MATH 263: Homework #3

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1. *Define the categories of left and right G-sets and show they are isomorphic.*

   Let **Set** be the category of sets, where $\mathrm{Hom}_{\mathbf{Set}}$ is the set of all set functions. Define the category of left G-sets $\mathbf{C}$ as
   $$\mathbf{C} = \{X \in \mathrm{Ob}(\mathbf{Set}) \mid G \text{ has a left action on } X\}$$
   with $\varphi \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ if $\varphi(ga) = g\varphi(a)$ for all $a \in A$. Define similarly $\mathbf{D}$ the category of right G-sets, with G having a *right* rather than left action on a set, and the morphisms preserving that action.

   Since for ever left action we can define a right action via $a \cdot g = g^{-1} \cdot a$, and vice versa, it follows that $\mathrm{Ob}(\mathbf{C}) = \mathrm{Ob}(\mathbf{D})$. Define $F : \mathbf{C} \to \mathbf{D}$ by $F(X) = X$ for all $X \in \mathrm{Ob}(\mathbf{C})$ and $F(\alpha) = \beta$ where $\beta(a \cdot g) = g^{-1} \cdot \alpha(a)$ and X is given the right action defined above.

   Let α_1, α_2 be morphisms in $\mathbf{C}$ and β_1, β_2 their respective images under F in $\mathbf{D}$. Then
   $$\begin{aligned}(F(\alpha_1) \circ F(\alpha_2))\,(a \cdot g) &= (\beta_1 \circ \beta_2)(a \cdot g)\\ &= \beta_1\left(g^{-1} \cdot \alpha_2(a)\right)\\ &= \beta_1(\alpha_2(a) \cdot g)\\ &= g^{-1} \cdot (\alpha_1 \circ \alpha_2)(a)\\ &= F(\alpha_1 \circ \alpha_2)(a \cdot g)\end{aligned}$$
   and
   $$F\left(1_A\right)(a \cdot g) = 1_A\left(g^{-1} \cdot a\right) = g^{-1} \cdot a = a \cdot g$$
   Hence $F : \mathbf{C} \to \mathbf{D}$ is a covariant functor. Let $G : \mathbf{D} \to \mathbf{C}$ be defined $G(\beta) = \beta'$ where $\beta'(g \cdot a) = \beta\left((a \cdot g^{-1}\right)$. Then
   $$(G \circ F)(\alpha) = G(\beta) = \beta'$$
   and $\beta'(g \cdot a) = \beta(a \cdot g^{-1}) = \alpha(g \cdot a)$. Hence $\beta' = \alpha$, and similarly for $F \circ G$. Therefore $G \circ F$ and $F \circ G$ are isomorphic to the identity functors on $\mathbf{C}$ and $\mathbf{D}$, respectively. Indeed, they are equal to the identity functor, so that choosing $\eta_A \in \mathrm{Hom}_{\mathbf{C}}(F(A), A)$ to be the identity morphism for each $A \in \mathrm{Ob}\,\mathbf{C}$ gives $(F \circ G) \circ \eta_A = \eta_B \circ 1$.

2. *Show that any G-set is the disjoint union of transitive G-sets and that any transitive right G-set is isomorphic to $H\backslash G$ for some subgroup H of G.*

   Let G act on a set X and define the orbit of the action as
   $$\mathcal{O}_x = \{g \cdot x \mid g \in G\}$$

The action restricted to $\mathcal{O}_x$ is clearly transitive since if $y, z \in \mathcal{O}_x$ then there exist g_1, g_2 such that $g_1 \cdot y = z = g_2 \cdot z$, and hence $g_1^{-1} g_2 \cdot z = y$. To see that the $\mathcal{O}_x$ partition X let $z \in \mathcal{O}_x \cap \mathcal{O}_y$ then

$$g_1 \cdot x = z = g_2 \cdot y \Rightarrow g_1^{-1} g_2 \cdot y = x$$

and hence $\mathcal{O}_x \subset \mathcal{O}_y$. The other direction is immediate, and, since every element of X is in at least one orbit, namely $\mathcal{O}_x$, it follows that the collection $\{\mathcal{O}_x \mid X \in x\}$ partitions X. The case for right group actions follows *mutatis mutandis*.

