MATH 257: Homework #7

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1. *Let $G = A_1 \times A_2 \times \cdots \times A_n$ and for each i let $B_i \trianglelefteq A_i$. Prove that $B_1 \times \cdots \times B_n \trianglelefteq G$ and that*

$$\frac{A_1 \times \cdots \times A_n}{B_1 \times \cdots \times B_n} \cong \frac{A_1}{B_1} \times \cdots \times \frac{A_n}{B_n}$$

Denote the direct product of the A_i, B_i, etc., by $\prod_{i=1}^n A_i$ and the n-tuple by $\prod_{i=1}^n (a_i)$, etc. Let $\prod_{i=1}^n (a_i) \in \prod_{i=1}^n A_i$ and $\prod_{i=1}^n (b_i) \in \prod_{i=1}^n B_i$, then

$$\begin{aligned}
\left(\prod_{i=1}^n (a_i)\right)\left(\prod_{i=1}^n (b_i)\right)\left(\prod_{i=1}^n (a_i)\right)^{-1} &= \left(\prod_{i=1}^n (a_i)\right)\left(\prod_{i=1}^n (b_i)\right)\left(\prod_{i=1}^n (a_i^{-1})\right) \\
&= \prod_{i=1}^n (a_i b_i a_I^{-1}) \\
&\in \prod_{i=1}^n B_i
\end{aligned}$$

And hence normality follows from the normality of each B_i. Define the map $\varphi : \prod_{i=1}^n A_i \to \prod_{i=1}^n \frac{A_i}{B_i}$ by $\prod_{i=1}^n (a_i) \mapsto \prod_{i=1}^n (a_i B_i)$. This map is clearly a surjective homomorphism, and

$$\begin{aligned}
\ker\varphi &= \{(a,b) \mid \varphi(a,b) = 1\} \\
&= \left\{\prod_{i=1}^n (a_i) \mid \prod_{i=1}^n (a_i B_i) = 1\right\} \\
&= \left\{\prod_{i=1}^n (a_i) \mid a_i B_i = B_i,\ i = 1, 2, \ldots, n\right\} \\
&= \left\{\prod_{i=1}^n (a_i) \mid a_i \in B_i,\ i = 1, 2, \ldots, n\right\} \\
&= \prod_{i=1}^n B_i
\end{aligned}$$

The second statement then follows from the first isomorphism theorem.

2. *Let G act on the set A. Prove that is $a, b \in A$ and b $g \cdot a$ for some $g \in G$ then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then the kernel of the action if $\cap_{g \in G} gG_ag^{-1}$.*

   Recall that if $b = g \cdot a$ then $g^{-1} \cdot b = a$. Let $h \in G_b$. Then $g^{-1}hg \in G_a$ since

$$(g^{-1}hg) \cdot a = (g^{-1}h) \cdot a = (g^{-1}h) \cdot b = g^{-1} \cdot (h \cdot b) = g^{-1} \cdot b = a$$

   Let $k = g^{-1}hg$ so that $h = gkg^{-1} \in gG_ag^{-1}$ since $k \in G_a$. Therefore $G_b \subseteq gG_ag^{-1}$. If $h \in gG_ag^{-1}$ then there exists $k \in G_a$ such that $h = gkg^{-1}$, but

$$h \cdot b = (gkg^{-1}) \cdot b = (gk) \cdot (g^{-1} \cdot b) = (gk) \cdot a = g \cdot (k \cdot a) = g \cdot a = b$$

   That is, $h \in G_b$ and therefore $gG_ag^{-1} \subseteq G_b$. If G acts transitively on A then there is one and only one orbit, i.e., $A = \{g \cdot a \mid G \in G\}$ for fixed a. This implies that if $b \in A$ then there is some g such that $b = g \cdot a$. From above we know that $G_b = gG_ag^{-1}$, and so the kernel of the action is

$$\bigcap_{b \in A} G_b = \bigcap_{g \in G} gG_ag^{-1}$$

   where $a \in A$ is fixed.

3. *Let G be a transitive permutation group on the finite set A. A* block *is a nonempty subset $B1$ of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$.*

   (a) *Prove that if B is a block containing the element a of A then the set G_b defined by $G_B = \{\sigma \in G \mid \sigma(B) = B\}$ is a subgroup of G containing G_a.*

   Since G is a group there is an identity map which fixes every element, and it is obviously in G_B. Moreover, since each σ is a permutation such that $\sigma(B) = B$, there exists an inverse σ^{-1} and $\sigma^{-1}(B) = B$. Finally, the composition of any two $\sigma_1, \sigma_2 \in G_B$ is in G_B since $\sigma_1(\sigma_2(B)) = \sigma_1(B) = B$. Therefore $G_B \leq G$.

   Let $a \in B$ and $\sigma \in G_a$ so that $\sigma(a) = a$. Then $a \in B$ and $a \in \sigma(B)$, so $\sigma(B) \cap B \neq \emptyset$. Because B is a block it must be the case that $\sigma(B) = B$ and therefore $\sigma \in G_B$, i.e., $G_a \subseteq G_B$.

