MATH 208: Homework #7

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1. *Show that f_v, the natural map from V^* to the ground field F, is in V^{**}.*

   Let $\alpha, \alpha_1, \alpha_2 \in V^*$ and $\beta \in F$, then

   $$f_v(\alpha_1 + \alpha_2) = (\alpha_1 + \alpha_2)(v) = \alpha_1(v) + \alpha_2(v) = f_v(\alpha_1) + f_v(\alpha_2)$$

   and

   $$f_v(\beta\alpha) = (\beta\alpha)(v) = \beta \cdot \alpha(v) = \beta \cdot f_v(\alpha)$$

   so f_v is linear.

   f_v is clearly bounded since, for fixed v and any α, $\|f_v(\alpha)\| \leq \|v\|\|\alpha\|$.

2. *Show that the map $v \mapsto f_v$ is a norm-preserving monomorphism.*

   By the previous problem we have that $\|f_v\| \leq \|v\|$, so all that remains to be shown is the other direction of the inequality.

   We have by the corolllary of the Hahn-Banach Theorem that for every $v \in V$ there exists $\alpha \in V^*$ such that $a(v) = 1$ and $\|a(v)\| = \|v\|$. Since $\|f_v(\alpha)\| = \|\alpha(v)\|$ by definition, $\|v\| \in \{\|f_v(\alpha)\|\}$, so $\|v\| \leq \|f_v\|$ and hence $\|f_v\| = \|v\|$.

   The linearity of the map trivially follows from the linearity of α, and so it is injective (since, if it were not, it would not be norm preserving). Therefore f_v is a norm-preserving monomorphism.

3. *Answer the following:*

   (a) *The set of matrices representing permutations on $l_n^p(F)$ is isomorphic to S_n.*

   Every element of S_n can be represented as a vector of n elements, each of the form $i \in \mathbb{N}$, $i \leq n$. Likewise, let A be a permutation of $l_n^p(F)$, then the i^{th} row as a 1 in the j^{th} column. Denote the set of all these matrices as M_p and define $\varphi : M_p \to S_n$ such that the i^{th} entry of the resultant vector has a value of of j.

   By the uniqueness of i, j this function is injective. $|M_p| = n! = |S_n|$, so the map is also surjective.

   Let $A, B \in M_p$ where A has a j in the i^{th} row and B has a k in the j^{th} row, then AB has a k in the i^{th} row so $\varphi(AB)$ has a k in the i^{th} entry. Similarly, $\varphi(A)$ has a j in the i^{th} entry and $\varphi(B)$ has a k in the j^{th} entry so $\varphi(A)\varphi(B)$ has a k in the i^{th} entry. Hence, $\varphi(AB) = \varphi(A)\varphi(B)$.

(b) *What are the matrices corresponding to even permutations?*

An even permutation is a permutation which can be arrived at with an even number of transpotitions, i.e., the exchange of only two elements. Since each permutation corresponds to a matrix with a 1 in each column and vector, a transposition corresponds to the exchange of only two rows. Hence, even permutations correspond to matrices in M_p with an even number of rows exchanged.

Since there is only one non-zero pattern (i.e., only one way in which we can multiply any n elements with unique columns and rows) in the matrix and it consists of all ones the determinant must be either 1 or -1. If the matrix has an even number of row exchanges then it has a positive determinant or, in this case, a determinant of 1. Therefore the set $\{A \in M_p \mid \det A = 1\}$ is the set of all even permutations.

4. *Find $Iso(l_n^p(\mathbb{C}))$.*

5. *Determine all isomorphisms from $l_n^p(F)$ to $l_n^{p'}(F)$, where n is fixed and $p \neq p'$.*

6. *Find $Iso(l^p(f))$.*

7. *$d'' := d + d'$ is a metric on $X \times X'$.*

Recall that $d''((x,x'),(y,y')) = d(x,y) + d'(x',y')$ where d and d' are the metrics on X and X' respectively. We must show that d'' is a metric on $X \times X'$.

(a) (Positive definite) If $(x,x') = (y,y')$ then $x = y$ and $x' = y'$, so $d''((x,x'),(y,y')) = d(x,x) + d'(x',x') = 0$. Likewise, if $d''((x,x'),(y,y') = 0$ then $d(x,y) = 0$ and $d'(x',y') = 0$, so $(x,x') = (y,y')$.

(b) (Transitivity)

$$\begin{aligned} d''((x,x'),(y,y')) &= d(x,y) + d'(x',y') \\ &= d(y,x) + d'(y',x') \\ &= d''((y,y'),(x,x')) \end{aligned}$$

(c) (Triangle inequality)

$$\begin{aligned} d''((x,x'),(y,y')) &= d(x,y) + d'(x',y') \\ &\leq d(x,z) + d(z,y) + d'(x',z') + d'(z',y') \\ &= d(x,z) + d'(x',z') + d(z,y) + d'(z',y') \\ &= d''((x,x'),(z,z')) + d''((z,z'),(y,y')) \end{aligned}$$

8. *Show the following:*

(a) *$GL_n(F)$ is a dense open subset of $M_n(F)$.*

Consider $M_n(F) \setminus GL_n(F)$, the set of all matrices with determinant zero. Take any convergent sequence (a_k) of $M_n(F) \setminus GL_n(F)$. Each a_i is a matrix with determinant zero, and since the determinant is a continuous function (a polynomial, in fact), $\lim_{k\to\infty} a_k = a$ must also be a matrix with determinant zero. Therefore $M_n(F) \setminus GL_n(F)$ is closed, and hence $GL_n(F)$ is open.

