MATH 270: Homework #7

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1. *Find all the values of*

   (a) $\log -i$

   For both of these exercises, the only values $\log z$ can take on are those such that $e^{\log z} = z$. Here
   $$\log -i = -i\frac{\pi}{2} + 2\pi i n$$
   for any $n \in \mathbb{Z}$

   (b) $\log(1+i)$

   Represented in polar form $1 + i = \sqrt{2}e^{i\frac{\pi}{4}} = e^{\log\sqrt{2}+i\frac{\pi}{4}}$, so
   $$\log(1+i) = \log\sqrt{2} + i\frac{\pi}{4} + 2\pi i n$$
   where the log function is defined as usual for real numbers and $n \in \mathbb{Z}$.

2. *Evaluate* $\int_\gamma \frac{dz}{z^2-2z}$ *where* γ *is the circle of radius* 1 *centered at* 2 *traveled once counterclockwise.*

   By Cauchy's theorem, since $z \mapsto \frac{1}{z}$ is holomorphic here,
   $$\frac{1}{z}I(\gamma, z) = \int_\gamma \frac{1}{\zeta}\frac{d\zeta}{\zeta - z} = \int_\gamma \frac{d\zeta}{\zeta^2 - z\zeta}$$
   This problem is a special case where $z = 2$. Since $I(\gamma, 2) = 1$, the integral is $\frac{1}{2}$.

3. *Prove that* $\mathbb{C} \setminus \{0\}$ *is not simply connected.*

   If $\mathbb{C}\setminus\{0\}$ were simply connected then every closed curve γ contained in it would be homotopic to a point and hence any function holomorphic on $\mathbb{C} \setminus \{0\}$ would have a 0 integral over γ by Cauchy's theorem. However, the function $z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and
   $$\int_{|z|=1} \frac{dz}{z} = 2\pi i$$
   This is a contradiction, and therefore $\mathbb{C} \setminus \{0\}$ cannot be simply connected.

4. *Prove that if the image of γ lies in a simply connected region A and if $z_0 \notin A$ then $I(\gamma, z_0) = 0$*

   First note that since γ is closed and in a simply connected region it divides the complex plane into two disjoint, open, connected sets, one of which is unbounded. Since $I(\gamma, z)$ is a continuous, integer-valued function with respect to z it must be constant on any connected set. However,

   $$\lim_{z \to \infty} I(\gamma, z) = \lim_{z \to \infty} \int_\gamma \frac{d\zeta}{\zeta - z} = 0$$

   So for sufficiently large z, $I(\gamma, z)$ is arbitrarily close to 0. Since $I(\gamma, z)$ is a continuous, integer-valued function this means it must be identically 0 on the unbounded region induced by γ. Therefore $I(\gamma, z) = 0$ for all $z \notin A$ since all points not in A are in this unbounded region.

5. *Let f be holomorphic on $A = \{z \in \mathbb{C} \mid |z| > 1\}$. Show that if γ_r is the circle of radius $r > 1$ and center 0 then $\int_{\gamma_r} f$ is independent of r.*

   Any two circles with radius r_1, r_2 are homotopic by the homotopy $H(t, \theta) = (1-t)r_1 e^{i\theta} + t r_2 e^{i\theta}$ where $t \in [0, 1]$ and $\theta \in [0, 2\pi]$. The desired result is a direct consequence of the deformation theorem, which states that if f is holomorphic on an open set (here A) then the integrals over any two homotopic curves in A are equal.

6. *Let f be holomorphic and non-vanishing on a region A. Let γ be a closed curve homotopic to a point in A. Show that*

   $$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

   Cauchy's formula for derivatives imply that f' is holomorphic on A. Since f does not vanish on A, $\frac{f'}{f}$ is also holomorphic on A. Because γ is holomorphic to a point Cauchy's formula yields the desired result.