MATH 263: Homework #6

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1. *Let $H^2 = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ and X be a surface with boundary.*

   (a) *Show that no point of H^2 of the form $(x,0)$ has a neighborhood in H^2 that is homeomorphic to an open set of $\mathbb{R}^2$.*

   This follows from the connectedness properties of H^2 and open subsets of $\mathbb{R}^2$, respectively. We may assume without loss of generality that any neighborhood of $(x,0) \in H^2$ is connected, since otherwise there exists an open connected subset of the neighborhood which is a refinement of the original. So, for any connected neighborhood U of a point $(x,0)$ we can remove a point, e.g., $(x,0)$, and U will remain simply connected. However, no point can be removed from any open set in $\mathbb{R}^2$ without it ceasing to be simply connected (viz., a small loop around the removed point will no longer be nullhomotopic).

   (b) *Show that $x \in \partial X$ if and only if there is a homeomorphism h mapping a neighborhood of x onto an open set of H^2 such that $h(x) \in \mathbb{R} \times \{0\}$.*

   Let $x \in \partial X$. Then any neighborhood U of x is homeomorphic to an open subset of H^2, but not an open subset of $\mathbb{R}^2$. If $h(x) \notin \mathbb{R} \times \{0\}$ then there would exist a ε-neighborhood of $h(x)$ homeomorphic to an open subset of $\mathbb{R}^2$ via inclusion whose preimage is a neighborhood of x, contradicting the hypothesis that $x \in \partial X$. Therefore $h(x) \in \mathbb{R} \times \{0\}$ if $x \in \partial X$.

   Assume there is a homeomorphism h mapping a neighborhood U of x onto an open set of H^2 such that $h(x) \in \mathbb{R} \times \{0\}$. Then $h(U)$ is homeomorphic to a neighborhood V of a point of the form $(y,0)$, which, from the first part, implies that V is not homeomorphic to an open subset of $\mathbb{R}^2$. Hence U is not homeomorphic to an open subset of $\mathbb{R}^2$ and $x \in \partial X$.

   (c) *Show that ∂X is a 1-manifold.*

2. *Show that the closed unit ball in $\mathbb{R}^2$ is a 2-manifold with boundary.*

   Let D be the closed unit ball in $\mathbb{R}^2$. If $U \subset D$ is a neighborhood of the point x contained in the interior of D then U is homeomorphic to an open subset of $\mathbb{R}^2$ by inclusion. Assume that U intersects the boundary of D. Then define a map $U \to H^2$ that fixes all interior points and sends elements of the boundary to their projection along the secant line defined by where U intersects D. This is a homeomorphism, though not with an open subset of H^2. However, we can create a second homeomorphism which takes the image of U and translates and rotates it so that the secant line defined above resides on the line $\mathbb{R} \times \{0\}$. The composition of these two homeomorphisms is the homeomorphism for which are are looking.

3. *Let X be a 2-manifold and let $\{U_1, \ldots, U_k\}$ be a collection of disjoint open sets in X. Suppose that for each i there is a homeomorphism h_i of the open unit ball B^2 with U_i. Let $\varepsilon = \frac{1}{2}$ and B_ε be the open ball of radius ε. Show that the space $Y = X \setminus \bigcup h_i(B_\varepsilon)$ is a 2-manifold with boundary and that ∂Y has k components.*

   Let $x \in Y$. Then either $x \in \partial h_i(B_\varepsilon)$ or not. If not then since X is a 2-manifold there exists a sufficiently small neighborhood U of x such that U is homeomorphic to $\mathbb{R}^2$. If $x \in \partial h_i(B_\varepsilon)$ for some i

then there exists a neighborhood V of x such that V does not intersect $\partial h_j(B_\varepsilon)$ for $i \neq j$, again since X is a 2-manifold, and such that V is homeomorphic to a subspace of $\mathbb{R}^2$. Treating V and $h_i(B_\varepsilon)$ as their corresponding subspaces in $\mathbb{R}^2$, then, we see that as a subset of $\mathbb{R}^2$ (not subspace), we can produce a homeomorphism of V with an open subset of H^2 by projecting the boundary of V in $\mathbb{R}^2$ onto the secant line defined by the points where V intersects $h_i(B_\varepsilon)$ and then composing it with a homeomorphism that rotates and translates this secant line onto $\mathbb{R} \times \{0\}$. Hence X is a 2-manifold with boundary.

