MATH 209: Homework #9

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Problem 1. *Show that* $\widehat{(\mathbb{Z}_n, +)} \cong (\mathbb{Z}_n, +)$

$\mathbb{Z}_n$ is compact, so all characters are unitary. Moreover, if $0 \neq m \in \mathbb{Z}_n$ is arbitrary then there exists $0 \neq n \in Z_n$ such that $mn = 0$. Let ψ be a character of $\mathbb{Z}_n$, then

$$1 = \psi(0) = \psi(mn) = \psi(m)^n$$

i.e., $\chi(m)$ is an n^{th} root of unity. Define

$$\psi_k(m) = \psi(km) = e^{\frac{i2\pi km}{n}}$$

Clearly $\{\psi_k \mid k \in \{0, 1, \ldots, n-1\}\} = \widehat{\mathbb{Z}}_n$, since, for any ψ we simply pick a root of unity to which $\psi(1)$ will map, and $\psi(m)$ is then generated by $\psi(1)$ to powers of $k \in \{0, 1, \ldots, n-1\}$. Consider the map from $\mathbb{Z}_n$ to $\widehat{\mathbb{Z}}_n$, $k \mapsto \psi_k$. From its definition this map is clearly surjective and homomorphic.

Let $\psi_k = \psi_j$. Then

$$\psi_{k-j} = \psi_k \psi_{-j} = \psi_k \psi_j^{-1} = 1 \Rightarrow e^{\frac{i2\pi(k-j)m}{n}} = 1 \Rightarrow k = j (\text{mod n}) \text{ or } m = 0$$

However, m is not identically zero, therefore $k = j$ and $k \mapsto \psi_k$ is an injection.

Problem 2. *Show that* $f \in \mathcal{S}(\mathbb{R})$ *if and only if* $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

To show sufficiency first note that the Schwartz condition is equivalent to

$$\sup_x |x^k f^{(n)}(x)| \leq C_{k,n} < \infty$$

for all $k, n \in \mathbb{N}$, where C is some constant that may depend on k and n. Moreover, every Schwartz function is Lebesgue integrable, so

$$\begin{aligned}
(t^k \widehat{f}^{(n)})(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^k (-ix)^n e^{-itx} f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{(-i)^k} \left(e^{-itx}\right)^{(k)} (-ix)^n f(x)\, dx \\
&= \frac{(-i)^k}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} \left((-ix)^n f(x)\right)^{(k)}
\end{aligned}$$

Since $f^{(n)} \in \mathcal{S}(\mathbb{R})$, this implies

$$\sup_t |t^k \widehat{f}^{(n)}(t)| \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| (x^n f(x))^{(k)} \right| < \infty$$

For the other direction, consider the inverse Fourier transform (which exists because $\widehat{f}$ is Schwartz)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(x) e^{itx}\, dx$$

If $\widehat{f}$ is Schwartz the same argument as above shows that f is Schwartz.

Problem 3. *Show that $f \mapsto \widehat{f}$ is a bijection of $\mathcal{S}(\mathbb{R})$.*

From the previous problem's arguments it is clear that f is Schwartz if and only if $\widehat{f}$ is Schwartz if and only if the inverse Fourier transform $\check{f}$ is Schwartz. Then, given arbitrary $g \in \mathcal{S}(\mathbb{R})$, there exists a $f \in \mathcal{S}(\mathbb{R})$ such that $\widehat{f} = g$, namely, $f = \check{g}$. Therefore $f \mapsto \widehat{f}$ is surjective.

For injectivity, let $\widehat{f} = \widehat{g}$. Then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(n) e^{inx} = \int_{\mathbb{R}} \widehat{g}(n) e^{inx} = g(x)$$

Therefore $f \mapsto \widehat{f}$ is also an injection, and hence a bijection.

Problem 4. *Find $f \in L^1(\mathbb{R})$ such that $\widehat{f} \notin L^1(\mathbb{R})$.*

Let $f(x) = e^{-x}\chi_{(0,\infty)}(x)$. Then f is the density function of a standard exponential distribution, and hence

$$\int_{\mathbb{R}} f(x)\, dx = \int_0^\infty e^{-x}\, dx = 1$$

Moreover,

$$\widehat{f}(t) = \int_0^\infty e^{-x(1+it)}\, dx = \frac{1}{1+it}$$

since $(1+it)e^{-x(1+it)}$ is the density function of an exponential distribution with parameter $(1+it)$. This function is clearly not integrable on the real line.

Problem 5. *Find the Fourier series of* $\varphi(e^{i\theta}) = \theta$*, for* $-\pi < \theta < \pi$*.*

Recall that

$$e^{in\theta} = \cos\theta + i\sin\theta \text{ and } e^{-in\theta} = \cos\theta - i\sin\theta \tag{1}$$

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}\,d\theta \\
&= \frac{1}{2\pi}\left(\frac{i\theta}{n}e^{-in\theta}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{i}{n}e^{-in\theta}\,d\theta\right) \\
&= \frac{1}{2\pi}\left(\frac{i\theta}{n}e^{-in\theta} + \frac{1}{n^2}e^{-in\theta}\Big|_{-\pi}^{\pi}\right) \\
&= \frac{1}{2\pi}\frac{e^{in\theta} + in\theta e^{-in\theta}}{n^2}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\frac{in\pi(e^{-in\pi} + e^{in\pi}) + e^{-in\pi} - e^{in\pi}}{n^2} \\
&= \frac{i(n\pi\cos(n\pi) - \sin(n\pi))}{\pi n^2} \text{ by (1)}
\end{aligned}$$

And $\widehat{f}(0) = 0$. Therefore

$$f(x) = \sum_{n\in\mathbb{Z}} \frac{i(n\pi\cos(n\pi) - \sin(n\pi))}{\pi n^2} e^{inx}$$

Problem 6. *Show that if* f *and* g *are smooth functions then* $f * g$ *is also smooth.*

Let $f, g \in C^\infty$. Then

$$(f*g)'(x) = \left(\int_G f(y)g(y^{-1}x)\,dy\right)' = \int_G f(y)g'(y^{-1}x)\,dy = (f*g')(x) = (f'*g)(x)$$

Inductively, if $f, g \in C^\infty$ then $(f*g)^{(n)}(x) = (f*g^{(n)})(x) = (f^{(n)}*g)(x)$. Since differentiability implies continuity, and the convolution is differentiable n times for any $n \in \mathbb{N}$, it follows that every derivative is continuous, i.e., $(f*g) \in C^\infty$.

Problem 7. *Show that* $f * g = g * f$*.*

Let G be a locally compact Abelian group with a Haar measure dy. Consider the map $y \mapsto y^{-1}x$. Then

$$(f*g)(x) = \int_G f(y)g(y^{-1}x)\,dy = \int_G f(y^{-1}x)g((y^{-1}x)^{-1}x)\,dy = \int_G f(y^{-1}x)g(y)\,dy = (g*f)(x)$$

Problem 8. *Show that if f is a smooth function then $\tilde{f}$ is also smooth.*

Problem 9. *Show that* $\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} = 2$.

Note that

$$(\sqrt{3}-1)^3 = 6\sqrt{3}-10 \text{ and } (\sqrt{3}+1)^3 = 6\sqrt{3}+10$$

Then

$$\begin{aligned}
\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} &= \sqrt[3]{6\sqrt{3}+10} - \sqrt[3]{6\sqrt{3}-10} \\
&= \sqrt[3]{(\sqrt{3}+1)^3} - \sqrt[3]{(\sqrt{3}-1)^3} \\
&= 2
\end{aligned}$$