Let A be a transitive right G-set and fix $x \in A$. Define $\varphi_x : G \to A$ by $g \mapsto x \cdot g$. Denote the stabilizer of x, the set of all elements of G which fix x, by G_x. We claim that φ_x is a well-defined isomorphism on $G_x \backslash G$. Let $G_x \cdot g = G_x \cdot g'$, then there exists a $g_0 \in G_x$ such that $g = g' g_0$ and

$$x \cdot g = x \cdot (g' g_0) = (x \cdot g') \cdot g = x \cdot g$$

Hence φ_x is well-defined. It is surjective by the transitivity of A, and is innjective since

$$x \cdot g = x \cdot g' \Rightarrow x \cdot (g' g^{-1}) = x \Rightarrow G_x \cdot (g' g^{-1}) = G_x$$

so that $G_x \cdot g' = G_x \cdot g$. Finally, φ_x is a homomorphism since

$$\varphi_x((G_x \cdot g) \cdot g') = \varphi_x(G_x(g \cdot g')) = x \cdot (gg') = (x \cdot g) \cdot g' = \varphi_x(g) \cdot g'$$

Therefore φ_x is an isomorphism of G-sets.

3. *Show that the stabilizer of Hx in $H \backslash G$ is $x^{-1} H x$.*

   The stabilizer of Hx is the set $\{g \in G \mid Hx \cdot g = Hx\}$. But

   $$Hx \cdot g = Hx \Leftrightarrow Hxgx^{-1} = H \Leftrightarrow xgx^{-1} \in H \Leftrightarrow g \in x^{-1} H x$$

   Thus the stabilizer of x in $H \backslash G$ is $x^{-1} H x$.

4. *Show that the G-set $H \backslash G$ is isomorphic to $K \backslash G$ if and only if H and K are conjugate subgroups in G.*

   Let $\varphi : H \backslash G \to K \backslash G$ be an isomorphism of G-sets. Then there exists some $Ka \in K \backslash G$ such that $\varphi(He) = Ka$ where $e \in G$ is the group identity. Hence, for all $h \in H$,

   $$Ka = \varphi(He) = \varphi(Hh) = Kah$$

   so that $aha^{-1} K = K$. Since this holds for all $h \in H$ it follows that $aHa^{-1} \subseteq K$. This is in fact true for any homomorphism of G-sets of this form. However, since φ is an isomorphism, it follows that

   $$\varphi^{-1}(Ke) = Ha^{-1}$$

   Then, *mutatis mutandis*, $a^{-1} K a \subseteq H$ or $K \subseteq aHa^{-1}$. Therefore $K = aHa^{-1}$.

5. *Show that if A is a retract of B^2 then every continuous map $f : A \to A$ has a fixed point.*

   Let $r : B^2 \to A$ be a retraction map and $f : A \to A$ be a continuous map. Define

   $$g = i \circ f \circ r$$

   where $i : A \hookrightarrow X$ is the inclusion map. $g : B^2 \to B^2$ is continuous and hence has a fixed point in B^2. But then $x = g(x) = (i \circ f \circ r)(x) = (f \circ r)(x)$, so that $x \in A$. Since r is a retract, $r(x) = x$, and therefore $x = (f \circ r)(x) = f(x)$.

6. *Show that if $h : S^1 \to S^1$ is nullhomotopic then h has a fixed point and maps some point x to its antipode $-x$.*

   If $h : S^1 \to S^1$ is nullhomotopic then it extends to a continuous function $\tilde{h} : B^2 \to S^1$. Let $i : S^1 \hookrightarrow B^2$ be the inclusion map. Then $i \circ \tilde{h}$ has a fixed point $x = (i \circ \tilde{h})(x) = \tilde{h}(x)$. But then $x \in S^1$ so that $x = \tilde{h}(x) = h(x)$. Similarly, since $-h$ is also nullhomotopic on S^1 it extends in the same way. Hence there exists some $x \in S^1$ such that $-h(x) = x$ or $h(x) = -x$.

