MATH 258: Homework #8

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1. *Let A be a commutative ring and M an A-module. Show that for all $a \in A$ the map $h_a : M \to M$ defined by $x \mapsto ax$ and for all $x \in M$ the map $t_x : A \to M$ defined by $a \mapsto ax$ are homomorphisms of the additive group of A. Deduce that h_a and t_x are A-module homomorphisms.*

   Both h_a and t_x are clearly well-defined, and are homomorphisms of the additive group associated with A be left and right distributivity, respectively. To see this, note that

   $$h_a(x+y) = a(x+y) = ax + ay = h_a(x) + h_a(y)$$

   and

   $$t_x(a+b) = (a+b)x = ax + bx = t_x(a) + t_x(b)$$

   To see that h_a and t_x are A-module homomorphisms, let $a, b \in A$. Then

   $$h_a(bx) = a(bx) = (ab)x = (ba)x = b(ax) = bh_a(x)$$

   and

   $$t_x(ba) = (ba)x = b(ax) = bt_x(a)$$

   The rest of the properties then follow trivially from the properties of homomorphisms. $0 \cdot x = t_x(0) = 0$, $a(x-y) = h_a(x-y) = h_a(x) - h_a(y) = ax - ay$, $a \cdot 0 = h_a(0) = 0$, and

   $$(-a)x = t_x(-a) = -t_x(a) = -(ax) = -h_a(x) = h_a(-x) = a(-x)$$

2. *Let A be a commutative ring and define $E_n = \{f \in A[x] \mid \deg f \leq n-1\}$. Show that E_n is an A-module isomorphic to A^n.*

   That E_n is an A-module follows by treating elements of A as elements of $A[x]$ of degree 0, and using the regular ring properties of $A[x]$. Define a map $\varphi : E_n \to A^n$ by

   $$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \mapsto (a_{n-1}, a_{n-2}, \ldots, a_0)$$

   This is obviously surjective, and injective since if $(a_{n-1}, \ldots, a_0) = (b_{n-1}, \ldots, b_0)$ then $a_i = b_i$ for $0 \leq i \leq n-1$, and hence the respective polynomials are equal. Letting $f, g \in A[x]$, where the coefficients of the polynomials are denoted by a_i and b_i respectively,(i.e., $f(x) = a_{n-1}x^{n-1} + \cdots + a_0$),

   $$\varphi(f+g) = (a_{n-1} + b_{n-1}, \ldots, a_0 + b_0) = (a_{n-1}, \ldots, a_0) + (b_{n-1}, \ldots, b_0) = \varphi(f) + \varphi(g)$$

   and for any $r \in A$

   $$\varphi(rf) = (ra_{n-1}, \ldots, ra_0) = r(a_{n-1}, \ldots, a_0) = r\varphi(f)$$

   Hence E_n and A^n are isomorphic as A-modules.

3. *Let A be a commutative ring.*

   (a) *Let M, N be A-modules. Show that $Hom_A(M,N)$ is an A-module.*
   Let $f, g \in \mathrm{Hom}_A(M,N)$ and let $a \in A$. Define $(f+g)(x) = f(x) + g(x)$ and $(af)(x) = af(x)$. These operations are well-defined on $\mathrm{Hom}_A(M,N)$ since

   $$(f+g)(x+y) = f(x+y) + g(x+y) = f(x) + f(y) + g(x) + g(y) = (f+g)(x) + (f+g)(y)$$

   and

   $$(af)(x+y) = af(x+y) = f(ax+ay) = f(ax) + f(ay) = af(x) + af(y) = (af)(x) + (af)(y)$$

   Since N is an A-module, $1 \cdot f(x) = f(x)$. Left-distributivity holds,

   $$a((f+g)(x)) = a(f(x)+g(x)) = af(x) + ag(x) = (af)(x) + (ag)(x)$$

   Right distributivity holds similarly, since N is an A-module. Finally, for $a, b \in A$,

   $$(ab)(f(x)) = a(bf(x)) = a(bf(x))$$

   since $f(x) \in N$ and N is an A-modules, which implies $(ab)f = a(bf)$.

