MATH 262: Homework #3

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1. *For the following topologies on $\mathbb{R}$ determine which of the others it contains. The standard topology, $\mathbb{R}_K$, the finite complement topology, the upper-limit topology, and the topology generated by the basis $(-\infty, a)$ for $a \in \mathbb{R}$.*

   Denote these topologies as $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$, respectively.

   $\mathcal{T}_1$ contains $\mathcal{T}_3$ since any finite set is closed in $\mathcal{T}_1$ and hence the complement is open. $\mathcal{T}_1$ also contains $\mathcal{T}_5$ since every basis element of $\mathcal{T}_5$ is also a basis element of $\mathcal{T}_1$.

   $\mathcal{T}_2$ contains $\mathcal{T}_1$ since every basis element of the latter is also a basis element of the former. From above, it also contains $\mathcal{T}_3$ and $\mathcal{T}_5$.

   $\mathcal{T}_3$ contains none of the other topologies.

   $\mathcal{T}_4$ contains $\mathcal{T}_1$, and hence contains $\mathcal{T}_3$ and $\mathcal{T}_5$. $\mathcal{T}_4$ also contains $\mathcal{T}_2$ since

   $$\mathbb{R} \setminus K = (-\infty, 0] \cup \bigcup_{n \in \mathbb{Z}_+} \left(\frac{1}{n+1}, \frac{1}{n} \right) \cup (1, \infty)$$

   is open in $\mathcal{T}_4$. Any set open set U in $\mathcal{T}_2$ is already open in $\mathcal{T}_1$ and hence open in $\mathcal{T}_4$, or is of the form $U \setminus K$ where U is open in the standard topology. In either case, since $\mathcal{T}_4$ contains both $\mathcal{T}_1$ and $\mathbb{R} \setminus K$, U is open in $\mathcal{T}_4$.

   Finally $\mathcal{T}_5$ contains none of the other topologies.

2. *If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?*

   Let $\mathcal{T} \subset \mathcal{T}'$ and denote the subspace topology inherited by Y from $\mathcal{T}$ and $\mathcal{T}$ as $\mathcal{T}_Y$ and $\mathcal{T}'_Y$, respectively. Then if $Y \cap U \in \mathcal{T}_Y$, $Y \cap U \in \mathcal{T}'_Y$ since $U \in \mathcal{T} \subset \mathcal{T}'$. It need not be strictly finer, however. Consider, for example, the standard topology on $\mathbb{R}$ and $\mathbb{R}_l$ restricted to the interval $[0, 1]$.

3. *If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$.*

   In the first case, either L is vertical or it is not. If it is then the subspace topology on L is simply the standard topology on $\mathbb{R}$. If L is not vertical then the topology is the lower-limit topology, since in this case L can intersect the closed edge of the basis "rectangle" $[a, b) \times (c, d)$ for some a, b, c, d.

In the seconds case, either L has positive slope or is vertical, or L has negative slope. In the first case the topology is again the lower-limit topology since L will only intersect at most one of the closed edges. If L has negative slope then the topology is the discrete topology since for any $(x, y) \in L$, $[x, x+1) \times [y, y+1)$ is a basis element containing (x, y). If every point is open, then the topology is necessarily the discrete topology.

4. *Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology I inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.*

   Denote these topologies as $\mathcal{T}_p$, $\mathcal{T}_o$, and $\mathcal{T}_s$, respectively, and note that $\mathcal{T}_p$ is the same as the subspace topology of $I \times I$ when $\mathbb{R}^2$ is given the topology induced by the usual metric, i.e., the l^2 norm. We will denote the pair (x, y) by $x \times y$ to avoid ambiguity.

   We claim that $\mathcal{T}_p \subsetneq \mathcal{T}_s$. Let $x \times y \in U \times B$, where $U \times V$ is a basis element of $\mathcal{T}_p$. Then $x \times y \in \{x\} \times V \subset U \times V$ and $\{x\} \times V$ is a basis element of $\mathcal{T}_s$, hence $\mathcal{T}_p \subset \mathcal{T}_s$. $\{1/2\} \times [0, 1]$ is an element of $\mathcal{T}_s$ not contained in $\mathcal{T}_p$, so the inclusion is proper.

