MATH 262: Homework #5

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1. (a) *Show that no two of the spaces* $(0,1)$*,* $(0,1]$*, and* $[0,1]$ *are homeomorphic.*

   Consider the separating properties of these sets. Removing any point from $(0,1)$ separates it, but it is possible to remove a point from $(0,1]$ and $[0,1]$ so that they both remain connected, viz., $\{1\}$. Likewise, the removal of any two points separates $(0,1]$, but it is possible to remove two points from $[0,1]$ and still remain connected, viz., $\{0,1\}$. Therefore none of these spaces are homeomorphic to any of the others.

   (b) *Suppose that there exist imbeddings* $f : X \to Y$ *and* $g : Y \to X$*. Show by means of an example that* X *and* Y *need not be homeomorphic.*

   Let $f(x) = \frac{1}{2}x + \frac{1}{4}$, $X = [0,1]$, and $Y = (0,1)$. Then $f(X) = \left[\frac{1}{4}, \frac{3}{4}\right] \subset Y$. f is clearly an imbedding of X in Y. Similarly, let $g(x) = x$, then $g(Y) = (0,1) \subset [0,1]$ is an imbedding of Y in X. From the previous part X and Y are not homeomorphic.

   (c) *Show that* $\mathbb{R}^n$ *and* $\mathbb{R}$ *are not homeomorphic if* $n > 1$*.*

   The removal of any one point of $\mathbb{R}$ separates it, but, this is not the case with $\mathbb{R}^n$ for $n > 1$. Indeed, not only is $\mathbb{R}^n \setminus \{a\}$ still connected, but it is path-connected. For any two points $x, y \in \mathbb{R}^n \setminus \{a\}$, if a is not on the line connecting x and y then that line connects x and y. Otherwise, if a is on that line, pick another point $z \in \mathbb{R}^n \setminus \{a\}$ distinct from x and y. Then the line connecting x to z joined with the line connecting z to y connects x and y. Hence $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for $n > 1$.

2. *Let* $f : X \to X$ *be continuous. Show that if* $X = [0,1]$ *there is a point* x *such that* $f(x) = x$*, What happens if* X *equals* $[0,1)$ *or* $(0,1)$*?*

   If $f(0) = 0$ or $f(1) = 1$ then we are done, so assume $f(0) > 0$ and $f(1) < 1$. Define $g(x) = f(x) - x$. Then $g(1) < 0 < g(0)$, and by the intermediate value theorm there exists some x such that $g(x) = 0$, which implies that $f(x) = x$.

   If X is $[0,1)$ or $(0,1)$ then this is not necessarily true because neither of these sets are compact. For example, $x \mapsto \frac{1}{2}(x+1)$ is a function which is continuous on X but has no fixed point there.

3. (a) *Let* X *and* Y *be ordered sets in the order topology. Show that if* $f : X \to Y$ *is order preserving and surjective then* f *is a homeomorphism.*

   If f is order preserving then $x < y$ implies $f(x) < f(y)$. f is an injection since if $f(x) = f(y)$ then it cannot be the case that either $x < y$ or $y < x$, since then $f(x) \neq f(y)$. Similarly, if $f(x) < f(y)$ then $x < y$, by the same argument *mutatis mutandis*. Hence f is a bijection and its inverse is also order-preserving. It is therefore sufficient to show that f is an open map, since it then follows that f^{-1} is also an open map and hence that f is a homeomorphism. This is fairly obvious since the order-preserving property of f guarantees that the image of a basis element (x, y) under f is $(f(x), f(y))$, which is still a basis element. f preserves unions since it is a bijection, and therefore the image of an open set under f is an open set.

(b) *Let $X = Y = \bar{\mathbb{R}}_+$. Given a positive integer n, show that the function $f(x) = x^n$ is order preserving and surjective. Conclude that its inverse is continuous.*

If x^n is order-preserving and surjective then by the previous part it is a homeomorphism, which by definition means that its inverse is continuous. If $x < y$ where $x > 0$, then $x^2 < xy < y^2$. Inductively it follows that $x^n < y^n$ for all positive integers n. This map is obviously surjective as $\sqrt[n]{x}$ is a well-defined real number for all $x > 0$ and $n \in \mathbb{Z}_+$.

