MATH 262: Homework #7

Jesse Farmer

08 March 2005

1. *Show that if X has a countable basis $\{B_n\}$ then every basis $\mathcal{C}$ for X contains a countable basis for X.*

   Since $\{B_n\}$ and $\mathcal{C}$ are bases for the same topology, it follows that for every $x \in X$ and m there exists a n and $C_{n,m}$ such that $x \in B_n \subset C_{n,m} \subset B_m$.

   Let $x \in X$ and let $x \in B_m \in \{B_n\}$. Then there exists some $C_{n,m}$ such that $x \in C_{n,m} \subset B_m$. If $x \in C_{n_1,m_1} \cap C_{n_2,m_2}$ then there exists n', B_{n_3}, C_{n',n_3} such that

   $$x \in C_{n',n_3} \subset B_{n_3} \subset B_{n_1} \cap B_{n_2} \subset C_{n_1,m_1} \cap C_{n_2,m_2}$$

   Hence the $\{C_{n,m}\}$ form a countable basis.

2. *Let X have a countable basis and let A be an uncountable subset of X. Show that uncountably many points of A are limit points of A.*

   Let $\{B_i\}$ be a countable basis for X and Z be the set of all points that are not limit points of X. For every $z \in Z$ there exists a neighborhood U_z of z which is disjoint from all other points of X, and there is some $x \in B_k \subset U_z$. But then the B_k must be disjoint, and hence $Z = \bigcup B_k$ is countable as it is the countable union of countable sets (singletons, in fact). If A is an uncountable subset of X then it must have an uncountable number of limit points since otherwise it would be the union of two countable sets, viz., the set of all limits points of A and $Z \cap A$ (which is countable from the argument above), and hence countable itself.

3. (a) *Show that every metrizable space with a countable dense subset X has a countable basis.*

   Let (X, ρ) be a metric space and $A \subset X$ a countable dense subset. For every x there exists a neighborhood U of x and a basis element $B_\epsilon(x)$ with $\epsilon < 1$ such that $x \in B_\epsilon(x) \subset U$. Since A is dense in X, there exists some $a \in B_{\frac{\epsilon}{3}}(x)$ so that $\rho(x, a) < \frac{\epsilon}{3}$. Then

   $$B_{\frac{2\epsilon}{3}}(a) \subset B_\epsilon(x)$$

   and choosing an integer n such that $\frac{\epsilon}{3} < \frac{1}{n} < \frac{2\epsilon}{3}$ (which is possible since the interval has length less than 1), it follows that

   $$x \in B_{\frac{1}{n}}(a) \subset B_{\frac{2\epsilon}{3}}(a) \subset B_\epsilon(x)$$

   Therefore the collection of sets $\left\{B_{\frac{1}{n}}(a) \mid n \in \mathbb{Z}_+, a \in A\right\}$ form a countable basis for X.

   (b) *Show that every metrizable Lindelöf space has a countable basis.*

   Let (X, ρ) be a metric Lindelöf space. Consider the open cover $\mathcal{B} = \left\{B_{\frac{1}{n}}(x) \mid x \in X\right\}$, for some fixed $n \in \mathbb{Z}_+$. Since X is Lindelöf there exists a countable subcover $\mathcal{B}'$. Define

   $$A_n = \left\{a \in X \mid B_{\frac{1}{n}}(a) \in \mathcal{B}'\right\}$$

and

$$A = \bigcup_{n \in \mathbb{Z}_+} A_n$$

Since the A_n are countable it follows that A is also countable. We claim that A is a dense, so that the condition from the previous part is satisfied and therefore X has a countable basis. Let $x \in X$ and U be a neighborhood of x containing the basis element $B_\epsilon(x)$. Choose $\frac{1}{n} < \epsilon$. Since A_n is a subcover there exists some a such that $x \in B_{\frac{1}{n}}(a)$. But then $a \in B_{\frac{1}{n}}(x) \subset B_\epsilon(x) \subset U$. Therefore x is in the closure of A, and hence A is dense in X. From the previous part X is second-countable.

4. *Show that if X has a countable dense subset then every collection of disjoint open sets in X is countable.*

   Let A be a countable dense subset of X and $\mathcal{O}$ a collection of nonempty disjoint open sets. If $U \in \mathcal{O}$ then there exists some $a \in A$ with $a \in U$. Since the sets in $\mathcal{O}$ are disjoint they cannot contain the same a, and therefore the cardinality of $\mathcal{O}$ at most the cardinality of A, i.e., $\mathcal{O}$ is countable.