   We claim that $\mathcal{T}_o \subsetneq \mathcal{T}_s$. Every basis element of $\mathcal{T}_o$ is in $\mathcal{T}_s$, but, for example $\{1/2\} \times (0, 1]$ is not in $\mathcal{T}_o$ since there is no basis element of the form $\{1/2\} \times (a, b)$ containing it as $b \leq 1$ by necessity.

   We claim that $\mathcal{T}_p$ and $\mathcal{T}_o$ are not comparable. As in the first case, $\{1/2\} \times (0, 1) \in \mathcal{T}_o$ but is not in $\mathcal{T}_p$. Next consider $\mathcal{T}_p$ and $\mathcal{T}_o$ neighborhoods around some point on the top edge of the box, e.g., $1/2 \times 1$. Then any $\mathcal{T}_o$ neighborhood contains points of the form $x \times 0$ where $x > 1/2$, but it is easy to see that the open ball of radius $1/2$ around $1/2 \times 1$ intersected with $I \times I$, which is open in $\mathcal{T}_p$, contains no such point.

5. *Let A, B be subsets of a space X, and $\{A_\alpha\}$ a family of subsets in X. Prove the following:*

   (a) *If $A \subset B$ then $\bar{A} \subset \bar{B}$.*
   $x \in \overline{A \cup B}$ if and only if every neighborhood U of x intersects $A \cup B$, i.e., $(A \cup B) \cap U \neq \emptyset$. But $(A \cup B) \cap U = (A \cap U) \cup (B \cap U)$, so that U intersects A or U intersects B. This is the case if and only if $x \in \bar{A} \cup \bar{B}$.

   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
   As above, $x \in \overline{A \cup B}$ if and only if every neighborhood U of x intersects $A \cap B$, i.e., $A \cap B \cap U \neq \emptyset$. But $A \cap B \cap U = (A \cap U) \cap (B \cap U)$, so that both $A \cap U$ and $B \cap U$ must be nonempty. This is the case if and only if $x \in \bar{A} \cap \bar{B}$.

   (c) $\bigcup \bar{A}_\alpha \subset \overline{\bigcup A_\alpha}$.
   If $x \in \bigcup \bar{A}_\alpha$ then every neighborhood U of x intersects some A_α. This means that for any neighborhood U, $U \cap \bigcup \bar{A}_\alpha = \bigcup \bar{A}_\alpha \cap U \neq \emptyset$, i.e., $x \in \overline{\bigcup A_\alpha}$.

6. *Criticize the following proof that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$. If $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$ then every neighborhood of U intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore $x \in \bigcup \bar{A}_\alpha$.*

   This proof assumes that there exists an α such that for any neighborhood U, $U \cap A_\alpha \neq \emptyset$, whereas it is actually the case that for every neighborhood there is *some* A_α such that $U \cap A_\alpha \neq \emptyset$. As an example, consider $\mathbb{Q} \subset \mathbb{R}$ with the usual topology. The union of all the singletons in $\mathbb{Q}$ is $\mathbb{Q}$, whose closure is $\mathbb{R}$. But the union of the closure of each singleton is just $\mathbb{Q}$, since points are closed in the usual topology.

7. *Show that X is Hausdorff if and only if $\triangle = \{x \times x \mid x \in X\}$ is closed in $X \times X$.*

   Assume $\triangle$ is closed, then $(X \times X) \setminus \triangle$ is open, i.e., for any $(x, y) \in (X \times X) \setminus \triangle$ there exists a basis element $U \times V$, where U and V are open in X, such that $(x, y) \in U \times V$. But $U \bigcap V = \emptyset$, since if $x \in U \cap V$ then $(x, x) \in U \times V$, i.e., $U \times V$ intersects $\triangle$. Hence $x \in U$, $y \in V$, and $U \bigcap V = \emptyset$, and X is Hausdorff.

