MATH 263: Homework #4

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1. *Find spaces whose fundamental groups are isomorphic to the following groups.*

   (a) $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

   The quotient space induced by the covering map $S^1 \to S^1$ given by $z \mapsto z^n$ has a fundamental group isomorphic to $\mathbb{Z}/n\mathbb{Z}$. This can be seen because S^1 is path-connected and, taking 1 as the base point, we have that there is a bijection between $\mathbb{Z}/p_*(\mathbb{Z})$ and $p^{-1}(1)$. Since each fiber has four elements and $p_*(\mathbb{Z})$ is a subgroup of $\mathbb{Z}$ (since p_* is easily seen to be a homomorphism), it follows that $\pi_1(p(S^1), 1) = Z/n\mathbb{Z}$.

   Writing $z \mapsto z^n$ as p_n then $p_n(S^1) \times p_m(S^1)$, i.e., the cartesian product of the quotient spaces of S^1 under p_n and p_m respectively, has a fundamental group isomorphic to the direct product of their respective fundamental groups, viz., $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

   (b) $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$

   By induction it is easy to see that $p_{n_1}(S^1) \times \cdots \times p_{n_k}(S^1)$ is such a space.

   (c) $\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$

   Using the exact argumentation as above it follows that the wedge product of two circles under the quotient maps p_n and p_m, respectively, have a free product of $\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$ by van Kampen.

   (d) $\mathbb{Z}/n_1\mathbb{Z} * \mathbb{Z}/n_2\mathbb{Z} * \cdots * \mathbb{Z}/n_k\mathbb{Z}$

   Again, inductively, the space $\bigvee_{i=1}^{k} p_{n_k}(S^1)$ is such a space.

2. *Find a presentation for the fundamental group of* $\mathbb{P}^2 \# T$.

   $\pi_1(\mathbb{P}^2)$ has a presentation of $\langle a \mid a^2 \rangle$ and $\pi_1(T)$ has a presentation of $\langle b, c \mid bcb^{-1}c^{-1} \rangle$. Since $\mathbb{P}^2 \# T$ can be viewed as the 6-sided polgon with labeling $aabcb^{-1}c^{-1}$ it follows that $\langle a, b, c \mid aabcb^{-1}c^{-1} \rangle$ is a presentation for $\pi_1(\mathbb{P}^2 \# T)$.

3. *Let X be the space obtained from a seven-sided polygonal region by means of the labelling scheme $abaaab^{-1}a^{-1}$. Show that the fundamental group of X is the free product of two cyclic groups.*

   The fundamental group of X is isomorphic to $\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. This can be seen by demonstrating a presentation of X. From 68.7, the fundamental group is isomorphic to the free product of $\mathbb{Z}/N_1 * \mathbb{Z}/N_2$ where N_1 and N_2 are two normal subgroups such that the normal subgroup N generated by $abaaab^{-1}a^{-1}$ is the smallest containing both of them. But in this case, we can write the labelling scheme as $aaab^{-1}a^{-1}ab$ so that N_2 is generated by aaa and N_2 is generated by $b^{-1}a^{-1}ab = 1$. Hence $N_1 = \{1\}$ and $N_2 = 3\mathbb{Z} = \langle a \mid a^3 \rangle$.

4. *Let K be the Klein Bottle.*

   (a) *Find a presentation for the fundamental group of K.*

   The Klein Bottle is homeomorphic to a square with labeling $baba^{-1}$, and under this quotient map all vertices are identified, so it has a presentation of $\langle a, b \mid baba^{-1} \rangle$.

(b) *Find a double covering map $p : T \to K$ where T is the torus. Describe the induced homomorphism of fundamental groups.*

5. (a) *Show that the Klein Bottle K is homeomorphic to $\mathbb{P}^2 \# \mathbb{P}^2$.*

   Recall that the fundamental group of the Klein Bottle has a presentation of $\langle a, b \mid baba^{-1} \rangle$. Cutting this along the diagonal with a line, labeled c, and gluing them together along the line labeled b gives a quotient space with labeling $ccaa$. Hence, under this quotient map, the fundamental group of the Klein Bottle has presentation $\langle c, a \mid ccaa \rangle$, which is also the presentation of $\mathbb{P}^2 \# \mathbb{P}^2$.

