MATH 207: Homework #8

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1. *Show that $\mathbb{Q}(\sqrt{2})$ is a field.*

   Let $a+b\sqrt{2}, c+d\sqrt{2}$ be arbitrary elements of $\mathbb{Q}(\sqrt{2})$. We see that $(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd)+(ad+bc)\sqrt{2}$, so we can treat $\mathbb{Q}(\sqrt{2})$ as the set of all ordered pairs of real numbers with addition and multiplication defined as $(a,b)+(c,d) = (a+b, b+d)$ and $(a,b)(c,d) = (ac+2bd, ad+bc)$.

   Since addition is coordinate-wise all the additive properties of $\mathbb{R}$ are inherited, so $(\mathbb{Q})\sqrt{2}), +)$ is an Abelian group. We only need look at multiplication and distributivity.

   - Multiplication:
     - Associativity
       $(a,b)((c,d)(e,f)) = (a,b)(ce+2df, cf+de) = (ace+2adf+2bcf+2bde, acf+ade+bce+2bdf) = (ac+2bd, ad+bc)(e,f) = ((a,b)(c,d))(e,f)$
     - Identity
       $(a,b)(1,0) = (1a+2(b0), 0a+1b) = (a,b)$
     - Inverse
       $(a,b)(\frac{a}{a-2b^2}, \frac{-b}{a-2b^2}) = (\frac{a^2}{a-2b^2} + 2\frac{-b^2}{a-2b^2}, \frac{-ab}{a-2b^2} + \frac{ab}{a-2b^2}) = (\frac{a-2b^2}{a-2b^2}, \frac{ab-ab}{a-2b^2}) = (1,0)$
   - Distributivity
     $(a,b)((c,d)+(e,f)) = (a,b)(c+e, d+f) = (ac+ae+2bd+2bf, ad+af+bc+be) = (ac+2bd, ad+bc)+(ae+2bf, af+be) = (a,b)(c,d)+(a,b)(e,f)$

   Therefore $\mathbb{Q}(\sqrt{2})$ is a field.

2. *Show that any non-empty set with the discrete metric is a metric space.*

   Let (X,ρ) be a set with the metric

   $$\rho(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

   - Positive definite: For any $x,y \in X$ we have either $\rho(x,y) = 1$ or $\rho(x,y) = 0$. Clearly this is always non-negative. Moreover, from the definition we get that $\rho(x,y) = 0$ if and only if $x = y$ directly.
   - Symmetric: As the equality relation is symmetric, so is the discrete metric on any set.
   - Triangle inequality:
     We need to guarantee that for any $x,y,s \in X$ we have $\rho(x,y) \leq \rho(x,s)+\rho(s,y)$. In this inequality, the left-hand side can be either 0 or 1 and the right-hand side can be one of 0,1, or 2. The only

combination where this statement7 is violated is when $\rho(x,y) = 1$ and $\rho(x,s) + \rho(s,y) = 0$, so we need to show this case is impossible.

If $\rho(x,s) + \rho(s,y) = 0$ then $\rho(x,s) = 0$ and $\rho(s,y) = 0$. But this implies that $x = s = y$ by positive definite. Hence, $\rho(x,y) = 0$. Therefore the triangle inequality holds in every case.

3. *Show that* $(\mathbb{R}^n, \rho_p)$ *is a metric space.*

   We define the metric as
   $$\rho_p(x,y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}}$$
   where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

   - Positive definite: Note that each term in the sum is positive. As this is the case, we cannot have a non-negative distance. If $\rho_p(x,y) = 0$ then $\sum_{k=1}^{n} |x_k - y_k|^p = 0$, but since each individual term in the sum is positive every term must be zero for this to occur, i.e., $|x_k - y_k|^p = 0$ for all k. Clearly, then, $|x_k - y_k| = 0$ for all k. That $x_k = y_k$ for all k follows directly from the properties of the absolute value on $\mathbb{R}^1$ and therefore, if $\rho_p(x,y) = 0$ then $x = y$. The converse is even more obvious.
   - Symmetric: This follows directly from the properties of the absolute value in $\mathbb{R}^1$.
   - Triangle inequality:

     Save for the statement that for $a, b \geq 0$ we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ if $\frac{1}{p} + \frac{1}{q} = 1$, I understand the proof in Kolmogorov and Fomin. Since this is more-or-less the crux of the Hölder Inequality it would be rather cheap of me to just copy it here.

