MATH 270: Homework #2

Jesse Farmer

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1. *Show that if $w \in \mathbb{C}$, then*

   (a) $|\Re w| \le |w|$
   Recall that $\Re w = \frac{w+\overline{w}}{2}$ and $|w| = |\overline{w}|$, so

   $$|\Re w| = \left|\frac{w+\overline{w}}{2}\right| \le \frac{|w| + |\overline{w}|}{2} = \frac{2|w|}{2} = |w|$$

   (b) $|\Im w| \le |w|$
   Since $\Im w = \frac{w-\overline{w}}{2i}$,

   $$|\Im w| = \left|\frac{w-\overline{w}}{2i}\right| \le \frac{|w| + |\overline{w}|}{|2i|} = \frac{2|w|}{2} = |w|$$

   (c) $|w| \le |\Re w| + |\Im w|$

   $$|w| = \left|\frac{w+\overline{w}+w-\overline{w}}{2}\right| \le \left|\frac{w+\overline{w}}{2}\right| + \left|\frac{w-\overline{w}}{2}\right| = \left|\frac{w+\overline{w}}{2}\right| + \left|\frac{w-\overline{w}}{2i}\right| = |\Re w| + |\Im w|$$

2. *For the following sets state whether or not the set is open or closed.*

   (a) $\{z \mid \Im z > 2\}$
   This set is open since for any z_0 in the set, $D(z_0, \Im z_0) \subset \{z \mid \Im z > 2\}$. This follows because

   $$|\Im z - \Im z_0| = |\Im(z - z_0)| \le |z - z_0| < \Im z_0$$

   The set is not closed because $z = (0, 2)$ limit point of the set, but not contained in the set.

   (b) $\{z \mid 1 \le |z| \le 2\}$
   Define $U = \{z \mid 1 \le |z|\}$ and $U' = \{z \mid |z| \le 2\}$. Consider U^c. If $z_0 \in U^c$ then $|z_0| < 1$, so there exists an $r > 0$ such that $|z_0| = 1 - r$. Let z be such that $|z| < r$, then

   $$|z_0 - z| \le |z_0| + |z| < 1 - r + r < 1$$

   And therefore U^c is open, and U is closed. Since the set is the intersection of U' and complement of the interior of U (which is closed by definition), it follows that the set itself is closed. It is not open since any neighborhood around the point $z = (2, 0)$ contains points in the complement of the set.

(c) $\{z \mid -1 < \Re z \leq 2\}$

This set is neither open nor closed, since -1 is a limit point of the set but not in the set, and any neighborhood around 2 contains points in the complement.

3. *Determine the sets on which the following functions are holomorphic, and compute their derivatives:*

(a) $(z+1)^3$

This function is a polynomial so it is holomorphic on all of $\mathbb{C}$ and its derivative is $3(z+1)^2$.

(b) $z + \frac{1}{z}$

This function is holomorphic on $\mathbb{C} \setminus \{0\}$, and its derivative is $1 - \frac{1}{z^2}$ there.

(c) $\left(\frac{1}{z-1}\right)^{10}$

This function is holomorphic on $\mathbb{C} \setminus \{1\}$, and its derivative is $-10\left(\frac{1}{z-1}\right)^{11}$ there.

(d) $\frac{1}{(z^3-1)(z^2+2)}$

Using de Moivre's formula for finding the roots of complex numbers shows that this function is holomorphic on $\mathbb{C}\setminus\{1, i\sqrt{2}, -i\sqrt{2}, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ and its derivative is $-\frac{(z^3-1)2z+3z^2(z^2+2)}{(z^3-1)^2(z^2+2)^2}$.

4. *Prove that $f(z) = |z|$ is not holomorphic.*

Let $z = x+iy$ and write $f(x,y) = |(x,y)| = \sqrt{x^2+y^2} = u(x,y)+iv(x,y)$. Then $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Since $v(x,y) = 0$ the partial derivative with respect to both x and y is zero, and hence the Cauchy-Riemann equations are not satisfied which implies $f(z) = |z|$ is not holomorphic.

5. *Find the radius of convergence of each of the following power series:*

(a) $\sum_{n=0}^{\infty} nz^n$

In all the following cases we use the fact that the radius of convergence r is given by $r = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

(b) $\sum_{n=0}^{\infty} \frac{z^n}{e^n}$

$$\lim_{n\to\infty} \frac{e^{n+1}}{e^n} = \lim_{n\to\infty} e = e$$

(c) $\sum_{n=1}^{\infty} n! \frac{z^n}{n^n}$

$$\lim_{n\to\infty} \frac{n!(n+1)^{n+1}}{(n+1)!n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

(d) $\sum_{n=1}^{\infty} \frac{z^n}{n}$

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

6. *Find the radius of convergence of each of the following power series:*

(a) $\sum_{n=0}^{\infty} n^2 z^n$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2 = 1$$

(b) $\sum_{n=0}^{\infty} \frac{z^{2n}}{4^n}$
Here write $z^2 = x$, then consider the series of x^n instead of z^{2n}. This gives a radius of convergence of

$$\lim_{n\to\infty}\frac{4^{n+1}}{4^n} = 4$$

However, $|x| \leq 4$ if and only if $|z| \leq 2$, hence the radius of convergence of the original series is 2.

