CMSC 277: Homework #2

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1. *Recall the definition of* $\mathrm{Subst}_{\theta,\gamma} : \mathrm{Form}_\mathrm{P} \to \mathrm{Form}_\mathrm{P}$. *If* $v : P \to \{0,1\}$ *is a truth assignment such that* $\bar{v}(\gamma) = \bar{v}(\theta)$ *then* $\bar{v}(\varphi) = \mathrm{Subst}_{\theta,\gamma}(\varphi)$ *for all* $\varphi \in \mathrm{Form}_\mathrm{P}$.

   For notatinal convenience denote $\bar{v}$ by w and $\mathrm{Subst}_{\theta,\gamma}$ by S. Define

   $$X = \{\varphi \in \mathrm{Form}_\mathrm{P} \mid w(\varphi) = w(S(\varphi))\}$$

   We proceed by induction. For the base case we wish to show that $P \subseteq X$. So let $\varphi \in X$ and recall that by definition $w(\varphi) = v(\varphi)$. Either $\varphi = \gamma$ or $\varphi \neq \gamma$. If the former is the case then

   $$w(\varphi) = w(\gamma) = w(\theta) = w(S(\varphi)$$

   and hence $\varphi \in X$. If the latter is the case then $w(\varphi) = w(S(\varphi))$ directly from the definition of S.

   Now let $\varphi \in X$. To show that X is closed under $h_\neg$ first assume $\gamma = (\neg\varphi)$. Then

   $$w(\neg\varphi) = w(\gamma) = w(\theta) = w(S(\neg\varphi))$$

   and hence $(\neg\varphi) \in \mathrm{Form}_\mathrm{P}$. If $\gamma \neq (\neg\varphi)$ then we have the following:

   $$\begin{aligned} w(S(\neg\varphi)) &= w[(\neg S(\varphi))] \\ &= \neg w(S(\varphi)) \\ &= \neg w(\varphi) \\ &= w(\neg\varphi) \end{aligned}$$

   where $\neg : \{0,1\} \to \{0,1\}$ is the map such that $\neg 0 = 1$ and $\neg 1 = 0$. Hence $(\neg\varphi) \in X$.

   The remainder of the proof essentially follows *mutatis mutandis*. Let $\varphi, \psi \in X$. First assume $\gamma = (\varphi \diamond \psi)$. Then

   $$w(\varphi \diamond \psi) = w(\gamma) = w(\theta) = w(S(\varphi \diamond \gamma))$$

   so that $(\varphi \diamond \psi) \in X$. Otherwise we have the following:

   $$\begin{aligned} w(S(\varphi \diamond \psi)) &= w[(S(\varphi) \diamond S(\psi))] \\ &= \diamond(w(S(\varphi)), w(S(\psi))) \\ &= \diamond(w(\varphi), w(\psi)) \\ &= w((\varphi \diamond \psi)) \end{aligned}$$

   where $\diamond : \{0,1\}^2 \to \{0,1\}$ is defined according to the behavior of w with respect to $\diamond$. In either case, $(\varphi \diamond \psi) \in X$. Since $X \subseteq \mathrm{Form}_\mathrm{P}$ by hypothesis and X is inductive, it follows that $X = \mathrm{Form}_\mathrm{P}$ and the proposition is proven.

2. *Let* $\mathrm{Form}_{\bar{\mathrm{P}}} = G(L^*, P, \{h_\neg, h_\vee, h_\wedge\})$ *and let* Dual *be defined as in the problem. Show that* $\mathrm{Dual}(\varphi)$ *is semantically equivalent to* $\neg\varphi$ *for all* $\varphi \in \mathrm{Form}_{\bar{\mathrm{P}}}$.

   Define
   $$X = \{\varphi \in \mathrm{Form}_{\bar{\mathrm{P}}} \mid \ \mathrm{Dual}(\varphi) \text{ is semantically equivalent to } \neg\varphi\}$$

   We will show that X is inductive. First, note that $P \subseteq X$ directly from the definition of Dual. So assume $\varphi, \psi \in X$. By the previous exercise we have that substituting a formula for something with which it is semantically equivalent preserves semantic equivalence. Writing $\equiv$ for semantic equivalence we have
   $$\mathrm{Dual}(\neg\varphi) \equiv \neg\,\mathrm{Dual}(\varphi) \equiv \neg(\neg\varphi) \equiv \varphi$$

   which follow by definition of Dual, the inductive hypothesis, and from the proof in class that $\neg\neg\varphi$ is syntactically equivalent to φ plus soundness and completeness, respectively.