   (b) *Show how to picture the 4-fold projective plane as an immersed surface in $\mathbb{R}^3$.*

   The 4-fold projective plane can be immersed in $\mathbb{R}^3$ by taking two Klein bottles, cutting a small hole in each surface, and identifying the boundaries of the two holes. In other words, the immersion is simply $K \# K$.

6. *Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.*

   First, X decomposes into an open-ended clinder and a distinct point with the boundaries of the circles at the end of the cylinder identified with the point. The cylinder deformation retracts to S^1, so that X is in fact a CW-complex consisting of one 0-cell and one 2-cell, i.e., the circle. Hence $\pi_1(X) = \mathbb{Z}$.

   This can also be seen by considering the space X as the wedge of S^2 and S^1. Since S^2 has trivial fundamental group, $\pi_1(X) \cong \{0\} \cong \mathbb{Z} = \mathbb{Z}$.

7. *Compute the fundamental group of the space optained from two tori $S^1 \times S^1$ by identifying the circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.*

   Write the first torus as $\langle a, b \mid aba^{-1}b^{-1} \rangle$ and the second as $\langle c, d \mid cdc^{-1}d^{-1} \rangle$. Identify the circle b with the circle d. Then this becomes a rectangle with labeling $(ac)b(ac)^{-1}d^{-1}$, but as $d = b$, this is really $(ac)b(ac)^{-1}b^{-1}$. Hence the fundamental group is again that of a torus, viz., $\mathbb{Z} \times \mathbb{Z}$, which can be seen by writing the group presentation as $\langle e, f \mid efe^{-1}f^{-1} \rangle$ for $e = ac$ and $f = b = d$.

8. *Consider the quotient space of a cube I^3 obtained by identifying each square face with the oppositce square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this show that $\pi_1(X) \cong Q_8$, the quaterntion group of order eight.*

9. *Let X bt the subspace of $\mathbb{R}^2$ that is the union of the circles C_n of radius n and center $(n, 0)$ for $n \in \mathbb{N}$. Show that $\pi_1(X)$ is the free group $\coprod_{n \in \mathbb{N}} \pi_1(C_n)$, the same as for the infinite wedge sum $\vee_\infty S^1$. Show that X and $\vee_\infty S^1$ are in fact homotopy equivalent but not homeomorphic.*

10. (a) *Show that $\mathrm{Hom}_G(H\backslash G, K\backslash G)$ can be identified with $K \backslash \{g \in G \mid gHg^{-1} \subset K\}$.*

    Every homomorphism of G-sets $\varphi \in \mathrm{Hom}_G(H\backslash G, K\backslash G)$ is completely determined by the coset to which it sends $He \in H\backslash G$. In otherwords, define $\varphi_a(xH) = xaK$. Then for $h \in H$,

    $$Ka = \varphi_a(He) = \varphi(Hh) = Kah$$

    so that $aha^{-1} \in K$, i.e., $aHa^{-1} \subset K$. Each such homomorphism is unique and every homomorphism is of this form (since it must send He to something). In other words, the map $a \mapsto \varphi_a$ is a bijection between $K \backslash \{g \in G \mid gHg^{-1} \subset K\}$ and $\mathrm{Hom}_G(H\backslash G, K\backslash G)$.

    (b) *Show that $Hom_G(H\backslash G, H\backslash G) = N_G(H)/H$.*

    This follows directly from the previous part since by definition $N_G(H) = \{g \in G \mid gHg^{-1} \subset H\}$. Since $N_G(H)$ is the largest subgroup of G in which H is normal, $N_G(H)/H = H\backslash N_G(H)$.