

# MATH 209: Homework #9

Jesse Farmer

31 May 2004

**Problem 1.** Show that  $(\widehat{\mathbb{Z}_n}, +) \cong (\mathbb{Z}_n, +)$

$\mathbb{Z}_n$  is compact, so all characters are unitary. Moreover, if  $0 \neq m \in \mathbb{Z}_n$  is arbitrary then there exists  $0 \neq n \in \mathbb{Z}_n$  such that  $mn = 0$ . Let  $\psi$  be a character of  $\mathbb{Z}_n$ , then

$$1 = \psi(0) = \psi(mn) = \psi(m)^n$$

i.e.,  $\psi(m)$  is an  $n^{\text{th}}$  root of unity. Define

$$\psi_k(m) = \psi(km) = e^{\frac{i2\pi km}{n}}$$

Clearly  $\{\psi_k \mid k \in \{0, 1, \dots, n-1\}\} = \widehat{\mathbb{Z}_n}$ , since, for any  $\psi$  we simply pick a root of unity to which  $\psi(1)$  will map, and  $\psi(m)$  is then generated by  $\psi(1)$  to powers of  $k \in \{0, 1, \dots, n-1\}$ . Consider the map from  $\mathbb{Z}_n$  to  $\widehat{\mathbb{Z}_n}$ ,  $k \mapsto \psi_k$ . From its definition this map is clearly surjective and homomorphic.

Let  $\psi_k = \psi_j$ . Then

$$\psi_{k-j} = \psi_k \psi_j^{-1} = \psi_k \psi_j^{-1} = 1 \Rightarrow e^{\frac{i2\pi(k-j)m}{n}} = 1 \Rightarrow k = j \pmod{n} \text{ or } m = 0$$

However,  $m$  is not identically zero, therefore  $k = j$  and  $k \mapsto \psi_k$  is an injection.

**Problem 2.** Show that  $f \in \mathcal{S}(\mathbb{R})$  if and only if  $\widehat{f} \in \mathcal{S}(\mathbb{R})$ .

To show sufficiency first note that the Schwartz condition is equivalent to

$$\sup_x |x^k f^{(n)}(x)| \leq C_{k,n} < \infty$$

for all  $k, n \in \mathbb{N}$ , where  $C$  is some constant that may depend on  $k$  and  $n$ . Moreover, every Schwartz function is Lebesgue integrable, so

$$\begin{aligned} (t^k \widehat{f^{(n)}})(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^k (-ix)^n e^{-itx} f(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{(-i)^k} (e^{-itx})^{(k)} (-ix)^n f(x) dx \\ &= \frac{(-i)^k}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} ((-ix)^n f(x))^{(k)} dx \end{aligned}$$

Since  $f^{(n)} \in \mathcal{S}(\mathbb{R})$ , this implies

$$\sup_t |t^k \widehat{f^{(n)}}(t)| \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |(x^n f(x))^{(k)}| < \infty$$

For the other direction, consider the inverse Fourier transform (which exists because  $\widehat{f}$  is Schwartz)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(x) e^{itx} dx$$

If  $\widehat{f}$  is Schwartz the same argument as above shows that  $f$  is Schwartz.

**Problem 3.** Show that  $f \mapsto \widehat{f}$  is a bijection of  $\mathcal{S}(\mathbb{R})$ .

From the previous problem's arguments it is clear that  $f$  is Schwartz if and only if  $\widehat{f}$  is Schwartz if and only if the inverse Fourier transform  $\check{f}$  is Schwartz. Then, given arbitrary  $g \in \mathcal{S}(\mathbb{R})$ , there exists a  $f \in \mathcal{S}(\mathbb{R})$  such that  $\widehat{f} = g$ , namely,  $f = \check{g}$ . Therefore  $f \mapsto \widehat{f}$  is surjective.

For injectivity, let  $\widehat{f} = \widehat{g}$ . Then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(n) e^{inx} = \int_{\mathbb{R}} \widehat{g}(n) e^{inx} = g(x)$$

Therefore  $f \mapsto \widehat{f}$  is also an injection, and hence a bijection.

