MATH 257: Homework #9

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1. *Prove that A_n contains a subgroup isomorphic to S_{n-2} for $n \geq 3$.*

2. *Prove that there are no simple groups of order 132.*

   Let G be a group such that $|G| = 132 = 2^2 \cdot 3 \cdot 11$. Consider n_3 and $n_1 1$. $n_3 = 1, 4, 22$ and $n_{11} = 1\,12$. If $n_3 = 4$, G acts on $\mathrm{Syl}_3(G)$ by conjugation, so there is a homomorphism to S_4. Since 24 does not divide 132, the kernel must be nontrivial which is necessarily normal. If $n_3 = 22$ and $n_{11} = 12$ then there are too many elements, so either $n_3 = 1$ or $n_{11} = 1$, but not both, and G has a nontrivial normal subgroup.

3. *Show that if $n_p \not\equiv 1 \mod p^2$ then there exist distinct Sylow p-subgroups P and Q such that $[P : P \cap Q] = [Q : Q \cap P] = p$.*

4. *Let $P \in Syl_p(G)$ and $N_G(P) \leq M \leq G$. Prove that $[G : M] \equiv 1 \mod p$.*

5. *Prove that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$.*

6. *Prove that if p is a prime and P is a non-abelian group of order p^3 then $|Z(P)| = p$ and $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

   From the class equation it follows immediately that $Z(P)$ is nontrivial. If $|Z(P)| = p^2$ then $G/Z(P)$ is cyclic and hence G is abelian. Therefore $|Z(P)| = p$. $P/Z(P)$ has order p^2 and cannot be cyclic, so $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

7. *Prove that every minimal normal subgroup of a finite solvable group is an elementary abelian p-group for some prime p.*

8. *Prove that every maximal subgroup of a finite solvable group has a prime power index.*