CMSC 277: Homework #3

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1. *For each $\varphi \in \mathrm{Form}_{\mathrm{P}}$ give a deduction showing that $\varphi \vdash \neg\neg\varphi$.*

$$\{\varphi, \neg\varphi\} \vdash \varphi \quad \text{(Assumption)} \tag{1}$$
$$\{\varphi, \neg\varphi\} \vdash \neg\varphi \quad \text{(Assumption)} \tag{2}$$
$$\varphi \vdash \neg\neg\varphi \quad \text{(Contr on (1) and (2))} \tag{3}$$

2. (a) *For each $\varphi, \psi \in \mathrm{Form}_{\mathrm{P}}$ give a deduction showing that $\neg\varphi \vdash \neg(\varphi \wedge \psi)$.*

$$\{\neg\varphi, \varphi \wedge \psi\} \vdash \neg\varphi \quad \text{(Assumption)} \tag{1}$$
$$\{\neg\varphi, \varphi \wedge \psi\} \vdash \varphi \wedge \psi \quad \text{(Assumption)} \tag{2}$$
$$\{\neg\varphi, \varphi \wedge \psi\} \vdash \varphi \quad (\wedge EL \text{ on (2)}) \tag{3}$$
$$\neg\varphi \vdash \neg(\varphi \wedge \psi) \quad \text{(Contr on (1) and (3))} \tag{4}$$

   (b) *For each $\varphi, \psi \in \mathrm{Form}_{\mathrm{P}}$ give a deduction showing that $\neg(\varphi \wedge \psi) \vdash (\neg\varphi) \vee (\neg\psi)$.*

$$\{\neg(\varphi \wedge \psi), \varphi, \psi\} \vdash \varphi \quad \text{(Assumption)} \tag{1}$$
$$\{\neg(\varphi \wedge \psi), \varphi, \psi\} \vdash \psi \quad \text{(Assumption)} \tag{2}$$
$$\{\neg(\varphi \wedge \psi), \varphi, \psi\} \vdash \varphi \wedge \psi \quad (\wedge I \text{ on (1) and (2)}) \tag{3}$$
$$\{\neg(\varphi \wedge \psi), \varphi, \psi\} \vdash \neg(\varphi \wedge \psi) \quad \text{(Assumption)} \tag{4}$$
$$\{\neg(\varphi \wedge \psi), \varphi\} \vdash \neg\psi \quad \text{(Contr on (3) and (4))} \tag{5}$$
$$\{\neg(\varphi \wedge \psi), \varphi\} \vdash (\neg\varphi) \vee (\neg\psi) \quad (\vee IR \text{ on (5)}) \tag{6}$$
$$\{\neg(\varphi \wedge \psi), \neg\varphi\} \vdash \neg\varphi \quad \text{(Assumption)} \tag{7}$$
$$\{\neg(\varphi \wedge \psi), \neg\varphi\} \vdash (\neg\varphi) \vee (\neg\psi) \quad (\vee IL \text{ on (7)}) \tag{8}$$
$$\neg(\varphi \wedge \psi) \vdash (\neg\varphi) \vee (\neg\psi) \quad (\neg PC \text{ on (6) and (8)}) \tag{9}$$

3. (a) *Show that if $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

   Let $\varphi \in \mathrm{Form}_{\mathrm{P}}$ be such that $\Gamma \vdash \varphi$ but $\Gamma \nvDash \varphi$. Then there exists a truth assignment $v : P \to \{0, 1\}$ such that $\bar{v}(\Gamma) = \{1\}$ and $\bar{v}(\varphi) = 0$. Hence Γ is satisfiable. Moreover, $\Gamma \cup \{\neg\varphi\}$ is also satisfiable since $\bar{v}(\neg\varphi) = 1$. Therefore by soundness both Γ and $\Gamma \cup \{\neg\varphi\}$ are consistent. However, as $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposition 3.53 and $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by assumption, it follows that $\Gamma \cup \{\neg\varphi\}$ is inconsistent – a contradiction.

   (b) *Show that every consistent set of formulas is satisfiable.*

   Assume Γ is consistent and that for every truth assignment $v : P \to \{0, 1\}$ there exists some $\varphi \in \Gamma$ such that $\bar{v}(\varphi) = 0$. Then $\Gamma \vDash \psi$ for all $\psi \in \mathrm{Form}_{\mathrm{P}}$, vacuously. In particular $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$. But then $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ by completeness, so that Γ is inconsistent – a contradiction.

4. *Suppose that $\theta \vdash \gamma$ and $\gamma \vdash \theta$. Show that if $\Gamma \vdash \varphi$ then $\Gamma \vdash \mathrm{Subst}_{\theta,\gamma}(\varphi)$.*

   Since θ and γ are syntactically equivalent, by soundness and completeness they are also semantically equivalent and hence for any truth assignment $v : P \to \{0,1\}$ we have that $\bar{v}(\theta) = barv(\gamma)$. If $\Gamma \vdash \varphi$ then by soundness $\Gamma \models \varphi$. From the first problem on the previous homework it follows that $\bar{v}(\varphi) = \mathrm{Subst}_{\theta,\gamma}(\varphi)$. Hence $\Gamma \models \mathrm{Subst}_{\theta,\gamma}(\varphi)$ and by completeness $\Gamma \vdash \mathrm{Subst}_{\theta,\gamma}(\varphi)$

5. *Suppose we eliminate the $\to E$ rule, the $\neg PC$ rule and the Contr rule. Show that the completeness theorem no longer holds.*

6. *Fix $k \in \mathbb{N}^+$. Let P be a poset such that such that every finite subset of P is the union of k chains. Show that P itself is the union of k chains.*

   Fix $k \in \mathbb{N}^+$ and define

$$\begin{aligned}\Gamma = &\{C_{a,1} \vee \cdots \vee C_{a,k} \mid a \in P\}\\ &\cup \{\neg(C_{a,i} \wedge C_{b,i}) \mid a \text{ and } b \text{ are incomparable}\}\end{aligned}$$

   Intuitively each $C_{a,i}$ corresponds to the element a being in one of k chains but incomparable elements being in different chains. We include the possibility of there only being “one” chain, e.g., a one element subset is the union of k chains, viz., itself k times. Let $\Gamma_0 \subset \Gamma$ be a finite subset and let $A = \{a_1, \ldots, a_n\}$ be the set of all $a \in P$ such that $C_{a,i}$ occurs in some element of Γ_0 for some $i \in \{1, \ldots, k\}$. By hypothesis we can write A as the union of k chains which corresponds to a truth assignment witnessing that Γ_0 is satisfiable.

   Since every such Γ_0 is satisfiable it follows from compactness that Γ itself is satisfiable, and hence P can be written as the union of k chains.