The only points of Y which have neighborhoods not homeomorphic to an open subset of $\mathbb{R}^2$ are those in $\partial h_i(B_\varepsilon)$, for each i. However, each $h_i(B_\varepsilon)$ is connected and $h_i(B_\varepsilon) \cap h_j(B_\varepsilon) = \emptyset$ for $i \neq j$. Hence if $x, y \in \partial Y$ then x and y are in the same component if and only if they are in $\partial h_i(B_\varepsilon)$. Since there are k of these there are precisely k components.

4. *Given a compact connected triangulable 2-manifold Y with boundary such that ∂Y has k components show that Y is homeomorphic to X-with-k-holes, where X is either S^2 or the n-fold torus T_n or the m-fold projective plane P_m.*

   Since each component of ∂Y is homeomorphic to a circle, Y can be written as the quotient space of a polygonal region in the plane with pairs of edges identified containing k holes. From the classification theorem, however, it follows that the polygonal region must be homeomorphic to S^2, T_n, or P_m, and hence Y is homeomorphic to one of these with k disjoint neighborhoods homeomorphic to B^2 removed from their surface.

5. *Let T be the torus.*

   (a) *Find a covering space of T corresponding to the subgroup $\mathbb{Z} \times \mathbb{Z}$ generated by the element $m \times \{0\}$, where $m \in \mathbb{Z}_+$.*

   This subgroup is precisely $m\mathbb{Z} \times \{0\} \cong m\mathbb{Z}$. Define $p : S^1 \times \mathbb{R}_+ \to T$ by

   $$p(z, x) = z^m \times (\cos 2\pi x, \sin 2\pi x)$$

   where S^1 is viewed as residing in the complex plane. This induces an injection of fundamental groups p_*, and the image is $p_*(\pi_1(S_1 \times \mathbb{R}_+)) = p_*(\pi_1(S^1)) \times p_*(\pi_1(\mathbb{R}_+)) = p_*(\pi_1(S^1)) \times \{0\} = m\mathbb{Z} \times 0 = \langle m \times 0 \mid m \in \mathbb{Z}_+ \rangle$ since $\mathbb{R}_+$ is simply connected.

   (b) *Find a covering space of T corresponding to the trivial subgroup of $\mathbb{Z} \times \mathbb{Z}$.*

   Let $p : \mathbb{R}_+ \to S^1$ be the usual covering map defined by $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. Then $p \times p : \mathbb{R}_+^2 \to T$ is a covering map, and the image of $\pi_1(\mathbb{R}_+^2)$ under p_* must be trivial since $\mathbb{R}_+^2$ is simply connected.

   (c) *Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $m \times \{0\}$ and $\{0\} \times n$, where $m, n \in \mathbb{Z}_+$.*

   Define a map $p : T \to T$ by $p(z, w) = z^m \times z^n$. Since this is the (Cartesian) product of two covering maps it is a covering map, and the corresponding subgroup is isomorphic to the image of $\pi_1(T)$ under p_*, which, in turn, is isomorphic to the Cartesian product of the fundamental group of S^1 under the image of the respective component monomorphisms of p_*. In other words, $p_*(\pi_1(T)) \cong m\mathbb{Z} \times n\mathbb{Z} = \langle m \times 0, 0 \times n \mid m, n \in \mathbb{Z}_+ \rangle$.

6. *Let G be a topological group with multiplication $m : G \times G \to G$ and identity element e. Let $p : \widetilde{G} \to G$ is a covering map. Show that given $\tilde{e}$ with $p(\tilde{e}) = e$ there is a unique multiplication operation on $\widetilde{G}$ that makes it into a topological group such that $\tilde{e}$ is the identity element and p is a homomorphism.*

7. *Let $p : \widetilde{G} \to G$ be a homomorphism of topological groups that is a covering map. Show that if G is abelian then so is $\widetilde{G}$.*