(c) $\sum_{n=0}^{\infty} n! z^n$

$$\lim_{n\to\infty}\frac{n!}{(n+1)!} = \frac{1}{n} = 0$$

(d) $\sum_{n=0}^{\infty} \frac{z^n}{1+2^n}$

$$\lim_{n\to\infty}\frac{1+2^{n+1}}{1+2^n} = \lim_{n\to\infty}\frac{\frac{1}{2^n}}{\frac{1}{2^n}}\frac{1+2^{n+1}}{1+2^n} = \lim_{n\to\infty}\frac{\frac{1}{2^n}+2}{\frac{1}{2^n}+1} = 2$$

7. *Prove that a power series converges absolutely everywhere or nowhere on its circle of convergence. Give an example to show that each case can occur.*

If a power series converges absolutely at a point z_0 on its radius of convergence, then for all $z \in \mathbb{C}$ such that $|z| = |z_0|$,

$$\sum_{i=0}^{\infty} |a_n||z|^n = \sum_{i=0}^{\infty} |a_n||z_0|^n < \infty$$

8. *Let*

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Show that f is a C^∞ function and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that f is not analytic at $x = 0$.

Let $p(x)$ be a n degree polynomial of $\frac{1}{x}$, that is

$$p(x) = \sum_{i=0}^{n} a_i \frac{1}{x^i}$$

We will first show that $f^{(n)}(x)$ is of the form $e^{\frac{-1}{x^2}}p(x)$ when $x > 0$. Clearly $f^{(0)}(x)$ is of this form, so assume

$$f^{(n)}(x) = e^{\frac{-1}{x^2}} \sum_{i=0}^{n} a_i \frac{1}{x^i}$$

Then applying the product rule and chain rule we get that

$$f^{(n+1)}(x) = e^{\frac{-1}{x^2}} \left(\frac{2}{x^3} \sum_{i=0}^{n} a_i \frac{1}{x^i} + \sum_{i=0}^{n} a_i \frac{-i}{x^{i-1}} \right)$$

which is still of the the form $e^{\frac{-1}{x^2}} p(x)$, for the appropriate $p(x)$.

Because the following is true

$$\begin{aligned} \lim_{x \to 0^+} e^{\frac{-1}{x^2}} \left(\sum_{i=0}^{n} a_i \frac{1}{x^i} \right) &= \lim_{x \to 0^+} \sum_{i=0}^{n} a_i \frac{\frac{1}{x^i}}{e^{\frac{1}{x^2}}} \\ &= \lim_{u \to +\infty} \sum_{i=0}^{n} a_i \frac{u^i}{e^{u^2}} \\ &= 0 \end{aligned}$$

we see that each $f^{(n)}(x)$ is continuous at zero since we define $f^{(n)}(x) = 0$ for all $x \leq 0$ (the left-hand limit is, of course, 0) All that remains to be shown is that each $f^{(n)}$ is differentiable at zero. Since $f^{(n)}(0) = 0$, it follows that

$$\begin{aligned} \lim_{x \to 0^+} \frac{e^{\frac{-1}{x^2}} \left(\sum_{i=0}^{n} a_i \frac{1}{x^i} \right)}{x} &= \lim_{x \to 0^+} \frac{\frac{1}{x} \left(\sum_{i=0}^{n} a_i \frac{1}{x^i} \right)}{e^{\frac{1}{x^2}}} \\ &= \lim_{x \to 0^+} \frac{\left(\sum_{i=0}^{n} a_i \frac{1}{x^{i+1}} \right)}{e^{\frac{1}{x^2}}} \\ &= 0 \end{aligned}$$

which is true by the same argument by which we showed continuity. Since $f^{(n)}(x) = 0$ for $x \leq 0$, the left-hand limit is obviously the same. Therfore $f^{(n)}(x)$ is differentiable at 0 and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, and hence f is also a C^∞ function. If f were analytic at 0 then from Theorem 1.9 in class f would be identically 0 in a neighborhood of 0, which is certainly not true – in fact, $f(x) \neq 0$ for all $x > 0$.