MATH 270: Homework #5

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1. *Evaluate the following integrals:*

   (a) $\int_\gamma \frac{z^2}{z-1}\, dz$, *where* γ *is a circle of radius* 2 *centered at* 0
   Let $f(z) = z^2$. f is holomorphic at 1 and $n(\gamma, 1) = 1$, so by Cauchy's integral formula

   $$2\pi i = 2\pi i f(1) = \int_\gamma \frac{z^2}{z-1}\, dz$$

   (b) $\int_\gamma \frac{e^z}{z^2}\, dz$, *where* γ *is the unit circle*
   Let $f(z) = e^z$. Then $f'(z) = e^z$ and $n(\gamma, 0) = 1$, so by Cauchy's integral formula for derivatives of holomorphic functions

   $$2\pi i = 2\pi i f'(0) = \int_\gamma \frac{e^\zeta}{(\zeta - 0)^2}\, d\zeta = \int_\gamma \frac{e^\zeta}{\zeta^2}\, d\zeta$$

2. *Let* f *be entire and assume that* $|f(z)| \leq M|z|^n$ *for large* z, *constant* M, *and some integer* n. *Show that* f *is a polynomial of degree* $\leq n$.

   Let γ be a circle with sufficiently large radius R so that by theorem 2.4.7

   $$|f^{(k)}(z)| \leq \frac{k!}{R^k} M \sup_{\zeta \in \gamma} |\zeta|^n$$

   But $\sup_{\zeta\in\gamma} |\zeta^n| = R^n$, so if $k > n$ then as $R \to \infty$ the right-hand side of the above inequality goes to 0 and we see that $f^k(z)$ is identically 0. That is, $f^{(k)}(z) = 0$ for all $k > n$. If f is not identically 0 itself then there exists some $j \in \mathbb{N}$ and constant c_i such that $f^{(j)} = c_j$. Inductively we see that there exist constants $c_0, \ldots, c_j$ such that $f(z) = \sum_{k=0}^{j} \frac{c_k}{k!} x^k$, and therefore f must be a polynomial.

3. *Let* f *be holomorphic on a region* Ω *and let* γ *be a closed curve in* Ω. *Show that for any* $z_0 \in \Omega \setminus \gamma$

   $$\int_\gamma \frac{f'(\zeta)}{\zeta - z_0}\, d\zeta = \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta$$

   From class we know that if f is holomorphic on Ω then $f^{(n)}(z)$ exists on Ω for all $n \in \mathbb{N}$ and

   $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

The above statement is a special case where $n = 1$. Note, I emailed the TA about this problem and whether or not we were allowed to assume this general theorem given to us by Sinan on Wednesday. Since I never received a response I will assume that we can use any general theorems given in class with complete proof before the homework was assigned.

4. *Evaluate the following integrals, where γ is a circle of radius 2 centered at the origin:*

   (a) $\int_\gamma \frac{dz}{z^2-1}$
   Using Cauchy's integral formula for $f(z) = 1$,

$$\int_\gamma \frac{1}{z^2-1}\,dz = \frac{1}{2}\left(\int_\gamma \frac{1}{z-1}\,dz - \int_\gamma \frac{1}{z+1}\,dz\right) = \pi i - \pi i = 0$$

   (b) $\int_\gamma \frac{dz}{z^2+z+1}$
   The roots of this polynomial are $\frac{-1\pm i\sqrt{3}}{2}$, which has a modulus of 1. Decomposing the polynomial into fractions gives that the original integral, as in the first part, is the difference of two functions with a 1 in the numerator times some real constant. By Cauchy's integral formula the integral for each of these must be equal, and hence the original integral is 0.

