CMSC 277: Homework #1

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1. *Using the Order Form of Recursion on $\mathbb{N}$, define X and $g : X^* \to X$ so that the corresponding f is the Fibonacci sequence.*

   Let $X = \mathbb{N}$ and $\sigma \in X^*$. Define $g : X^* \to X$ as follows:

   $$g(\sigma) = \begin{cases} \sigma(|\sigma|) + \sigma(|\sigma| - 1) & |\sigma| \geq 2 \\ 1 & |\sigma| < 2 \end{cases}$$

   Thus $f(0) = g(\lambda) = 1$ and $f(1) = g(f[0]) = 1$, but $f(n) = g(f[0] * \cdots * f[n-1]) = f(n-1) + f(n-2)$ for all $n \geq 2$.

2. *Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{5\}$, and $\mathcal{H} = \{h_1, h_2\}$ where h_1 and h_2 are given as in the homework sheet.*

   (a) *Calculuate V_3 and W_3.*
   Let $V_0 = B = \{5\}$. Then we have the following

   $$\begin{aligned} &V_0 = \{5\} \\ &V_1 = V_0 \cup \{h_1(5)\} \cup \{h_2(5,5)\} = \{5, 7\} \\ &V_2 = V_1 \cup \{h_1(7)\} \cup \{h_2(7,7), h_2(5,7), h_2(7,5)\} = \{4, 5, 7\} \\ &V_3 = V_2 \cup \{h_1(4)\} \cup \{h_2(4,4), h_2(4,7), h_2(7,4), h_2(4,5), h_2(5,4)\} = \{4, 5, 7\} \end{aligned}$$

   Since $B = \{5\}$, the only witnessing sequence of length 1 is 5. As $h_1(5) = 5$ and $h_2(5,5) = 7$ we have that the only witnessing sequences of length two are 55 and 57. Hence the only witnessing sequences of length three are 555, 577, 575, and 574 since $h_1(7) = 4$ and all other combinations of h_k and elements with witnessing sequences of length two, viz., 5 and 7, are already one of $\{4, 5, 7\}$. Hence $W_3 = \{4, 5, 7\}$.

   (b) *Calculate $G(A, B, \mathcal{H})$.*
   In general if $V_n = V_{n+1}$ for some $n \in \mathbb{N}$ then it is easy to see that $G(A, B, \mathcal{H}) = V_n$ since then

   $$\begin{aligned} V_{n+2} &= V_{n+1} \cup \{h(a_1, \ldots, a_k) \mid h \in \mathcal{H}_k, a_i \in V_{n+1}\} \\ &= V_n \cup \{h(a_1, \ldots, a_k) \mid h \in \mathcal{H}_k, a_i \in V_n\} \\ &= V_{n+1} \end{aligned}$$

   and hence $V_k = V_n$ for all $k \geq n$ by induction. Since $V_2 = V_3$ it follows that $G(A, B, \mathcal{H}) = \{4, 5, 7\}$.

3. *Let $A = \mathbb{R}$, $B = \left\{\frac{1}{2}, 3\right\}$, and $\mathcal{H} = \{h_2, h_3\}$ given by*

   $$h_2(x, y) = \frac{x + y}{2}$$

and

$$h_3(x,y,z) = \begin{cases} |x-y| & x \neq y \\ \sqrt{z} & x = y \end{cases}$$

Show that $G(A,B,\mathcal{H}) \subseteq [0,3]$.

Let $C \subset [0,3]$. We will show that $C \cup h_2(C^2) \cup h_3(C^3) \subset [0,3]$ so that, by induction, the proposition follows immediately. Let $x, y \in C$, then

$$0 \leq \frac{x+y}{2} \leq \frac{3+3}{2} = 3$$

so that $h_2(x,y) \in [0,3]$ for all $x, y \in C$. Now let $a_1, a_2, a_3 \in C$. By the definition of $[0,3]$ it follows that $|a_i - a_j| \in [0,3]$ for all $i, j \in \{1,2,3\}$ so it suffices to show that $h(x) = \sqrt{x} \in [0,3]$ for all $x \in C$. But this follows immediately, too, from the monotonicity of h, i.e., $0 \leq h(x) \leq 3$ for all $x \in C \subseteq [0,3]$.

Since $B \subset [0,3]$ we have $V_1 \subset [0,3]$, and that if $V_n \subseteq [0,3]$ then $V_{n+1} \subseteq [0,3]$. Hence $V_n \subseteq [0,3]$ for all $n \in \mathbb{N}$ and therefore the proposition follows by the definition of $G(A,B,\mathcal{H})$.

4. *Let* $A = \mathbb{R}^+$, $B_1 = \left\{\sqrt{2}\right\}$, $B_2 = \left\{\sqrt{2}, 16\right\}$, *and* $\mathcal{H} = \{h\}$ *where* $h : A \to A$ *is defined by* $x \mapsto x^2$.

   (a) *Describe* $G(A,B_1,\mathcal{H})$ *and* $G(A,B_2,\mathcal{H})$ *explicitly.*

   Let $G_1 = G(A,B_1,\mathcal{H})$ and $G_2 = G(A,B_2,\mathcal{H})$. Define $X = \left\{2^{2^{k-1}} \mid k \in \mathbb{N}\right\}$. We claim that $X = G_1 = G_2$. First, we show that X is inductive with respect to both B_1 and B_2. $\sqrt{2} = 2^{2^{-1}}$ and $16 = 2^{2^2}$, so $B_1, B_2 \subset X$. Let $x \in X$, then $x^2 = \left(2^{2^{k-1}}\right)^2 = 2^{2^k} \in X$. Hence $G_1, G_2 \subseteq X$.