(c) *Let X be the subspace $(-\infty, -1) \cup [0, \infty)$ of $\mathbb{R}$. Show that the function $f : X \to \mathbb{R}$ defined by setting $f(x) = x + 1$ if $x < -1$ and $f(x) = x$ if $x \geq 0$ is an order-preserving surjection. Is f a homeomorphism? Compare with (a).*

Let $A = (-\infty, -1)$ and $B = [0, \infty)$. If $x, y \in A$ then $f(x) = x + 1 < y + 1$, and similarly for B. If $x \in A$ and $y \in B$ then $x < y$ and $f(x) < 0$ and $f(y) \geq 0$, so that $f(x) < f(y)$. Let $a \in \mathbb{R}$. If $a \geq 0$ then $f(a) = a$. Otherwise, $f(a - 1) = a$, so f is surjective. f cannot be a homeomorphism because $A \cup B$ is disconnected, but this is precisely what makes the implication fail, i.e., there is no guarantee that (a) holds if X is not connected.

4. *What are the components and path components of $\mathbb{R}_l$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_l$?*

Lemma 1. *$\mathbb{R}_l$ is totally disconnected.*

Proof. Let $A \subset \mathbb{R}_l$ be a connected subset such that there are at least two distinct points $x, y \in A$ and assume without loss of generality that $x < y$. $(-\infty, y)$ and $[y, \infty)$ are both closed and open in $\mathbb{R}_l$. $A \setminus (-\infty, y)$ and $A \setminus [y, \infty)$ therefore separate A, i.e., A is not connected. □

From this lemma is follows directly that the only components and therefore path-components are the singletons of $\mathbb{R}_l$. If f is continuous then $f(\mathbb{R})$ must be connected since $\mathbb{R}$ is connected. Since the only connected sets are the singletons, it follows that f must be constant. Every constant function is continuous, and therefore $f : \mathbb{R} \to \mathbb{R}_l$ is continuous if and only if it is constant.

5. (a) *What are the components and path components of $\mathbb{R}^\omega$ in the product topology?*

Lemma 2. *Let $\{X_\alpha\}$ be a family of path-connected spaces indexed by an arbitrary set J. Then the Cartesian product of all these spaces is also path-connected under the product topology.*

Proof. Let $X = \prod_{\alpha \in J} X_\alpha$. Let $x = (x_\alpha)$ and $y = (y_\alpha)$ be two points of X. By hypothesis there exists a continuous function $\gamma_\alpha : [0, 1] \to X_\alpha$ for each $\alpha \in J$ such that $\gamma_\alpha(0) = x_\alpha$ and $\gamma_\alpha(1) = y_\alpha$. Define $\gamma = (\gamma_\alpha)_{\alpha \in J}$. This function is continuous from the properties of the product topology, and by construction connects x and y. □

Since $\mathbb{R}$ is path-connected it follows directly from the above lemma that $\mathbb{R}^\omega$ is path-connected when endowed with the product topology, and hence the only path-component (and component) is $\mathbb{R}^\omega$.

(b) *Consider $\mathbb{R}^\omega$ with the uniform topology. Show that $\vec{x}$ and $\vec{y}$ lie in the same component if and only if the sequence*

$$\vec{x} - \vec{y} = (x_1 - y_1, x_2 - y_2, \ldots)$$

is bounded.

Lemma 3. *Let $\mathbb{R}^\omega$ have the uniform topology. Let A be the set of all bounded sequences and let B be the set of all unbounded sequences. Then A and B separate $\mathbb{R}^\omega$.*

Proof. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Let $\vec{x} = (x_1, x_2, \ldots) \in A$ and $\vec{y} = (y_1, y_2, \ldots) \in B$ There exists some real $R > 0$ such that $|x_n| < R$ for all $n \in \mathbb{N}$ by hypothesis, and, since y is unbounded, there must exist some $k \in \mathbb{N}$ such that $|y_k| > R + 1$. But then $|x_n - y_k| > 1$ and $\overline{\rho}(\vec{x}, \vec{y}) = 1$. Therefore, for any $r < 1$, the ball of radius r about a point in one of A or B does not contain any elements of the other set and hence both are open. Moreover, since $A = \mathbb{R}^\omega \setminus B$, it follows that A and B separate $\mathbb{R}^\omega$. □