7. *Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x then g is surjective.*

   Suppose for contradiction that g is not surjective, and pick $x_0 \in S^2 \setminus \operatorname{im} g$. Then $g\mid_{S^2 \setminus \{x_0\}}$ is continuous and there exists a homeomorhpism $h : S^2 \setminus \{x_0\} \to \mathbb{R}^2$. Define $k = h \circ g$. Then $k(x) \neq k(-x)$ for all $x \in S^2$, since otherwise $h(g(x)) = h(g(-x))$ and hence $g(x) = g(-x)$. But this contradicts Borsuk-Ulam, and therefore g must be surjective.

8. *Let $h : S^1 \to S^1$ be continuous and antipode-preserving with $h(b_0) = b_0$. Show that h_* carries a generator of $\pi_1(S^1, b_0)$ to an odd power of itself.*

   Let q be the quotient map that identifies antipodal points on the sphere. Then the fundamental group of this quotient space has only two classes of loop, namely, those that end at b_0 and those that end at $-b_0$, and is therefore isomorphic to the only group of order 2: $\mathbb{Z}/2\mathbb{Z}$. If $h : S^1 \to S^1$ preserves antipodal points then h can be extended to this quotient space via $\tilde{h}([x]) = [h(x)]$, where $[x]$ denotes the equivalence class of x under this identification. Then q_* is the natural homomorphism from $Z \to \mathbb{Z}/2\mathbb{Z}$ and we have $q_* \circ h_* = \tilde{h}_* \circ q_*$. If h sent a generator of $\mathbb{Z}$ to an even number then $q \circ h$ would be the trivial homomorphism, which would also imply that $\tilde{h}_* \equiv 0$. Hence $\tilde{h}$ would be nullhomotopic, which is impossible by theorem 57.1.

9. (a) *Show that $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.*

   Let $h : \mathbb{R}^n \to R^1$ be a homeomorphism, and pick $x \in \mathbb{R}^n$. Then $h\mid_{\mathbb{R}^n \setminus \{x\}}$ is still a homeomorphism, but its image is disconnected, viz., it separates $\mathbb{R}^1$ into $(-\infty, h(x))$ and $(h(x), \infty)$. But this is impossible as connectedness is a topological property and $\mathbb{R}^n \setminus \{x\}$ is still connected.

   (b) *Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 2$.*

   Pick $x \in \mathbb{R}^n$ and let $h : \mathbb{R}^n \to \mathbb{R}^2$ be a homeomorphism. Then $h' = h\mid_{\mathbb{R}^n \setminus \{x\}}$ is still a homeomorphism. However, $\operatorname{im} h' = \mathbb{R}^2 \setminus \{y\}$ for some y, which has a fundamental group isomorphic to the integers, and $\mathbb{R}^n \setminus \{x\}$ still has a trivial fundamental group. Then h'_* is an isomorphism between these two groups, which is absurd. Therefore h cannot be a homeomorphism.

10. *Use the covering space given by figure 60.3 to show that the fundamental group of the figure eight is not abelian.*

    Label the three points of intersection in the covering space e_1, e_0, and e_2 in that order from left to right. Let $\tilde{g}$ be the path traversing the upper-half of B_1 from e_0 to e_1 and $\tilde{f}$ be the path along the lower-half of A_1 from e_0 to e_2. Let $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$. Then these two functions are loops around A and B, respectively, based at x_0.

    Consider the lift of $f * g$. This function first traverses the lower-half of A_1 and then the B_0, ending at e_2. Similarly, $g * f$ first traverses the upper-half of B_1 followed by A_0, ending at e_1. Since these lifts end at different points it cannot be the case that $f * g$ is path homotopic to $g * f$, and therefore the figure eight is not abelian.

11. *State and prove a universal property for free products.*

    Let $\{G_\alpha\}_{\alpha \in I}$ be a family of groups. Then the free product can be defined as a set C and a family of group homomorphisms $\{f_\alpha : G_\alpha \to C\}_{\alpha \in I}$ such that for any group H and family homomorphisms $\{g_\alpha : G_\alpha \to D\}$ there exists a uniqe homomorphism $h : C \to H$ such that $h \circ f_\beta = g_\beta$ for all $\beta \in I$. The free product of the $\{G_\alpha\}_{\alpha \in I}$ satisfy this property, and any other group C which satisfies this property is isomorphic to the free product.

