# MATH 262: Homework #2

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1. *Prove that $\mathbb{A}$ is countable and that $\mathbb{R} \setminus \mathbb{A}$ is uncountable.*

   First, we show that $\mathbb{Z}$ and $\mathbb{Q}$ are countable. Consider the map $\varphi : \mathbb{Z}_+ \to \mathbb{Z}$ where

$$\varphi(n) = \begin{cases} -\frac{n}{2} & \text{if } 2 \mid n \\ \frac{n-1}{2} & \text{if } 2 \nmid n \end{cases}$$

   Then $\varphi$ is clearly a bijection between these two sets, and hence $\mathbb{Z}$ is countable. Let $A = \{(p, q) \mid p, q \neq 0, p \text{ and } q \text{ are relatively prime}\} \cup \{0\} \subset \mathbb{Z} \times \mathbb{Z}$, which is countable since $\mathbb{Z} \times \mathbb{Z}$ is. Then there is a natural bijection from $A$ to $\mathbb{Q}$ given by $(p, q) \mapsto \frac{p}{q}$ and $0 \mapsto 0$, and hence $\mathbb{Q}$ is countable.

   Denote the set of polynomials of degree $n$ with with rational coefficients by $P_n$. We may assume the leading coefficient $a_n$ of the polynomial is 1, since, if it is not, we may divide through by $a_n$ and have a polynomial with exactly the same roots but 1 as the leading coefficient. There is a bijection from $P_n \to \mathbb{Q}^n$ defined by

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mapsto (a_{n-1}, \ldots, a_0)$$

   so that $P_n$ is countable.

   If $P$ is the set of all polynomials with rational coefficients then

$$P = \bigcup_{n \in \mathbb{N}} P_n$$

   where $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. $\mathbb{N}$ is countable, and hence $P$ is also countable as it is the countable union of countable sets.

   From the Fundamental Theorem of Algebra we know there exist at most $n$ distinct real roots of a polynomial $p$ of degree $n$. Denote the set of all real roots of $p$ by $R_p$. Then

$$\mathbb{A} = \bigcup_{p \in P} R_p$$

   But $P$ is countable, as is $R_p$, and hence $\mathbb{A}$ is also countable.

   We know that $\mathbb{R}$ is not countable. If $\mathbb{R} \setminus \mathbb{A}$ were countable, then $\mathbb{R} = \mathbb{A} \cup (\mathbb{R} \setminus \mathbb{A})$ would also be countable, a contradiction. Hence $\mathbb{R} \setminus \mathbb{A}$, the set of transcendental numbers, is uncountable.

2. *Determine whether or not each of the following sets is countable:*

   (a) *The set $A$ of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.*
   There is a bijection between $A$ and $\mathbb{Z}_+ \times \mathbb{Z}_+$ given by

   $$f \mapsto (f(0), f(1))$$

   and hence $A$ is countable.

   (b) *The set $B_n$ of all functions $f : \{1, \ldots, n\} \to \mathbb{Z}_+$.*
   As above, there is a bijection between $B_n$ and $\mathbb{Z}_+^n$ given by

   $$f \mapsto (f(1), \ldots, f(n))$$

   and hence $B_n$ is countable for all $n \in \mathbb{Z}_+$.

   (c) *The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$.*
   $C$ is the countable union of countable sets and is therefore countable.

   (d) *The set $D$ of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.*
   Every function from $\mathbb{Z}_+$ to $\{0, 1\}$ is also a function from $\mathbb{Z}_+$ to $\mathbb{Z}_+$. From the following exercise it follows that this set must be uncountable, since the set of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ is uncountable and any set cannot have a proper subset with cardinality greater than the set itself.

   (e) *The set $E$ of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.*
   This set is uncountable since there is a bijection from $E$ to $\wp(\mathbb{Z}_+)$ given by

   $$f \mapsto \{n \in \mathbb{Z}_+ \mid f(n) = 1\}$$

   This is obviously injective, since if two functions $f$ and $g$ are 1 on the same subset of $\mathbb{Z}_+$ then they must be 0 everywhere else and hence equal on all of $\mathbb{Z}_+$. It is surjective since, if $A \in \wp(\mathbb{Z}_+)$ we can define

   $$f(n) = \begin{cases} 1 & n \in A \\ 0 & n \in \mathbb{Z}_+ \setminus A \end{cases}$$

   Then $f \mapsto A$.

   (f) *The set $F$ of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are eventually zero.*
   If $f$ is eventually zero then are are a finite number of $n \in \mathbb{Z}_+$ such that $f(n) = 1$. Define

   $$F_n = \{f \in F \mid f(n) = 1 \text{ and } f(x) = 0 \text{ for all } x \geq n\}$$

   Then $F_n$ is finite, and in fact it is easy to see by counting that $|F_n| = 2^{n-1}$. But

   $$F = \bigcup_{n \in \mathbb{Z}_+} F_n$$

   So $F$ is countable.