**Problem 4.** Find  $f \in L^1(\mathbb{R})$  such that  $\widehat{f} \notin L^1(\mathbb{R})$ .

Let  $f(x) = e^{-x} \chi_{(0,\infty)}(x)$ . Then  $f$  is the density function of a standard exponential distribution, and hence

$$\int_{\mathbb{R}} f(x) dx = \int_0^{\infty} e^{-x} dx = 1$$

Moreover,

$$\widehat{f}(t) = \int_0^{\infty} e^{-x(1+it)} dx = \frac{1}{1+it}$$

since  $(1+it)e^{-x(1+it)}$  is the density function of an exponential distribution with parameter  $(1+it)$ . This function is clearly not integrable on the real line.

**Problem 5.** Find the Fourier series of  $\varphi(e^{i\theta}) = \theta$ , for  $-\pi < \theta < \pi$ .

Recall that

$$e^{in\theta} = \cos \theta + i \sin \theta \text{ and } e^{-in\theta} = \cos \theta - i \sin \theta \quad (1)$$

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta} d\theta \\ &= \frac{1}{2\pi} \left( \frac{i\theta}{n} e^{-in\theta} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{i}{n} e^{-in\theta} d\theta \right) \\ &= \frac{1}{2\pi} \left( \frac{i\theta}{n} e^{-in\theta} + \frac{1}{n^2} e^{-in\theta} \Big|_{-\pi}^{\pi} \right) \\ &= \frac{1}{2\pi} \frac{e^{in\theta} + in\theta e^{-in\theta}}{n^2} \Big|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \frac{in\pi(e^{-in\pi} + e^{in\pi}) + e^{-in\pi} - e^{in\pi}}{n^2} \\ &= \frac{i(n\pi \cos(n\pi) - \sin(n\pi))}{\pi n^2} \text{ by (1)} \end{aligned}$$

And  $\widehat{f}(0) = 0$ . Therefore

$$f(x) = \sum_{n \in \mathbb{Z}} \frac{i(n\pi \cos(n\pi) - \sin(n\pi))}{\pi n^2} e^{inx}$$

**Problem 6.** Show that if  $f$  and  $g$  are smooth functions then  $f * g$  is also smooth.

Let  $f, g \in C^\infty$ . Then

$$(f * g)'(x) = \left( \int_G f(y) g(y^{-1}x) dy \right)' = \int_G f(y) g'(y^{-1}x) dy = (f * g')(x) = (f' * g)(x)$$

Inductively, if  $f, g \in C^\infty$  then  $(f * g)^{(n)}(x) = (f * g^{(n)})(x) = (f^{(n)} * g)(x)$ . Since differentiability implies continuity, and the convolution is differentiable  $n$  times for any  $n \in \mathbb{N}$ , it follows that every derivative is continuous, i.e.,  $(f * g) \in C^\infty$ .

**Problem 7.** Show that  $f * g = g * f$ .

Let  $G$  be a locally compact Abelian group with a Haar measure  $dy$ . Consider the map  $y \mapsto y^{-1}x$ . Then

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy = \int_G f(y^{-1}x) g((y^{-1}x)^{-1}x) dy = \int_G f(y^{-1}x) g(y) dy = (g * f)(x)$$

**Problem 8.** *Show that if  $f$  is a smooth function then  $\tilde{f}$  is also smooth.*

**Problem 9.** *Show that  $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} = 2$ .*

Note that

$$(\sqrt{3} - 1)^3 = 6\sqrt{3} - 10 \text{ and } (\sqrt{3} + 1)^3 = 6\sqrt{3} + 10$$

Then

$$\begin{aligned}\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} &= \sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10} \\ &= \sqrt[3]{(\sqrt{3} + 1)^3} - \sqrt[3]{(\sqrt{3} - 1)^3} \\ &= 2\end{aligned}$$