4. *Show that* $(\mathbb{C}^n, \rho_p)$ *is a metric space.*

5. *Show that* $(\mathbb{R}^n, \rho_\infty)$ *is a metric space.*

   - Positive definite: This trivially follows from the same property in $\mathbb{R}$.
   - Symmetric: This trivially follow from the same property in $\mathbb{R}$.
   - Triangle inequality:

     It suffices to show that $\rho(a,b) \leq \rho(a) + \rho(b)$. If j is the index at which $max_{1 \leq k \leq n} |x_k - y_k|$ is attained, then clearly $max_{1 \leq k \leq n} |x_k + y_k| = |x_j + y_j| \leq |x_j| + |y_j| = max_{1 \leq k \leq n} |x_k| + max_{1 \leq k \leq n} |y_k|$ by this very property in $\mathbb{R}$.

6. *Show that the unit circles in* $l_2^p(\mathbb{R})$ *cover the entire space between* $p = 1$ *and* $p = \infty$.

   The function, $f(p) = x^p + y^p$, which describes the unit circle in l_2^p is continuous and thus by the Intermediate Value Theorem takes on every value from its minimum ($p = 1$) to its maximum ($p = \infty$). Moreover, since the derivative is always positive it takes on each value once if we restrict x, y to be "non-corner" points.

7. *Show that the open ball is an open set.*

   Let (X, ρ) be a metric space, $x \in X$ be arbitrary, and $B_\varepsilon(x)$ be an open neighborhood around x. If $y \in B_\varepsilon(x)$ then $\rho(x,y) < \varepsilon$. Hence, there is some $r > 0$ such that $\rho(x,y) = \varepsilon - r$.

   Then, for all points $z \in X$ such that $\rho(y,z) < h$ we have $\rho(x,z) \leq \rho(x,y) + \rho(y,z) < r - h + h = r$. Therefore $s \in B_\varepsilon(x)$, i.e., $B_\varepsilon(x)$ is open.

8. *Show that a subset of a metric space is closed if and only if it contains all its accumulation points.*

   Let (X,ρ) be a metric space and $S \subseteq X$. If S is closed then S^c is open. Assume for contradiction that $x_0 \in S^c$ is an accumulation point for S. Every neighborhood around x_0 has a point in S, but because S^c is open some of these neighborhoods will also be contained in S^c, a contradiction.

   Conversely, if S is not closed then S^c is not open. That is, there exists some $x_0 \in S^c$ such that for any $\varepsilon > 0\, B_\varepsilon(x_0) \nsubseteq S^c$. So there is some $x \in S$ such that $x \in B_\varepsilon(x_0)$ for all $\varepsilon > 0$ and x_0 is an accumulation point of S not in S.

   Therefore a subset of a metric space is closed if and only if it contains all its accumulation points.

9. *Find the following under the standard metric in* $\mathbb{R}$:

   (a) $\partial\mathbb{Q} = \mathbb{R}$

   (b) $\partial(\mathbb{R}\backslash\mathbb{Q}) = \mathbb{R}$

   (c) $\partial([0,1]) = \{0,1\}$

10. *Show the following*:

    (a) $\partial A = \overline{A}\backslash\mathring{A}$

    Recall that $x \in \partial A$ if and only if every neighborhood around x intersects both A and A^c.

    First we show that $\partial A \subseteq \overline{A}\backslash\mathring{A}$. If $x \in \partial A$ and $x \in \mathring{A}$ then there is a neighborhood around x which is contained in A, a contradiction since every neighborhood around x must also intersect the complement of A. If $x \in \partial A$ and $x \notin \overline{A}$ then there is some closed set B which contains A that does not contain x. Thus $x \in X\backslash B \subseteq X\backslash A$ is open, and so there exists a neighborhood around x completely contained in both $X\backslash B$ and $X\backslash A$. This is a contradiction since every neighborhood must intersect A.

