MATH 270: Homework #3

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1. *Show, by changing variables, that the Cauchy-Riemann equations in terms of polar coordinates become*
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

   By the chain rule,
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \text{ and } \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

   And similarly for $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$. Since $x = r\cos\theta$ and $y = r\sin\theta$ we have
$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta & \qquad \frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta & \qquad \frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta \end{aligned}$$

   The Cauchy-Riemann equations give $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Making these substitutions into the partials of u and v with respect to θ gives
$$\begin{aligned} \frac{\partial u}{\partial \theta} &= -\frac{\partial v}{\partial y}r\sin\theta - \frac{\partial v}{\partial x}r\cos\theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial u}{\partial y}r\sin\theta + \frac{\partial u}{\partial x}r\cos\theta \end{aligned}$$

   Comparing these two equations with the first equations derived for the partials of u and v with respect to r gives
$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$
   which is the deired result.

2. *Define the symbol $\partial f/\partial z$ by*

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$$

(a) *If $f(z) = z$, show that $\frac{\partial f}{\partial z} = 1$ and $\frac{\partial f}{\partial \overline{z}} = 0$.*
If $f(z) = z = x + iy$ then $\frac{\partial f}{\partial x} = 1$ and $\frac{\partial f}{\partial y} = i$, so

$$\frac{\partial f}{\partial z} = \frac{1}{2}(1 + \frac{1}{i}i) = 1$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(1 - \frac{1}{i}i) = 0$$

(b) *If $f(z) = \overline{z}$, show that $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial \overline{z}} = 1$.*
If $f(z) = \overline{z} = x - iy$ then $\frac{\partial f}{\partial x} = 1$ and $\frac{\partial f}{\partial y} = -i$, so

$$\frac{\partial f}{\partial z} = \frac{1}{2}(1 - \frac{1}{i}i) = 0$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(1 + \frac{1}{i}i) = 1$$

(c) *Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ obey the sum, product, and scalar multiple rules for derivatives.*
Seriously? Ugh. These all follow obviously from the fact that the partials of with respect to x and y obey these rules, but here goes:

$$\begin{aligned}
\frac{\partial(f+g)}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial g}{\partial x}\right) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial g}{\partial x} + \frac{1}{i}\frac{\partial g}{\partial x}\right) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial z}\right) + \frac{1}{2}\left(\frac{\partial g}{\partial z}\right)
\end{aligned}$$

It follows *mutatis mutandis* for $\frac{\partial f}{\partial \overline{z}}$.

$$\begin{aligned}
\frac{\partial(f \cdot g)}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial x}f + \frac{1}{i}\frac{\partial f}{\partial y}g + \frac{1}{i}\frac{\partial g}{\partial y}f\right) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x}g + \frac{1}{i}\frac{\partial f}{\partial y}g\right) + \frac{1}{2}\left(\frac{\partial g}{\partial x}f + \frac{1}{i}\frac{\partial g}{\partial y}f\right) \\
&= \frac{\partial f}{\partial z}g + \frac{\partial g}{\partial z}f
\end{aligned}$$

It follows *mutatis mutandis* for $\frac{\partial f}{\partial \overline{z}}$.

Let $\alpha \in \mathbb{C}$, then

$$\begin{aligned} \alpha\frac{\partial f}{\partial z} &= \frac{\alpha}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\alpha\frac{\partial f}{\partial x} + \frac{1}{i}\alpha\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial \alpha f}{\partial x} + \frac{1}{i}\frac{\partial \alpha f}{\partial y}\right) \\ &= \frac{\partial \alpha f}{\partial z} \end{aligned}$$

It follows *mutatis mutandis* for $\frac{\partial f}{\partial \overline{z}}$.

(d) *Show that the expression* $\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm}z^n\overline{z}^m$ *is a holomorphic function of* z *if and only if* $a_{mn} = 0$ *when* $m \neq 0$.

Because $\frac{partial}{\partial \overline{z}}$ preserves additivity, we must show that this statement is true for each of the terms in the double sum. Let $n, m \in \mathbb{N}$ be arbitrary, then

$$\frac{\partial f}{\partial x} = a_{nm}(nz^{n-1}\overline{z}^m + mz^n\overline{z}^{m-1})$$

and

$$\frac{\partial f}{\partial y} = a_{nm}(inz^{n-1}\overline{z}^m - imz^n\overline{z}^{m-1})$$

This function is holomorphic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$, which means if and only if

$$\frac{\partial f}{\partial \overline{z}} = na_{nm}z^{n-1}\overline{z}^m = 0$$

Since every term but a_{nm} cannot be identically zero, it must be the case that $a_{nm} = 0$ when $m \neq 0$.

3. (a) *Let* $f(z) = u(x, y) + iv(x, y)$ *be a holomorphic function defined on an open, connected set* Ω. *If* $au(x, y) + bv(x, y) = c$ *on* Ω, *where* a, b, c *are real constants not all* 0, *prove that* $f(z)$ *is constant on* Ω.

