CMSC 277: Homework #5

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1. *Let $\mathfrak{L} = \{R\}$ where R is a binary relation symbol and let $\mathfrak{M}$ be a finite $\mathfrak{L}$-structure. Show that there exists a $\sigma \in \mathrm{Sent}_{\mathfrak{L}}$ such that for all $\mathfrak{L}$-structures $\mathfrak{N}$ we have*

   $$\mathfrak{N} \vDash \sigma \textit{ if and only if } \mathfrak{M} \cong \mathfrak{N}$$

   If $\mathfrak{N} \cong \mathfrak{M}$ then $\mathfrak{N} \equiv \mathfrak{M}$, so simply choose $\sigma \in \mathrm{Th}(\mathfrak{M})$.

   To see the converse, let

   $$\tau_i = \bigwedge_{\substack{k=1 \\ k \neq i}}^{n} (v_i \neq v_k)$$

   and define

   $$\sigma_n = (\exists v_1 \cdots \exists v_n((\tau_1) \wedge (\tau_2) \wedge \cdots \wedge (\tau_n))) \wedge (\forall v)((v = v_1) \vee (v = v_2) \vee \cdots \vee (v = v_n))$$

   Then are precisely n elements in any $\mathfrak{L}$-structure which models σ_n. Define

   $$\upsilon = \bigwedge_{(a_i, a_j) \in R^{\mathfrak{M}}} R v_i v_j$$

   Then let $\sigma = \sigma_n \wedge \upsilon$. If $\mathfrak{N} \vDash \sigma$ and s is some variable assignment, then the map $M \to N$ given by $a_i \mapsto s(v_i)$ is an isomorphism between $\mathfrak{M}$ and $\mathbb{N}$.

2. *Let $\mathfrak{L} = \{f\}$ where f is a binary function symbol. Let $\mathfrak{M} = \{0,1\}^*$ and $f^{\mathfrak{M}} : \mathfrak{M}^2 \to \mathfrak{M}^2$ be the concatenation operation.*

   (a) *Show that $\{\lambda\} \subset M$ is definable in $\mathfrak{M}$.*
   Define
   $$\varphi(x) = \forall y(fxy = y)$$
   This expresses the statement that $\tau\sigma = \sigma$ if and only if $\tau = \lambda$.

   (b) *Show that for each $n \in \mathbb{N}$ the set $\{\sigma \in \mathfrak{M} \mid |\sigma| = n\}$ is definable.*
   Let $X_n = \{\sigma \in \mathfrak{M} \mid |\sigma| = n\}$. By the previous part we have that X_0 is definable. We proceed by strong induction. Assume that each X_k for $k < n$ is definable. Let $\varphi_k(x) \in \mathrm{Sent}_{\mathfrak{L}}$ be the sentence which defines X_k. Then define

   $$\varphi_n(x) = \exists y \exists z(\varphi_1(y) \wedge \varphi_{n-1}(z) \wedge (fyz = x))$$

   Then $X_n = \{x \in M \mid \mathfrak{M} \vDash \varphi_n(x)\}$, since every element of X_n can be written uniquely as the concatenation of a sequence of length 1 and a sequence of length $n - 1$.

(c) *Find all automorphisms of* $\mathfrak{M}$.

First note that $(M, f^{\mathfrak{M}})$ is isomorphic to the free monoid on two generators. Hence any automorphism of $\mathfrak{M}$ is specified completely by its action on the two generators. Likewise, any permutation $\sigma \in S_2$ induces an automorphism given by

$$h_\sigma(\tau) = \sigma(\tau(1)) * \cdots * \sigma(\tau(|\tau|))$$

Hence $\mathrm{Aut}(\mathfrak{M}) = \{h_\sigma \mid \sigma \in S_2\}$.

(d) *Show that* $\{\sigma \in M \mid \sigma$ *contains no* $1s\}$ *is not definable in* $\mathfrak{M}$.

Let $X = \{\sigma \in \mathfrak{M} \mid \sigma \text{ contains no 1s}\}$.

Recall that any definable set X in $\mathfrak{M}$ is closed under the natural action of elements of $\mathrm{Aut}(\mathfrak{M})$. Hence it suffices to construct an automorphism under which X is not closed.

Since there are only two automorphisms of $\mathfrak{M}$ the choice is pretty obvious: let $\sigma = (1\,2)$, using cycle notation. Then $h_\sigma(X) = \{\sigma \in M \mid \sigma \text{ contains no 0s}\} \not\subset X$.

3. *Let* $\mathfrak{L} = \{f\}$ *where* f *is a binary function symbol. Let* $\mathfrak{M}$ *be the* $\mathfrak{L}$*-structure* $(\mathbb{N}, \cdot)$.

(a) *Show that* $\{0\}$ *is definable in* $\mathfrak{M}$.

Define

$$\varphi(x) = \forall y(fxy = x)$$

Since this defines a left zero in $\mathbb{N}$, and $\mathbb{N}$ has a unique zero, it follows that the set defined by $\varphi(x)$ is precisely $\{0\}$.

