MATH 208: Homework #3

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1. *Let $X \subseteq \mathbb{R}^n$ be a compact subset. Prove that every continuous real-valued function on X can be approximated by real polynomials in n variables, uniformly on X.*

   Every real-valued polynomial in $\mathbb{R}^n$ has terms of the form $c\prod_{k=1} x^{n_k}$, for some $n_k \in \mathbb{N}$. It is clear that this is a subset of $C(X,\mathbb{R})$ and that it is closed under addition and multiplication. Addition is obvious (i.e., any sum of these terms plus any other sum of these terms is surely a sum of these terms), and multiplication follows from repeated application of the distributive law. Since this set is closed, it follows that $\mathbb{R}^n[x]$ is a subring of $C(X,\mathbb{R})$.

   Trivially, every constant function is a polynomial of this kind. Let $a, b \in X$ such that $a \neq b$. Then $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$. Define $K = \{k \in \mathbb{N} \mid a_k \neq b_k\}$. Since $a \neq b$ we know that $K \neq \varnothing$. Consider $f(x_1, \ldots, x_n) = \prod_{k \in K}^n (x_k - a_k)$. It is clear that $f(a) = 0$ and $f(b) \neq 0$, and that this is a real-valued polynomial in n variables. This then separates points of X, and is therefore, by the Stone-Weierstrass theorem, dense in $C(X,\mathbb{R})$. It follows immediately that any continuous function in $\mathbb{R}$ can be approximated by a polynomial of n variables.

2. *Prove that every continuous complex-valued function on the unit circle can be approximated uniformly on the unit circle by Laurent polynomials.*

   Every Laurent polynomial is continuous on the unit circle, since $|z| = 0$ if and only if $z = 0$. Additive and multiplicative closure require nothing more than the distributive property. As this is an algebraically closed subset of the ring $C(S^1,\mathbb{C})$, it is also a subring.

   Define $p(z) = \sum_{j=-N}^{N} a_j z^j$, for $a_j \in \mathbb{C}$ and $N \in \mathbb{N}$ and let $\mathbb{C}[z, z^{-1}]$ denote the set of all Laurent polynomials.

   If we consider the Laurent polynomial where $N = 0$, we see that all complex constant functions are of this form.

   We know that $z\overline{z} = |z| = 1$, so $\overline{z} = \frac{1}{z}$. So

$$\overline{p}(z) = \overline{\sum_{j=-N}^{N} a_j z^j} = \sum_{j=-N}^{N} \overline{a_j z^j} = \sum_{j=-N}^{N} \overline{a_j}\,\overline{z^j} = \sum_{j=-N}^{N} b_j \left(\frac{1}{z}\right)^j \in \mathbb{C}[z, z^{-1}]$$

   Therefore this is stable under complex conjugation.

   The identity function is a Laurent polynomial which trivially separates points on S^1.

   By the Stone-Weierstrass theorem we have that $\mathbb{C}[z, z^{-1}]$ is dense in $C(S^1,\mathbb{C})$, so any continuous function on S^1 can be approximated by Laurent polynomials.

3. *Let A be the dense subring provided by the Stone-Weierstrass theorem and B be the closure of A. Prove that B is also a subring of $C(X, \mathbb{R})$.*

   Let $f \in B$ be arbitrary. If $f \in A$, then we are done since A itself is algebraically closed. We can thus consider only the limits of functions in A, since B is the union of A and the accumulation points of A. From the previous homework we know that the unform limit of continuous function is continuous (the "$\frac{\epsilon}{3}$" argument), so $B \subseteq C(X, \mathbb{R}$. Since the limit of the sum of the sum of the limits and the limit of the product of the product of the limits, we see that B is algebraically closed and thus that B is a subring of $C(X, \mathbb{R})$.

4. *Show that for any $\epsilon > 0$ there exists a real polynomial $p(y)$ in one variable such that $|p(y) - |y|| < \epsilon$ for all $y \in [-M, M]$.*

5. *Show that $A_{\mathbb{R}}$ is a subring of $C(X, \mathbb{R})$ that satisfies the conditions of the Stone-Weierstrass theorem on $\mathbb{R}$.*

   Clearly every constant function is a real-valued complex function. We only need to show that $A_{\mathbb{R}}$ separates points.

   We know that A is stable under complex conjugation, so $\overline{f} \in A$. If $x, y \in X$ such that $x_1 \neq x_2$ then there exists $f \in A$ such that $y_1 = f(x_1) \neq f(x_2) = y_2$. It is the case that $\Re(y_1) \neq \Re(y_2)$ or $\Im(y_1) \neq \Im(y_2)$. If it is the former, consider $u = \frac{f+\overline{f}}{2}$. Then $u(x_1) \neq u(x_2)$. If it is the latter, consider $v = \frac{i(\overline{f}-f)}{2}$. Then $v(x_1) \neq v(x_2)$. Both u, v are real valued functions, one of which will separate points for any $x_1, x_2 \in \mathbb{X}$. Moreover, $A_{\mathbb{R}}$ is trivially closed under addition and multiplication, since any sum or product of real-valued functions could never be a complex valued-function.