It is sufficient to consider the case where $\vec{y} = 0$ since $\vec{x}$ and $\vec{y}$ lie in the same component if and only if $\vec{x} - \vec{y}$ and 0 lie in the same component, and $\vec{x} \mapsto \vec{x} - \vec{y}$ is a homeomorphism of $\mathbb{R}^\omega$ for fixed $\vec{y}$

Assume $\vec{x}$ is bounded, then $|x_n| < R$ for some real $R > 0$ and all $n \in \mathbb{N}$. Define

$$\gamma(t) = t\vec{x} = (tx_1, tx_2, \ldots)$$

For any $\epsilon > 0$ let $\delta = \frac{\epsilon}{R}$. Then $\gamma(B_\delta(t)) \subset B_\epsilon(\gamma(t))$, and hence γ is continuous. Thus γ connects 0 and $\vec{x}$, so that they must lie in the same path component and hence the same component.

Conversely, if $\vec{x}$ is unbounded then by Lemma 3 it is in a different component from 0.

(c) *Consider $\mathbb{R}^\omega$ with the box topology. Show that $\vec{x}$ and $\vec{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence $\vec{x} - \vec{y}$ is eventually zero.*

As in the previous exercise we may assume that $\vec{y} = 0$. If $\vec{x}$ is eventually 0 then there exists some $N \in \mathbb{N}$ such that $\vec{x} \in \mathbb{R}^N \times \{0\} \times \{0\} \times \cdots \subset \mathbb{R}^\omega$, which is homeomorphic to $\mathbb{R}^N$. Since we know $\mathbb{R}^N$ is connected, it follows that $\vec{x}$ and 0 are in the same component.

Let $\vec{x}$ have infinitely many non-zero terms. Define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by

$$h_n(\vec{z}) = \begin{cases} s_n z_n & x_n \neq 0 \\ z_n & x_n = 0 \end{cases}$$

where s_n is a sequence of real numbers such that $s_n|x_n| \to \infty$ as $n \to \infty$, e.g., $s_n = \frac{n}{|x_n|}$. This map is a bijection, and also a homeomorphism. Consider the open set $U_1 \times U_2 \times \cdots$. Then $f_n(U_n)$ is either U_n if $x_n = 0$, or $s_n U_n$ if $x_n \neq 0$, but both of these are open. It is the same for f^{-1}, so that h is a homeomorphism.

Moreover, $h(0) = 0$ and $h(\vec{x})$ is unbounded, so that they lie in different components from the previous part (the box topology is finer than the uniform topology). Since h is a homeomorphism $\vec{x}$ and 0 must be in different components.

6. *Let X be locally path connected. Show that every connected open set in X is path connected.*

Let $A \subset X$ be open and connected and fix $a \in A$. Define

$$P_a = \{x \in A \mid \text{There exists a path connecting } x \text{ to } a\}$$

To show that A is path connected it is sufficient to show that P_a is both open and closed in A since A is connected. To show that P_a is open let $y \in A$. Then by hypothesis there exists a neighborhood in A of y, call it U_y, which is path-connected. But then $P_a = \bigcup_{y \in A} U_y$, and hence P_a is open.

Likewise, to show that P_a is closed, let $y \in A$ be a limit point of P_a. Then there exists a path-connected neighborhood of y, U_y, which intersects with P_a. Let $z \in P_a \cap U_y$. There is a path connecting any $x \in P_a$ to z, and, since U_y is path connected, another path connecting z to y, and therefore a path connecting x to y, viz., the combination of these two paths. Hence $y \in P_a$, and P_a is closed. It follows that $P_a = A$.

7. (a) *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. If $\mathcal{T}'$ is finer than $\mathcal{T}$ what does the compactness of X under one of these topologies imply about compactness under the other?*

If X is compact with respect to $\mathcal{T}'$ then it is compact with respect to $\mathcal{T}$. This is obvious since any cover $\mathcal{C}$ in $\mathcal{T}$ is also a cover in $\mathcal{T}'$, and hence has a finite subcover. The converse does not hold. As an example, let $X = [0, 1]$. If $\mathcal{T}'$ is the discrete topology and $\mathcal{T}$ is the usual topology, then we know that X is compact with respect to $\mathcal{T}$. However, it is not compact with respect to $\mathcal{T}'$ since the union of all the singletons is a cover of X with no finite subcover.

