MATH 263: Homework #2

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1. *Let $p : E \to B$ be a continuous and surjective map. Suppose that U is an open set of B that is evenly covered by p. Show that if U is connected then the partition of $p^{-1}(U)$ into slices is unique.*

   Let $\{V_\alpha\}$ be a family of open sets homeomorphic to U which partition $p^{-1}(U)$. Since U is connected each V_α is. Consider a subset $A \subset p^{-1}(U)$, where A is connected. A cannot be contained in more than one V_α since then V_α and the remaining elements of the partition would separate A into two disjoint open sets, contradicting the fact that A is connected. Therefore the $\{V_\alpha\}$ correspond exactly to the connected components of $p^{-1}(U)$, which implies any such decomposition is unique.

2. *Let $p : E \to B$ be a covering map, where B is connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$ then $p^{-1}(b)$ has k elements for every $b \in B$.*

   Let $f : I \to B$ be a path connecting b_0 to an arbitrary b, and label the k points in $p^{-1}(b_0)$ as $\{e_1, \ldots, e_k\}$ so that $p(e_j) = b_0$ for $1 \leq j \leq k$. For each k there exists a uniqe map $\tilde{f} : I \to E$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(e_k) = b_0$. Call this map $\tilde{f}_k$. Then $p \circ \tilde{f}_k(1) = f(1) = b$ which implies $\tilde{f}_k(1) \in p^{-1}(b)$ for all k, i.e., there are at least k elements in $p^{-1}(b)$. The converse follows *mutatis mutandis* by considering a path g that connects an arbitrary b to b_0, and defining g_k similarly.

3. *Let $q : X \to Y$ and $r : Y \to Z$ be covering maps and define $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$ then p is a covering map.*

   Since $r^{-1}(z)$ if finite, by the previous problem, for all $z \in Z$ there exist $\{y_1, \ldots, y_n\}$ such that $r^{-1}(z) = \{y_1, \ldots, y_n\}$. Let U_z be an evenly covered neighborhood of z by $V_1, \ldots, V_n$ via r, and let W_i be an evenly covered neighborhood of y_i via q. Since $r^{-1}(z)$ is finite and r is open on $V_i \bigcup W_i$ it follows that

   $$U'_z = \bigcup_{i=1}^{n} r\left(V_i \cup W_i\right)$$

   is an open neighborhood of z evenly covered by r. However, now, each $r^{-1}(U'_z)$ is also evenly covered by q. Writing the slices of W_i as $\{O_{i,j}\}$, it follows that

   $$(r \circ q)^{-1}\left(U'_z\right) = \bigcup_{j} \bigcup_{i=1}^{n} \left(O_{i,j} \cap q^{-1}(V_i)\right)$$

   which partition $(r \circ q)^{-1}\left(U'_z\right)$ by construction. $r \circ q$ is a homeomorphism over the sets on the right-hand side since the composition of homeomorphisms is a homeomorphism, and these are restrictions of sets on which r and q are homeomorphic to sets on which they both are. Hence $r \circ q$ is a covering map.

4. *For a path-connected space X show that $\pi_1(X)$ is abelian if and only if all base-point change homomorphisms β_h depend only on the endpoints of the path h.*

   Assume $\pi_1(X, x_0)$ and $\pi_1(X, x_0)$ are abelian and let α, β be two paths connecting x_0 to x_1.

   $$[\beta * f * \bar{\beta}] = [\beta * \bar{\beta} * f] = [f] = [\alpha * \bar{\alpha} * f] = [\alpha * f * \bar{\alpha}]$$

That is, $\alpha_f = \beta_f$ for all such α, β. Assume the converse, then

$$[p * f * f * \overline{(p * f)}] = [p * g * f * \overline{(p * g)}]$$

by hypothesis, i.e., $p * g$ and $p * f$ induce the same homomorphism since they share the same endpoints. But since $[\overline{(p * f)}] = [\overline{f} * \overline{p}]$ it follows that $[f] = [g * f * \bar{g}]$, and hence $[f]$ and $[g]$ commute.

5. *Show that for a space X the following three conditions are equivalent:*

   (a) *Every map $S^1 \to X$ is homotopic to a constant map, with image a point.*

   (b) *Every map $S^1 \to X$ extends to a map $D^2 \to X$.*

   (c) *$\pi_1(X, x_0) = 0$ for all $x_0 \in X$.*

   Deduce that a space X is simply connected if and only every map $S^1 \to X$ are homotopic.

   Assume f is homotopic to a point y_0 via H. Then, denoting the Euclidian (l_2) norm by $\|\cdot\|$, define

$$\tilde{f}(\vec{x}) = \begin{cases} H\left(f\left(\frac{\vec{x}}{\|\vec{x}\|}\right), 1 - \|\vec{x}\|\right) & \vec{x} \neq 0 \\ y_0 & \vec{x} = 0 \end{cases}$$

   From the continuity of f, this function is continuous on all of D^2, and when $\|\vec{x}\| = 1$, i.e., when $\vec{x} \in S^1$, this function is precisely $f(\vec{x})$. To see the converse assume f extends continuously to D^2 and define

$$H(\vec{x}, t) = \tilde{f}((1-t)\vec{x} + t x_0)$$

   where $x_0 \in S^1$. This is a homotopy between f and x_0, as $\tilde{f}$ is continuous and equals f on S^1.

6. *Let $\Phi : \pi_1(X, x_0) \to [S^1, X]$ be the map obtained by "forgetting" the basepoint of a homotopy class. Show that Φ is onto if X is path-connected, and that $\Phi([g]) = \Phi([f])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.*

   If X is path connected then any two basepoints can be connected via a path, and hence the image of $[f]$ under Φ is precisely its homotopy class. If $[f]$ and $[g]$ are conjugate then there exists a path which connects their basepoints and hence f and g are homotopic by the homotopy which "slides" the basepoint of f along the line which connects it to the basepoint of g. If f and g are in the same homotopy class then there exists a homotopy $F : S^1 \times I \to X$. Fixing $x_0 \in S^1$, define a path by $h(t) = F(x_0, t)$. Then f and g are conjugated via h.

7. *Show that every homomorphism of $\pi_1(S^1)$ can be realized as the induced homomorphism φ_* for some $\varphi : S^1 \to S^1$.*

   It can be seen that $\mathrm{Hom}(\mathbb{Z}) \cong \mathbb{Z}$ by considering the map $\varphi \mapsto \varphi(1)$ for all $\varphi \in \mathrm{Hom}(\mathbb{Z})$. Since $\pi_1(S^1) \cong \mathbb{Z}$, it follows that any homomorphism $\psi_n : \pi_1(S^1) \to \pi_1(S^1)$ is actually of the form $\psi : \mathbb{Z} \to n\mathbb{Z}$ for some $n \in \mathbb{Z}$. This follows from basic algebra – the image of a group homomorphism is a subgroup of the range, and the only subgroups of $\mathbb{Z}$ are those of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$. In terms of $\pi_1(S^1)$, this sends all elements with lifting correspondence m to nm. For any $[f] \in \pi_1(S^1)$ we can pick a path homotopy class representative of the form $e^{2\pi imt}$, where m is the image of $[f]$ under its lifting correspondence. Under ψ the image is therefore $[e^{2\pi imnt}]$.

   Now consider the map $\zeta_n : S^1 \to S^1$ defined by $z \mapsto z^n$. Then $(\zeta_n)_*$ takes loops in the class of $e^{2\pi imt}$ to the class of $e^{2\pi imnt}$, i.e., $\psi_n = (\zeta_n)_*$.