MATH 262: Homework #6

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1. *Give* $[0,1]^\omega$ *the uniform topology. Find an infinite subset of this space that has no limit point.*

   Let $\bar{\rho}$ be the uniform metric. Let e_i be the ω-vector with a 1 in the i^{th} coordinate and 0 in every other coordinate. Then $\bar{\rho}(e_i, e_j) = 1$ for $i \neq j$. In particular, a ball of radius 1/2 around any of the e_i intersects no other e_i, so that $\{e_i\}_{i\in\mathbb{Z}_+}$ is an infinite subset of $[0,1]^\omega$ without a limit point.

2. *Show that* $[0,1]$ *is not limit point compact as a subspace of* $\mathbb{R}_l$.

   Let $A = \left\{1 - \frac{1}{n} \mid n \in \mathbb{Z}_+\right\} \subset [0,1]$. In $\mathbb{R}$ (with the usual topology) the only limit point of this set is 1. Since $\mathbb{R}_l$ is finer than $\mathbb{R}$, 1 is the only possible limit point this sequence. For any $\epsilon > 0$ the neighborhood $[1, \epsilon)$ of 1 does not intersect A, so 1 cannot be a limit point of A and therefore $[0,1]$ is not limit point compact in $\mathbb{R}_l$.

3. *Let* (X, ρ) *be a metric space. Show that if* $f : X \to X$ *is a isometry and* X *is compact then* f *is a homeomorphism.*

   Any isometry of a metric space, compact or not, is injective since if $f(x) = f(y)$ then

   $$0 = \rho(f(x), f(y)) = \rho(x, y)$$

   which implies $x = y$.

   Since X is compact its image under f, $f(X)$, is also compact. Assume for contradiction that f is no surjective, then there exist some $a \in X \setminus f(X)$. Choose a ball around a of radius ϵ such that it does not intersect $f(X)$ – this is possible since X is metric and hence Hausdorff. Define a sequence recursively as follows. Let $x_0 = a$ and let $x_n = f(x_{n-1})$. By our choice of ϵ, it follows that $\rho(a, f(x)) > \epsilon$ for every $x \in X$. In particular, $\rho(a, x_n) > \epsilon$ for all $n \geq 1$. Since f is an isometry it follows by induction that

   $$\rho(x_n, x_m) = \rho(x_{n-1}, x_m) = \cdots = \rho(a, x_m) > \epsilon$$

   That is, $\rho(x_n, x_m) > \epsilon$ for all $n \neq m$. But this means $\{x_n\}_{n\in\mathbb{N}}$ is not limit point compact, as a ball of radius $\epsilon_0 < \epsilon$ around any x_n intersects none of the other x_n – contradicting the compactness of X. Therefore f must be surjective.

   That it is bicontinuous is almost too trivial for words, but it follows directly from the fact that f is an isometry. The image of any open ball $B_r(x)$ under f will just be $B_r(f(x))$, and likewise for f^{-1}. Therefore both f and f^{-1} are continuous, and, since it is also a bijection, f is a homeomorphism of X.

4. *Let* (X, ρ) *be a metric space.*

   (a) *Show that if* f *is a contraction mapping and* X *is compact then* f *has a unique fixed point.*

   Define recursively $A_1 = f(X)$ and $A_n = f(A_{n-1})$. Since f is continuous (obviously any contraction mapping is – simply choose $\delta = \epsilon/\alpha$ in the definition of continuity) each A_n is compact. Define

   $$A = \bigcap_{i\in\mathbb{Z}_+} A_n$$

A is nonempty since $\{A_n\}$ is a nested sequence of compact sets. If $x \in A$ then x is a limit of the sequence $f^{(n)}(x)$, i.e., f composed with itself n times, as $n \to \infty$ (Do we need X to be first countable? It is since it's a metric space, but I'm not sure this implication holds otherwise.) But for any such sequence

$$\lim_{n\to\infty} f^{(n)}(x) = x \Rightarrow x = \lim_{n\to\infty} f^{(n+1)}(x) = f(x)$$

Hence $f(x) = x$ for any $x \in A$. If x and y are distinct fixed points of f then

$$\rho(x, y) = \rho(f(x), f(y)) \leq \alpha\rho(x, y) < \rho(x, y)$$

which is a contradiction, and hence $x = y$.

(b) *Show that if f is a shrinking map and X is compact then f has a unique fixed point.*
Suppose $f(x) \neq x$ for every $x \in X$. Define

$$g(x) = \rho(f(x), x)$$

It follows that g is continuous and strictly positive since f is continuous. Because X is compact g attains its minimum on X, and so there is some $x_0 \in X$ such that $0 < g(x_0) \leq g(x)$ for all $x \in X$. But then

$$g(f(x_0)) = \rho(f(x_0), f(f(x_0))) < \rho(x_0, f(x_0) = g(x_0)$$

contradicting the minimality of $g(x_0)$. Therefore there exists some $x \in X$ such that $f(x) = x$. As above, if x, y are two fixed points then

$$\rho(x, y) = \rho(f(x), f(y)) < \rho(x, y)$$

which is absurd. Therefore it must be the case that $x = y$.

