MATH 263: Homework #7

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1. *Let Y be the quasicircle obtained by adjoining the topologist's sine curve to the unit circle, and collapsing the portion on the y axis to a poin. Let $f : Y \to S^1$ be this quotient map. Show that f does not list to the covering space $\mathbb{R} \to S^1$ even though $\pi_1(Y) = 0$.*

   I think "arc" here means something precise, and I am not quite sure what.

2. *Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \cong Y$ then $\widetilde{X} \cong \widetilde{Y}$.*

   Let $f : X \to Y$ be a homeomorphism and $p : \widetilde{X} \to X$ a covering map. Then certainly $f \circ p$ is a covering map of Y by $\widetilde{X}$, as it simply identifies every neighborhood in Y with its homeomorphic copy in X. So both $\widetilde{X}$ and $\widetilde{Y}$ are covering spaces of Y via $f \circ p$ and p', say, with trivial fundamental group. Hence $(f \circ p)_*(\pi_1(\widetilde{X})) = 0 = p'_*(\pi_1(\widetilde{Y}))$ and therefore $\widetilde{X} \cong \widetilde{Y}$.

3. *Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.*

   The fundamental group of $X = S^1 \vee S^1$ is $\mathbb{Z} * \mathbb{Z}$. We therefore wish to find subgroups of $G = \mathbb{Z} * \mathbb{Z}$ with index 2 and 3, respectively, up to isomorphism. Since then $|G/H| = 2, 3$, it follows that $G/H \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$. In particular this means H is normal, so, up to isomorphism, there is only one connected 2-sheeted and 3-sheeted covering space.

   For the 2-sheeted case, consider $S^1 \vee S^1 \vee S^1$ where the left and right circles are identified with the left circle in X and each hemisphere of the center circle is identified with the right circle. Each fiber has 2 points and therefore the image under the induced map of this covering map of the fundamental group of the quotient space has index 2 in $Z * \mathbb{Z}$.

   Similarly, consider $S^1 \vee S^1 \vee S^1 \vee S^1$ where the second the third circles are identified 2-fold (i.e., as above, where each hemisphere is mapped to one of the circles) with the left and right circles in X, respectively, and the first and fourth circles are identified directly with the left and right circles in X, again, respectively. Each fiber consists of 4 points, and therefore this is a 4-sheeted covering space.

4. *Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.*

   The fundamental group of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

5. *Given maps $X \to Y \to Z$ such that both $Y \to Z$ and $X \to Z$ are covering spaces. show that $X \to Y$ is a covering space if Z is locally path connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.*

   Let $p_1 : X \to Y$ and $p_2 : Y \to Z$ and assume Z is locally path connected. Then by hypothesis there exist, for all $z \in Z$, evenly covered neighborhoods U_z and U'_z by p_2 and $p_2 \circ p_1$, respectively. Then $U = U_z \cap U'_z$ is a neighborhood evenly covered by *both* p_2 and $p_2 \circ p_1$.

   Let $y \in Y$. Since Z is locally path connected we can choose a suitable neighborhood U of $p(y)$ such that U is evenly covered by p_2 and $p_2 \circ p_1$. Let $y \in V$ where $p_2 |_V$ is a homeomorphism from V to U. Then

$$p_1^{-1}(V) = \left(p_1^{-1} \circ p_2^{-1} \circ p_2 |_V\right)(V) = \left(p_1^{-1} \circ p_2^{-1}\right)(U)$$

Hence p_1 is a covering map since U is evenly covered by $p_2 \circ p_1$ by hypothesis.

Identifying $\pi_1(X)$ and $\pi_1(Y)$ with their isomorphic copies (i.e., identify $\pi_1(X) \leq \pi_1(Y)$ via p_{1_*} and $\pi_1(Y) \leq \pi_1(Z)$ via p_{2_*}) in $\pi_1(Z)$ gives

$$\pi_1(X) \leq \pi_1(Y) \leq \pi_1(Z)$$

If $\pi_1(X) \trianglelefteq \pi_1(Z)$ then certainly $\pi_1(X) \trianglelefteq \pi_1(Y)$, and hence X is a normal covering of Y if X is a normal covering of Z.

6. *Given a covering space action of a group G on a path-connected, locally path-connected space X, then each subgroup $H \subset G$ determines a composition of covering spaces $X \to X/H \to X/G$. Show:*

   (a) *Every path-connected covering space between X and X/G is isomorphic to X/H for some subgroup $H \leq G$.*

   (b) *Two such covering spaces X/H_1 and X/H_2 of X/G are are isomorphic if and only if H_1 and H_2 are conjugate subgroups of G.*

   (c) *The covering space $X/H \to X/G$ is normal if and only if $H \trianglelefteq G$.*

7. *Let G be a discrete group.*

   (a) *Show that the category of right transitive G-sets is isomorphic to the category with objects $H\backslash G$, where we take one H for each conjugacy class subgroups of G and morphisms G-maps.*

   Let $\mathbf{C}$ be the category of transitive right G-sets and $\mathbf{D}$ the second category defined in the exercise. From a previous homework we know that for any $A \in \mathrm{Ob}(\mathbf{C})$ there exists an isomorphism of G-sets $\varphi_x : G/G_x \to A$ defined by $x \mapsto x \cdot g$, where G_x denotes the stabilizer of x under the action of G. Furthermore, we showed that the stabilizer of Hx in $H\backslash G$ is $x^{-1}Hx$ and that $H\backslash G$ and $K\backslash G$ are isomorphic as G-sets if and only if H and K are conjugate in G.

   Define a functor on $\mathbf{C}$ by sending $A \in \mathrm{Ob}(\mathbf{C})$ to its image under φ_x, modulo conjugacy classes of subgroups of G. For any $A, B \in \mathrm{Ob}(\mathbf{C})$ there exist φ_A, φ_B such that φ is an isomorphism between A and $H\backslash G$ for some $H \leq G$ and φ_B is an isomorphism between B and $K\backslash G$. If $\eta : A \to B$ is a G-map then define $F(\eta) = \varphi_B \circ \eta \circ \varphi_A^{-1}$. This is clearly a well-defined G-map from $H\backslash G$ to $K\backslash G$, as G-maps are stable under composition.

   From the first paragraph it is clear that $F : \mathbf{C} \to \mathbf{D}$ so defined is an isomorphism of categories,odulo conjugacy classes of subgroups of G, the functor F sends each object to a *conjugacy class*, i.e., the image of each object is unique up to isomorphism.

   (b) *Describe explicitly the category of transitive S_3-sets.*

   Since $|S_3| = 6$, there exist a Sylow 2-subgroup and a Sylow 3-subgroup. $\mathrm{Syl}_p(G)$ is stable under conjugation, in general, so that the objects consists of the trivial group, a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a group isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and all of S_3.

   (c) *Do the same for $\mathbb{Z}/10\mathbb{Z}$-sets.*

   As above, since $|\mathbb{Z}/10\mathbb{Z}| = 10$ there exist Sylow 2 and Sylow 5 subgroups, which are unique modulo conjugation. Hence the transitive $\mathbb{Z}/10\mathbb{Z}$ consist of objects isomorphic to the trivial group, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$, and $\mathbb{Z}/10\mathbb{Z}$.

   (d) *Show that the category of finite dimensional real vector spaces is equivalent to a category whose objects are the natural numbers $\mathbb{N}$.*

   For each V, a real vector space, consider $V \to \dim V$.