(g) *The set $G$ of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually* $1$.
Similarly, define

$$G_n = \{f \in G \mid f(n) = 1 \text{ and } f(x) = 0 \text{ for all } x \geq n\}$$

There is a bijection between $G_n$ and all the functions from $A = \{1, \dots, n-1\}$ to $\mathbb{Z}_+$ given by

$$f \mapsto f\mid_A$$

where $f_A$ is $f$ restricted to $A$. This is easily seen to be a bijection since for all $x > n-1$, $f(x) = g(x)$ for any $f, g \in G_n$. $G$ is the union of all the $G_n$ over $\mathbb{Z}_+$, and is therefore countable.

(h) *The set $H$ of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.*
Define

$$H_n = \{f \in H \mid f \text{ is eventually } n\}$$

Each $H_n$ is countable by the previous part, i.e., the constant 1 from the previous part was completely arbitrary. Then $H$ is the union of all these $H_n$ over $\mathbb{Z}_+$ and hence is countable.

(i) *The set $I$ of all two-element subsets of $\mathbb{Z}_+$.*
As the set of all finite subsets of $\mathbb{Z}_+$ is countable and as $I$ is a subset of this set, $I$ is also countable.

(j) *The set $J$ of all finite subsets of $\mathbb{Z}_+$.*
Let

$$J_n = \{A \subset J \mid |A| = n\}$$

Then there exists a surjection from $\mathbb{Z}_+^n$ to $J_n$ given by

$$(a_1, a_2, \dots, a_n) \mapsto \{a_1, a_2, \dots, a_n\}$$

Hence $J_n$ is countable since $\mathbb{Z}_+^n$ is countable. But

$$J = \bigcup_{n \in \mathbb{Z}_+} J_n$$

so that $J$ is also countable.

3. (a) *Show that if $B \subset A$ and if there is an injection $f : A \to B$ then $A$ and $B$ have the same cardinality.*
Let $C_0 = A \setminus B$ and define recursively $C_{n+1} = f(C_n)$. Note that the $f(C_j)$ are pairwise disjoint. Assume there is a counterexample, then there is a minimal counterexample, i.e., minimal $i, j$ with $i \neq j$ such that $C_i \cap C_j \neq \emptyset$. $C_0$ is disjoint with respect to all the other $C_j$, so that if they are disjoint $j > 0$. Then

$$\emptyset \neq C_i \cap C_j = f(C_{i-1}) \cap f(C_{j-1}) \supset f(C_{i-1} \cap C_{j-1})$$

Hence $C_{i-1} \cap C_{j-1} \neq \emptyset$, contradicting the minimality of $i$ and $j$. Let $C = \bigcup_{i=1}^{\infty} C_i$ and define

$$h(x) = \begin{cases} f(x) & x \in C \\ x & x \notin C \end{cases}$$

$h$ is injective since it cannot be the case that if $f(x) = f(y)$ then $x \in C$ and $y \notin C$, or vice versa. Let $b \in B$. If $b \notin C$ then $h(b) = b$, otherwise, if $b \in C$ then there is some $C_k$ such that $b \in C_k$. Then $b \in C_k = f(C_{k-1})$, and hence there is some $a \in C_{k-1}$ with $h(a) = f(a) = b$. Therefore $h$ is a bijection, and $A$ and $B$ have the same cardinality.

(b) *Two sets $A, C$ have the same cardinality if there exist injective functions $f, g$ with $f : A \to C$ and $g : C \to A$.*

Let $f$ and $g$ be as stated, then $g \circ f : A \to g(C)$ is an injection. By the previoius part there exists a bijection $h : A \to g(C)$. Since $g$ is injective there also exists a bijection $g^{-1} : g(C) \to C$. Define a bijection by from $A$ to $C$ by

$$\varphi(x) = (g^{-1} \circ h)(x)$$

The composition of two bijections is a bijection, so $A$ and $C$ have the same cardinality.

4. *Let $X$ be a topological space and let $A \subset X$. Show that $A$ is open in $X$ if for every $x \in A$ there is an open set $U$ containing $x$ such that $U \subset A$.*

Let $U_x$ be an open set containing an arbitrary point $x \in A$. Then, since $U_x \subset A$ for all $x \in A$,

$$A = \bigcup_{x \in A} U_x$$

Since each $U_x$ is open $A$ itself is open.