5. *Show that if X is normal then every pair of disjoint closed sets have neighborhoods whose closures are disjoint.*

   Let A and B be disjoint closed sets. Since X is normal there exist disjoint neighborhoods U_1 and V_1 such that $A \subset U_1$ and $B \subset V_1$. But then $X \setminus U_1$ is closed with $V_1 \subset X \setminus U_1$, and similarly, $U_1 \subset X \setminus V_1$. Again, by the normality of X there exist disjoint neighborhoods U_2 and V_2 such that $V_1 \subset X \setminus U_1 \subset U_2$ and $U_1 \subset X \setminus U_1 \subset V_2$. Since $X \setminus V_1$ and $X \setminus U_1$ are closed it follows that $\overline{U}_1 \subset V_2$ and $\overline{V}_1 \subset U_2$. Hence U_2 and V_2 are precisely the neighborhoods for which we were looking.

6. *Let $f, g : X \to Y$ be continuous and Y a Hausdorff space. Show that $\{x \in X \mid f(x) = g(x)\}$ is closed in X.*

   Let $f, g : X \to Y$ be any two continuous functions and Y be Hausdorff. Then the set

   $$C = \{x \in X \mid f(x) = g(x)\}$$

   is closed. This follows from the fact that $x \in X \setminus C$ then there exist disjoint neighborhoods U, V of $f(x)$ and $g(x)$ respectively, and $x \in f^{-1}(U) \cup f^{-1}(V)$, i.e., $X \setminus C$ is open.

7. *Show that a closed subspace of a normal space is normal.*

   Let X be normal and $Y \subset X$ a closed subspace. Let A and B be two closed subsets in Y. From the definition of the subspace topology, $A = Y \cap A'$ and $B = Y \cap B'$ where A' and B' are closed in X, and therefore A and B are also closed. Since X is normal there exist disjoint U and V such that $A \subset U$ and $B \subset V$. But then $U \cap Y$ and $V \cap Y$ are open sets in Y which separate A and B, and hence Y is normal.

8. *Show that every regular Lindelöf space is normal.*

   Let A and B be disjoint closed subsets of a regular Lindelöf space X. Since they are closed they are also Lindelöf as subspaces of X. By the regularity of X choose a neighborhood U_x for every point $x \in A$ such that $x \in U_x \subset \overline{U}_x \subset X \setminus B$, which is open. Then there exist a countable subcover of $\{U_x\}$, call it $\{U_n\}$ where $n \in \mathbb{Z}_+$. Similarly, there is a countable cover of B by open sets $\{V_n\}$. Define

   $$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i$$

   and similarly define a V'_n. The $\{U'_n\}$ and $\{V'_n\}$ are open as they are the set different between an open set and a closet set. Moreover, they are still covers of A and B, respectively. Let $U = \bigcup_{n \in \mathbb{Z}_+} U'_n$ and $V = \bigcup_{n \in \mathbb{Z}_+} V'_n$. These are open neighborhoods of A and B.

   Assume for contradiction that there exists an $x \in U \cap V$. Then $x \in U'_k \cap V'_j$ for some j, k. If $j = k$ then $x \in U_k \cap V_k$, which is impossible by our construction from the regularity of X. So assume without loss of generality that $j < k$. Then V'_j is disjoint from both U'_k and U_k by construction, again, which is also a contradiction. Therefore there exist disjoint neighborhoods around A and B, i.e., X is normal.

9. *Is $\mathbb{R}^\omega$ normal in the product topology? In the uniform topology?*

   $\mathbb{R}^\omega$ is metrizable in both topologies, and therefore normal in both topologies.

10. (a) *Show that a connected normal space having more than one point is uncountable.*

    Let X be a connected normal space with at least two distinct points $x, y \in X$. From Urysohn's lemma there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. Since X is connected and f is continuous, $f(X) = [0,1]$, and therefore the cardinality of X must be at least that of $[0,1]$, i.e., X is uncountable.

    (b) *Show that a connected regular space having more than one point is uncountable.*

    Let X be a connected regular space. If X is countable then X is Lindelöf. Since X is also regular, from a previous problem, it follows that X is normal, and hence from the previous part X is uncountable – a contraction.

11. *Give a direct proof of the Urysohn lemma for a metric space (X, ρ) by setting*

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}$$

It follows directly that $0 \le \rho(x, A) \le \rho(x, A) + \rho(x, B)$ and therefore that $0 \le f(x) \le 1$ for all $x \in X$. If $x \in A$ then $\rho(x, A) = 0$ and $f(x) = 0$. If $x \in B$ then $\rho(x, B) = 0$ and $f(x) = 1$. Since no point is in both A and B, $\rho(x, A) + \rho(x, B)$ never vanishes, and hence f is continuous.