   (b) *Let M, N, P be A-modules. Show that the map $\eta : Hom_A(M,N) \to Hom_A(M,P)$ given by $f \mapsto g \circ f$ and $\varphi : Hom_A(P,M) \to Hom_A(N,M)$ given by $f \mapsto f \circ g$ are A-module homomorphisms.*
   Let $g \in \mathrm{Hom}_A(N,P)$, and $f, h \in \mathrm{Hom}_A(M,N)$. Then to show that $\eta(f+h) = \eta(f) + \eta(h)$, the following suffices

   $$g \circ (f+h)(x) = g(f(x)+h(x)) = g(f(x)) + g(h(x)) = (g \circ f)(x) + (g \circ h)(x)$$

   and, for $a \in A$,

   $$g \circ (af)(x) = g(af(x)) = ag(f(x) = a(g \circ f)(x)$$

   so that $\eta(af) = a\eta(f)$. The above works because $f(x) \in N$ and $g \in \mathrm{Hom}_A(N,P)$. It follows *mutatis mutandis* for φ.

4. *Let A be a commutative ring and M an A-module. Define $t_M : A \to End_A(M)$ by $a \mapsto \tilde{a}$ where $\tilde{a}(x) = ax$. Show that $\tilde{a}$ and t_M are a ring homomorphisms and that* $\ker t_M = \{a \in A \mid aM = 0\}$.

   $\tilde{a}$ is a homomorphism from the first exercise, and t_M is a homomorphism since

   $$t_M(a+b)(x) = (a+b)x = ax + bx = \tilde{a}(x) + \tilde{b}(x) = t_M(a)(x) + t_M(b)(x)$$

   and

   $$\ker t_M = \{a \in A \mid t_M(a) = 0\} = \{a \in A \mid \tilde{a} = 0\} = \{a \in A \mid ax = 0, x \in M\} = \{a \in A \mid aM = 0\}$$

5. *Let A be a commutative ring. Show that any cyclic A-module M is isomorphic to the A-module A/I, for some ideal I of A.*

   Let $t_x : A \to M$ be defined by $t_x(a) = ax$. From the first exercise t_x is a homomorphism of A-modules. Since M is a cyclic A-module there exists some $x \in M$ such that $M = Ax$, and hence $t_x(A) = M$. By the first isomorphism theorem

   $$\frac{A}{\ker t_x} \cong t_x(A) = M$$

6. *Let A be a commutative ring. Show that M is a simple A-module if and only if M is isomorphic to A/P, where P is a maximal ideal in A.*

   Every simple A-module is cyclic, since if there were a nonzero element $x \in M$ such that $M \neq Ax$, then Ax would be a proper submodule of M – contradicting the fact that M is simple. Thus ths previous exercise applies.

   Assume that $A/P \cong M$ where P is a maximal ideal of A and M is an R-module. Since P is maximal the only ideals (and hence submodules when A/P is treated as an A-module) are 0 and A/P, and hence M is simple. For the other direction, note that the submodules of A/P are of the form N/P where $P \subset N$ is an A-module. If the only submodules of A/P are 0 and A/P then the only such N are A and P, i.e., P is maximal in A.

7. *Let A be a commutative ring and $\{M_i\}$ for $1 \leq i \leq n$ a family of A-modules. Let N be any A-module. Let*

$$\varphi : Hom_A(N, M_1 \times \cdots \times M_n) \to \prod_{i=1}^{n} Hom_A(N, M_i)$$

   be defined by $f \mapsto (f_1, \ldots, f_n)$ where $f_i = \pi_i \circ f$ and π_i is the natural projection from $M_1 \times \cdots \times M_n$ to M_i. Show that φ is an isomorphism of A-modules.

   φ is surjective since for any $(f_1, f_2, \ldots, f_n)$ we have $f \mapsto (f_1, \ldots, f_n)$ where $f(x) = (f_1(x), \ldots, f_n(x))$. It is injective since the kernel of φ are all f such that $\pi_i \circ f = 0$ for $1 \leq i \leq n$, but this means precisely that f sends any $x \in N$ to $(0, 0, \ldots, 0)$ and hence $f \equiv 0$. Therefore $\ker \varphi = 0$.

   From previous exercises we know that π_i is an A-module homomorphism, so that $\pi_i \circ (f + g) = \pi_i \circ f + \pi_i \circ g$, and, similarly, $\pi_i \circ (af) = a(\pi_i \circ f)$. It follows immediately that φ is a homomorphism, and hence an A-module isomorphism.

