MATH 209: Homework #1

Jesse Farmer

05 April 2004

1. *Generalize the system of polar coordinates in $\mathbb{R}^2$ to $\mathbb{R}^n$.*

   Pick a point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We claim that the polar representation of this is

$$x_k = \begin{cases} r_{n-1} \cos\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} & k = 1 \\ r_{n-1} \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} & k = 2 \\ r_{n-1} \cos\theta_{k-1} \sin\theta_k \cdots \sin\theta_{n-1} & k \geq 3 \end{cases}$$

   For $\mathbb{R}^2$ we know that (x_1, x_2) is represented as $r_1(\cos\theta_1, \sin\theta_2)$ and for $\mathbb{R}^3$ that (x_1, x_2, x_3) is represented as $r_2(\cos\theta_1 \sin\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_2)$. For induction, assume that this is true for $\mathbb{R}^n$ and consider (x_{n+1}, r_{n-1}). Converting this to polar coordinates,

$$\begin{aligned} x_{n+1} &= r_n \cos\theta_n \\ r_{n-1} &= r_n \sin\theta_n \end{aligned}$$

   We have expressions for all x_k with $k < n$, each containing an r_{n-1}. Substituting in the above for r_{n-1} yields

$$x_k = \begin{cases} r_n \cos\theta_1 \sin\theta_2 \cdots \sin\theta_n & k = 1 \\ r_n \sin\theta_1 \sin\theta_2 \cdots \sin\theta_n & k = 2 \\ r_n \cos\theta_{k-1} \sin\theta_k \cdots \sin\theta_n & k \geq 3 \end{cases}$$

   Which proves the original statement.

2. *Find the volume element of $\mathbb{R}^n$ in spherical coordinates.*

   We claim that the volume element for $\mathbb{R}^n$ with $n \geq 3$ is

$$r_{n-1}^{n-1} \sin\theta_2 \sin^2\theta_3 \cdots \sin^{n-2}\theta_{n-1} dr_{n-1} d\theta_1 \cdots d\theta_{n-1}$$

   Given that the volume element for $\mathbb{R}^2$ is $r dr_1 d\theta_1$, we will assume for induction that the above is true for $\mathbb{R}^n$. For $\mathbb{R}^{n+1}$ the volume element

$$dx_1 dx_2 \cdots dx_{n+1}$$

   can be written by assumption as

$$r_{n-1}^{n-1} \sin\theta_2 \sin^2\theta_3 \cdots \sin^{n-2}\theta_{n-1} dr_{n-1} d\theta_1 \cdots d\theta_{n-1} dx_{n+1}$$

which is equivalent to

$$r_{n-1}^{n-1} r_n \sin\theta_2 \sin^2\theta_3 \cdots \sin^{n-2}\theta_{n-1} dr_n d\theta_1 \cdots d\theta_{n-1} d\theta_n$$

But from Problem 1 we know

$$r_{n-1} = r_n \sin\theta_n$$

which implies the volume element for $\mathbb{R}^{n+1}$ is

$$r_n^n \sin\theta_2 \sin^2\theta_3 \cdots \sin^{n-2}\theta_{n-1} \sin^{n-1}\theta_n dr_n d\theta_1 \cdots d\theta_{n-1} d\theta_n$$

This proves our original statement.

3. *Show that* $\lim_{n\to\infty} V(B^n) = 0$.

Consider the ratio between the volumes of the n-sphere and the n-dimensional unit hypercube

$$R(n) = \frac{V(B^n)}{V(C^n)} = \frac{V(B^n)}{2^n}$$

It is then sufficient to show

$$\lim_{n\to\infty} 2^n R(n) = 0$$

$R(n)$ is also the probability of choosing n independent and identically distributed points from $[-1, 1]$ and having them lie in the unit sphere. If five or more of these choises are greater than $\frac{1}{\sqrt{5}}$ in magnitude then the point rests outside the unit sphere. The interval $[-1, 1]$ has length 2 and the interval $\left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$ has length $\frac{2}{\sqrt{5}}$, so the probability of picking a point in the latter interval is $\frac{1}{\sqrt{5}}$. We use $\sqrt{5}$ because 5 is the first number whose square root is greater than 2, but any larger number would also suffice. Moreover, any number greater than 1 could be used to show that

$$\lim_{n\to\infty} R(n) = 0$$

Because we can only pick a finite number of points outside $\left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$, for large n we have that $R(n)$ decreases by about a factor of $\sqrt{5}$. But $\sqrt{5} > 2$, so

$$\lim_{n\to\infty} 2^n R(n) \approx \lim_{n\to\infty} \left(\frac{2}{\sqrt{5}}\right)^n = 0$$

4. *Do the following problems about the gamma function*

   (a) *Find* $x_0 \in (0, 1)$ *such that* $\Gamma(s)$ *assumes a minimum at at* x_0.