5. *Let $X$ be a set and let $\mathcal{T}_c$ be the collection of all subsets $U$ of $X$ such that $X \setminus U$ is either countable or all of $X$. Show that $\mathcal{T}_c$ is a topology on $X$. Is the collection $\mathcal{T}_\infty$, the collection of all subsets $U$ of $X$ such that $X \setminus U$ is infinite, empty, or all of $X$, a topology on $X$?*

Clearly $\emptyset, X \in \mathcal{T}_c$ since $X \setminus U$ is all of $X$ if and only if $U = \emptyset$, and certainly $X \setminus X = \emptyset$ is countable. So we need only consider the case where $X \setminus U$ is countable. Let $\{U_\beta\}$ be a collection of open sets, then fix $\beta'$ and

$$X \setminus \bigcup_\beta U_\beta = \bigcap_\beta (X \setminus U_\beta) \subset X \setminus U_{\beta'}$$

Hence the set is closed under arbitrary union. Likewise, the complement of a finite intersection is a finite union of intersections and if each such intersection is countable then so is that finite union, i.e., $\mathcal{T}_c$ closed under finite intersection and is therefore a topology.

$\mathcal{T}_\infty$ is not a topology since the finite intersection of two infinite sets might be finite, e.g., two open sets whose complement is infinite but only share one element would have a finite intersection. An explicit example of this would be $\mathbb{Z}$. In $\mathcal{T}_\infty$ every singleton is open but $\mathbb{Z} \setminus \bigcup_{i \neq 2} \{i\}$ is finite.

6. (a) *If $\{T_\alpha\}$ is a family of topologies on $X$ show that $\bigcap \mathcal{T}_\alpha$ is a topology on $X$. Is $\bigcup \mathcal{T}_\alpha$ a topology on $X$?*

Let $\{U_\beta\} \subset \bigcap \mathcal{T}_\alpha$. Then $U_\beta \in \mathcal{T}_\alpha$ for all $\alpha, \beta$ in an arbitrary indexing set. But then $\bigcup_\beta U_\beta \in \mathcal{T}_\alpha$ for all $\alpha$, and hence $\bigcup_\beta U_\beta \in \bigcap \mathcal{T}_\alpha$. Similarly, the finite intersection of any of the $\{U_\beta\}$ is an element of $\bigcap \mathcal{T}_\alpha$ since by hypothesis the finite intersection is an element of every $\mathcal{T}_\alpha$. $X$ and $\emptyset$ are also in every $\mathcal{T}_\alpha$, so $\bigcap \mathcal{T}_\alpha$ is a topology on $X$.

$\bigcup \mathcal{T}_\alpha$ need not be a topology on $X$. For example consider $X = \{1, 2, 3\}$ and two topologies $\{\emptyset, X, \{1\}, \{1, 2\}\}$ and $\{\emptyset, X, \{1\}, \{2, 3\}\}$. Their union contains $\{1, 2\}$ and $\{2, 3\}$, two subsets whose intersection is not in the union. Hence a union of arbitrary topologies is not necessarily a topology. It is, however, a subbasis for a topology.

(b) *Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on $X$. Show that there is a unique smallest topology on $X$ containing all the collections $\mathcal{T}_\alpha$ and a unique largest topology contained in all $\mathcal{T}_\alpha$.*

If there is a largest or smallest such topology then it must be unique, since any other topology satisfying these conditions must be comparable to such a topology by definition.

First, we show that $\bigcap \mathcal{T}_\alpha$ is the largest topology contained in all the $\mathcal{T}_\alpha$. That this is contained in all the topologies is clear, since it is the intersection of all those topologies. Let $\mathcal{T}'$ be a topology contained in all the $T_\alpha$. If $x \in \mathcal{T}'$ then $x \in \mathcal{T}_\alpha$ for all $\alpha$, and certainly $x \in \bigcap \mathcal{T}_\alpha$ from the definition of an arbitrary intersection. Hence $\bigcap \mathcal{T}_\alpha$ is the largest topology contained in all the $\mathcal{T}_\alpha$.

Second, we show that the topology $\mathcal{T}$ generated by the subbasis $\bigcup \mathcal{T}_\alpha$ is the smallest topology containing all the $\mathcal{T}_\alpha$. It clearly contains all the topologies since it contains their union. Let $T'$ be another topology containing all the $\mathcal{T}_\alpha$ and let $U \in \mathcal{T}$.

$$U = \bigcup_\beta \left( \bigcap_{i=1}^{n} U_{i,\beta} \right)$$

where $U_{i,b} \in \bigcup \mathcal{T}_\alpha \subset T'$. Hence $T \subset T'$, and $T$ is the smallest topology containing all the $\mathcal{T}_\alpha$.

(c) *If $X = \{a, b, c\}$ let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ and the the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.*

The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_1$ is $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\{\emptyset, X, \{a\}\}$.

