MATH 270: Homework #6

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1. *Consider the function* $f(x) = \frac{1}{z^2}$.

   (a) *f satisfies $\int_\gamma f(z)\, d = 0$ for all closed contours not passing through the origin but is not holomorphic at $z = 0$. Does this statement contradict Morera's Theorem?*

   No, because Morera's theorem requires that the integral be zero along any closed curve, not only those avoiding any singularities.

   (b) *f is founded as $z \to \infty$ but it is not a constant. Does this statement contradict Liouville's theorem?*

   No. Liouville's theorem requires that f be bounded on all of $\mathbb{C}$, not eventually bounded, i.e., bounded for z with sufficiently large moduli.

2. *Let $f(z)$ be entire and let $|f(z)| \geq 1$ for all $z \in \mathbb{C}$ Prove that f is constant.*

   Since $|f(z)| \geq 1$ on $\mathbb{C}$, f never vanishes and $\frac{1}{f(z)}$ is also entire. Then $\left|\frac{1}{f(z)}\right| \leq 1$, so by Liouville's theorem it is constant, and so f must also be constant.

3. *Let f be entire and let $|f(z)| \leq M$ for z in $\gamma = \{z \mid |z| = R\}$, for fixed R. Prove that*

$$f^{(k)}(re^{i\theta}) \leq \frac{k!M}{(R-r)^k}$$

   for all $k \in \mathbb{N}$ and $0 \leq r < R$.

   Pick any point $z_0 \in \mathbb{C}$ with $|z_0| = r < R$. Then there exists a neighborhood around z_0 of radius $R - r$ contained in γ. f is entire so by the maximum modulus principle $|f(z)| \leq M$ on this circle, too. Cauchy's inequality implies that $|f(z_0)^{(k)}| \leq \frac{k!}{(R-r)^k} M$ for $k \in \mathbb{N}$, but since z_0 was arbitrary this is true for all z with $|z| < R$ and the result follows.

4. *Let f and g be holomorphic on a region A with $g'(z) \neq 0$ for all $z \in A$. Furthermore, let g be injective and γ be any closed curve in A. Show that for $z \notin \gamma$,*

$$f(z)I(\gamma, z) = \frac{g'(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{g(\zeta) - g(z)}\, d\zeta$$

Since g is injective and g' never vanishes g^{-1} exists and $\frac{dg^{-1}(w)}{dw} = \frac{1}{g'(z)}$ where $z = g^{-1}(w)$. Therefore, writing $\zeta = g^{-1}(w)$ and $w_0 = g(z)$, by Cauchy's theorem

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{g(\zeta)-g(z)}\,d\zeta &= \frac{1}{2\pi i}\int_\gamma \frac{dg^{-1}(w)}{dw}\frac{f\left(g^{-1}(w)\right)}{w-w_0}\,dw \\
&= \frac{dg^{-1}(w_0)}{dw} f\left(g^{-1}(w_0)\right) I\left(\gamma, z\right) \\
&= \frac{f(z)}{g'(z)} I\left(\gamma, z\right)
\end{aligned}$$

5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R and let $A = \{z \in \mathbb{C} \mid |z| < R\}$. Show that $\int_\gamma f = 0$ for every closed curve γ in A where $A = \{z \in \mathbb{C} \mid |z| < R\}$.*

   Every polynomial is entire, so by Cauchy's theorem $\int_\gamma a_n z^n = 0$ for any closed curve γ in $\mathbb{C}$, assuming a_n is defined there. Since integrals are finitely additive, if γ is a closed curve in A

   $$\int_\gamma \left(\sum_{n=0}^{k} a_n z^n\right) dz = \sum_{n=0}^{k}\int_\gamma a_n z^n\,dz = 0$$

   Certainly, then, the integral is 0 in the limit, since the above statement is true for every $k \in \mathbb{N}$.

6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$. If $0 < r < R$ show that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$ where $z = re^{i\theta}$ and*

   $$a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} f\left(re^{i\theta}\right) e^{-in\theta}\,d\theta$$

   and

   $$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta = \sum_{n=0}^{\infty}|a_n|^2 r^{2n}$$

   From Taylor's theorem the coefficients a_n are $a_n = \frac{f^{(n)}(0)}{n!}$. Parameterize the circle of radius r by $\gamma(\theta) = re^{i\theta}$ for $\theta \in [0, 2\pi]$, then by Cauchy's theorem

   $$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} rie^{i\theta}\,d\theta = \frac{1}{2\pi r^n}\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta$$

   As for the second equation, consider that

   $$\begin{aligned}
f\left(re^{i\theta}\right)\overline{f\left(re^{i\theta}\right)} = \left|f\left(re^{i\theta}\right)\right|^2 &= \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta}\right)\left(\sum_{m=0}^{\infty}\overline{a_m} r^m e^{-im\theta}\right) \\
&= \sum_{n=0}^{\infty}\left[\left(\sum_{m=0}^{\infty}\overline{a_m} r^m e^{-im\theta}\right) a_n r^n e^{in\theta}\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(a_n\overline{a_m} r^{n+m} e^{i(n-m)\theta}\right)
\end{aligned}$$

   But note that if $n = m$ then $\int_0^{2\pi} a_n\overline{a_m} r^{n+m} e^{i(n-m)\theta} = 2\pi|a_n|^2 r^{2n}$, and if $n \neq m$ this integral is 0. Hence

   $$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta = 2\pi\sum_{n=0}^{\infty}|a_n|^2 r^{2n}$$

7. *Suppose f is a nonvanishing function on $\overline{D(0,1)}$ such that $f\mid_{D(0,1)}$ is holomorphic. Prove that if $|f(z)| = 1$ when $|z| = 1$ then f is constant.*

Define

$$\tilde{f}(z) = \begin{cases} f(z) & |z| \leq 1 \\ \frac{1}{\overline{f(\overline{z})}} & |z| > 1 \end{cases}$$

Since f never vanishes $\tilde{f}$ is defined and analytic for $|z| > R$ with a power series

$$\frac{1}{\overline{f(\overline{z})}} = \frac{1}{\sum_{n=0}^{\infty} \overline{a_n} z^n}$$

This power series converges since it has a radius of convergence of $\frac{1}{R}$ where R is the radius of convergence of the power series representation of f. Hence, if $R > 1$ then $\frac{1}{R} < 1$ and the series converges. By the same reasoning as in the Schwarz reflection principle $\tilde{f}$ is entire and $\tilde{f}\mid_{\overline{D(0,1)}} = f$. In particular this means $\left|\tilde{f}(z)\right| = 1$ when $|z| = 1$. The only analytic functions with constant moduli on any set are constant functions, so $\tilde{f}$ must be constant on the boundary of the unit disc. Cauchy's formula implies that $\tilde{f}$ is the same constant for any z with $|z| < 1$. Finally, $\tilde{f}$ must be constant since the extension is unique.