(b) *Show that* $\{1\}$ *is definable in* $\mathfrak{M}$.

Define

$$\varphi(x) = \forall y(fxy = y)$$

Since identities are unique in a monoid, it follows that the set defined by $\varphi(x)$ is precisely $\{1\}$.

(c) *Show that* $\{p \in \mathbb{N} \mid p$ *is prime*$\}$ *is definable in* $\mathfrak{M}$.

Let $\varphi(x)$ be as in the previous part. Define

$$\psi(x) = ((\exists m \exists n) fmn = x) \rightarrow (\varphi(m) \vee \varphi(n))$$

(d) *Find all automorphisms of* $\mathfrak{M}$.

Since $(\mathbb{N}, \cdot)$ is a free commutative monoid with countably many generators (i.e., the primes), it follows that any automorphism of this monoid (and hence, by our construction, any automorphism of $\mathfrak{M}$) is completely determined by its action on the generators. Likewise, any permutation of the generators σ induces an automorphism h_σ by

$$h_\sigma(n) = \sigma(p_1)^{a_1} \cdot \cdots \cdot \sigma(p_k)^{a_k}$$

where $n = p_1^{a_1} \cdots p_k^{a_k}$ by the fundamental theorem of arithmetic.

Therefore $\mathrm{Aut}(\mathfrak{M}) = \{h_\sigma \mid \sigma \in S_P\}$, where P is the set of all primes in $\mathbb{N}$ and S_P denotes the permutation group of the set P.

(e) *Show that* $\{n\}$ *is not definable in* $\mathfrak{M}$ *for* $n \geq 2$.

Recall that if a set $X \subset M^k$ is definable then it is closed under the natural action of any automorphism. Hence to prove that such an X is not definable it suffices to find an automorphism under which X is not closed.

But this is not hard. Fix $n \in \mathbb{N}$ and choose a $\sigma_n \in S_P$, where P is the set of all primes in $\mathbb{N}$, which swaps every prime appearing in the prime decomposition of n with a prime not appearing in the decomposition. Then certainly $h_\sigma(n) \neq n$, since n has a unique prime decomposition up to powers and commutativity.

(f) *Show that* $\{(k,m,n) \in \mathbb{N}^3 \mid k+m=n\}$ *is not definable in* $\mathfrak{M}$.

Let $X = \{(k,m,n) \in \mathbb{N}^3 \mid k+m=n\}$ and pick some element, say, $y = (2,4,6)$. Let $\sigma \in S_P$ be, in cycle notation, $\sigma = (2\,3\,5)$. Then $h_\sigma(2) = 3$, $h_\sigma(4) = 9$ and $h_\sigma(6) = 15$. Hence h_σ sends y outside X, and so X is not definable.

4. (a) *Give an example of a language* $\mathfrak{L}$ *together with a* $\varphi \in \mathrm{Form}_{\mathfrak{L}}$ *and* $x, y \in \mathrm{Var}$ *such that* $(\varphi^y_x)^x_y \neq \varphi$.

Let $L = \{R\}$ where R is a binary relation. Define

$$\varphi = \forall y \forall x = RxyRyx$$

Then

$$(\varphi^y_x)^x_y = (\forall y \forall y = RyyRyy)^x_y = (\forall x \forall x = RxxRxx) \neq \varphi$$

(b) *Suppose* $\mathfrak{L}$ *is a language and* $x, y \in \mathrm{Var}$. *Show that for every* $\varphi \in \mathrm{Form}_{\mathfrak{L}}$ *with* $y \notin \mathrm{OccurVar}(\varphi)$ *we have* $(\varphi^y_x)^x_y = \varphi$.

Denote by φ', $(\varphi^y_x)^x_y$ and let $X = \{\varphi \in \mathrm{Form}_{\mathfrak{L}} \mid y \notin \mathrm{OccurVar}(\varphi) \Rightarrow \varphi' = \varphi\}$. We proceed by induction (thrice!) to show that $X = \mathrm{Form}_{\mathfrak{L}}$.

First note that if $y \in \mathrm{OccurVar}(\varphi)$ then $\varphi \in X$ vacuously, so it suffices to consider only those $\varphi \in \mathrm{Form}_{\mathfrak{L}}$ with $y \notin \mathrm{OccurVar}(\varphi)$.