   (b) *Find* $\Gamma(x_0)$.

   (c) *Show that* $\Gamma(s)$ *is monotonic decreasing on* $(0, x_0)$ *and monotonic increasing on* $(x_0, 0)$.

   (d) *Graph* $\Gamma(s)$ *for* $s > 0$.

   (e) *Extend* $\Gamma(s)$ *to* $(\infty, 0) \setminus \{-1, -2, -3, \ldots\}$.

5. *Show that any monotone function $f : [a, b] \to \mathbb{R}$ is almost everywhere differentiable.*

In this proof will will denote the left-hand and right-hand limits of f at x_0 as $f(x_0 - 0)$ and $f(x_0 + 0)$, respectively.

First we show that any monotone function as above is almost everywhere continuous. Let $x \in [a, b]$ and let $\{x_n\}$ be any sequence such that $x_n < x$ and $x_n \to x$. Then $f(x_n)$ is a nondecreasing sequence which is bounded above by $f(x_0)$, and hence $f(x-0)$ exists. Likewise for $f(x+0)$. Therefore, if f has a discontinuity it is a jump discontinuity. The sum of the sizes of these jumps can be nore more than $f(b) - f(a)$. Let J_n be the set of all jumps greater than $\frac{1}{n}$ and let J be the set of all jumps. Then

$$J = \bigcup_{n=1}^{\infty} J_n$$

where each J_n is finite. Therefore J is at most countably infinite, and thus measure zero.

We will first prove the theorem for continuous monotone functions, and use the above to extend it to every monotone function.

Definition: *Let f be a continuous function defined on an interval $[a, b]$. A point $x_0 \in [a, b]$ is called* invisible from the right *if there is a point ξ such that $x_0 < \xi \le b$ and $f(x_0) < f(\xi)$, and* invisible from the left *if there is a point ξ such that $a \le \xi < x_0$ and $f(\xi) < f(x_0)$.*

Lemma 1: *The set of all points invisible from the right with respect to a function f continuous on $[a, b]$ is the union of no more than countably many pairwise disjoint open intervals (a_k, b_k) such that*

$$f(a_k) \le f(b_k) \quad (k = 1, 2, \ldots) \tag{1}$$

If x_0 is invisible from the right then this is true of any point sufficiently close to x_0, and hence any point within an open neighborhood of x_0. So the set G of all these points is open, and can therefore be written as the countable union of pairwise disjoint intervals (a_k, b_k). Let (a_k, b_k) be one of these intervals and suppose $f(b_k) < f(a_k)$. There is a point $x_0 \in (a_k, b_k)$ such that $f(x_0) > f(b_k)$. Of the points $x \in (a_k, b_k)$ such that $f(x) = f(x_0)$, let x^* be the one with the largest x-coordinate. Since x^* belongs to (a_k, b_k) and hence is invisible from the right, there is a point $\xi > x^*$ such that $f(\xi > f(x^*)$. Clearly ξ cannot belong to (a_k, b_k) from our choice of x^*. Likewise, $\xi > b_k$ is impossible, since it implies $f(b_k) < f(x_0) < f(\xi)$ despite the fact that b_k is not invisible from the right. Since $\xi \ne b_k$, we have a contradiction. Therefore $f(a_k) \le f(b_k)$.

Lemma 1': *The set of all points invisible from the left with respect to a function f continuous on $[a, b]$ is the union of no more than countably many pairwise disjoint open intervals (a_k, b_k) such that*

$$f(a_k) \ge f(b_k) \quad (k = 1, 2, \ldots)$$

The proof of this statement follow *mutatis mutandis* from Lemma 1.

We denote the upper and lower limits from the left and right sides as $\lambda_L, \Lambda_L, \lambda_R, \Lambda_R$, respectively.

Lemma 2: *Let f be a continuous nondecreasing function on $[a,b]$ with λ_L, Λ_R as defined above. Given any numbers c, C, ρ such that*

$$0 < c < C < \infty, \qquad \rho = \frac{c}{C}$$

let

$$E_\rho = \{x \mid \lambda_l < c, \Lambda_R > C\}$$

Then

$$\mu\{x \mid x \in E_\rho \cap (\alpha, \beta)\} \leq \rho(\beta - \alpha)$$

for every $(\alpha, \beta) \subset [a,b]$.