(b) *Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$ then either $\mathcal{T} = \mathcal{T}'$ or they are not comparable.*

It is sufficient to show that if $\mathcal{T}$ and $\mathcal{T}'$ are comparable then they are equal. Assume $\mathcal{T}$ and $\mathcal{T}'$ are comparable and $\mathcal{T}' \subset \mathcal{T}$ without loss of generality. Denote by X' and X the same underlying set

X with the topologies $\mathcal{T}'$ and $\mathcal{T}$ respectively. Then the inclusion map $i : X' \to X$ is a continuous bijection. Since X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, by Theorem 26.6, it follows that i is a homeomorphism and hence $\mathcal{T} = \mathcal{T}'$.

8. *Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B respectively.*

   Lemma 4. *Let Y be a compact subspace of the Hausdorff space X. If $x \in X \setminus Y$ then there exist disjoint neighborhoods U and V containing x and Y, respectively.*

   Proof. Let $x \in X \setminus Y$. For every $y \in Y$ there exist disjoint neighborhoods U_y and V_y of x and y, respectively. Then $\bigcup_{y \in Y} V_y$ is a covering of Y. Since X is compact there exist $V_{y_1}, \ldots, V_{y_n}$ such that $V = \bigcup_{i=1}^n V_i$ covers Y. Let $U = \bigcap_{i=1}^n U_{y_i}$. Then U is open, $x \in U$ since $x \in U_{y_i}$ for every i, and $U \cap V = \emptyset$. □

   Let A and B be disjoint compact subspaces of the Hausdorff space X and let $x \in A$ be arbitrary. From the lemma there exist U_x and V_x containing x and B, respectively. As in the lemma, $\{U_x\}$ cover A, and so there is a finite subset that also covers A, call it $U_{x_1}, \ldots, U_{x_n}$ and their union U. Then the intersection of the corresponding $\{V_{x_i}\}$ is open, contains B, and is disjoint from U.

9. *Show that if Y is compact then the projection $\pi : X \times Y \to X$ is a closed map.*

   Let $A \subset X \times Y$ be closed. We want to show that $\pi(A)$ is closed. Let $x \in X \setminus \pi(A)$. Then

   $$\pi^{-1}(x) = x \times Y \subset (X \times Y) \setminus A$$

   and therefore $\pi^{-1}(A)$ is open. By Lemma 26.8 (the tube lemma) there exists a $W \subset X$ such that $x \times W$ and $W \times Y \subset (X \times Y) \setminus A$. Hence $W \cap \pi(A) = \emptyset$ and $x \in W \subset X \setminus \pi(A)$, so that $X \setminus \pi(A)$ is open, i.e., $\pi(A)$ is closed.

10. *Let $f : X \to Y$ and Y be compact Hausdorff. Show that f is continuous if and only if the* ***graph*** *of f*

    $$G_f = \{x \times f(x) \mid x \in X\}$$

    is closed in $X \times Y$.

    To begin, assume that f is continuous. We will show that $(X \times Y) \setminus G_f$ is open. Let $(x, y) \in (X \times Y) \setminus G_f$ so that $y \neq f(x)$. Since Y is Hausdorff there exist disjoint neighborhoods U and V of $f(x)$ and y, respectively. It follows that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \times V \subset (X \times Y) \setminus G_f$. Since f is continuous $f^{-1}(U)$ is open, and therefore $(X \times Y) \setminus G_f$ is open, i.e, G_f is closed.

    Now suppose G_f is closed. Let $V \subset Y$ be open. Since $(X \times Y) \setminus (X \times (Y \setminus V)) = X \times V$ is open, $X \times (Y \setminus V)$ is closed. Then $A = G_f \cap (X \times (Y \setminus V))$ is also closed. But

    $$A = \{x \times f(x) \mid f(x) \in Y \setminus V\}$$

    From the previous problem $\pi(A) = \{x \mid x \in Y \setminus V\} = Y \setminus f^{-1}(V)$ is closed, and hence $f^{-1}(V)$ is open and f is continuous.