MATH 209: Homework #6

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1. *Show that if $f \in L^\infty(X)$ then $|f| \le \|f\|_\infty$ almost everywhere.*

   Recall that $\|f\|_\infty = \inf A$ where $A = \{c > 0 \mid |f| \le c \text{ almost everywhere}\}$. By definition, for any $\epsilon > 0$ there exists a $c \in A$ such that $c < \|f\|_\infty + \epsilon$. Moreover, from the definition of A,

   $$|f| \le c < \|f\|_\infty + \epsilon$$

   But this means that $|f| \le \|f\|_\infty$ whenever $|f| \le c$, i.e., almost everywhere.

2. *Show that if f is a nonnegative, measurable function such that $\int_X f\,d\mu = 0$, then $f = 0$ almost everywhere.*

   To prove the contrapositive, assume that there exists a $\epsilon > 0$ such that $f > \epsilon$ on a set $A \subset X$ of positive measure. Then

   $$0 < \epsilon\mu(A) \le \int_X f\,d\,u$$

3. *Let p, q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ then $fg \in \mathcal{L}^1$ and $\|fg\|_1 \le \|f\|_p \cdot \|g\|_q$.*

   Let $f \in L^p(X)$ and $g \in L^q(X)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

   Lemma *For any two numbers $x, y \ge 0$ and $\frac{1}{p} + \frac{1}{q} = 1$*

   $$xy \le \frac{x^p}{p} + \frac{y^q}{q} \tag{1}$$

   Proof Let $f(x) = xy - \frac{x^p}{p}$. Then

   $$f'(x) = y - x^{p-1}$$

   which, since the function is monotonic increasing, implies that the maximum is at $x = y^{\frac{1}{p-1}}$. Therefore

   $$\begin{aligned} xy - \frac{x^p}{p} &\le y^{\frac{p}{p-1}} \frac{y^{\frac{p}{p-1}}}{p} \\ &= \frac{y^{\frac{p}{p-1}}(p-1)}{p} \\ &= \frac{y^q}{q} \end{aligned}$$

Pointwise, this is true for arbitrary nonnegative functions. Using (1) with $x = \frac{|f|}{\|f\|_p}$ and $y = \frac{|g|}{\|g\|_q}$, it follows from the monotonicity of the Lebesgue integral and the definition of $\|\cdot\|_p$ that

$$\begin{aligned}\int_X \frac{|fg|}{\|f\|_p\|g\|_q}\,d\mu &\leq \frac{1}{p\|f\|_p^p}\int_X |f|^p\,d\mu + \frac{1}{q\|g\|_q^q}\int_X |g|^q\,d\mu \\ &= \frac{1}{p}+\frac{1}{q} \\ &= 1\end{aligned}$$

Which is equivalent to

$$\|fg\|_1 \leq \|f\|_p\|g\|_q < \infty$$

From this, then, it follows that if $f \in L^p(X)$ and $g \in L^q(X)$, then $fg \in L^1(X)$.

4. *Show that if $f, g \in \mathcal{L}^p$ then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

   Assume $p > 1$, since otherwise the above follows trivially from the triangle inequality. Let $q = \frac{p}{p-1}$ so that $\frac{1}{p}+\frac{1}{q} = 1$. Let $f, g \in L^2(X)$, then

$$|f+g|^p = |f+g||f+g|^{p-1} \leq |f||f+g|^{p-1} + |g||f+g|^{p-1}$$

   and

$$\int_X |f+g|^p\,d\mu \leq \int_X |f||f+g|^{p-1}\,d\mu + \int_X |g||f+g|^{p-1}\,d\mu \tag{2}$$

   From the previous problem it follows that

$$\int_X |f||f+g|^{p-1}\,d\mu \leq \left(\int_X |f|^p\right)^{\frac{1}{p}}\left(\int_X |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \tag{3}$$

$$\int_X |g||f+g|^{p-1}\,d\mu \leq \left(\int_X |g|^p\right)^{\frac{1}{p}}\left(\int_X |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \tag{4}$$

   Applying (2) to the sum of (3) and (4) and noting that $(p-1)q = p$ and $1 - \frac{1}{q} = \frac{1}{p}$ yields

$$\int_X |f+g|^p\,d\mu \leq \left(\int_X |f+g|^p\right)^{\frac{1}{q}}\left[\left(\int_X |f|^p\right)^{\frac{1}{p}} + \left(\int_X |g|^p\right)^{\frac{1}{p}}\right]$$

   which is equivalent to

$$\left(\int_X |f+g|^p\,d\mu\right)^{1-\frac{1}{q}} = \left(\int_X |f+g|^p\,d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\right)^{\frac{1}{p}} + \left(\int_X |g|^p\right)^{\frac{1}{p}}$$

5. *Finish the proof that $\mathcal{L}^p$ is complete by extending it to $p > 1$.*

   If $p = \infty$ then let $\{f_n\}$ be a Cauchy sequence. It follows that for n, m sufficiently large

$$|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty < \epsilon$$

   It follows immediately that $\|\lim_{n\to\infty} f_n\|_\infty < \infty$, and therefore $L^\infty(X)$ is complete.

