MATH 207: Problem Set 6

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1. *Let F be a field and $A \in M_2(F)$. Show that A has a inverse if and only if $det(A) \neq 0$*

   By question 2, A is invertible if and only if $det(A)$ is invertible in F. But since F is a field the only element that is not invertible is 0. Therefore A is invertible if and only if $det(A) \neq 0$;

2. *Let R be a commutative ring with one and $A \in M_2(R)$. When does A have an inverse?*

   Claim: A is invertible if and only if $det(A)$ is invertible in R.

   Assume A is invertible, then there exists a $B \in M_2(R)$ such that $AB = I$. But this means $det(AB) = det(I) = 1$. Since the determinent is distributive, $1 = det(AB) = det(A)det(B) \Rightarrow det(B) = det(A)^{-1}$. That is, $det(A)$ is invertible.

   Assume $det(A)$ is invertible and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We see that $det(A) = ad - bc$, and $A \cdot \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. That is, A is invertible.

   Therefore A is invertible if and only if $det(A)$ is invertible in R.

3. *What are the zero divisors of $M_2(R)$?*

4. *Let $(R, +, \cdot, <)$ be an ordered integral domain. Show that R has a subring which is order isomorphic to $\mathbb{Z}$.*

5. *Show that $\mathbb{Q}$ does not satisfy the least upper bound property.*

   Let $S = \{p \in \mathbb{Q} | p^2 < 2\}$. We know that $p^2 = 2 \Rightarrow p \notin \mathbb{Q}$. So we want to show that this number is the upper bound. Let $\alpha = \sup S \in \mathbb{Q}$.

   - Assume $\alpha^2 > 2$

     $\alpha^2 > 2 \Leftrightarrow \alpha^2 + 2\alpha > 2 + 2\alpha \Leftrightarrow \alpha > \frac{2\alpha+2}{\alpha+2}$

     $$\begin{aligned} (\tfrac{2\alpha+2}{\alpha+2})^2 > 2 &\Leftrightarrow 4\alpha^2 + 8\alpha + 4 > 2\alpha^2 + 8\alpha + 8 \\ &\Leftrightarrow 4\alpha^2 > 2\alpha^2 + 4 \\ &\Leftrightarrow 2\alpha^2 > 4 \\ &\Leftrightarrow \alpha^2 > 2 \end{aligned}$$

     So $\frac{2\alpha+2}{\alpha+2}$ is an upper bound but less than α, contradicting the assumption that $\alpha = \sup S$.

- Assume $\alpha^2 < 2$

  $\alpha^2 < 2 \Leftrightarrow \alpha^2 + 2\alpha < 2 + 2\alpha \Leftrightarrow \alpha < \frac{2\alpha+2}{\alpha+2}$

  $$\begin{aligned} (\tfrac{2\alpha+2}{\alpha+2})^2 < 2 &\Leftrightarrow 4\alpha^2 + 8\alpha + 4 < 2\alpha^2 + 8\alpha + 8 \\ &\Leftrightarrow 4\alpha^2 < 2\alpha^2 + 4 \\ &\Leftrightarrow 2\alpha^2 < 4 \\ &\Leftrightarrow \alpha^2 < 2 \end{aligned}$$

  So $\frac{2\alpha+2}{\alpha+2} \in S$ but greater than α, contradicting the assumption that $\alpha = \sup S$.

Hence the supremum must satisfy $\alpha^2 = 2$, but no rational number does this. Therefore we have a bounded set, S, whose supremum is not in $\mathbb{Q}$. That is, $\mathbb{Q}$ does not satisfy the least upper bound property.

6. *Show that there is a bijection from the normal subgroups of $\frac{G}{H}$ and the normal subgroups of G containing H.*

7. *Let G be a finite group and H be a subgroup of G with index k. Show that there exists a set of elements $x_1, x_2, \ldots, x_k$ in G which can serve as complete coset representatives for both left and right cosets of H.*

8. *Find all possible areas of lattice squares in $\mathbb{R}^2$.*

9. *Find all positive integers which can be the length of the hypotenuse of a right triangle with legs of integer length.*

10. *Define a polyhedron in $\mathbb{R}^n$.*

11. *Find a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.*

12. *Show that if $(a_n), (b_n)$ are Cauchy sequences then $(a_n + b_n)$ is a Cauchy sequence.*

    We have $\forall r > 0 \exists N_1 \in \mathbb{N} \ni m,n > N_1 \Rightarrow |a_n - a_m| < \frac{r}{2}$ and $\forall r > 0 \exists N_2 \in \mathbb{N} \ni m,n > N_2 \Rightarrow |b_n - b_m| < \frac{r}{2}$

    Let $N = max(N_1, N_2)$, then $\forall r > 0, |(a_n + b_n) - (a_m + b_m)| \leq |a_n - a_m| + |b_n - b_m| < \frac{r}{2} + \frac{r}{2} = r$

    Therefore if (a_n) and (b_n) are Cauchy sequences then so is $(a_n + b_n)$.

13. *Let $(R, +, \cdot)$ be a ring with one. Show that $(R^x, \cdot)$ is a group.*

    - Associativity is inhereted.
    - Each element has an inverse by definition of $(R^x, \cdot)$.
    - $1 \cdot 1 = 1$, so there is an identity in R^x
    - Let $a, b \in R^x$, then a^{-1}, b^{-1} exist. $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}b = 1$. So the operation is an internal law of composition.

14. *Describe $(\mathbb{Z}_n^x, \cdot)$ for $2 \leq n \leq 16$.*

15. *Let C be the set of all Cauchy sequences in $\mathbb{Q}$. Show that $\{(a_n) \in C | a_n \to 0\}$ is a maximal ideal.*