MATH 270: Homework #1

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1. *Express the following complex number in the form $a + ib$:*

   (a) $(2+3i)+(4+i)$

   $$(2+3i)+(4+i) = 6+4i$$

   (b) $\frac{2+3i}{4+i}$

   $$\frac{2+3i}{4+i} = \frac{2+3i}{4+i}\frac{4-i}{4-i} = \frac{1}{17}(2+3i)(4-i) = \frac{11}{17}+\frac{10}{17}i$$

   (c) $\frac{1}{i}+\frac{3}{1+i}$

   $$\frac{1}{i}+\frac{3}{1+i} = -i+\frac{3}{1+i}\frac{1-i}{1-i} = \frac{3-3i}{2}-i = \frac{3}{2}-\frac{5}{2}i$$

2. *Find the real and imaginary parts of the following, where $z = x + iy$:*

   (a) $\frac{1}{z^2}$

   For $z = x+iy$, $z^2 = x^2-y^2+2xyi$. So $\frac{1}{z^2} = \left(\frac{\overline{z}}{|z|^2}\right)^2$, and

   $$\Re\frac{1}{z^2} = \frac{\left(\frac{\overline{z}}{|z|^2}\right)^2+\left(\frac{z}{|z|^2}\right)^2}{2} = \frac{1}{2|z|^4}(\overline{z}^2+z^2) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

   Using the same formula for squaring a complex number as before (only subtracting instead of adding), yields

   $$\Im\frac{1}{z^2} = \frac{\left(\frac{\overline{z}}{|z|^2}\right)^2-\left(\frac{z}{|z|^2}\right)^2}{2i} = \frac{1}{2i|z|^4}(\overline{z}^2-z^2) = -\frac{2xy}{(x^2+y^2)^2}$$

   (b) $\frac{1}{3z+2}$

   $$\frac{1}{3z+2} = \frac{1}{(3x+2)+3yi} = \frac{1}{(3x+2)+3yi}\frac{(3x+2)-3yi}{(3x+2)-3yi} = \frac{(3x+2)-3yi}{(3x+2)^2+9y^2}$$

   so

   $$\Re\frac{1}{3z+2} = \frac{3x+2}{(3x+2)^2+9y^2} \text{ and } \Im\frac{1}{3z+2} = \frac{-3y}{(3x+2)^2+9y^2}$$

3. *Define* $\phi_z(w) = zw$.

   (a) *Prove that the matrix of* ϕ_z *is given by* $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

   Choose $\{(1,0),(0,1)\}$ as the basis for both $\mathbb{C}$ as the domain and range. The map ϕ_z is obviously linear, so to find a matrix representation we look at how the transformation affects the basis elements. Write $z = (x,y)$. Then

   $$\phi_z\left((1,0)\right) = (x,y)(1,0) = (x,y) \text{ and } \phi_z\left((0,1)\right) = (x,y)(0,1) = (-y,x)$$

   Therefore the matrix representation of ϕ_z is precisely as was given in the statement of the problem.

   (b) *Show that* $\phi_{z_1 z_2} = \phi z_1 \circ \phi_{z2}$.

   Let $w \in \mathbb{C}$ be arbitrary.

   $$\phi_{z_1 z_2}(w) = (z_1 z_2)w = z_1(z_2 w) = z_1 \phi_{z_2}(w) = \phi_{z_1}(\phi_{z_2}(w)) = (\phi_{z_1} \circ \phi_{z_2})(w)$$

4. *Let* $a + ib = \frac{x-iy}{x+iy}$. *Show that* $a^2 + b^2 = 1$.

   Let $z = x + iy$ and define $w = \frac{\overline{z}}{z}$. Then, since $|\overline{z}| = |z|$ and $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$,

   $$|w| = 1 \Rightarrow |w|^2 = 1 \Rightarrow (\Re w)^2 + (\Im w)^2 = 1 \Rightarrow a^2 + b^2 = 1$$

5. *Solve the following equations:*

   (a) $z^6 + 8 = 0$

   The roots are of the form $\sqrt[6]{8}\left[\cos\frac{\pi+2\pi k}{6} + \sin\frac{\pi+2\pi k}{6}\right]$ for $k = 0, 1, \ldots, 5$. Simply calculating these yields

   $$\frac{z}{\sqrt[6]{8}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i, -i, \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

   (b) $z^3 - 4 = 0$

   The roots are of the form $\sqrt[3]{6}\left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right]$ for $k = 0, 1, 2$. Calculating these yields

   $$\frac{z}{\sqrt[3]{6}} = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

6. *Let* w *be an* n*th root of unity,* $w \neq 1$. *Show that* $1 + w + \cdots + w^{n-1} = 0$.

   By hypothesis $w^n - 1 = 0$, so

   $$0 = w^n - 1 = (w-1)(1 + w + \cdots w^{n-1})$$

   Since $w \neq 1$, $1 + w + \cdots w^{n-1} = 0$.

7. *Show that the roots of a polynomial with real coefficients occur in conjugate pairs.*

   It is easy to see that because the complex numbers are a field and conjugation preserves both addition and multiplication for two elements, that conjugation preserves addition and multiplication for any number of elements. Assume z_0 is a root of a polynomial $p(z) = a_n z^n + \cdots a_1 z + a_0$. Then since $a_i \in \mathbb{R}$,

   $$0 = \overline{p(z_0)} = \overline{\sum_{i=0}^{n} a_n z_0^n} = \sum_{i=0}^{n} \overline{a_n z_0^n} = \sum_{i=0}^{n} \overline{a_n}\,\overline{z_0^n} = \sum_{i=0}^{n} a_n \overline{z_0}^n = p(\overline{z_0})$$

8. *Assuming either* $|z| = 1$ *or* $|w| = 1$ *and* $\overline{z}w \neq 1$, *prove that*

$$\left|\frac{z-w}{1-\overline{z}w}\right| = 1$$

The problem is equivalent to showing that $|z - w| = |1 - \overline{z}w|$. Assume $|z| = 1$. Then $|\overline{z}| = 1$ and

$$|z - w| = |\overline{z}||z - w| = |\overline{z}z - \overline{z}w| = ||z|^2 - \overline{z}w| = |1 - \overline{z}w|$$

If $|w| = 1$ then

$$|z - w| = |\overline{w}||z - w| = |z\overline{w} - 1| = \overline{|1 - z\overline{w}|} = |\overline{1 - z\overline{w}}| = |1 - \overline{z\overline{w}}| = |1 - \overline{z}w|$$

9. *Does* $z^2 = |z|^2$? *If so, prove this equality. If not, for what* z *is it true?*

This is not in general true, e.g., if $z = i$. For $z = x+iy$, $z^2 = x^2 - y^2 + 2iyx$ and $|z|^2 = x^2 + y^2$. So clearly these two are equal if and only if $y = 0$, i.e., $z \in \mathbb{R}$.