MATH 209: Homework #4

Jesse Farmer

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1. *Let $f, f_i, g : \mathbb{R}^n \to [-\infty, \infty]$ be measurable. Prove the following statements.*

   We will assume the sets on which each of these functions takes the values $\pm\infty$ is 0. In showing that the sets are measurable for $-\infty < f(x) < \infty$, etc., we can take the union those sets and the values for which the function is $\pm\infty$ and still retain a measurable set. This restriction can be dealt with in other ways, e.g., redefining addition and multiplication of functions in the case where both of f and g are infinity.

   (a) *$f + g$ is measurable.*
   Let $a, t \in \mathbb{Q}$ be arbitrary, then $\{x \mid f(x) > t\}$ and $\{x \mid g(x) > a - t\}$ are measurable and hence $\{x \mid f(x) > t\} \cap \{x \mid g(x) > a - t\}$ is measurable. However,

   $$\{x \mid (f+g)(x) > a\} = \bigcup_{t \in \mathbb{Q}} \big(\{x \mid f(x) > t\} \cap \{x \mid g(x) > a - t\}\big)$$

   Therefore $f + g$ is measurable.

   (b) *$f \cdot g$ is measurable.*
   Let f and g be measurable functions. Since the continuous function of a measurable function is measurable, it follows that $-g$ is measurable. By part (a), $f - g$ is measurable. Also, $(f + g)^2$ and $(f - g)^2$ are measurable. Therefore

   $$fg = \frac{(f+g)^2 + (f-g)^2}{4}$$

   is measurable.

   (c) *$f \vee g = \sup\{f, g\}$ is measurable.*
   Clearly $\sup\{f(x), g(x)\} > a$ if and only if $f(x) > a$ or $g(x) > a$. Hence

   $$\{x \mid (f \vee g)(x) > a\} = \{x \mid f(x) > a\} \cup \{x \mid g(x) > a\}$$

   is measurable, and therefore $f \vee g$ is measurable.

   (d) *$f \wedge g = \inf\{f, g\}$ is measurable.*
   Clearly $\inf\{f(x), g(x)\} < a$ if and only if $f(x) < a$ or $g(x) < a$. Hence

   $$\{x \mid (f \wedge g)(x) < a\} = \{x \mid f(x) < a\} \cup \{g(x) < a\}$$

(e) $\bigvee_{i \in \mathbb{N}} f_i$ *is measurable.*
Let $g(x) = \bigvee_{i \in \mathbb{N}} f_i(x)$. Then

$$\{x \mid g(x) > a\} = \bigcup_{i \in \mathbb{N}} \{x \mid f_i(x) > a\}$$

is measurable, and therefore g is measurable.

(f) $\bigwedge_{i \in \mathbb{N}} f_i$ *is measurable.*
Let $g(x) = \bigwedge_{i \in \mathbb{N}} f_i(x)$. Then

$$\{x \mid g(x) < a\} = \bigcup_{i \in \mathbb{N}} \{x \mid f_i(x) < a\}$$

is measurable, and therefore g is measurable.

(g) $\limsup_i f_i$ *is measurable.*
Let $h(x) = \inf\{g_m(x)\}$ where $g_m(x) = \sup\{f_n(x)\}$ such that $n \geq m$. Each g_m is measurable by the above problems, and hence h is measurable since the functions (in this case h) defined by the countable "cap" of a set of measurable functions is measurable.

(h) $\liminf_i f_i$ *is measurable.*
This is the same problem as above, except that $h(x) = \sup\{g_m(x)\}$ where $g_m = \inf\{f_n(x)\}$ such that $n \geq m$. The rest of the proof is the same as the previous problem.

2. *Let X be a measure space and Y, Z metric spaces. Show that if $f : X \to Y$ is measurable and $g : Y \to Z$ is continuous then $g \circ f : X \to Z$ is measurable.*

   Let $A \subseteq Z$ be open. By the continuity of g, $g^{-1}(A)$ is open and by the measurability of f $f^{-1}(g^{-1}(A)) = (f^{-1} \circ g^{-1})(A)$ is measurable. Therefore $g \circ f$ is measurable.

3. *Let X, Y, Z be both measure spaces and metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are measurable, is $g \circ f : X \to Z$ measurable?*

4. *Show that if $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ pointwise almost everywhere then f is measurable.*

   If $f_n(x) \to f(x)$ pointwise, then the $\limsup$ and $\limsup$ of the $f_i(x)$ exist and are both equal to $f(x)$. By above, therefore, f is measurable.

5. *Show that simple functions are measurable.*

   Let $s : X \to Y$ be a simple function, then there exist finitely many non-zero α_i and (at most) countably many measurable A_i such that

$$s = \sum_{i \in N} \alpha_i \chi_{A_i}$$

   Take $E \subseteq Y$ to be open. Then E consists of finitely many points, since Y itself has only finitely many points. Then $s^{-1}(E)$ is the union of a finite number of the A_i, and hence measurable since the A_i are measurable by hypothesis. Therefore s is measurable.

6. *Show that if f is measurable then f is the limit of a sequence of simple functions almost everywhere.*

   Let f be a measurable function. Define a sequence of simple functions as follows. If $|f(x)| < n$ then let $f_n(x) = \frac{m}{n}$ for $\frac{m}{n} \leq f(x) < \frac{m+1}{n}$ for $m \in \mathbb{Z}$. If $|f(x)| \geq n$ then define $f_n(x) = \frac{x}{|x|}n$. Each f_n takes on no more than $2(n^2+1)$ values, and hence they are simple. Moreover, as $n \to \infty$ the first condition (i.e., $|f(x)| < n$) is satisfied for more and more values of x, so $f_n \to f$ uniformly.

7. *Do Kaplan #1-6*

   See attached sheets of paper.

8. *Compute $\int_0^\infty \frac{\sin x}{x} dx$.*

   The computer says $\frac{\pi}{2}$, but I have no idea how to show it.

9. *Show that the uniform limit of Riemann integrable functions is not necessarily Riemann integrable.*

   Let $\{r_1, r_2, \ldots\}$ be an ordering of the rationals on $[0, 1]$. Define

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in \{r_1, \ldots, r_n\} \\ 0 & x \notin \{r_1, \ldots, r_n\} \end{cases}$$

   Each f_i is Riemann integrable since it is continuous except on a set of measure zero. Moreover,

$$\int_0^1 f_n(x)\, dx = 0$$

   This approaches uniformly the function

$$f(x) = \begin{cases} \frac{1}{n} & x = r_n \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   which is nowhere continuous, since in any neighborhood of an irrational point there will be rational points outside any ϵ-neighborhood, and is therefore not Riemann integrable.

10. *Check what happens if the interval is infinite.*

    Consider

$$f_n(x) = x^{-\left(1+\frac{1}{n}\right)}$$

    For every $n \in \mathbb{N}$ this function is Riemann integrable on $[1, \infty)$. Moreover, this approaches $f(x) = \frac{1}{x}$ uniformly. However, $f(x)$ is *not* Riemann integrable on $[1, \infty)$.