0 < x_1$ *is an order relation on* $\mathbb{R}^2$*. Describe it geometrically.*

    Clearly $(x_0, y_0) < (x_0, y_0)$ is never true, since that would imply $x_0 < x_0$. Transitivity follows from the fact that both equality, i.e., the case where $y_0 - x_0^2 = y_1 - x_1^2$ occurs, and the usual order on $\mathbb{R}$ are transitive.

    The sets with elements that have preceeding elements are parabola described by the equivalence relation in (8), i.e., we could redescribe the equivalence relation by saying that two elements are equivalent if there is a chain of preceeding elements from one to the other and it would have the same equivalence classes as in (8).

11. *Consider the following order relations on* $\mathbb{Z}_+ \times \mathbb{Z}_+$*:*

    (i) *The dictionary order.*

    (ii) $(x_0, y_0) < (x_1, y_1)$ *if either* $x_0 - y_0 < x_1 - y_1$*, or* $x_0 - y_0 = x_1 - y_1$ *and* $y_0 < y_1$*.*

    (iii) $(x_0, y_0) < (x_1, y_1)$ *if either* $x_0 + y_0 < x_1 + y_1$*, or* $x_0 + y_0 = x_1 + y_1$ *and* $y_0 < y_1$*.*

    In these order relations, which elements have immediate predecessors? Does that set have a smallest element? Show that all three order types are different.

    The elements that have immediate predecessors are, in the three cases respectively, those elements whose first coordinates are equal, those coordinates such that $x_0 - y_0 = x_1 - y_1$, and those elements such that $x_0 + y_0 = x_1 + y_1$. In each case there is a smallest element, though that would not be true if we were dealing with $\mathbb{Z}$ instead of $\mathbb{Z}_+$.

12. *Prove that if an ordered set* A *has the least upper bound property then it has the greatest lower bound property.*

    Let A_0 be a nonempty subset of A that is bounded below and define B as the set of all lower bounds of A_0. By construction $B \neq \emptyset$, and, moreover, any element of A_0 is an upper bound for B so it is bounded above. By hypothesis, then, there exist a supremum α of B. To show that this α is the greatest lower bound for A_0 it suffices to show that α is just a lower bound since by definition $\alpha \geq b$ for any $b \in B$. Suppose for contradiction that α were not a lower bound for A_0. Then there exists $x \in A_0$ with $x < \alpha$. Since α is the least upper bound of B there must be some $y \in B$ with $x < y < \alpha$, but then y is not a lower bound for A_0 – a contradiction. Therefore α is the greatest lower bound for A_0.

13. (a) *Prove that for any $n \in \mathbb{Z}_+$, every nonempty subset of $\{1, \ldots, n\}$ has a largest element.*
    Clearly this is true for $n = 1$, so assume $n > 1$. Let $\emptyset \neq B \subset \{1, \ldots, n+1\}$. If $B \subset \{1, \ldots, n\}$ then B has a largest element by assumption. Otherwise $n+1 \in B$, and $n+1$ is the largest element of B.

    (b) *Explain why you cannot conclude from the above that every nonempty subset of $\mathbb{Z}_+$ has a largest element.*
    The above statement is about finite ordinals, while such a statement about $\mathbb{Z}_+$ would be about about an infinite ordinal (i.e., ω).

14. *Show that every positive number a has exactly one square root.*

    First we show that for any number $x^2 < a$ there exists another number b such that $x < b$ and $b^2 < a$, and similarly for $x^2 > a$. Then we show that the set $\{x \in \mathbb{R} \mid x^2 < a\}$ is nonempty and bounded above, and so has a supremum α. If $\alpha^2 < a$ then there would be another number b such that $\alpha^2 < b^2 < a$, contradicting the fact that α is an upper bound. If $\alpha^2 > a$ then there would be another element with $b < \alpha$ and $b^2 > a$, contradicting the fact that α is the least upper bound. The uniquenss of the square root of a positive number follows then from the uniqueness from the fact that if $b < c$ then $b^2 < bc < c^2$, so that $b^2 \neq c^2$. Similarly for $b > c$.

15. *Let $m, n \in \mathbb{Z}_+$ and $X \neq \emptyset$.*

    (a) *If $m \leq n$ find an injective map $f : X^m \to X^n$.*
    Fix $x'_1, \ldots, x'_n \in X$, and define
    $$f(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_m, x'_1, \ldots, x'_{m-n})$$

    (b) *Find a bijective map $g : X^m \times X^n \to X^{m+n}$.*
    Define
    $$g((x_1, x_2, \ldots, x_m), (x'_1, \ldots, x'_n)) = g(x_1, x_2, \ldots, x_m, x'_1, \ldots, x'_n)$$

    (c) *Find an injective map $h : X^n \to X^\omega$.*
    Fix $x \in X$ and define
    $$h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x, x, \ldots)$$

    (d) *Find a bijective map $k : X^n \times X^\omega \to X^\omega$.*
    Define
    $$k((x_1, \ldots, x_n), (x'_1, x'_2, \ldots)) = (x_1, \ldots, x_n, x'_1, x'_2, \ldots)$$

    (e) *Find a bijective map $l : X^\omega \times X^\omega \to X^\omega$.*
    Define
    $$l((x_1, x_2, \ldots), (x'_1, x'_2, \ldots)) = (x_1, x'_1, x_2, x'_2, \ldots)$$

    (f) *If $A \subset B$, find an injective map $m : (A^\omega)^n \to B^\omega$.*
    Define
    $$m\left((x^1_1, x^1_2, \ldots), (x^2_1, x^2_2, \ldots), \ldots (x^n_1, \ldots)\right) = (x^1_1, x^2_1, \ldots, x^n_1, x^1_2, \ldots, x^n_2, x^1_3, \ldots)$$