MATH 257: Homework #5

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03 November 2004

1. *Let A be an abelian group and let $B \leq A$. Prove that A/B is abelian. Give an example of a non-abelian group containing a proper normal subgroup N such that G/N is abelian.*

   Since A is abelian B is also abelian and therefore normal in A, so the group operation on A/B is well-defined. Let $a_1, a_2 \in A$ be arbitrary, then

   $$a_1Ba_2B = a_1a_2B = a_2a_1B = a_2Ba_1B$$

   so A/B is abelian. The converse is not true. Take $D_6 = \{1, r, r^2, f, fr, fr^2\}$ where r and f represent rotations and "flips" of the corresonding polygon respectively. Define $R = \{1, r, r^2\}$. Every element of D_6 can be written as $f^j r^k$ for some $j, k \in \mathbb{N}$, so $f^j r^k R r^{-k} f^{-j} = \{1, r^{-1}, r^{-2}\}$, but $r^{-1} = r^2$ and $r^{-2} = r$, so $R \trianglelefteq D_6$ and D_6/R is well-defined. However, $|R| = 3$, so $[D_6 : R] = 2$ and therefore G/R must be abelian.

2. *Let G be a group and $N \trianglelefteq G$. Denote $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$.*

   In general if $H \leq G$ then $aH = bH$ if and only if $b^{-1}a \in H$ by Proposition 3.4. Therefore

   $$\begin{aligned} \overline{xy} = \overline{yx} &\Leftrightarrow xNyN = yNxN \\ &\Leftrightarrow xyN = yxN \\ &\Leftrightarrow (yx)^{-1}xy \in N \\ &\Leftrightarrow x^{-1}y^{-1}xy \in N \end{aligned}$$

3. *Let G be a group and let $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$. Show that $N \trianglelefteq G$ and G/N is abelian.*

   $N \leq G$ by construction. Since N is normal and $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$ it follows from the previous problem that G/N is abelian.

4. *Prove that if $N \trianglelefteq G$ where $|G| < \infty$ and $(|N|, [G : N])) = 1$ then N is the unique subgroup of G of order $|N|$*

   From the previous homework we know that if $H \leq G$ and $(|H|, [G : N]) = 1$ then $H \leq N$. Assume $M \trianglelefteq G$ and $|M| = |N|$. Then $(|M|, [G : N]) = 1$, so $M \leq N$. However,

   $$[G : M] = \frac{|G|}{|M|} = \frac{|G|}{|N|} = [G : N]$$

   so $(|N|, [G : M]) = 1$ and $N \leq M$. Therefore $M = N$, i.e., N is unique.

5. *Prove that if $H \trianglelefteq G$ with prime index p then for all $K \leq G$ either $K \leq H$ or $G = HK$ and $[K : K \cap H] = p$.*

   Either $K \leq H$ or not, so assume not. If $G = HK$ then by the second isomorphism theorem

   $$p = \left|\frac{G}{H}\right| = \left|\frac{HK}{H}\right| = \left|\frac{K}{K \cap H}\right| = [K : K \cap H]$$

   So it is sufficient to show that $G = HK$.

   Since $H \trianglelefteq G$, $(Hg)^n = Hg^n$. In particular this means for any $g \in G \setminus H$, $H = (Hg)^p = Hg^p$. That is, $g^p \in H$ for any $g \in G \setminus H$. p must be the smallest integer such that this is true since otherwise we could pick a $g \in G \setminus H$ and $k \neq 1$ such that $(gH)^k = H$ where $k \mid p$, a contradiction. Hence each of $1, g, g^2, \ldots, g^{p-1}$ are distinct coset representatives. By hypothesis there are p such representatives and therefore every coset representative can be repesented as g^j for $0 \leq j \leq p - 1$ if $g \in G \setminus H$. Assuming K is not a subgroup of H, then there exists a $k \in K \setminus H \subseteq G \setminus H$ and $\{1, k, k^2, \ldots, k^{p-1}\}$ is a complete set of coset representatives.

   Let $g \in G$ be arbitrary. Then there exists $x \in G$ such that $g \in Hx$. However, from above, $x = k^j$ for the appropriate j, and hence $g \in Hk^j \subseteq HK$, i.e., $G \subseteq HK$. That $HK \subseteq G$ is obvious, so $G = HK$.

6. *Let $C \trianglelefteq A$ and $D \trianglelefteq B$. Prove that $C \times D \trianglelefteq A \times B$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.*

   Let $(c, d) \in C \times D$ and $(a, b) \in A \times B$, then for some $c' \in C$ and $d' \in D$

   $$(a, b)(c, d)(a, b)^{-1} = (a, b)(c, d)(a^{-1}, b^{-1}) = (aca^{-1}, bdb^{-1}) = (c', d') \in C \times D$$

   Therefore $C \times D \trianglelefteq A \times B$. Define the map $\varphi : A \times B \to (A/C) \times (B/D)$ by $\varphi(a, b) = (aC, bD)$. This map is obviously surjective and

   $$\begin{aligned} \ker \varphi &= \{(a, b) \mid \varphi(a, b) = 1\} \\ &= \{(a, b) \mid (aC, bD) = 1\} \\ &= \{(a, b) \mid aC = C, bD = D\} \\ &= \{(a, b) \mid a \in C, b \in D\} \\ &= C \times D \end{aligned}$$

   By the first isomorphism theorem

   $$(A \times B)/(C \times D) = (A \times D)/\ker \varphi \cong \varphi(A \times B) = (A/C) \times (B/D)$$

7. *Let p be a prime and let G be a group of order $p^a m$ where $p \nmid m$. Assume $P \leq G$ with $|P| = p^b$ and $N \trianglelefteq G$ with $|N| = p^a n$, where $p \nmid n$. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$.*

   From the previous homework we know $P \cap N \trianglelefteq P$ and therefore $P/(P \cap N)$ is a group. By Lagrange's theorem $|P \cap N| \mid |P|$, but $|P| = p^b$, so there exists some $k \leq b$ such that $|P \cap N| = p^k$. From the second isomorphism theorem (in the book, or the fourth in class)

   $$PN/N \cong P/(P \cap N) \Rightarrow |PN/N| = \frac{|P|}{|P \cap N|} = \frac{p^a}{p^k} = p^{a-k}$$

   Hence it is sufficient to show that $k \geq b$ so that $k = b$. Because $PN \leq G$ the largest power of p in $|PN|$ is p^a. From the second isomorphism theorem we know

   $$|PN| \cdot |P \cap N| = |P| \cdot |N| = p^{a+b} n$$

   Hence the power of p in $|P \cap N|$ must be at least p^b, i.e., $k \geq b$, and hence $k = b$.