MATH 209: Homework #5

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1. *Let s, t be simple functions. Show that $\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$*

   Because s and t are simple there exist α_i, β_i such that

$$\int_X (s+t)\,d\mu = \sum_{i=1}^{\infty}(\alpha_i+\beta_i)\mu\left(A_i\right) = \sum_{i=1}^{\infty}\alpha_i\mu\left(A_i\right) + \sum_{i=1}^{\infty}\beta_i\mu\left(A_i\right) = \int_X s\,d\mu + \int_X t\,d\mu$$

2. *What sets are measurable with respect to the measure ν?*

   Recall that

$$\nu(A) = \int_A f\,d\mu$$

   for f measurable. Clearly A must be at least Lebesgue measurable, since otherwise the definition does not make any sense. If A is Lebesgue measurable then for any $E \subseteq \mathbb{R}^n$, $\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$. I am concerned about the measurability of $E \cap A$ and $E \setminus A$, since if they are not measurable then $\nu(E)$ wouldn't make sense. However, under certain restrictions it seems to me that the additivity of ν and the fact that it is define in terms of the Lebesgue integral would guarantee that if A is Lebesgue measurable then it is *nu*-measurable.

   What confuses me is that we defined *measurable* in terms of the outer measure, but we have nothing akin to that for arbitrary measure as far as I know.

3. *Let $f : X \to \mathbb{C}$ be measurable. Show that $|\int_X f\,d\mu| \leq \int_X |f|\,d\mu$.*

   From the properties of complex numbers it follows that $|u| \leq |f|$, $|v| \leq |f|$, and $|f| \leq |u| + |v|$. Moreover, for any Lebesgue measurable function there exists $c \in \mathbb{C}$ with $|c| = 1$ such that

$$c\int_X f\,d\mu \geq 0$$

   Let $cf = u + iv$ Then $\int_X cf\,d\mu$ is real because

$$\int_X cf\,d\mu = c\int_X f\,d\mu = \left|\int_X f\,d\mu\right|$$

   Therefore

$$\left|\int_X f\,d\mu\right| = \left|\int_X u\,d\mu\right| \leq \int_X |cf|\,d\mu = |c|\int_X |f|\,d\mu = \int_X |f|\,d\mu$$

4. *Do Stewart #1-12*

   See attached papers.

5. *Write a proof for the Lebesgue Dominated Convergence Theorem.*

   Let E be a measurable set and $\{f_n\}$ a sequence of measurable functions such that $f_n \to f$ pointwise on E and let g be a measurable function on E such that $|f_n(x)| \leq g(x)$. Clearly f and f_n are measurable on E, so $g + f_n \geq 0$. By Fatou's Lemma,

$$\int_E (f+g)\,d\mu \leq \liminf_{n\to\infty} \int_E (f_n + g)\,d\mu$$

   and hence

$$\int_E f\,d\mu \leq \liminf_{n\to\infty} \int_E f_n\,d\mu \tag{1}$$

   Moreover, by hypothesis $g - f_n \geq 0$, so again by Fatou's Lemma

$$\int_E (g-f)\,d\mu \leq \liminf_{n\to\infty} \int_E (g - f_n)\,d\mu \Rightarrow -\int_E f\,d\mu \leq \liminf_{n\to\infty} -\int_E f_n\,d\mu$$

   Which is equivalent to

$$\limsup_{n\to\infty} \int_E f_n\,d\mu \leq \int_E f\,d\mu \tag{2}$$

   Inequalities (1) and (2) together imply

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$