MATH 208: Homework #8

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1. *Show that the following are locally Euclidian:*
   (a) $SL_n(F)$
   (b) $O(n, F)$
   (c) $SO(n, F)$
   (d) $U(n)$
   (e) $SU(n)$

2. *Show that the following are Lie groups:*
   (a) $U(n)$
   (b) $SU(n)$

3. *Show that the derivative L exists at x_0 if and only if* $\lim_{h\to 0}\left|\frac{f(x_0+h)-f(x_0)-Lh}{h}\right| = 0$.

   Let
   $$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = L$$
   Then for any $\epsilon > 0$ we have there is some δ such that for $0 < |h| < \delta$
   $$\left|\frac{f(x_0+h)-f(x_0)-Lh}{h}\right| < \epsilon$$
   But
   $$\left|\frac{f(x_0+h)-f(x_0)-Lh}{h}\right| = \left|\left|\frac{f(x_0+h)-f(x_0)-Lh}{h}\right|\right|$$
   So if either of these is less than ϵ then the other is, which guarantees that the derivative L exists at x_0 if and only if $\lim_{h\to 0}\left|\frac{f(x_0+h)-f(x_0)-Lh}{h}\right| = 0$.

4. *Find the center of $GL_n(F)$.*

   The center of $GL_n(F)$ is the set of elements which commute with every element in $GL_n(F)$.

Consider the $n \times n$ matrix

$$B = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and let A be an arbitrary matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} a_{1n} & a_{11} & \cdots & a_{1,n-1} \\ a_{2n} & a_{21} & & a_{2,n-1} \\ \vdots & & \ddots & \vdots \\ a_{nn} & a_{n1} & \cdots & a_{n,n-1} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ \vdots & & \ddots & \vdots \\ a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

Look at AB then BA we see $a_{11} = a_{22}$, but finding a_{22} in AB we see that it corresponds to a_{33} in CA. Likewise for a_{33}, so continuing this process we get that $a_{11} = a_{22} = \cdots = a_{nn}$.

Now consider the $n \times n$ matrix

$$S = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$$

Requiring that $SA = AS$ for any A in the center we see that, in general

$$a_{ij} = a_{(n-i+1),(n-j+1)} \tag{1}$$

Define S_k as the matrix S with a -1 in the k^{th} row. For each $k \leq n$ we see AS_k is AS with the k^{th} column having a minus sign and S_kA is AS with the k^{th} row having a minus sign. From (1) we see that the only elements that must not satisfy $a_{ij} = -a_{ij}$ are precisely the diagonals of A. Therefore the center of $GL_n(F)$ consists precisely of those matrices of the form αI where $\alpha \in F^{\times}$, since we know already all of these matrices commute.

5. *Let*

$$f_r(x) = \begin{cases} \frac{1}{q^r} & \frac{p}{q} = x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that f_r is not differentiable anywhere if $1 \leq r \leq 2$.

First, f_r is not differentiable at any non-zero rational point since f_r is not continuous there. $0 \neq x \in \mathbb{R} \setminus \mathbb{Q}$. We know that there exists infinitely many $\frac{p}{q}$ such that $|x - \frac{p}{q}| < \frac{1}{q^r}$ by the next problem, but between each of these points and x the slope is at at most -1, i.e., in any interval we can find infinitely many rational numbers such that the slope of the secant is bounded away from zero by a constant. Hence, the derivative, which is the limit of secant slopes, does not exist.

6. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Show that there exists infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.*

Let α be arbitrary and irrational and $N \in \mathbb{N}$. For brevity's sake we will denote $[\alpha] = \alpha - \lfloor \alpha \rfloor$. Consider the intervals

$$\left[0, \frac{1}{N}\right), \left[\frac{1}{N}, \frac{2}{N}\right), \ldots, \left[\frac{N-1}{N}, 1\right)$$

Clearly since α is irrational we have $0 \leq [q\alpha] < 1$ for any irrational α and rational q. Consider $[0], [\alpha], [2\alpha], [3\alpha], \ldots, [N\alpha]$. There are $N+1$ of these and each is less than 1, but only N intervals above. Therefore there exist $q_1, q_2, S \in \mathbb{N}$ such that

$$[q_1\alpha], [q_2\alpha] \in \left[\frac{S}{N}, \frac{S+1}{N}\right)$$

Letting $q = |q_1 - q_2|$ we get that for some $p \in \mathbb{Z}$

$$|q\alpha - p| < \frac{1}{N}$$

or

$$|\alpha - \frac{p}{q}| < \frac{1}{Nq} \leq \frac{1}{q^2}$$

Assume for contradiction that there are only a finite such $\frac{p_i}{q_i}$ satisfying this condition. Since α is irrational this difference is never exactly 0 and so there exists some N' such that for all $i = 0, \ldots, N$

$$|\alpha - \frac{p_i}{q_i}| > \frac{1}{N'}$$

But we can apply the original argument for this N', producing a $\frac{p}{q}$ which is within $\frac{1}{q^2}$ of α.

7. *Let*

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For all $n \in \mathbb{N}$ show that $f^{(n)}(0) = 0$.

Let $p(x)$ be a n degree polynomial of $\frac{1}{x}$, that is

$$p(x) = \sum_{i=0}^{n} a_i \frac{1}{x^i}$$

We will first show that $f^{(n)}(x)$ is of the form $e^{\frac{-1}{x^2}}p(x)$ when $x \neq 0$. Clearly $f^{(0)}(x)$ is of this form, so assume

$$f^{(n)}(x) = e^{\frac{-1}{x^2}} \sum_{i=0}^{n} a_i \frac{1}{x^i}$$

Then applying the product rule and chain rule we get that

$$f^{(n+1)}(x) = e^{\frac{-1}{x^2}} \left(\frac{2}{x^3} \sum_{i=0}^{n} a_i \frac{1}{x^i} + \sum_{i=0}^{n} a_i \frac{-i}{x^{i-1}} \right)$$

which is still of the the form $e^{\frac{-1}{x^2}}p(x)$, for the appropriate $p(x)$.

Because the following is true

$$\begin{aligned}
\lim_{x \to 0} e^{\frac{-1}{x^2}} \left(\sum_{i=0}^{n} a_i \frac{1}{x^i} \right) &= \lim_{x \to 0} \sum_{i=0}^{n} a_i \frac{\frac{1}{x^i}}{e^{\frac{1}{x^2}}} \\
&= \lim_{u \to \pm\infty} \sum_{i=0}^{n} a_i \frac{u^i}{e^{u^2}} \\
&= 0
\end{aligned}$$

we see that each $f^{(n)}(x)$ is continuous at zero since we define $f^{(n)}(0) = 0$. All that remains to be shown is that each $f^{(n)}$ is differentiable at zero. Since $f^{(n)}(0) = 0$, it follows that

$$\begin{aligned}
\lim_{x \to 0} \frac{e^{\frac{-1}{x^2}} \left(\sum_{i=0}^{n} a_i \frac{1}{x^i} \right)}{x} &= \lim_{x \to 0} \frac{\frac{1}{x} \left(\sum_{i=0}^{n} a_i \frac{1}{x^i} \right)}{e^{\frac{1}{x^2}}} \\
&= \lim_{x \to 0} \frac{\left(\sum_{i=0}^{n} a_i \frac{1}{x^{i+1}} \right)}{e^{\frac{1}{x^2}}} \\
&= 0
\end{aligned}$$

which is true by the same argument by which we showed continuity. Therfore $f^{(n)}(x)$ is differentiable at 0 and $f^{(n)}(0) = 0$.

8. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be such that* $f(x, y) = (x \sin(y), e^x + y^2)$.

   (a) *Is* f *differentiable at any point?*

   f is differentiable at every point in $\mathbb{R}^2$.