   (c) $\int_\gamma \frac{dz}{z^2-8}$
   Since both roots of the polynomial in the denominator are outside γ, the winding number of γ around both points is 0 and by Cauchy's integral formula

$$\int_\gamma \frac{1}{z^2-8}\,dz = \frac{1}{2}\left(\int_\gamma \frac{1}{z-2\sqrt{2}}\,dz - \int_\gamma \frac{1}{z+2\sqrt{2}}\,dz\right) = 0$$

   (d) $\int_\gamma \frac{dz}{z^2+2z-3}$
   Since the winding number of γ about 3 is 0,

$$\int_\gamma \frac{dz}{z^2+2z-3} = \frac{1}{4}\left(\int_\gamma \frac{dz}{z-1}\,dz - \int_\gamma \frac{dz}{z+3}\right) = \frac{1}{4}(2\pi i - 0) = \frac{\pi i}{2}$$

5. *Let f be analytic on A and $f'(z_0) \neq 0$. Show that if γ if a sufficiently small circle centered at z_0 then*

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{dz}{f(z) - f(z_0)}$$

For a sufficiently small curve γ f^{-1} exists and is differentiable by the Inverse Mapping Theorem with derivative given by $\frac{df^{-1}(w)}{dw} = \frac{1}{f'(z)}$ where $w = f(z)$. Substituting z for $f^{-1}(w)$ and $w_0 = f(z_0)$ gives, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{f(z)-f(z_0)} = \frac{1}{2\pi i}\int_\gamma \frac{df^{-1}(w)}{dw}\frac{dw}{w-w_0} = \frac{df^{-1}(w_0)}{dw} = \frac{1}{f'(z_0)}$$

6. *Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{\sqrt{2\pi}}{4}$.*

Let $f(z) = e^{-z^2}$. Consider the contour γ defined by the line segment joining 0 to R, the arc from R to $Re^{i\frac{\pi}{4}}$ and the line segment joining $Re^{i\frac{\pi}{4}}$ to 0 again. f is holomorphic here so by Cauchy's theorem the integral over γ is 0. Moreover, these three segments can be parameterized by $\gamma_1(t) = t$ for $t \in [0, R]$, $\gamma_2(t) = Re^{it\frac{\pi}{4}}$ for $t \in [0, 1]$, and $-\gamma_3(t) = Re^{i\frac{\pi}{4}}$ for $t \in [0, R]$. Hence

$$0 = \int_0^R e^{-x^2}\,dx + \int_0^1 e^{-R^2 e^{it\frac{\pi}{4}}} iR\frac{\pi}{4} e^{it\frac{\pi}{4}}\,dt - \int_0^R e^{t^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}}\,dt$$

As $R \to \infty$ the first integral is well-known to converge to $\frac{\sqrt{\pi}}{2}$. This is easy to proove by squaring it to produce a double-integral and then changing to polar coordinates. The second integral goes to 0 since $\frac{R}{e^{R^2}} \to 0$ as $R \to \infty$. Therefore

$$\begin{aligned}
\frac{\sqrt{\pi}}{2} &= \lim_{R\to\infty} \int_0^R e^{-t^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}}\,dt \\
&= \lim_{R\to\infty} \int_0^R e^{-it^2} e^{i\frac{\pi}{4}}\,dt \\
&= \lim_{R\to\infty} \int_0^R \frac{(1-i)(\cos t^2 - i\sin t^2)}{\sqrt{2}}\,dt \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2}} \int_0^R \cos t^2 + \sin t^2\,dt + i\frac{1}{\sqrt{2}} \lim_{R\to\infty} \int_0^R \cos t^2 - \sin t^2\,dt
\end{aligned}$$

The only way this quantity can be real-valued is if the imaginary part above is 0, which gives a very simple system of equations of the form $A + B = C$ and $A - B = 0$, where $A = \frac{1}{\sqrt{2}} \int_0^R \cos t^2\,dt$ and $B = \frac{1}{\sqrt{2}} \int_0^R \sin t^2\,dt$. This implies $A = B = \frac{C}{2}$, and therefore, as $R \to \infty$

$$\int_0^\infty \sin t^2\,dt = \int_0^\infty \cos t^2\,dt = \frac{\sqrt{2}}{2}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4}$$