7. (a) *Show that the collection $\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$ is a basis that generates the standard topology on $\mathbb{R}$.*

$\mathcal{B}$ is a collection of open sets in $\mathcal{T}$, the standard topology on $\mathbb{R}$. Suppose $U$ is open in the standard topology and $x \in U$, then there exist $c, d \in \mathbb{R}$ with $c < d$ such that $x \in (c, d) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exist $a, b \in \mathbb{Q}$ with $a < b$ and $x \in (a, b) \subset (c, d) \in U$. By Lemma 13.2, $\mathcal{B}$ forms a basis for $\mathcal{T}$.

(b) *Show that the collection $\mathcal{C} = \{[a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$ is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.*

$\mathcal{C}$ is clearly a basis for a topology for all the same reasons the basis for $\mathbb{R}_l$ is. Moreover, every element of $\mathcal{C}$ is open in the lower-limit topology and hence the topology which it generates must be coarser than the lower-limit topology. That it is strictly coarser follows from considering $[\sqrt{2}, 2)$, which is open in $\mathbb{R}_l$. $\sqrt{2} \in [\sqrt{2}, 2)$, so suppose there exist $a, b \in \mathbb{Q}$ with $a < b$ and $\sqrt{2} \in [a, b)$. Since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, it must be the case that $\sqrt{2} \in (a, b)$. But then $a \notin [\sqrt{2}, 2)$ and $[a, b) \not\subset [\sqrt{2}, 2)$, so that $[\sqrt{2}, 2)$ is not open in the topology generated by $\mathcal{C}$. Hence this topology is strictly coarser than the lower-limit topology on $\mathbb{R}$.

8. *Show that if $Y$ is a subspace of $X$ and $A$ is a subset of $Y$ then the topology $A$ inherits as a subspace of $Y$ is the same as the topology it inherits as a subspace of $X$.*

Denote the topology $A$ inherits from $Y$ or $X$ as $\mathcal{T}_{A,Y}$ or $\mathcal{T}_{A,X}$ respectively. Then since $A \subset Y \subset X$,

$$\mathcal{T}_{A,Y} = \{A \cap U \mid U \in \mathcal{T}_Y\} = \{A \cap (Y \cap V) \mid V \in \mathcal{T}\} = \{A \cap V \mid V \in \mathcal{T}\} = \mathcal{T}_{A,X}$$

9. *Show that* $\pi_1 : X \times Y \to X$ *defined by* $(x, y) \mapsto x$ *and* $\pi_2 : X \times Y \to Y$ *defined by* $(x, y) \mapsto y$ *are both open maps.*

   Let $U \times V$ be open in the product topology, i.e., $U$ is open in $X$ and $V$ is open in $Y$. Then

   $$\pi_1 (U \times V) = \{\pi_1(x, y) \mid (x, y) \in U \times V\} = \{x \mid (x, y) \in U \times V\} = U$$

   which is by definition open in $X$. That $\pi_2$ is an open map follows *mutatis mutandis.*

10. *Let* $X$ *be a countable set. Find an infinite number of non-isomorphic well-orderings of* $X$. *How many well-orderings of* $X$ *are there?*

    Since $X$ is countable there exists a bijection $\varphi : X \to \mathbb{Z}_+$ and a well-ordering can be defined on $X$ by

    $$x <_\varphi y \Leftrightarrow \varphi(x) < \varphi(y)$$

    where $<$ is an arbitrary well-ordering on $\mathbb{Z}_+$. Hence it is sufficient to talk about well-orderings of the positive integers, rather than $X$ itself. Indeed, insofar as ordering is concerened, $X$ and $\mathbb{Z}_+$ are the same sets.

    We can create an infinite class of non-isomorphic well-orderings on $\mathbb{Z}_+$ by picking some $n \in \mathbb{Z}_+$ and saying that $x <_n y$ if and only if $y \leq n < x$, or $x < y \leq n$, or $n < x < y$. This is equivalent to ordering the integers as

    $$\{n + 1, n + 2, \ldots, 1, 2, \ldots, n\}$$

    Every integer in $\{1, 2, \ldots, n\}$ is greater than every integer in its complement, but within this set and its complement we use the normal ordering on $\mathbb{Z}_+$. To see that these well-orderings are not isomorphic consider $(\mathbb{Z}_+, <_m)$ and $(\mathbb{Z}_+, <_n)$ where $m < n$. If there were an order-preserving bijection between these two ordered sets then any such bijection would have to send some subset of $n$ integers in $(\mathbb{Z}_+, <_m)$ to $\{1, \ldots, n\}$ in $(\mathbb{Z}_+, <_n)$. After choosing $m$ such integers, however, the $m + 1$ such integer would necessarily be out of order.

    Denote the set of all well-orderings of the integers by $W$, then $W$ is uncountable. Assume $W$ were countable for contradiction, then there exists a $1 - 1$ correspondence with the integers, i.e., a list of well-orderings. But this list itself defines a well-ordering which cannot be in the list since, if it were, $W$ would be an element of itself.