   Similarly, by deMorgan's laws as proven in class,
   $$\mathrm{Dual}(\varphi \vee \psi) \equiv \mathrm{Dual}(\varphi) \wedge \mathrm{Dual}(\psi) \equiv \neg\varphi \wedge \neg\psi \equiv \neg(\varphi \vee \psi)$$

   It follows *mutatis mutandis* for $h_\wedge$, and hence $X = \mathrm{Form}_{\bar{\mathrm{P}}}$.

3. (a) *Show that the following are equivalent for* $\Gamma_1, \Gamma_2 \subseteq \mathrm{Form}_{\mathrm{P}}$*:*

      i. Γ_1 *and* Γ_2 *are semantically equivalent.*

      ii. *For all* $\theta \in \mathrm{Form}_{\mathrm{P}}$, $\Gamma_1 \vDash \theta$ *if and only if* $\Gamma_2 \vDash \theta$.

      That the latter implies the former is trivial, since if it holds for all $\theta \in \mathrm{Form}_{\mathrm{P}}$ it certainly holds for subsets of $\mathrm{Form}_{\mathrm{P}}$. Define
      $$X = \{\theta \in \mathrm{Form}_{\mathrm{P}} \mid \text{If } \Gamma_1, \Gamma_2 \text{ are semantically equivalent then } \Gamma_1 \vDash \theta \text{ if and only if } \Gamma_2 \vDash \theta.\}$$

      I'm not sure where to go from here. The inductive steps seems easy enough, e.g., if $\varphi, \psi \in X$ then $\Gamma_1 \vDash \varphi \wedge \psi$ implies $\Gamma_1 \vDash \varphi$ and $\Gamma_1 vDash \psi$. Hence by the inductive hypothesis $\Gamma_2 \vDash \varphi$ and $\Gamma_2 vDash \psi$, so that $\Gamma_2 \vDash \varphi \wedge \psi$. Why this holds for the base case, however, I don't know.

   (b) *Show that if* Γ *is finite then* Γ *has an independent semantically equivalent subset.*

      We proceed by induction on the size of Γ. If $|\Gamma| = 0$ then the statement is vacuously true, so let $|\Gamma| = k$ for some finite k. If Γ is independent then we are done, so assume there exists φ such that $\Gamma \setminus \{\varphi\} \vDash \varphi$. By the inductive hypothesis there exists $\Gamma_0 \subseteq \Gamma \setminus \{\varphi\}$ such that Γ_0 is indepednent and semantically equivalent to $\Gamma \setminus \{\varphi\}$. But $\Gamma \setminus \varphi$ is semantically equivalent to Γ since $\Gamma \setminus \varphi \vDash \Gamma$ by hypothesis and $\Gamma \vDash \Gamma \setminus \{\varphi\}$ trivially. Therefore Γ_0 is independent and semantically equivalent to Γ.

   (c) *Show that there exists a set* P *and an infinite set* $\Gamma \subseteq \mathrm{Form}_{\mathrm{P}}$ *which has no independent semantically equivalent subset.*

4. *Let* $P = \{A_1, \ldots, A_7\}$. *Show that there exists a boolean function* $f : \{0,1\}^7 \to \{0,1\}$ *such that* $\mathrm{Depth}(\varphi) \geq 5$ *for all* $\varphi \in \mathrm{Form}_{\mathrm{P}}$ *with* $B_\varphi = f$.

   Recall that the map $\varphi \mapsto B_\varphi$ is surjective from $\mathrm{Form}_{\mathrm{P}}$ to the set of all boolean functions over P.

   Consider all formulas with $\mathrm{Depth}(\varphi) \leq 4$. In general for a formula with depth n there will be 2^n possible positions for atomic formulas, so that for $|P| = 7$ there are $7^{(2^4)} \cdot 3^1 5$ formulas of depth at most 4 (since any formula of depth less than 4 can incrase their depth by using $A = (A \wedge A)$), where $3^1 5$ is counting the use of the connectives. There are $2^{(2^7)}$ boolean functions in all. Hence there are at most $7^{(2^4)} \cdot 3^1 5$ functions represented by these elements. Since $7^{(2^4)} \cdot 3^1 5 < 2^{(2^7)}$ it follows that there must exist at least one function which can be represented only by a formula of depth at least 5.