For $1 \leq p < \infty$ we will augment the proof in class. Let $\{f_n\}$ be a Cauchy sequence and pick a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\| < \epsilon 2^{-k}$ for any $\epsilon > 0$. Define

$$g_k = \sum_{i=1}^{k} |f_{n_{k+1}} - f_{n_k}|$$

From Minkowski's inequality

$$\|g_k\|_p = \left\| \sum_{i=1}^{k} |f_{n_{k+1}} - f_{n_k}| \right\|_p \leq \sum_{i=1}^{k} \|f_{n_{k+1}} - f_{n_k}\|_p < 1$$

Fatou's Lemma guarantees that $\|\lim_{k\to\infty} g_k\|_p$ is finite almost everywhere since

$$\left\| \lim_{k\to\infty} g_k \right\|_p = \left(\int_X \left(\lim_{k\to\infty} g_k \, d\mu \right)^p \right)^{\frac{1}{p}} \leq \liminf_{k\to\infty} \left(\int_X g_n^p \, d\mu \right)^{\frac{1}{p}} \leq 1$$

As in class, define

$$f = f_{n_1} + \sum_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k} \right)$$

Since the right-hand side is absolutely convergent, f is well-defined (almost everywhere). Moreover, from the above definition it follows that $f(x) = \lim_{k\to\infty} f_{n_k}(x)$, again, almost everywhere. To show that $f_n \to f$ in $L^p(X)$, let $\epsilon > 0$ and n, m sufficiently large so that $\|f_n - f_m\| < \epsilon$. By Fatou's Lemma,

$$\int_X |f - f_m|^p \, d\mu \leq \liminf_{k\to\infty} \int_X |f_{n_k} - f_m|^p \, d\mu \leq \epsilon^p$$

6. *Show that if $f, g \in L^p(X)$ then $(f \mid g) = \int_X f\bar{g} \, d\mu$ is well-defined and a positive-definite Hermitian form.*

This is well-defined since if we consider the product of the component real-valued functions of these two complex-valued functions, which are measurable, we get an integral of the same form (i.e., the component functions of the products are also integrable).

Positive definite

$$(f \mid f) = \int_X f\bar{f} \, d\mu = \int_X |f|^2 \geq 0$$

From Problem 2 it follows that equality holds if and only if $f = 0$ almost everywhere. **Sesquilinear**

$$\begin{aligned} (\alpha f_1 + \beta f_2 \mid g) &= \int_X (\alpha f_1 + \beta f_2)\bar{g} \, d\mu \\ &= \alpha \int_X f_1 \bar{g} \, d\mu + \beta \int_X f_2 \bar{g} \, d\mu \\ &= \alpha (f_1 \mid g) + \beta (f_2 \mid g) \end{aligned}$$

Since the conjugate preserves multiplication and addition, it follows that the conjugate scalars pull out and is linear in the second variable.

Complex symmetric Since integration preserves conjugation (i.e., the conjugate of the integral is the integral of the conjugate),

$$\begin{aligned}
\overline{(f \mid g)} &= \overline{\int_X f\bar{g}\,d\mu} \\
&= \int_X \overline{f\bar{g}}\,d\mu \\
&= \int_X \bar{f}g\,d\mu \\
&= (g \mid f)
\end{aligned}$$

7. *Show that $L^p(X) \subseteq L^1(X)$ for $1 \le p < \infty$ and $\mu(X) < \infty$.*

   Let p, q be such that $\frac{1}{p} + \frac{1}{q} = 1$, $g = 1$, and $f \in L^p(X)$. Then clearly $g \in L^q(X)$, so by Hölder's inequality,
   $$\|f\| = \|fg\| \le \|f\|_p \|g\|_q < \infty$$

8. *Show that an orthonormal basis is linearly independent.*

   Let $\{v_n\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, $\{v_{n_i}\}$ be an arbitrary finite collection of basis elements, and
   $$\alpha_1 v_{n_1} + \alpha_2 v_{n_2} + \cdots + \alpha_m v_{n_m} = 0$$

   where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are scalars. For each $i = 1, 2, \ldots, m$ it follows from the properties of an inner product on a Hilbert space that

   $$0 = (\alpha_1 v_{n_1} + \alpha_2 v_{n_2} + \cdots + \alpha_m v_{n_m} \mid v_{n_i}) = \sum_{k=1}^{m} \alpha_k (v_{n_k} \mid v_{n_i}) = \alpha_i (v_{n_i} \mid v_{n_i}) = \alpha_i$$

   This proves the linear independence of arbitrary $\{v_{n_i}\}$, and therefore of $\{v_n\}$.