   Let $x = 2^{2^{k-1}} \in X$ for some $k \in \mathbb{N}$. Then

   $$\sqrt{2} * 2 * 2^2 * \cdots * 2^{2^{k-2}} * 2^{2^{k-1}}$$

   is a witnessing sequence for x in both G_1 and G_2. Hence $X = G_1 = G_2$.

   (b) *Show that* $G(A,B_1,\mathcal{H})$ *is free but* $G(A,B_2,\mathcal{H})$ *is not free.*

   G_1 satisfies condition three vacuously since $|\mathcal{H}| = 1$, and condition two trivially since $x \mapsto x^2$ is injective on $\mathbb{R}^+$. Since $h(G_1) = \left\{2^{2^k} \mid k \in \mathbb{N}\right\} = G_1 \setminus \left\{\sqrt{2}\right\}$, the first condition is also satisfied.

   G_2 is not free since $16 \in h(G_2)$, so it does not satisfy the first condition.

   (c) *Define* $\iota : B_2 \to \mathbb{R}$ *by* $\iota(\sqrt{2}) = 0$ *and* $\iota(16) = \frac{7}{2}$. *Define* $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *by* $(a,x) \mapsto \log_2 a + x$. *Show that there exists a function* $f : G(A,B_2,\mathcal{H}) \to \mathbb{N}$ *such that* $f(b) = \iota(b)$ *for all* $b \in B_2$ *and* $f(h(a)) = g(a, f(a))$ *for all* $a \in G_2$.

   I am going to assume we are not simply taking $f : G \to \mathbb{N}$, since $\frac{7}{2}$ is not in $\mathbb{N}$, making the problem statement trivially false. Instead, we'll take it to be $\mathbb{Q}$.

   If $f(16) = g(4, f(4)) = \iota(16)$ then f is well-defined since 16 is generated by h and B_1, so that every subsequent element of G is accounted for. But this is precisely what happens:

   $$\begin{aligned} f(\sqrt{2}) &= 0 \\ f(2) &= \log_2 \sqrt{2} + 0 = \frac{1}{2} \\ f(4) &= \log_2 2 + \frac{1}{2} = \frac{3}{2} \\ f(16) &= \log_2 4 + \frac{3}{2} = \frac{7}{2} = \iota(16) \end{aligned}$$

   Every subsequence value of f is therefore determined uniquely by g.

5. *Let $A = \mathbb{N}^+$, $B = \{3, 7\}$, and $\mathcal{H} = \{h_1, h_2\}$ where $h_1(n) = 20n + 1$ and $h_2(n, m) = 2^n(2m + 1)$. Show that $G(A, B, \mathcal{H})$ is free.*

   Let $G = G(A, B, \mathcal{H})$. The first condition is easily satisfied since since $h_2(n, m)$ is even for all $n, m \in A$ and $7 < 13 < h_1(n)$ for all $n \in A$. h_1 is injective over all $\mathbb{R}$ (in fact, it is bijective), so it is certainly injectiver over G.

   To see that h_2 is injective let $(n_1, m_1), (n_1, m_1) \in A^2$ and assume $2^{n_1}(2m_1 + 1) = 2^{n_2}(2m_2 + 1)$. If $n_1 < n_2$ then it must be that $2^{n_2 - n_1} \mid 2m_1 + 1$, which is absurd as $2m_1 + 1$ is always odd. The case for $n_2 < n_1$ follows *mutatis mutandis*, and hence $n_1 = n_2$. Therefore $2m_1 + 1 = 2m_2 + 1$, and hence $m_1 = m_2$. Therefore h_2 is injective over A^2 (and, in particular, G^2).

   The third condition is satisfied since h_1 is always odd and h_2 is always even, and therefore G is free.

6. *Let $(A, B, \mathcal{H})$ be a generating system that is not free. Show that there exists a set X and functions $\iota : B \to X$ and $g_h : (A \times X)^k \to X$ such that there is no function $f : G \to X$ satisfying $f \mid_B = \iota$ and $f(h(a_1, \ldots, a_k)) = g_h(a_1, f(a_1), \ldots, a_k, f(a_k))$.*

   We will proceed case-by-case, assume that f is a function satisfying both the above properties.

   (a) Assume there exists $h \in \mathcal{H}^k$ and $(a_1, \ldots, a_k) \in G^k$ such that $h(a_1, \ldots, a_k) = b_0 \in B$.
   Let $X = \mathbb{N}$ and define $\iota(b) = 1$ for all $b \in B$ and $g_h(a'_1, f(a'_1), \ldots, a'_j, f(a'_j)) = 0$ for all $(a'_1, \ldots, a'_j) \in G^k$ and $h \in H^j$, for all j.. Then

   $$1 = \iota(b_0) = f(b_0) = f(h(a_1, \ldots, a_k)) = g_h(a_1, f(a_1), \ldots, a_k, f(a_k)) = 0$$

   which is absurd.

   (b) Assume there exists some $h \in \mathcal{H}_k$ and distinct $\vec{a} \in G^k, \vec{b} \in G^k$ such that $h(\vec{a}) = h\vec{b})$.
   Let $X = A^*$. Define $\iota(b) = b$ and $g_h(a_1, f(a_1), \ldots, a_k, f(a_k)) = a_1 * \cdots * a_k$. Then

   $$\begin{aligned} b_1 * \cdots b_k &= g_h(b_1, f(b_1), \ldots, b_k, f(b_k)) \\ &= f(h(\vec{b})) \\ &= f(h(\vec{a})) \\ &= g_h(a_1, f(a_1), \ldots, a_k, f(a_k)) \\ &= a_1 * \cdots * a_k \end{aligned}$$

   which implies $\vec{a} = \vec{b}$, a contradiction.

   (c) Case 3