For the base case we will show that $\mathrm{AtomicForm}_{\mathfrak{L}} \subset X$. We proceed by induction here, too. To show that $\mathrm{Term}_{\mathfrak{L}} \subset X$, let $\varphi \in \mathcal{C} \cup \mathrm{Var}$. Since $y \notin \mathrm{OccurVar}(\varphi)$, from the definition of substitution it follows that $\varphi' = \varphi$. Now let $\varphi_1, \ldots, \varphi_k \in X \cap \mathrm{Term}_{\mathfrak{L}}$ with $y \notin \mathrm{OccurVar}(\varphi_i)$ for all i. Then from the definition of substitution it follows that

$$h_f(\varphi_1, \ldots, \varphi_k)' = f\varphi_1'\varphi_2' \cdots \varphi_k' = f\varphi_1 \cdots \varphi_k = h_f(\varphi_1, \ldots, \varphi_k)$$

Hence $\mathrm{Term}_{\mathfrak{L}} \subset X$. Now, assuming we have $\varphi_1, \ldots, \varphi_k \in \mathrm{Term}_{\mathfrak{L}}$ with $y \notin \mathrm{OccurVar}(\varphi_i)$ for all i, it follows directly from the definition of substitution that

$$(=\varphi_1\varphi_2)' = (=\varphi_1'\varphi_2') = (=\varphi_1\varphi_2)$$

and

$$(R\varphi_1 \cdots \varphi_k)' = (R\varphi_1' \cdots \varphi_k') = (R\varphi_1 \cdots \varphi_k)$$

so that $\mathrm{AtomicForm}_{\mathfrak{L}} \subset X$. The final proof is really identical to all the above. If $\varphi, \psi \in X$, with y not occurring in either, then $(\neg\varphi)' = \neg\varphi' = \neg\varphi$, $(\varphi \diamond \psi)' = (\varphi' \diamond \psi') = (\varphi \diamond \psi)$, and so forth. Hence $\mathrm{Form}_{\mathfrak{L}} = X$ and the proposition follows.

5. *Let* $\mathfrak{L} = \{P\}$ *where* P *is a binary relation symbol, and let* $x, y \in \mathrm{Var}$ *be distinct.*

(a) *Give a deduction showing that* $\forall x \forall y Pxy \vdash \forall y \forall x Pxy$.

In place of subscripts or superscripts to denote substitution, I will replace the actual variable with the appropriate letter. Fix $t, u \in \mathrm{Var}$ not occurring in any of the formulas below.

$$\begin{array}{rlr} \{\forall x \forall y Pxy, \neg\forall y \forall x Pxy\} \vdash \forall x \forall y Pxy & (\textit{Assumption}) & (1) \\ \{\forall x \forall y Pxy, \neg\forall y \forall x Pxy\} \vdash \forall y Puy & (\forall E \text{ on } 1) & (2) \\ \{\forall x \forall y Pxy, \neg\forall y \forall x Pxy\} \vdash Put & (\forall E \text{ on } 2) & (3) \\ \{\forall x \forall y Pxy, \neg\forall y \forall x Pxy\} \vdash \neg\forall x \forall y Pxy & (\textit{Assumption}) & (4) \\ \{\forall x \forall y Pxy, \neg\forall y \forall x Pxy\} \vdash \neg\forall y Puy & (\forall E \text{ on } 4) & (5) \\ \{\forall x \forall y Pxy, \neg\forall y \forall x Pxy\} \vdash \neg Put & (\forall E \text{ on } 5) & (6) \\ \forall x \forall y Pxy \vdash \forall y \forall x Pxy & (\textit{Contr} \text{ on 3 and 6}) & (7) \end{array}$$

(b) *Give a deduction showing that* $\exists x \forall y Pxy \vdash \forall y \exists x Pxy$.

In place of subscripts or superscripts to denote substitution, I will replace the actual variable with the appropriate letter. Fix $t, u \in \mathrm{Var}$ not occurring in any of the formulas below.

$$\begin{aligned}
\{\forall y Puy, \neg\forall y \exists x Pxy\} &\vdash \forall y Puy && (\mathit{Assumption}) && (1)\\
\{\forall y Puy, \neg\forall y \exists x Pxy\} &\vdash Put && (\forall E \text{ on } 1) && (2)\\
\{\forall y Puy, \neg\forall y \exists x Pxy\} &\vdash \exists x Pxt && (\exists I \text{ on } 2) && (3)\\
\{\forall y Puy, \neg\forall y \exists x Pxy\} &\vdash \neg\forall y \exists x Pxy && (\mathit{Assumption}) && (4)\\
\{\forall y Puy, \neg\forall y \exists x Pxy\} &\vdash \neg\exists x Pxt && (\forall E \text{ on } 4) && (5)\\
\forall y Puy &\vdash \forall y \exists x Pxy && (\mathit{Contr} \text{ on 3 and 5}) && (6)\\
\exists x \forall y Pxy &\vdash \forall y \exists x Pxy && (\exists I \text{ on } 6) && (7)
\end{aligned}$$

(c) *Show that* $\forall y \exists x Pxy \nvdash \exists x \forall y Pxy$.

Using completeness/soundess we can pass to semantics, and come up with a specific example. Let $\mathfrak{L} = \{P\}$ and $\mathfrak{M}$ be (G, P) where G is a nontrivial group and $P = \{(a,b) \mid ab = e\}$. Then it is true that for all y there exists an x such that $yx = e$, viz., $x = y^{-1}$. However, it is not true that there exists an x such that for every y, $xy = e$ since then x is idempotent so that $x = e$ and G is trivial.