(c) *Let $X = [0, 1]$. Show that $f(x) = x - x^2/2$ maps X into X and is a shrinking map that is not a contraction.*
Let $x, y \in [0, 1]$ and assume without loss of generality that $x < y$. Then since $1 - (y - x) < 1$,

$$|f(x) - f(y)| = \left| x - \frac{x^2}{2} - y + \frac{y^2}{2} \right| < |(x - y)(1 - (y - x))| < |x - y|$$

Assume that f is a contraction mapping with contration factor $\alpha < 1$. Taking $x = 0$ and $y < 2(1 - \alpha)$, so that $1 - y/2 > \alpha$, we get

$$y\left(1 - \frac{y}{2}\right) = \left| y - \frac{y^2}{2} \right| \leq \alpha y$$

By our choice of y, this is a contradiction. Therefore f cannot be a contraction mapping.

5. *Show that $[0, 1]^\omega$ is not locally compact in the uniform topology.*

Let C be a compact subspace of X that contains some ϵ-neighborhood of 0. As in the first problem define e_i as the ω-vector with ϵ in the i^{th} coordinate and 0 in all other coordinates. Then the $\{e_i\}$ are an infinite subset of C but contain no limit point, contradicting the fact that C is compact (and hence limit point compact).

6. *Show that the one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1.*

The real line $\mathbb{R}$ is homeomorphic to $(0, 2\pi)$, so it is sufficient to show that the one-point compactification of $(0, 2\pi)$ is the circle. Let $\{\infty\}$ be the additional point in the one-point compactification and define $f : (0, 2\pi) \cup \{\infty\} \to \mathbb{R}^2$ by

$$f(t) = \begin{cases} (\cos t, \sin t) & t \in (0, 2\pi) \\ (1, 0) & t = \infty \end{cases}$$

From the properties of $\sin t$ and $\cos t$ on $(0, 2\pi)$, this function is continuous and bijective. For open sets not containing $(1, 0)$ the preimage under f is open, so let U be a neighborhood of $(1, 0)$. Without loss of generality write

$$U = \{(\cos t, \sin t) \mid t \in (-\epsilon, \epsilon)\}$$

for some $\epsilon > 0$. Then

$$f^{-1}(U) = (0, \epsilon) \cup \{\infty\} \cup (2\pi - \epsilon, 2\pi)$$

The complement of this is $[\epsilon, 2\pi - \epsilon]$, which is compact, and hence the preimage of any neighborhood around $(1, 0)$ is open with respect to the topology on the one-point compactification. It follows, then, that f is a homeomorphism.

7. *Show that if $f : [0, 1] \to \mathbb{R}^\omega$ is continuous with respect to the box topology then all but finitely many of the coordinate functions are constant.*

   The image of $[0, 1]$ under f must be compact, so it is sufficient to prove that the only compact subsets of $\mathbb{R}^\omega$ are those that have all but finitely many component sets consisting of one element. This is not hard to see. Let $C_1 \times C_2 \times C_3 \times \cdots$ be a compact set in $\mathbb{R}^\omega$. At the very least each C_i must be compact since otherwise we could product a covering of one single C_i that has no finite subcover, and hence the direct product of all these sets could have no finite subcover. Assume without loss of generality that each C_i can be covered by two open sets whose intersection is strictly contained in both, but not empty (we wish to avoid the case where C_i is a single point, so that a single open set must contain the whole C_i or none of it). We will show that no finite subcover exists, and hence cannot exist in the case where each C_i is covered by more than two such open sets. Identify these two open sets with 0 and 1, respectively. Create a cover of $C_1 \times C_2 \times \cdots$ by taking all sequences that consist of a single 1 and everything else 0, and identifying this sequence with the associated direct product of open sets. For a finite subset of this to cover the entire direct product it must contain an infinite number of 1s, which is impossible by our construction. Thus, for there to be a finite subcover, it must be the case that all but finitely many C_i satisfy the property that any open set either contains C_i or none of C_i. But this only occurs with C_i consists of a single point.

8. *Let $(X, <)$ be an ordered set. Show that every infinite sequence in X has an infinite monotone subsequence. Give an example where the sequence has arbitrarily large increasing subsequences, but no infinite increasing subsequence.*

   Call $n \in \mathbb{N}$ a "peak" of the sequence $\{x_n\}$ if $a_m < a_n$ for all $m > n$. There are two cases. If there are infinitely many peaks $n_1, n_2, \ldots$, then $x_{n_1} > x_{n_2} > \ldots$, and $\{x_{n_k}\}$ is an increasing subsequence. If there are finitely many peaks let n_1 be greater than all the peaks. Since n_1 is not a peak there is some n_2 with $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$. Since n_2 is also not a peak there is some $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$. Continuing in this manner we produce a monotonic sequence.