CMSC 277: Homework #4

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1. *Let $\mathfrak{L} = \{<, +, F\}$ where $<$ is a binary relation symbol, $+$ is a binary function symbol, and F is a unary function symbol. Find a $\sigma \in \mathrm{Sent}_{\mathfrak{L}}$ such that for all $f : \mathbb{R} \to \mathbb{R}$, $(\mathbb{R}, <, +, f) \vDash \sigma$ if and only if f is continuous on $\mathbb{R}$.*

   Fix $f : \mathbb{R} \to \mathbb{R}$. Define an $\mathfrak{L}$-structure $\mathfrak{M}$ as follows. Let $M = \mathbb{R}$, $F^{\mathfrak{M}} = f$,

   $$<^{\mathfrak{M}} = \left\{(x, y) \in \mathbb{R}^2 \mid x < y\right\}$$

   and

   $$(+t_1t_2)^{\mathfrak{M}} = t_1^{\mathfrak{M}} + t_2^{\mathfrak{M}}$$

   for $t_1, t_2 \in \mathrm{Term}_{\mathfrak{L}}$. Now define σ as

   $$\sigma = \forall c \forall x \forall < 0 \varepsilon \to \exists \delta \wedge \wedge < c + x\delta < x + \delta \neg = xc \wedge < fc + x\varepsilon < fx + \varepsilon fc$$

   which in human-readable format is simply

   $$\sigma = (\forall c)(\forall x)(\forall \varepsilon > 0)(\exists \delta)(((x \neq c) \wedge (c < x+\delta) \wedge (x < \delta + c)) \to ((f(c) < f(x)) + \varepsilon) \wedge (f(x) < f(c) + \varepsilon))$$

   or more succinctly

   $$\sigma = (\forall c)(\forall x)(\forall \varepsilon > 0)(\exists \delta)(0 < |x - c| < \delta \to |f(x) - f(c)| < \epsilon)$$

2. *Let $\mathfrak{L} = \{R\}$ where R is a binary relation symbol. Show that the class of DAGs is weak elementary class in the language $\mathfrak{L}$.*

   First note that digraphs correspond bijectively with the class of $\mathfrak{L}$-structures, since a digraph is by definition a pair (V, E) where $E \subset V^2$. If $\mathfrak{M}$ is an $\mathfrak{L}$-structure then the pair $(M, R^{\mathfrak{M}})$ is a directed graph. Similarly, if G is a digraph then it can be realized as an $\mathfrak{L}$-structure by interpreting R as $E(G)$. Define

   $$\sigma_n = \neg(\exists v_1 \exists v_2 \cdots \exists v_n((Rv_1v_2) \wedge (Rv_2v_3) \wedge \cdots \wedge (Rv_nv_1))$$

   and let $\Sigma = \{\sigma_n \mid n \in \mathbb{N}^+\}$. Then the class of acyclic digraphs is precisely $\mathrm{Mod}(\Sigma)$.

3. *Let $\mathfrak{L} = \{F\}$ where F is a binary relation symbol.*

   (a) *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Let g, h be defined as in the statement of the problem. Show that $(A, g) \not\equiv (B, h)$.*

   Define

   $$\sigma = (\exists z \forall y \exists x)(f(x, y) = z)$$

   $(A, g) \vDash \sigma$ since $g(2, 1) = g(2, 2) = g(4, 3) = g(2, 4) = 2$, but $(B, h) \nvDash \sigma$ as there is not even an element of B contained in every column of the table.

(b) *Show that* $(\mathbb{N},+) \not\equiv (\mathbb{Z},+)$.

The most clear difference between the additive structure of $\mathbb{N}$ and $\mathbb{Z}$ is that the former is a monoid while the latter is a group. However, since we have no constant symbol with which to express the identity we will have to find a less direct way. Let F be interpreted as addition on $\mathbb{N}$ and $\mathbb{Z}$, respectively, and define σ as follows, in Polish notation

$$\sigma = (\forall n \exists m \forall p = p + +nmp)$$

That is, for all n there exists an m such that for all p, $n+m+p=p$. $\sigma \in \mathrm{Th}(\mathbb{Z})$ since every $n \in \mathbb{Z}$ has an additive inverse, while $\sigma \notin \mathrm{Th}(\mathbb{N})$ since 0 is the only element of $\mathbb{N}$ which has an inverse. Hence $\mathrm{Th}(\mathbb{N}) \neq \mathrm{Th}(\mathbb{Z})$ and therefore $(\mathbb{N},+) \not\equiv (\mathbb{Z},+)$.

4. (a) *Let* $\mathfrak{L} = \emptyset$. *For each* $n \in \mathbb{N}^+$ *find a* $\sigma \in \mathrm{Sent}_{\mathfrak{L}}$ *such that* $\mathrm{Spec}(\sigma) = \{n\}$.