   Thus $A_{\mathbb{R}}$ satisfies the hypotheses of the real Stone-Weierstrass theorem.

6. *Show that $C_c(X, F)$ is contained in the space $BC(X, F)$. Determine whether or not $C_c(X, F)$ is closed in $BC(X, F)$ or not. When is $C_c(X, F)$ a Banach space?*

   Let $f \in C_c(X, F)$. Since the support of f is compact and the continuous image of a compact set is compact and thus bounded, $C_c(X, F) \subseteq BC(X, F)$.

   Define

$$f_n(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ \frac{1}{x} & 1 \leq x \leq n \\ \frac{-x+1}{n} + 1 & n < x < n+1 \\ 0 & n+1 \leq x \end{cases}$$

   Clearly each f_n is compactly supported by the $\frac{1}{x}$ section, and that as $n \to \infty$, f_n approaches the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ \frac{1}{x} & 1 \leq x \end{cases}$$

   This function is not compactly supported since it is non-zero on an unbounded set, which means the support of f is not compact. Therefore $C_c(X, F)$ is not topologically closed.

(When is $C_c(X,F)$ a Banach space. Iff X is compact?)

7. *Show that $C_c(X,F) \subseteq C_0(X,F) \subseteq BC(X,F)$. Prove that $C_0(X,F)$ is a Banach space. Give examples showing that, in general, all three inclusions are strict.*

   Any $f \in C_c(X,F)$ must eventually be zero as its support is bounded. So for any $\epsilon > 0$ simply choose its support, and outside of that $|f(x)| < \epsilon$. Therefore $C_c(X,F) \subseteq C_0(X,F)$. Take $g \in C_0(X,F)$, then there exists a compact set outside of which g is arbitrarily small, and certainly bounded. Since g is continous, its image is bounded on that compact set. Therefore $C_0(X,F) \subseteq BC(X,F)$.

   (Banach space)

   Let

$$f(x) = \begin{cases} \frac{1}{x^2} & |x| \geq 1 \\ 1 & |x| < 1 \end{cases}$$

   Then $f \in C_0(X,F)$ but $f \notin C_c(X,F)$. Any constant function strictly satisfies the second inclusion.

8. *Show that $X = \{(a_n) \mid a_i \in \mathbb{R}$ and all but finitely many are zero$\} \subseteq l^\infty(\mathbb{R})$ is of the first kind.*

   Define $A_k = \{(a_n) \in X \mid \text{Exactly } k \ a_i \text{ are zero}\}$. Clearly $X \subseteq \bigcup_{k \in \mathbb{N}} A_k$. Then, let $(b_k) \in A_k$ be such that $b_j = 0$ for some $j \in \mathbb{N}$.

   For any $\epsilon > 0$ define a new sequence as

$$c_n = \begin{cases} b_n & n \neq m \\ \frac{\epsilon}{2} & n = m \end{cases}$$

   Then $||a_k - b_k|| = \frac{\epsilon}{2}$, so $(b_n) \in B_\epsilon((a_n))$ but $(b_n) \notin A_k$ since it has $k+1$ non-zero terms. This implies that the interior of $\overline{A}_k$ is empty, i.e., A_k is nowhere dense. Therefore X is of the first kind, by definition.

9. *Let $T : V \to W$ be a linear transformation. Show that the following are equivalent:*

   (a) *T is bounded*

   (b) *T is continuous at zero*

   (c) *T is continuous for every $v \in V$*

($a \Rightarrow b$) Assume that there exists some $C > 0$ such that $\|T(v)\| \leq C\|v\|$ for all $v \in V$. Let $\delta = \frac{\epsilon}{C}$. Since $T(0) = 0$ have

$$\|T(v) - T(0)\| = \|T(v)\| \leq C\|v\| < \epsilon$$

($b \Rightarrow c$) Assume T is continuous at 0, then there is some $\delta > 0$ such that $\|v\| < \delta \Rightarrow \|T(v)\| < \epsilon$ for all $v \in V$, since $T(0) = 0$. So, as $v - a \to 0$ we have $\|T(v) - T(a)\| \leq \|T(v-a)\| < \epsilon$, i.e., T is continuous for all $v \in V$.

($c \Rightarrow a$) Since T is a continous mapping from $V \to W$ we know that $T^{-1}(B_1(0))$ is open and contains 0 since $T(0) = 0$. So there exists $r > 0$ such that $B_r(0) \subset T^{-1}(B_1(0))$. Let $\epsilon = \frac{r}{2\|v\|}$, with $v \neq 0$. Then

$$\|\epsilon v\| = |\epsilon|\|v\| = \frac{r}{2} < r$$

so $\epsilon v \in B_r(0)$ and $T(\epsilon v) \in B_1(0)$.
Finally,

$$|\epsilon| \|T(v)\| = \|\epsilon T(v)\| = \|T(\epsilon v)\| < 1 = \frac{r}{2\|v\|} \cdot \frac{2\|v\|}{r} = |\epsilon| \frac{2\|v\|}{r}$$

Letting $C = \frac{2}{r}$, we get that $\|T(v)\| \leq C\|v\|$ for all $v \in V$, where C does not depend on v.