It is easy to see that the free product satisfies this property. Let $i_\alpha : G_\alpha \hookrightarrow \coprod_{\beta \in I} G_\beta$ be the inclusion map, i.e., $i_\alpha(g) = g$ for all $g \in G_\alpha$. Then for any family of homomorphisms $\{\varphi_\alpha : G_\alpha \to D\}$ for some group D define

$$\varphi\left(a_{\alpha_1}^{\epsilon_1} \cdots a_{\alpha_k}^{\epsilon_k}\right) = \varphi_{\alpha_1}\left(a_{\alpha_1}\right)^{\epsilon_1} \cdots \varphi_{\alpha_k}\left(a_{\alpha_k}\right)^{\epsilon_k}$$

where $a_{\alpha_k} \in G_{\alpha_k}$ for some k. This is a homomorphism by construction, and is unique since for any other homomorphism satisfying this, say φ', we have $\varphi'(g_\alpha) = \varphi_\alpha(g_\alpha)$ for g_α in G_α. Hence $\varphi \equiv \varphi'$ on each G_α, and therefore on all of $\coprod_{\beta \in I} G_\beta$.

Let C be any other group which satisfies this property, and let $\{f_\beta : G_\beta \to C\}$ be a family of homomorphisms. Let $\left\{i_\beta : G_\beta \hookrightarrow \coprod_{\beta \in I} G_\beta\right\}$ be the inclusion maps. Then there exists a unique homomorphism $f : \coprod_{\beta \in I} G_\beta \to C$ such that $f_\beta = f \circ i_\beta$ for all $\beta \in I$. We claim this is in fact a group isomorphism, and to prove this claim we will demonstrate a two-sided inverse.

By hypothesis C also satisfies this universal property, and therefore there exists a unique homomorphism $g : C \to \coprod_{\beta \in I} G_\beta$ such that $i_\beta = g \circ f_\beta$ for all $\beta \in I$. From above we have that $f_\beta = f \circ i_\beta$, so that $i_\beta = g \circ f \circ i_\beta$. But since i_β is the inclusion map it follows that $g \circ f(g_\beta) = g_\beta$ for all $\beta \in G_\beta$, and for all $\beta \in I$. Since $g \circ f$ is a homomorphism it follows immediately that $g \circ f$ is the identity on $\coprod_{\beta \in I} G_\beta$. Similarly, since $f_\beta = f \circ i_\beta$, it follows that $f_\beta \equiv f$ on G_β. Then by exchanging the roles of C and $\coprod_{\beta \in I} G_\beta$ we get that $f_\beta = f \circ g \circ f_\beta$. Hence $f \circ g$ is the identity on C by the fact that $f \circ g$ is a homomorphism and that they are both unique. Therfore f is, in fact, an isomorphism of groups.

12. *Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.*

Let g, h be nonidentity elements in G and H, respectively. Then for any reduced and non-empty word $w \in G*H$ there exists a word w', which might be the empty word, such that $w = w'g'$ for some $g' \in G$, or $w = w'h'$ for some $h' \in H$. Assume for former is the case. Then $hw \neq wh$ since hw ends in an element of G and wh ends in an element of h. Similarly, if $w = w'h'$, then $gw \neq wg$. Therefore, for any reduced nom-empty $w \in G * H$, there exists an element with which it does not commute, and hence cannot be in the center, i.e., $|Z\left(G * H\right)| = 1$.

If $w = w'gw'^{-1}$ for some $w' \in G * H$ then $w^n = w'g^n w'^{-1}$. Therefore any $G * H$-conjugate of a finite order element in G (or H) is of finite order. To see the converse, assume $w \in G * H$ is of finite order. w cannot be of even length since if $w = w_1 \cdots w_{2k}$ then w_{2k} and w_1 are in different groups and w^n is a reduced word of length $(2k)^n$, which can obviously never be 1. So assume w is of odd length. In order to allow any reduction of length in w^n it must be the case that $w_{2k+1} = w_1^{-1}$, and $w_j = w_{2k+2-j}$ in general. Hence $w = w'w_{k+1}w'^{-1}$ where $w' = w_1 \cdots w_k$.