   (b) *Show that if B is a block and $\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B)$ are all the distinct images of B under the elements of G then these form a partition of A.*

   (c) *A (transitive) group G on a set A is said to be* primitive *if the only blocks in A are the trivial ones: the sets of size 1 and A itself. Show that S_4 is primitive on $A = \{1, 2, 3, 4\}$. Show that D_8 is not primitive as a permutation group on the four vertices of a square.*

   Assume $B \subset A$ and consider the action of S_4 on B. There is an $a \in B \setminus A$. Since B is not only one element we can pick a σ which fixes every element in B except a, and then sends a to some element in $A \setminus B$ and sends that element to A, so that $a \in \sigma(B) \cap B$, but $\sigma(B) \neq B$.

   The action of the dihedral group preserves the relative position of opposite vertices of a n-gon, so for D_8 the set consisting of a vertex and its opposite form a block.

   (d) *Prove that the transitive group G has a primitive A if and only if for each $a \in A$ the only subgroups of G containing G_a are G_a and G.*

4. *Let H and K be subgroups of the group G. For each $x \in G$ define the HK* double coset *of x in G to be the set*

$$HxK = \{hxk \mid h \in H, k \in K\}$$

(a) *Prove that HxK is the union of the left cosets $x_1K, x_2K, \ldots, x_nK$ where $\{x_1K, \ldots, x_nK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.*

The action is defined by $h \cdot (xK) = (hx)K$. If these are all the orbits, then for every $y \in HxK$ there exists some $h \in H$ such that $y \in hxK$. However, hx is in one of the above orbits and therefore $y \in x_1K$. This is both necessary and sufficient, so $y \in HxK$ if and only if $y \in (hx)K = x_iK$ for some $h \in H$ and the corresponding i.

(b) *Prove that HxK is a union of right cosets of H.*

For the same HxK is a union of left cosets it is also the union of right cosets. A different argument that the one above, though, is that if there is some element g with $Hg \cap HxK$, then there exist $h, h' \in H$ and $k \in K$ such that $hg = h'xk \Rightarrow g = h^{-1}h'xk \Rightarrow g \in HxK \Rightarrow Hg \subseteq HxK$. So if any right coset has anything in common with the double coset then it is complete contained within the double coset, and so the union over all the K of Hxk must be HxK.

(c) *Show that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G.*

Since H, K are subgroups of G they contain the identity, and therefore every x is in at least one double coset (i.e., HxK). To show that the HxK partition G it is sufficent to show that if two double cosets have any element in common then they must be equal. Let $x, y \in G$ and assume there is a $w \in G$ such that $w \in HxK$ and $w \in HyK$. There must exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that

$$h_1xk_1 = w = h_2yk_2$$

But then

$$x = h_1^{-1}h_2yk_2k_1^{-1} \text{ and } y = h_2^{-1}h_1xk_1k_2^{-1}$$

So every element l of HxK can be expressed as an element of HyK and vice versa by using the above equalities, and therefore $HxK = HyK$ and the set of double cosets partitions G. It is obvious that the union of all double cosets is the entire set since both H and K contain the identity element.

(d) *Prove that $|HxK| = |K| \cdot \left[H : H \cap xKx^{-1}\right]$.*

xKx^{-1} is a subgroup of G. Consider the map $HxK \to HxKx^{-1}$ defined by $hxk \mapsto hxkx^{-1}$. This map is clearly a bijection, so that $|HxK| = |HxKx^{-1}|$. Moreover, if the sets are finite (since otherwise the question doesn't even make sense),

$$|HxK| = |HxKx^{-1}| = \frac{|H||xKx^{-1}|}{|H \cap xKx^{-1}|}$$

But G acting on K by conjugation is a permutation so that $|xKx^{-1}| = |K|$, and the original statement is proven.

(e) *Prove that $|HxK| = |H| \cdot \left[H : K \cap x^{-1}Hx\right]$.*

This follows *mutatis mutandtis* from the previous part – change the map from $hxk \mapsto hxkx^{-1}$ to $hxk \mapsto x^{-1}hxk$, and note that $x^{-1}Hx \leq G$.

5. *Prove that if $H \leq G$ has finite index n then there is a normal subgroup K of G with $K \leq H$ and $[G : K] \leq n!$.*

   $\frac{G}{K}$ is a group which acts on $\frac{G}{H}$ by the action $gK \cdot g'H = (gg')H$ since $K \leq H$ (and so $HK = H$). This has a permutation representation by Cayley's theorem, and since $\frac{G}{H}$ is a set of n elements, the largest order it could have is $n!$ which is the order of S_n, the set of all possible permutation of n elements. Hence $[G : K] \leq n!$.

6. *Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.*

   Let p be the smallest prime dividing n. $1 < p < n$ since n is composite, and by hypothesis there exists a subgroup of order $\frac{n}{p}$ and hence of index p. By Corollary 5 this subgroup is normal and non-trivial, and so G is not simple.

7. *Let Q_8 be the quaternion group of order 8.*

   (a) *Prove that Q_8 is isomorphic to a subgroup of S_8.*

   By Cayley's theorem every group of order n is isomorphic to a subgroup of S_n.

   (b) *Prove that Q_8 is not isomorphic to a subgroup of S_n for any $n \leq 7$.*