(b) *$GL_n(F)$ is a locally compact group.*

9. $(G, \cdot, d)$ *is a topological group if the map* $(x, y) \mapsto x^{-1}y$ *is continuous.*

   Since $f(x, y) = x^{-1}y$ is continuous for any $x, y \in G$ it follows that $h(x) = f(x, e) = x^{-1}$ is continuous as a function from $G \to G$. Because composition of functions preserves continuity, $(f \circ h)(x) = xy$ is also continuous. Therefore $(G, \cdot, d)$ is a topological group.

10. $\mathbb{Q}_p(\sqrt{p})$ *is a field.*

    Let $a + b\sqrt{p}, c + d\sqrt{p}$ be arbitrary elements of $\mathbb{Q}_p(\sqrt{p})$. We see that $(a + b\sqrt{p})(c + d\sqrt{p}) = (ac+pbd)+(ad+bc)\sqrt{p}$, so we can treat $\mathbb{Q}_p(\sqrt{2})$ as the set of all ordered pairs of p-adic numbers with addition and multiplication defined as $(a, b) + (c, d) = (a + b, b + d)$ and $(a, b)(c, d) = (ac + pbd, ad + bc)$.

    Since addition is coordinate-wise all the additive properties of $\mathbb{Q}_p$ are inherited, so $(\mathbb{Q}_p)\sqrt{p}), +)$ is an Abelian group. We only need look at multiplication and distributivity.

    - Multiplication:
      - Associativity

$$\begin{aligned}(a, b)((c, d)(e, f)) &= (a, b)(ce + pdf, cf + de)\\ &= (ace + padf + pbcf + pbde, acf + ade + bce + pbdf)\\ &= (ac + pbd, ad + bc)(e, f)\\ &= ((a, b)(c, d))(e, f)\end{aligned}$$

      - Identity

$$(a, b)(1, 0) = (1a + p(b0), 0a + 1b) = (a, b)$$

      - Inverse

$$\begin{aligned}(a, b)\left(\frac{a}{a - pb^2}, \frac{-b}{a - pb^2}\right) &= \left(\frac{a^2}{a - pb^2} + p\frac{-b^2}{a - pb^2}, \frac{-ab}{a - pb^2} + \frac{ab}{a - pb^2}\right)\\ &= \left(\frac{a - pb^2}{a - pb^2}, \frac{ab - ab}{a - pb^2}\right)\\ &= (1, 0)\end{aligned}$$

    - Distributivity

$$\begin{aligned}(a, b)((c, d) + (e, f)) &= (a, b)(c + e, d + f)\\ &= (ac + ae + pbd + pbf, ad + af + bc + be)\\ &= (ac + pbd, ad + bc) + (ae + pbf, af + be)\\ &= (a, b)(c, d) + (a, b)(e, f)\end{aligned}$$

    Therefore $\mathbb{Q}_p(\sqrt{p})$ is a field.

11. *Show that a closed subgroup of a locally compact group is itself locally compact.*

    Let $G' < G$ be a closed subgroup of G, where G is locally compact. Take any point x in G', then there exists some neighborhood of $B(x)$ in G whose closure in compact since x is also in G. Let $B(x)' = B(x) \cap G'$, then $B'(x)$ is clearly a neighborhood of x in G', so we will show that its closure is sequentially compact in G' and hence compact in G'.

Take any sequence (a_n) in $\overline{B'(x)}$. This is also a sequence in $\overline{B(x)}$, and so must have a convergent subsequence (a_{n_k}) by the local compactness of G. By the choice of (a_n) each a_{n_k} is in $\overline{B'(x)}$, and must converge in there since the limit is an accumulation point of G', which is closed. Therefore, G' is locally compact.

12. *Show that $Iso(l_n^2(\mathbb{R})) = O(n, \mathbb{R})$.*

    Let $A = (a_{ij}) \in Iso(l_n^2(\mathbb{R}))$. Since an isometry preserves the dot product and the dot product of any two basis elements is zero, the dot product of any two distinct rows or columns is 0 and the dot product of any identical rows or columns is 1. Let $A^T A = B$, so $b_{ij} = \sum_{k=1}^n a_{ki} a_{kj}$. If $i = j$ then there is a 1 in that position (since $\sum_{k=1}^n a_{kj}^2 = 1$), otherwise there is a 0 by the fact that the dot product is preserved. Therefore $A^T A = I$, since $i = j$ are exactly the diagonal points. Therefore $Iso(l_n^2(\mathbb{R})) \subseteq O(n, \mathbb{R})$

    Likewise, if $A^T A = I$ we get that $a_{kk} = 1$ for $1 \leq k \leq n$ and $a_{kj} = 0$ for $k \neq j$. Clearly, then $\sum_{k=1}^n a_{kj}^2 = 1$. It is then sufficient to show that each row or column of Ahas only one 1 or -1 – I am not precisely sure how.

13. *Around what line does the reflection matrix $A = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$*

    First we note that $\det A = -\cos^2\theta - \sin^2\theta = -1$, so A is indeed a reflection. To find the line around which this matrix reflects we must find the points which are not affected by the transformation, i.e., the points $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

    In general,

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x\cos\theta + y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

So we need

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x\cos\theta + y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

But this means that

$$x = -x\cos\theta + y\sin\theta \tag{1}$$

and

$$y = x\sin\theta + y\cos\theta \tag{2}$$

Therefore $y = x\frac{1+\cos\theta}{\sin\theta}$ and $y = x\frac{\sin\theta}{1-\cos\theta}$. It is easy to check that these are both satisfied irrespective of x or y (multiply the first by $\frac{1-\cos\theta}{1-\cos\theta}$ to see that they are equivalent).

We can reduce this to

$$y = x\frac{1+\cos\theta}{\sin\theta} = x\frac{2\cos^2\frac{\theta}{2}}{\sin\theta} = x\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = x\cot\frac{\theta}{2}$$

This is the line around which A reflects.