# MATH 270: Homework #4

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29 October 2004

1. *Show that the region* $G = \{z \mid |\Re z| < 1$ *and* $|\Im z| < 3\} \cup \{z \mid |\Re z| < 3$ *and* $|\Im z| < 1\}$ *is star-shaped.*

   Let $z_0 \in G$, then the line segment joining 0 and $z_0$ is parameterized by $\gamma(t) = tz_0$ for $t \in [0, 1]$. Then $|\Re tz_0| = t|\Re z_0| \le |\Re z_0|$ and $|\Im tz_0| = t|\Im z_0| \le |\Im z_0|$. If $z_0 \in \{z \mid |\Re z| < 1 \text{ and } |\Im z| < 3\}$ then $|\Re\gamma(t)| < 1$ and $|\Im\gamma(t)| < 3$, so $\gamma(t)$ is contained in the same set for all $t \in [0, 1]$ (and likewise if $z_0$ is in the other rectangle). Therefore $G$ is a star-shaped region about 0 since every point in $G$ is connected to 0 through a line segment.

2. *Evaluate the following integrals without performing an explicit computation:*

   (a) $\int_\gamma \frac{1}{z}\,dz$ *where* $\gamma(t) = \cos t + 2i\sin t$ *for* $t \in [0, 2\pi]$.
   $\gamma$ parameterizes an ellipse, which is homotopic to a circle which can be parameterized by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Hence

   $$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} i = 2\pi i$$

   (b) $\int_\gamma \frac{1}{z^2}\,dz$ *where* $\gamma$ *is as in (a).*
   This function is entire and therefore the integral is 0 since $\gamma$ is closed.

   (c) $\int_\gamma \frac{e^z}{z}\,dz$ *where* $\gamma(t) = 2 + e^{it}$ *for* $t \in [0, 2\pi]$.
   $\gamma$ is closed and the function is entire, so the integral is 0.

   (d) $\int_\gamma \frac{1}{z^2-1}\,dz$ *where* $\gamma$ *is a circle of radius* 1 *centered about* 1.
   We can decomposed this into partial fractions by writing

   $$\frac{1}{z^2-1} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$$

   However, $\frac{1}{z+1}$ is holomorphic on this disc and so its contribution to the integral is 0. $\frac{1}{z-1}$ around the circle centered at 1 is the same as the integral of $\frac{1}{\zeta}$ around the unit disc centered at 0 through a simple change of variables. From part $(a)$ this integral is $2\pi i$, so

   $$\int_{C(1,1)} \frac{1}{z^2-1}\,dz = \frac{1}{2}\int_{C(0,1)} \frac{1}{\zeta}\,d\zeta = \pi i$$

3. *Evalutate* $\int_\gamma \frac{1}{z}\, dz$ *where* $\gamma$ *is the line segment joining* $1$ *to* $i$*.*

   By Cauchy's theorem the integral along this line joined with the quarter of the unit circle in the first quadrant is 0. The quarter-circle is parameterized by $\gamma(\theta) = e^{i\theta}$ for $\theta \in \left[0, \frac{\pi}{2}\right]$, hence

$$\int_\gamma \frac{dz}{z} = \int_0^{\frac{\pi}{2}} i\, dz = \frac{i\pi}{2}$$

4. *Evaluate* $\int_\gamma \sin z\, dz$ *where* $\gamma$ *is the unit circle.*

   $\sin z$ is holomorphic since $\sin z = \Re e^{iz} = \frac{e^{iz} - e^{-iz}}{2i}$, and $e^{iz}$ is certainly holomorphic. Hence the integral is 0.