   (b) *Is the previous result still valid if* a, b, c *are complex constants?*

4. *Let* f *be holomorphic on the set* $A = \{z \mid \Re z > 1\}$ *and* $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ *on* A. *Show that there is a real constant* a *and a complex constant* d *such that* $f(z) = -icz + d$ *on* A.

   By the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, so $2\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0$. This means u depends only on y and v depends only on x. Because u and v depend only on y and x, respectively, $\frac{\partial u}{\partial y}$ depends only on y and $\frac{\partial v}{\partial x}$ depends only on x. However, since the Cauchy-Riemann equations give $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, neither can depend on x or y so they must be equal to some real constant, c.

   Together this means $u(x, y) = cy + d'$ and $v(x, y) = -cx + d''$, and therefore

$$f(z) = cy + d' - icx + id'' = -ic\left(x - \frac{y}{i}\right) + d' + d'' = -ic(x + iy) + d' + d'' = -icz + d$$

5. *Let f be holomorphic on Ω. Define $g : \Omega \to \mathbb{C}$ by $g(z) = \overline{f(z)}$. When is g holomorphic?*

   If $f(z) = u + iv$ for real-valued functions u and v, then $g(z) = u + iv'$ where $v' = -v$. g is holomorphic if and only if it satisfies the Cauchy-Riemann equations (and is differentiable in the real sense). Hence, it is holomorphic if and only if f is differentiable in the real sense and

$$\frac{\partial u}{\partial x} = \frac{\partial v'}{\partial y} = -\frac{\partial v}{\partial y}$$

   and

$$\frac{\partial u}{\partial y} = -\frac{\partial v'}{\partial x} = \frac{\partial v}{\partial x}$$

6. *Evaluate the following:*

   (a) $\int_\gamma y\, dz$, *where γ is the union of the line segments joining 0 to i then to $i + 2$.*

   (b) $\int_\gamma \sin 2z\, dz$, *where γ is the line segment joining $i + 1$ to $-i$.*

   This function has an antiderivative, namely, $-\frac{\cos 2z}{2}$. Hence the integral is $-\frac{\cos(-i2)}{2} + \frac{\cos(2i+2)}{2} = \frac{1}{2}\left[\cos(2i + 2) - \cos(2i)\right]$

   (c) $\int_\gamma z e^{z^2}\, dz$, *where γ is the unit circle.*

   This function has an antiderivative, namely $\frac{e^{z^2}}{2}$, and since γ is a closed path the integeral is 0.

7. *Does $\Re\left\{\int_\gamma f\, dz\right\} = \int_\gamma \Re f\, dz$?*

   No. Let $f(z) = z$ and let $\gamma(t) = it$ for $t \in [0, 1]$ so that $\Re\gamma = 0$. Then

$$\Re \int_\gamma f\, dz = \Re \int_0^1 iti\, dt = \Re \int_0^1 -t = -\frac{1}{2}$$

   but

$$\int_\gamma \Re f\, dz = \int_0^1 0 \cdot i\, dt = 0$$

8. *Evaluate the following integrals:*

   (a) $\int_\gamma \overline{z}\, dz$, *where γ is the unit circle traversed once in a counterclockwise direction.*

   We can parameterize the unit circle by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Hence

$$\int_\gamma \overline{z}\, dz = \int_0^{2\pi} e^{-i\theta} e^{i\theta} i\, d\theta = \int_0^{2\pi} i = i2\pi$$

   (b) $\int_\gamma (x^2 - y^2)\, dz$, *where γ is the straight line from 0 to i.*

   We can parameterize this line by $\gamma(t) = it$ for $t \in [0, 1]$. Note that $\Re\gamma = 0$ and $\Im\gamma = t$. Hence

$$\int_\gamma x^2 - y^2\, dz = \int_0^1 -it^2\, dt = -\frac{i}{3}$$

9. *Evaluate the following:*

(a) $\int_{|z|=1} \frac{dz}{z}$, $\int_{|z|=1} \frac{dz}{|z|}$, $\int_{|z|=1} \frac{|dz|}{z}$, $\int_{|z|=1} \left|\frac{dz}{z}\right|$

(b) $\int_\gamma z^2\, dz$, *where* γ *is the curve given by* $\gamma(t) = e^{it}\sin^3 t$ *for* $0 \le t \le \frac{\pi}{2}$.

z^2 has a primitive, namely $F(z) = \frac{z^3}{3}$. Noting that $e^{i\frac{\pi}{2}} = i$ and $\sin\frac{\pi}{2} = 1$, we get

$$\int_\gamma z^2\, dz = F\left(\gamma\left(\frac{\pi}{2}\right)\right) - F(\gamma(0)) = \frac{i^3}{3} = -\frac{i}{3}$$

10. *Show that every disk is convex.*

For any two points within a disk one can characterize the segment joining them by

$$\gamma_{x,y} = \{(1-t)x + ty \mid t \in [0,1]\}$$

It is sufficient to show this for a disc of arbitrary radius r about the origin. So let $x, y \in D(r, 0)$, then $|x|, |y| < r$, and

$$|(1-t)x + ty| \le (1-t)|x| + t|y| < r(1-t+t) = r$$

hence $\gamma_{x,y} \subset D(r, 0)$, i.e., $D(r, 0)$ is convex.