    If $x \notin \mathring{A}$ then every interval intersects the complement of A and if $x \in \overline{A}$ every interval intersects A itself. Therefore if both of these conditions hold $x \in \partial A$.

    Therefore $\partial A = \overline{A}\backslash\mathring{A}$.

    (b) $\mathring{A} = X\backslash(\overline{X\backslash A})$

    First we show that $\mathring{A} \subseteq X\backslash(\overline{X\backslash A})$. Let $x \in \mathring{A}$. That $x \in X$ is trivial. As $x \in \mathring{A}$, there exists some open set $B \subseteq A$ such that $x \in B$. We then have $x \notin X\backslash B$ and $X\backslash A \subseteq X\backslash B$. But $X\backslash B$ is a closed set which contains $X\backslash A$, so $x \notin \overline{X\backslash A}$

    Second, we show that $\mathring{A} \supseteq X\backslash(\overline{X\backslash A})$. If $x \in (\overline{X\backslash A})$ then there is closed set $x \notin B \supseteq X\backslash A$, but this means that $x \in B\backslash X \supseteq A$ and that $B\backslash X$ is open. Therefore $x \in \mathring{A}$ as it is in an open set contained in A.

    Therefore $\mathring{A} = X\backslash(\overline{X\backslash A})$.

    (c) $\overline{A} = X\backslash(X\backslash A)^o$

    First we show that $\overline{A} \subseteq X\backslash(X\backslash A)^o$. Let $x \in \overline{A}$. That $x \in X$ is trivial. Assume for contradiction that $x \in (X\backslash A)^o$, then there is some open set $x \in B \subseteq X\backslash A$. We then have that $x \notin X\backslash B$ and $X\backslash B \supseteq A$ is closed. Hence, $C = \overline{A} \cap (X\backslash B)$ is closed and $x \notin C$. Moreover, $A \subseteq C$. This is a contradiction since C is a closed set which contains A and, as $x \in \overline{A}$, it must be in every such set.

    Second we show that $\overline{A} \supseteq X\backslash(X\backslash A)^o$. If $x \notin (X\backslash A)^o$ then x is not in any open set in $X\backslash A$. But if a closed set contains A then its complement is one of these sets, and thus contains x. Therefore $x \in \overline{A}$.

    Therefore $\overline{A} = X\backslash(X\backslash A)^o$.

11. (a) *Show that $\mathring{A}$ is the set of all interior points of A.*

If $x \in \mathring{A}$ then there is an open set $S \subseteq A$ such that $x \in S$. But since S is open there must be an open neighborhood around x, i.e., x is an interior point of S. This same neighborhood, however, is contained in A, so x is also an interior point of A.

Conversely, if x is an interior point of A then there is some open neighborhood around x contained in A. Since an open neighborhood is also an open set and this neighborhood is contained in A, it is also in $\mathring{A}$ by definition of the interior.

Therefore $x \in \mathring{A}$ if and only if x is an interior point of A. That is, $\mathring{A}$ is the set of all interior points of A.

(b) *Show that $\overline{A}$ is union of A and the set of all accumulation points of A.*

We will denote the set of all accumulation points of A as A'.

First we show that $\overline{A} \subseteq A \cup A'$. If $x \notin A \cap A'$ then there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \emptyset$. But as $B_\varepsilon(x)$ is open, $B_\varepsilon^c(x)$ is closed and does not contain x. Moreover, $A \subseteq B_\varepsilon^c(x)$ since they are disjoint. It is clearly impossible for x to then be in the intersection of all closed sets that contain A as $x \notin B_\varepsilon^c(x)$.

Now we show $A \cup A' \subseteq \overline{A}$. If $x \in A$ then x is certainly in any set, closed or open, which contains A. Thus $x \in \overline{A}$. Let $x \in A'$ and assume for contradiction that $x \notin \overline{A}$. Then there exists a a closed set $B \supseteq A$ with $x \notin B$ which implies $x \notin A$. x is an accumulation point for A and so is also an accumulation point for any superset, including B. As B is closed it contains all its accumulation points, specifically it contains x, a contradiction.

Therefore $\overline{A}$ is union of A and the set of all accumulation points of A.