MATH 208: Homework #5

Jesse Farmer

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1. *Prove the following statements about the adjoint map.*

   (a) $(T_1 + T_2)^* = T_1^* + T_2^*$
   Let $\alpha \in V^*$ and $v \in V$ be arbitrary, then

$$\begin{aligned}((T_1 + T_2)^*\alpha)(v) &= \alpha((T_1 + T_2)v) \\ &= \alpha(T_1 v + T_2 v) \\ &= \alpha(T_1 v) + \alpha(T_2 v) \\ &= (T_1^*\alpha)(v) + (T_2^*\alpha)(v)\end{aligned}$$

   (b) $(zT)^* = zT^*$
   Let $\alpha \in V^*$ and $z \in F$ be arbitrary, then

$$\begin{aligned}((zT)^*\alpha)v &= \alpha(z(Tv)) \\ &= z\alpha(Tv) \\ &= z(T^*\alpha)(v)\end{aligned}$$

   (c) $(I)^* = I^*$
   Let $\alpha \in V^*$ and $v \in V$ be arbitrary, then

$$\begin{aligned}((I)^*\alpha)v &= \alpha(Iv) \\ &= \alpha(v) \\ &= (I^*(\alpha))(v)\end{aligned}$$

   (d) $(T_1 \circ T_2)^* = T_2^* \circ T_1^*$
   Let $\alpha \in V^*$ and $v \in V$ be arbitrary, then

$$\begin{aligned}((T_1 \circ T_2)^*(\alpha))(v) &= \alpha((T_1 \circ T_2)(v)) \\ &= (\alpha \circ T_1)(T_2(v)) \\ &= (T_2^* \circ \alpha)(T_1(v)) \\ &= ((T_2^* \circ T_1^*)(\alpha))(v)\end{aligned}$$

2. *Find the matrix corresponding to the adjoint map.*

   Let $T = (a_{ij})$, $T^* = (b_{ij})$, and v be a column vector and α a row vector.

   Then

$$\begin{aligned} Tv &= (a_{ij}) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= (a_{i*} \cdot v) \end{aligned}$$

   Consider the effect of α,

$$\begin{aligned} \alpha(Tv) &= (\alpha_1 \cdots \alpha_n)(a_{i*} \cdot v) \\ &= \alpha_1(a_1 \cdot v) + \cdots + \alpha_n(a_1 \cdot v) \\ &= \alpha_1(a_{11}v_1 + \cdots + a_{1n}v_n) + \cdots + \alpha_n(a_{n1}v_1 + \cdots + a_{nn}v_n) \end{aligned}$$

   As α is a row vector, we see then that $(\alpha T^*)(v) = \alpha_1(b_{11}v_1 + \cdots + b_{n1}v_n) + \cdots + \alpha_n(b_{1n}v_1 + \cdots + b_{nn}v_n)$. By the definition of the adjoint these two quantities are equal, which implies that $b_{ij} = a_{ji}$, i.e., $T^* = T^T$.

3. *Prove that T is an open map if and only if there exists $r > 0$ such that $T(B_1(0)) \supset B_r(0)$.*

   Let T be an open map. Then $T(B_1(0))$ is open and, since $0 = T(0) \in B_r(0)$, there exists $r > 0$ such that $B_r(0) \subset T(B_1(0))$.

   Let $V_0 \subset V$ be open and $v \in V_0$ be arbitrary. Then there exists $r > 0$ such that $B_r(v) \subset V_0$. Since $B_r(v) = B_r(0) + v$ and $B_r(v) = rB_1(v)$, we have $B_1(0) \subset \frac{V_0 - v}{r}$ and $T(B_1(0)) \subset T(\frac{V_0-v}{r}) = \frac{T(V_0)-T(v)}{r}$. By hypothesis there exists $r' > 0$ such that $B_{r'}(0) \subset T(B_1(0))$. Hence $B_{r'}(0) \subset \frac{T(V_0)-T(v)}{r}$ which implies $B_{rr'}(T(v)) \subset T(V_0)$. As v was arbitrary, it follows that T takes open sets to open sets, i.e., T is an open map.

4. *Prove that the inverse of a bounded bijective linear map is a bounded linear map.*

   Let $T \in B(V, W)$ be bijective. Then we know T^{-1} exists. By the bijectivity of T we have that for every $w_1, w_2 \in W$ there exists a unique $v_1, v_2 \in V$ such that $Tv_1 = ww_1$ and $Tv_2 = w_2$. We have

$$\begin{aligned} T^{-1}(w_1 + w_2) &= T^{-1}(Tv_1 + Tv_2) \\ &= T^{-1}(T(v_1 + v_2)) \\ &= v_1 + v_2 \\ &= T^{-1}(w_1) + T^{-1}(w_2) \end{aligned}$$

   Let $w \in W$, $\alpha \in F$ be arbitrary, and $v \in V$ be such that $Tv = w$, then

$$\begin{aligned} T^{-1}(\alpha w) &= T^{-1}(\alpha T v) \\ &= T^{-1}(T(\alpha v)) \\ &= \alpha v \\ &= \alpha T^{-1}(w) \end{aligned}$$

By the Open Mapping Theorem T is an open map, i.e., if $U \subset V$ is open then $T(U)$ is open. Consider $T^{-1} : W \to V$. We have for every open $U \subset V$ that $T(U) = (T^{-1})^{-1}(U)$ is open. Therefore T^{-1} is continuous and hence bounded.

5. *Show that you can replace $\overline{T(B_1)}$ with $T(B_1)$ in the proof of open mapping theorem.*

   We have that $\|w\| < r$ implies $\overline{T(B_1)}$, and in general $\|w\| < r2^{-n}$ implies $w \in \overline{T(B_{2^{-n}})}$. It is sufficient to show that there exists $v \in B_1$ such that $Tv = w$.

   There exists a $v_1 \in B_{\frac{1}{2}}$ such that $\|w - Tv_1\| < \frac{r}{4}$. And, in general, there exists $v_n \in B_{2^{-n}}$ such that $\|w - \sum_{j=1}^{n} Tv_j\| < r2^{-n-1}$. Because V is a Banach space it follows that $\sum_{j=1}^{\infty} Tv_j = v \in V$, where $Tv = w$. Note that $\|v\| < \sum_{n=1}^{\infty} 2^{-n}$, so $B_{\frac{r}{2}}(0) \subset T(B_1)$.