Let

$$\tau_i = \bigwedge_{\substack{k=1 \\ k \neq i}}^{n} (v_i \neq v_k)$$

and define

$$\sigma_n = (\exists v_1 \cdots \exists v_n ((\tau_1) \wedge (\tau_2) \wedge \cdots \wedge (\tau_n))) \wedge (\forall v)((v = v_1) \vee (v = v_2) \vee \cdots \vee (v = v_n))$$

Then $\mathrm{Spec}(\sigma_n) = \{n\}$ since *sigma* is satisfiable if and only if there are precisely n elements in any $\mathfrak{L}$-structure which models σ_n.

(b) *Give an example of a finite language* $\mathfrak{L}$ *and a* $\sigma \in \mathrm{Sent}_{\mathfrak{L}}$ *such that* $\mathrm{Spec}(\sigma) = \{2^n \mid n \in \mathbb{N}^+\}$.

I cannot think of how to do this with a finite language. At a glance the most obvious thing to do is to choose a language $\mathfrak{L}$ in which it is possible to construct a sentence such that

$$\mathfrak{M} \vDash \sigma \text{ if and only if } A = \wp(M) \text{ for some finite set } A$$

or something similar using the fact that there are 2^n boolean functions on $\{0,1\}^n$. However, I cannot think of how to do this without including an item in the language for *every* n, making $\mathfrak{L}$ infinite.

5. (a) *Let* $\mathfrak{L} = \{P\}$ *where* P *is a unary relation symbol. Calculate* $I(\mathrm{Cn}(\emptyset), n)$ *for all* $n \in \mathbb{N}^+$.

Let $\mathfrak{M}$ be an $\mathfrak{L}$-structure of cardinality n. We may assume without loss of generality that $M = \{1, \ldots, n\}$. Note that there are only $n+1$ possible values for $P^{\mathfrak{M}}$, viz.,

$$P^{\mathfrak{M}} \in \{\emptyset, \{1\}, \ldots, \{n\}\}$$

If $\mathfrak{M}, \mathfrak{M}'$ are two $\mathfrak{L}$-structures of cardinality n and both $P^{\mathfrak{M}}$ and $P^{\mathfrak{M}'}$ are nonempty then one can define an isomorphism between the two by sending the single element in $P^{\mathfrak{M}}$ to the single element in $P^{\mathfrak{M}'}$, and permuting the remaining elements. If one of $P^{\mathfrak{M}}$ or $P^{\mathfrak{M}'}$ is empty, however, the two $\mathfrak{L}$-structures are not even elementarily equivalent since the sentence asserting that Px is true can be true in in the nonempty structure, depending on the variable assignment, but will never be true in the empty structure.

Hence

$$I(\mathrm{Cn}(\emptyset), n) = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases}$$

(b) *Let* $\mathfrak{L} = \{F\}$ *where* F *is a unary function symbol. Let*

$$\sigma = (\forall x \forall y ((fx = fy) \to (x = y))) \wedge (\forall z \exists x (fx = z))$$

Show that $I(\mathrm{Cn}(\sigma), n) = p(n)$ *for all* $n \in \mathbb{N}^+$ *where* $p(n)$ *is the partition function.*

Let $\mathfrak{M}$ be an $\mathfrak{L}$-structure of cardinality n. Then $\mathfrak{M} \models \sigma$ if and only if F is interpreted as a permutation of $\mathfrak{M}$, of which there are $n!$. Consider these permutations as elements of the group S_n acting on $\mathfrak{M}$ and let $\mathfrak{M}_\sigma$ be the model corresponding to $\sigma \in S_n$. It is fairly clear that $\mathfrak{M}_\sigma \cong \mathfrak{M}_\tau$ if and only if σ and τ are conjugate in S_n. This follows since if $\alpha\sigma\alpha^{-1} = \tau$ for some $\alpha \in S_n$ then α as a permutation of the underlying set M induces an isomorphism between $\mathfrak{M}_\sigma$ and $\mathfrak{M}_\tau$. Similarly, if $\mathfrak{M}_\sigma \cong \mathfrak{M}_\tau$ via β then it follows directly from the definition of an isomorphism of models that $\beta\sigma\beta^{-1} = \tau$. Hence $I(\mathrm{Cn}(\sigma), n)$ is equal to the number of conjugacy classes of S_n.

It is a basic result from group theory that two elements of S_n are conjugate if and only if they have the same cycle type. It is easy to see that the number of cycle types of elements in S_n is exactly $p(n)$, where $p(n)$ is the number of partitions of n, since each cycle type corresponds to exactly one partition of n. Hence $I(\mathrm{Cn}(\sigma), n) = p(n)$.