   (b) *What is* $Df(x, y)$*?*

   Consider f as $f(x, y) = (f_1(x, y), f_2(x, y))$. For any point $(x, y) \in \mathbb{R}^2$ the linear transformation which best approximates f at (x, y) is

$$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \sin(y) & x\cos(y) \\ e^x & 2y \end{pmatrix}$$

9. *Show that if $Df(x)$ exists then it is unique.*

   Let f be differentiable at x_0.

   Assume there exists a $D_1 f(x_0), \delta_1$ such that for $0 < |h| < \delta_1$

   $$\left| \frac{f(x_0 + h) - f(x_0) - D_1 f(x_0) h}{h} \right| < \frac{\epsilon}{2}$$

   and a $D_2 f(x_0), \delta_2$ such that for $0 < |h| < \delta_2$

   $$\left| \frac{f(x_0 + h) - f(x_0) - D_1 f(x_0) h}{h} \right| < \frac{\epsilon}{2}$$

   Assume $D_1 f(x_0) \neq D_1 f(x_0)$ since otherwise we are done and let $\delta = \min\{\delta_1, \delta_2\}$, then for $0 < |h| < \delta$ we have

   $$\left| \frac{f(x_0 + h) - f(x_0) - D_1 f(x_0) h}{h} \right| + \left| \frac{f(x_0 + h) - f(x_0) - D_1 f(x_0) h}{h} \right| < \epsilon$$

   Assuming ...

   I've tried to work through this, and all I know is that $D_1 \neq D_2$ iff $\|D_1 - D_2\| > 0$, but this always produces an inequality in the wrong "direction."

10. *Let $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$.*

    (a) *What conditions are required for a "μ-product rule?"*

    We require that $\mu(a, b + c) = \mu(a, b) + \mu(a, c)$, i.e., distributivity, $\mu(0, a) = \mu(a, 0) = 0$, $\alpha\mu(a, b) = \mu(\alpha a, b) = \mu(a, \alpha b)$, i.e., associativity with respect to "normal" multiplication, and commutativity. We also require that $|\mu(a, b)| = \alpha |a||b|$ for some $\alpha \in \mathbb{R}^k$. Finally μ must be continuous.

    I don't like all these conditions, especially the second-to-last, however I see now way to avoid it. The proof for the other product rules work because we can talk about what $|a \cdot b|$ and $|a \times b|$ mean, and the result comes from the fact that they are related to the "normal" absolute value. I think if we require this we can drop continuity, since I originally included it to guarantee that $\lim_{x \to 0} \mu(x, x) = 0$ and things of this form.

    (b) *Prove the "μ-product rule" under these conditions.*

    $$\lim_{x \to a} \left| \frac{\mu(f(x), g(x)) - \mu(f(a), g(a)) - \big(\mu(g(a), Df(a)(x - a)) + \mu(f(a), Dg(a)(x - a))\big)}{x - a} \right|$$

    is equivalent to the following after adding and subtracting the same thing a few times, several applications of commutativity, and three applications of distributivity (the line is broken because otherwise it is too long)

    $$\begin{aligned} & \lim_{x \to a} \left| \frac{\mu(g(a), f(x) - f(a) - Df(a)(x - a))}{x - a} \right| \\ + & \lim_{x \to a} \left| \frac{\mu(f(a), g(x) - g(a) - Df(a)(x - a))}{x - a} \right| \\ + & \lim_{x \to a} \left| \frac{\mu(f(x) - f(a), g(x) - g(a))}{x - a} \right| \end{aligned}$$

For the first we get $\alpha|g(a)|\frac{|f(x)-f(a)-Df(a)(x-a)}{|x-a|}$, which goes to zero as $x \to a$ since f is assumed to be differentiable. The second follows from the same facts, but with f and g switched.

Ths third we can write as $\alpha|g(x)-g(a)|\frac{|f(x)-f(a)|}{|x-a|}$, which also goes to zero as $x \to a$.

Therefore if f, g are differentiable at a then $D\mu(f(a), g(a))(a) = \mu(g(a), Df(a)) + \mu(f(a), Dg(a))$.