MATH 270: Homework #8

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1. *Find the residues of the following functions at the indicated points:*

   (a) $\frac{e^z-1}{\sin z}$ *at* $z=0$
   Both e^z-1 and $\sin z$ have zeros of order 1 at $z=0$, so the residue at $z=0$ is 0.

   (b) $\frac{1}{e^z-1}$ *at* $z=0$
   e^z-1 has a zero of order 1 at $z=0$, but all derivatives are non-zero, so the residue at $z=0$ is given by $1/e^z$ evaluated at $z=0$, i.e., it has a residue of 1.

   (c) $\frac{z+2}{z^2-2z}$ *at* $z=0$
   $\lim_{z\to 0} z\frac{z+2}{z^2-2z}=-1$, so this is the desired residue.

   (d) $\frac{1+e^z}{z^4}$ *at* $z=0$
   z^4 has a zero of order 4, so the residue is $\phi^{(3)}(0)/3!$ where $\phi(z)=1+e^z$. Evaluated at $z=0$ this is $\frac{1}{6}$.

2. *Evaluate* $\int_\gamma \frac{dz}{(z+1)^3}$ *for the following curves:*

   (a) γ *is a circle of radius* 2 *centered at* 0.
   Let $f(z)=\frac{1}{(z+1)^3}$ and $\phi(z)=1$, then

   $$\mathrm{Res}_{z=-1}\frac{1}{(z+1)^3}=\frac{\phi^{(z)}(1)}{2!}=0$$

   By the Residue Theorem the integral is therefore 0.

   (b) γ *is a square with vertices at* 0, 1, $1+i$, *and* i.
   This region contains no singularities and so the integral is 0.

3. *Show using Cauchy's inequalities that the Laurent-Taylor series of a holomorphic funciton on* $\Omega\setminus\{z_0\}$, *where* z_0 *is a removable singularity or a pole, converges in* $0<|z-z_0|<R$ *where* $D(z_0,R)\subset\Omega$, *and that*

   $$f(z)=\sum_{k=-n}^{\infty}a_k(z-z_0)^k$$

This question confuses me because it seems like exactly what DDSF proved in class, namely, that if f has a pole or a removable singularity at z_0 then on $D(z,0) \setminus \{z_0\}$ there is some holomorphic function h such that

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + h(z)$$

The first (finite) number of terms do not affect the convergence of anything since they are finite, and $h(z)$ is holomorphic precisely because it can be expressed as a convergent power-series. At what point are Cauchy's inequalities necessary?

4. *Show the convergence of the series*

$$\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

on $D(z_0, R) \setminus \{z_0\}$ *where*

$$a_k = \frac{1}{2\pi i} \int_{\partial D(z_0, R/2)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta$$

I do not understand why it is sufficient to prove this statement for $R/2$, so instead I will do it for any two curves below.

5. *Deduce Laurent's theorem from these results.*

If I remember to draw a picture of what I'm talking about in the morning, then that will go a long way toward explaining what I'm about to try and articulate – I'm pretty sure this is a normal thing to do anyhow. Let γ_1 and γ_2 be two circles with radius r_1 and r_2 such that $r_1 < |z - z_0| < r_2$. Fix z in this region and let γ_3 be a segment joining γ_1 and γ_2 that does not pass through z. Let γ be the path that first traverses γ_2, then γ_3, then γ_1 in reverse order, and back again along γ_3 in reverse order. Clearly γ as constructed like this, i.e., $\gamma = \gamma_2 + \gamma_3 - \gamma_1 - \gamma_3$, is homotopic to a neighborhood γ' of z as the points of self-intersection are homotopic to a point. f is holomorphic in this neighborhood, so by Cauchy's integral formula

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\gamma'} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta
\end{aligned}$$

But $\frac{f(\zeta)}{\zeta - z}$ itself is also holmorphic on γ_2, so that

$$\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\gamma_2} \sum_{k=0} a_k (z-z_0)^k\, d\zeta = \sum_{k=0}^{\infty} \left[\int_{\gamma_2} a_k\, d\zeta \right] (z-z_0)^k$$

since $(z - z_0)^k$ is bounded on γ_2. From Cauchy's formula we know that $a_k = \frac{f(\zeta)}{(\zeta - z)^{k+1}}$, and that this series converges uniformly on γ_2, so the portion consisting of positive powers of n is proven for the laurent expansion.

Anyhow, it's getting late. The portion with the negative parts is done the same way, except that the fact that the powers of n are negative guarantee that it converges *outside* the radius r_1, i.e., if a power-series in z has a radius of convergence R then the corresponding power-series in $1/z$ converges in $1/R$, but since R is arbitrarily small, this new power-series converges outside the disc rather than inside – it has a lower limit of $1/R$ and no upper-limit in terms of convergence.

I don't understand why part 2 is different from the extra-credit, though. This part shows that the Laurent series with the given coefficients converges to f in particular, while part 2 seems to just say that it converges, but not to what specifically. Why not simply prove the former, which implies the latter?