   Similarly, assume X is Hausdorff. Then for any distinct $x, y \in X$ there exist disjoint neighborhoods U, V of x and y, respectively. Then (x, y) is in some basis element for $X \times X$, namely, $U \times V$. Since U and V are disjoint, $(U \times V) \setminus \triangle = \emptyset$, and therefore $(X \times X) \setminus \triangle$ is open, i.e., $\triangle$ is closed.

8. *Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.*

   (a) *Show that i is continuous if and only if $\mathcal{T}'$ is finer than $\mathcal{T}$.*
   Assume i is continuous and let $U \in \mathcal{T}$. Then $U = i(U) = i^{-1}(U) \in \mathcal{T}'$ and hence $\mathcal{T}'$ is finer than $\mathcal{T}$. Assume $\mathcal{T}'$ is finer than $\mathcal{T}$ and let $U \in T$. Then $U = i(U) = i^{-1}(U) \in \mathcal{T} \subset \mathcal{T}'$ and hence i is continuous.

   (b) *Show that i is a homeomorphism if and only if $\mathcal{T}' = \mathcal{T}$.*
   i is a homeomorphism if and only if it is bijective, continuous, and has a continuous inverse. Since $i^{-1} = i$, from the previous part, we have i is a homeomorphism if and only if $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is finer than $\mathcal{T}'$, i.e., if and only if $\mathcal{T}' = \mathcal{T}$.

9. *Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.*

   Define

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   The density of the rationals in the irrationals and vice versa guarantees that this function is continuous at nowhere except $x = 0$.

10. *Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.*

    (a) *Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.*
    First, a small lemma: every order topology is Hausdorff. Let $x, y \in X$ be distinct where X has the order topology. Then either there is some third element $a \in X$ with $x < a < y$, in which case the basis elements $(-\infty, a)$ and (a, ∞) are disjoint neighborhoods containing x and y, respectively. If there is no such element then $(-\infty, y)$ and (x, ∞) are disjoint basis elements containing x and y, respectively. Either way, X is Hausdorff. Let $x \in X$ be such that $f(x) > g(x)$. Then, since Y is Hausdorff, there exist disjoint neighborhoods U and V of $f(x)$ and $g(x)$. Since these neighborhoods are disjoint and Y has the order topology, then every element of U is greater than every element of V, as there is one element of U, $f(x)$, greater than one element of V, $g(x)$. Furthermore, the continuity of f implies that $W = f^{-1}(U) \bigcap f^{-1}(V)$ is an open neighborhood of $x \in X$ such that $f(w) > g(w)$ for all $w \in W$. Hence $\{x \mid f(x) > g(x)\}$ is open in X, and therefore $\{x \mid f(x) \leq g(x)\}$ is closed in X.

    (b) *Show that $h(x) = \min\{f(x), g(x)\}$ is continuous.*
    Let $A = \{x \mid f(x) \leq g(x)\}$ and $B = \{x \mid f(x) \geq g(x)\}$. By the previous part both of these are closed, and $A \bigcap B$ is precisely the set where $f = g$. From Theorem 18.3 (the pasting lemma) it follows that h is continuous on $A \cup B = X$.

11. *Let $A \subset X$ and $f : A \to Y$ continuous, where Y is Hausdorff. Show that any continuous extension of f to $\bar{A}$ is unique.*

    Let $f, g : Z \to Y$ be any two continuous functions and Y be Hausdorff. Then the set

    $$C = \{x \in Z \mid f(x) = g(x)\}$$

    is closed. This follows from the fact that $x \in X \setminus C$ then there exist disjoint neighborhoods U, V of $f(x)$ and $g(x)$ respectively, and $x \in f^{-1}(U) \cup f^{-1}(V)$, i.e., $X \setminus C$ is open.

    Let g, g' be two continuous extensions of f to $\bar{A}$, where $f : A \to Y$ is continuous. Then C from above is closed, using $Z = \bar{A}$. But this set contains A and hence contains $\bar{A}$, since $\bar{A}$ is by definition the smallest closed set containing A. Hence g and g' agree on $\bar{A}$.