8. *Using the notation from the previous problem, let $\eta_i : M_i \to M_1 \times \cdots \times M_n$ map x to $(0, \ldots, x, \ldots, 0)$, where x is in the i^{th} place. Show that the map*

$$\psi : Hom_A(M_1 \times \cdots \times M_n, N) \to \prod_{i=1}^{n} Hom_A(M_i, N)$$

   given by $f \mapsto (f \circ \eta_1, \ldots, f \circ \eta_n)$ is an isomorphism of A-modules.

   ψ is surjective since $f(x_1, \ldots, x_n) = x_1 + \cdots x_n$ maps to $(f_1, f_2, \ldots, f_n)$ by ψ. It is also injective since if $f, g \in \mathrm{Hom}_A(M_1 \times \cdots \times M_n, N)$ they map 0 to 0, and hence if $f_i = g_i$ for all i, then $f(\vec{x}) = g(\vec{x})$ for all $\vec{x} \in \prod M_i$ since this means precisely that they agree as functions of each coordinate.

   Each η_i is a homomorphism of modules, so that ψ is also a homomorphism, and hence an isomorphism.

9. *Let $\{M_i\}$ with $1 \leq i \leq n$ be a family of A-modules, and $N_i \subset M_i$ sub-modules. Consider the map*

$$\theta : M_1 \times \cdots \times M_n \to M_1/N_1 \times \cdots \times M_n/N_n$$

   given by $\theta(x_1, \ldots, x_n) = (\bar{x}_1, \ldots, \bar{x}_n)$. Show that θ is an A-module epimorphism. Deduce that

$$\frac{M_1 \times \cdots \times M_n}{N_1 \times \cdots \times N_n} \cong \frac{M_1}{N_1} \times \cdots \times \frac{M_n}{N_n}$$

   That θ is an A-module homomorphism follows from the fact that addition and multiplication in M_i/N_i as an A-module is well-defined, i.e., $\overline{x_i + y_i} = \bar{x}_i + \bar{y}_i$ and $\overline{ax_i} = \bar{a}\bar{x}_i$. It is surjective since for any $(\bar{x}_1, \ldots, \bar{x}_n)$ we can choose coset representatives $x_1, \ldots, x_n$ such that $(x_1, \ldots, x_n) \mapsto (\bar{x}_1, \ldots, \bar{x}_n)$. Finally, since $\bar{x}_i = 0$ if and only if $x_i \in N_i$, it follows that

$$\ker \theta = \left\{ \vec{x} \in \prod M_i \mid \bar{x}_i = 0, 1 \leq i \leq n \right\} = \left\{ \vec{x} \in \prod N_i \right\} = \prod N_i$$

   and the relation follows from the first isomorphism theorem.

10. *Let A be a commutative ring and $\{I_i\}$ a family of mututally comaximal ideals for $1 \leq i \leq n$. Let M be an A-module and define $\varphi : M \to M/(I_1 M) \times \cdots \times M/(I_n M)$ by*

$$\varphi(x) = (\theta_1(x), \ldots, \theta_n(x))$$

where $\theta_i(x) = x + I_i M$. Show that φ is a surjective A-module homomorphism. Deduce that

$$\frac{M}{(I_1 \cap \cdots \cap I_n)M} \cong \frac{M}{I_1 M} \times \cdots \times \frac{M}{I_n M}$$

φ is obviously a ring homomorphism since each θ_i is a homomorphism (which follows directly from the fact that addition and module multiplication are well-defined on each submodule). Since the $\{I_i\}$ are comaximal we can choose e_i which is congruent to 1 modulo I_i, and congruent to 0 modulo I_j for all $j \neq i$, since certainly if $I_i + I_j = A$ then $I_1 + \cdots + I_n = 1$. Consider $(\bar{x}_1, \ldots, \bar{x}_n)$ and let x_i be a representative of $\bar{x}_i$. Then $x = \sum_{i=1}^n e_i x_i$ is congruent to x_i modulo $I_i M$ (as $e_j = 0$ modulo I_j for $j \neq i$) so that $\varphi(x) = (\bar{x}_1, \ldots, \bar{x}_n)$. Hence φ is surjective.

Certainly $I_1 M \cap \cdots \cap I_n M = (I_1 \cap \cdots \cap I_n)M$ since the latter is precisely thos elements of the form $\sum a_i x_i$ where the sum is finite, $a_i \in I_j$ for all j, and $x_i \in M$. The kernel of φ will be x such that $x \in I_i M$ for all i, i.e., $x \in I_1 M \cap \cdots I_n M$, so that $\ker \varphi = (I_1 \cap \cdots \cap I_n)M$, and the result follows.