5. *Let P be a set not containing the symbols $($, $\neg$, $\wedge$, $\vee$, and $\rightarrow$. Let $L = P \cup \{(, \neg, \wedge, \vee, \rightarrow\}$. Define a unary function $h_\neg$ and binary functions $h_\wedge$, $h_\vee$, and $h_\rightarrow$ on L^* as follows:*

$$\begin{aligned} h_\neg(\varphi) &= (\neg\varphi \\ h_\wedge(\varphi, \psi) &= (\varphi \wedge \psi \\ h_\vee(\varphi, \psi) &= (\varphi \vee \psi \\ h_\rightarrow(\varphi, \psi) &= (\varphi \rightarrow \psi \end{aligned}$$

Show that $(L^*, P, \mathcal{H})$ is free where $\mathcal{H} = \{h_\neg, h_\wedge, h_\vee, h_\rightarrow\}$.

Let $S = \{\neg, \wedge, \vee, \rightarrow\}$. Define $K : L^* \rightarrow \mathbb{Z}$ as follows. For $\diamond \in S$ let $K(\diamond) = -1$ and $K(() = 1$. For all other symbols $\varphi \in L$ let $K(\varphi) = 0$. Let $K(\lambda) = 0$ and for $\sigma \in L^*$ let $K(\sigma) = \sum_{i=1}^{|\sigma|} K(\sigma(i))$. In other words, we are going to exploit the fact that in this language there are as many left parantheses as there are connectives in a well-formed formula.

Lemma 0.1. *$K(\varphi) = 0$ for all $\varphi \in \mathrm{Form_P}$.*

Proof. We proceed by induction. Define

$$X = \{\varphi \in \mathrm{Form_P} \mid K(\varphi) = 0\}$$

Certainly $P \subset \varphi$ since $K(A) = 0$ for all $A \in P$. Let $\varphi, \psi \in X$. Then

$$\begin{aligned} K((\neg\varphi) &= K(() + K(\neg) + K(\varphi) \\ &= 1 + -1 + K(\varphi) \\ &= 0 \end{aligned}$$

For $\diamond \in \{\vee, \wedge, \rightarrow\}$,

$$\begin{aligned} K((\varphi \diamond \psi) &= K(() + K(\varphi) + K(\diamond) + K(\psi) \\ &= 1 + K(\varphi) + -1 + K(\psi) \\ &= 0 \end{aligned}$$

Since $X \subseteq \mathrm{Form_P}$ and X is inductive, $X = \mathrm{Form_P}$. This means all well-formed formulas have the same number of left parentheses as connectives. ■

Here is a second lemma.

Lemma 0.2. *If $\varphi \in \mathrm{Form_P}$ and $\lambda \neq \sigma \subset \varphi$ then either $K(\sigma) \leq -1$ or σ ends in a connective.*

Proof. We cannot simply say that $K(\sigma) \leq -1$ for all proper initial segments because then things such as $(\neg$ would become well-formed formulas, which we do not want. Let X be the set of all $\varphi \in \mathrm{Form_P}$ satisfying this property. This is trivially satisfied by elements of P since the only proper initial segments are λ and $K(\lambda) = 0$.

Let $\varphi \in \mathrm{Form_P}$. Then $\sigma \in \{(, (\neg, (\neg\tau\}$ where $\tau \subset \varphi$. If $\sigma = ($ then $K(\sigma) = -1$. If $\sigma = (\neg$ then σ ends in a connective. If τ does not end in a connective then $K(\tau) \leq -1$ by hypothesis and hence $K(\sigma) = K(() + K(\neg) + K(\tau) \leq -1$. Otherwise, σ ends in a connective since τ does.

The proofs for binary connetives, $\diamond$, is the same. If $\varphi, \psi \in X$ then one must consider σ as one of $($, $(\tau$, $(\varphi$, $(\varphi\diamond$, or $(\varphi \diamond \tau'$ where $\tau \subset \varphi$ and $\tau' \subset \varphi$. The same analysis proves that either there are too few connectives in σ or σ ends in a connective. ■

Hence $\varphi \in \mathrm{Form_P}$ is a well-formed formula if and only if it has the same number of left parentheses as connectives and does not end in a connective. Together these two lemmas prove that no proper initial segment of a well-formed formula is itself a well-formed formula. From this the injectivity of the various functions in the definition of a free generating system follows since we have that $\varphi, \psi \in \mathrm{Form_P}$ implies $\varphi \not\subset \psi$ (and $\lambda \notin \mathrm{Form_P}$.