Let $x_0 \in (\alpha, \beta)$ such that $\lambda_L < c$. Then there exists a point $\xi < x$ such that

$$\frac{f(\xi) - f(x_0)}{\xi - x_0} < c$$

That is,

$$f(\xi) - c\xi > f(x_0) - cx_0$$

Therefore x_0 is invisible from the left with respect to the function $f(x) - cx$. By Lemma 1', the set of all such x_0 is the union of no more than countably many pairwise disjoint open intervals $(\alpha_k, \beta_k) \subset (\alpha, \beta)$, where

$$f(\beta_k) - f(\alpha_k) \leq c(\beta_k - \alpha_k) \tag{2}$$

Let G_k be the set of points in (α_k, β_k) such that $\Lambda_R > C$. Then by the same argument as above, replacing Lemma 1' with Lemma 1, we get that G_k is the union of no more than countably many pairwise disjoint open intervals $(\alpha_{k_n}, \beta_{k_n})$ where

$$C(\beta_{k_n} - \alpha_{k_n}) \leq f(\beta_{k_n}) - f(\alpha_{k_n}) \tag{3}$$

Clearly E_ρ is covered by this system of intervals and from (2) and (3) it follows that

$$\begin{aligned}
C\sum_{k,n}(\beta_{k_n} - \alpha_{k_n}) &\leq C\sum_{k,n}[f(\beta_{k_n}) - f(\alpha_{k_n})] \\
&\leq \sum_k [f(\beta_k) - f(\alpha_k)] \\
&\leq c\sum_k(\beta_k - \alpha_k) \\
&\leq c(\beta - \alpha)
\end{aligned}$$

which implies

$$\sum_{k,n}(\beta_{k_n} - \alpha_{k_n}) \leq \rho(\beta - \alpha)$$

Theorem: *If $f : [a,b] \to \mathbb{R}$ is a monotonic function then f is differentiable almost everywhere on $[a,b]$.*

We may assume f is nondecreasing, since otherwise we merely have to consider $-f$. We may also assume f is continuous, since this condition can be dropped afterwards. It is sufficient to show that $\Lambda_R < +\infty$ and $\lambda_L \geq \Lambda_R$ almost everywhere on $[a,b]$, since if we consider $g(x) = -f(-x)$ we get that g is continuous and nondecreasing, like f. Moreover, $\lambda_L^g = \lambda_R$ and $\Lambda_R^g = \Lambda_L$. Therefore

$$\Lambda_R \leq \lambda_l \leq \Lambda_L \leq \lambda_R \leq \Lambda_R$$

i.e., f is differentiable almost everywhere on $[a,b]$.

Let $\Lambda = +\infty$. Then there is some point x_0 such that for every $C > 0$ there exists $\xi > x_0$ such that

$$\frac{f(\xi) - f(x_0)}{\xi - x_0} > C$$

That is, x_0 is invisible from the right with respect to $f(x) - Cx$. By Lemma 1, the set of all these x_0 can be covered by a countable union of pairsie disjoint intervals (a_k, b_k), whose end points satisfy

$$f(b_k) - f(a_k) \geq C(b_k - a_k)$$

Dividing by C and summing, we get

$$\sum_k (b_k - a_k) \leq \sum_k \frac{f(b_k) - f(a_k)}{C} \leq \frac{f(b) - f(a)}{C}$$

By making C arbitrarily large we get that the set of x_0 where $\Lambda_R = +\infty$ is measure zero.

To prove the other half of the condition, let c, C, E_ρ be as in Lemma 2. To show that $\lambda_L \geq \Lambda_R$ almost everywhere it is sufficient to prove that E_ρ is measure zero, since the set of points where $\lambda_L < \Lambda_R$ can be covered by countably many sets of the same form as E_ρ by choosing ρ appropriately. Let t be the measure of E_ρ, then for any $\epsilon > 0$ there is any open set G equal to the union of at most countably many open intervals (a_k, b_k) such that $E_\rho \subset G$ and

$$\sum_k (b_k - a_k) < t + \epsilon$$

If

$$t_k = \mu[E_\rho \cap (a_k, b_k)]$$

then

$$t = \sum_k t_k$$

But by Lemma 2, $t_k \leq \rho(b_k - a_k)$. Hence

$$t \leq \rho \sum_k (b_k - a_k) < \rho(t + \epsilon)$$

which implies $t \leq \rho t$, but since $0 < \rho < 1$, t $= 0$. Therefore $\lambda_L \geq \Lambda_R$ almost everywhere, as asserted.

To drop the requirement of continuity we note that because every discontinutity of a monotone function is a jump discontinuity, there are still neighborhoods around points in which invisibility from the left or right is retained. Define $G(x) = \max\{f(x-0), f(x), f(x+0)\}$. Replace (1) with the statement

$$f(a_k + 0) \leq G(b_k)$$

This suffices to replace the lemmas where f is monotonic and discontinuous.