CMSC 277: Homework #7

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1. *Fix a nonstandard model of analysis $^*\mathfrak{R}$. Let $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$. Show that r is in the closure of a if and only if there exists $a \in (A^*) \cap \mathcal{F}$ such that $\mathrm{st}(a) = r$.*

   Let $r \in \overline{A}$. Then the sentence

   $$\forall \varepsilon > 0 \exists s \in \underline{A}(|\underline{r} - s| < \varepsilon)$$

   is in $\mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(^*\mathfrak{R})$. Let ε be a positive infinitesimal, then there exists some (finite) $a \in {}^*A$ such that $|a - r| < \varepsilon$ so that $a \approx r$. But as $r \in \mathbb{R}$ it follows that $\mathrm{st}(a) = r$.

   Similarly, if $a \in {}^*A \cap \mathcal{F}$ such that $\mathrm{st}(a) = r$ then the sentence

   $$\forall \varepsilon > 0 \exists s \in \underline{A}(|\underline{r} - s| < \varepsilon)$$

   is in $\mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(^*\mathfrak{R})$ since substituting $\underline{a}$ for s satisfies it. That $r \in \overline{A}$ follows immediately.

2. *Fix a nonstandard model of analysis $^*\mathfrak{R}$ and let $f : \mathbb{R} \to \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$ if and only if for all $a, b \in {}^*\mathbb{R}$ with $a \approx b$ we have $^*f(a) \approx {}^*f(b)$*

   We proceed using the contrapositive.

   Assume that there exist $a, b \in {}^*\mathbb{R}$ such that $a \approx b$ but $^*f(a) \not\approx {}^*f(b)$. Then there exists some $\varepsilon > 0$ such that $|^*f(a) - {}^*f(b)| > \varepsilon$. The sentence

   $$\forall \delta > 0 \exists x, y \in \mathbb{R}(|x - y| < \delta \wedge |\underline{f}(x) - \underline{f}(y)| > \underline{\varepsilon}$$

   is in $\mathrm{Th}(^*\mathfrak{R}) = \mathrm{Th}(\mathfrak{R})$ since substituting a for x and b for y satisfies it. But this sentence says precisely that f is not uniformly continuous.

   Assume that f is not uniformly continuous and fix $\varepsilon > 0$. Then the sentence

   $$\forall \delta > 0 \exists a, b \in \mathbb{R}(|a - b| < \delta \wedge |\underline{f}(a) - \underline{f}(b)| > \underline{\varepsilon}$$

   is in $\mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(^*\mathfrak{R})$. Letting δ be a positive infinitesimal we then have that $a \approx b$ but $^*f(a) \not\approx {}^*f(b)$ so that the second property does not hold.

3. *Let $\mathfrak{L}$ be the empty language. For each $n \in \mathbb{N}^+$ let $\sigma_n \in \mathrm{Sent}_{\mathfrak{L}}$ be*

   $$\exists x_1 \cdots \exists x_n \left(\bigwedge_{1 \le i < j \le n} x_i \neq x_j \right)$$

   Let $\Sigma = \{\sigma_n \mid n \in \mathbb{N}^+\}$ and define $T = \mathrm{Cn}(\Sigma)$. Show that T has QE and is complete.

   First note that any model of T is necessarily infinite since any model of σ_n has a cardinality of at least n. Hence any model of T cannot be finite.

To show that T has quantifier elimination let $\exists y(\alpha_1 \wedge \cdots \wedge \alpha_m)$ be as in Proposition 7.11. Then each α_i is of the form $x_j = y$ or $x_j \neq y$ for some j. If there exists an i such that $\alpha_i = (x_j = y)$ then

$$T \vDash \exists y(\alpha_1 \wedge \cdots \wedge \alpha_m) \leftrightarrow (\alpha_1 \wedge \cdots \wedge \alpha_m)_y^{x_j}$$

If $\alpha_i = (x_j \neq y)$ for all i and some j then it is always the case that $T \vDash \exists y(\alpha_1 \wedge \cdots \wedge \alpha_m)$ since any model of T is necessarily infinite. Simply pick $\varphi(x_1, \ldots, x_k)$ to be any tautology, e.g., $x_1 = x_1$.

Completeness is trivial: the $\mathfrak{L}$-structure consisting of a single point can be embedded in any model of T. Since T has QE it follows directly from Proposition 7.15 that T is complete.

4. (a) *Show that the theory of DLO has QE and is complete.*

   Recall that any model of DLO is infinite. Let $\exists y(\alpha_1 \wedge \cdots \wedge \alpha_m)$ be as in Proposition 7.11. As above, if there exists an i such that $\alpha_i = (x_j = y)$ then

   $$DLO \vDash \exists y(\alpha_1 \wedge \cdots \wedge \alpha_m) \leftrightarrow (\alpha_1 \wedge \cdots \wedge \alpha_m)_y^{x_j}$$

   Hence we may assume that each α_i is of the form $x_j \neq y$, $x_j < y$, or $x_j \nless y$ for some $j \leq k$. But each of these is tautologically satisfied by DLO since the theory contains the axioms that there are no endpoints, i.e., $\forall x \exists y(y < x)$ and $\forall x \exists y(x < y)$.

   To see that DLO is complete take the $\mathfrak{L}$ structure consisting of two points in x, y with $x < y$. Then this structure can be embedded in any model of DLO by mapping x anywhere and mapping y to some element in the model greater than the image of x, which exists by the DLO axioms. It follows from Proposition 7.15 that DLO is complete.

   (b) *How many definable subsets of $\mathbb{R}^2$ are there in the model $(\mathbb{R}, <)$ of DLO?*

   $\mathbb{R}^2$ and $\emptyset$ are definable as witnessed by $x = x$ and $\neg(x = x)$, respectively. Furthermore, the set of $(x, y) \in \mathbb{R}^2$ satisfying the sentences $x < y$, $y < x$, $x = y$, and $x \neq y$, plus their negations and pairwise conjunction and disjunction (note that some of these are contradictory or redundant, e.g., $(x < y) \wedge (x = y)$, or $(x < y) \wedge (x \neq y)$).

   Using the fact that DLO has QE and is complete we can show that these are the only such definable sets.