MATH 208: Homework #6

Jesse Farmer

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1. *Prove the Banach-Steinhaus Theorem*

   Let V, W be normed linear spaces with V complete, and $\{T_\alpha : V \to W\}$ be a family of bounded linear maps such that $\{\|T_\alpha v|\}$ is bounded. Define $F_n := \{v \in V \mid \|T_\alpha v\| \leq n\|v\|\}$.

   Each F_n is closed and $\bigcup_{n\in\mathbb{N}} F_n = V$. The Baire Category Theorem implies that there exists some $k \in \mathbb{N}$ such that F_k has a nonempty interior, so there exists $v_0 \in V$ such that for $\epsilon > 0$ we have $B_\epsilon(v_0) \subset F_k$.

   Let $v \in V$ be such that $\|v\| = 1$. Then

$$\begin{aligned} \|T_\alpha v\| &\leq \epsilon^{-1}(\|T_\alpha(v_0 + \epsilon v)\| + \|T_\alpha v_0\|) \\ &\leq \epsilon^{-1}k(\|v_0 + \epsilon v\| + \|v_0\|) \\ &\leq \epsilon^{-1}k(2\|v_0\| - \epsilon) \end{aligned}$$

2. *Prove the Open Mapping Theorem*

   Let V, W be Banach space and $T \in B(V, W)$ be surjective. For $n \in \mathbb{N}$ and $0 \in V$ define $B_n := B_n(0)$. Clearly $V = \bigcup_{n\in\mathbb{N}} B_n$.

   By the surjectivity of T, $T(V) = W$, so $W = \bigcup_{n\in\mathbb{N}} T(B_n)$ and since W is complete, $\overline{T(B_1)}^o \neq \varnothing$. Note that if $T(B_1)$ is nowhere dense then $T(B_n)$ is nowhere dense, and vice versa, since we can simply scale B_1 by multiplying or dividing by n.

   We want to show that there exists $r > 0$ such that $B_r(0) \subset T(B_1(0))$, which implies that T is an open map.

   Let $w_0 \in \overline{T(B_1)}$ be such that $B_{4r}(w_0) \subset \overline{T(B_1)}$ for some $r > 0$. Then take w_1 such that $\|w_1 - w_0\| < 2r$. We can pick $v_1 \in V_1$ such that $w_1 = Tv_1$ (by density). Consider $B_{2r}(w_1) \subset B_{4r}(w_0) \subset \overline{T(B_1)}$.

   $B_{2r}(0) = -w_1 + B_{2r}(w_1)$. Suppose $w \in B_{2r}(0)$, then $w \in -w_1 + B_{2r}(w_1) \subset \overline{-w_1 + B_{2r}(w_1)}$.

   If $w \in -w_1 + T(B_1)$ then for some $v \in B_1$ we have

$$\begin{aligned} w &= -w_1 + T(v) \\ &= -T(v_1) + T(v) \\ &= T(v - v_1) \end{aligned}$$

So $\|T(v - v_1\| < 2$ implies $w \in \overline{T(B_2)}$. This statement follows *mutatis mutandis* for $\overline{T(B_2)}$. If $\|w\| < r$ then $w \in \overline{T(B_1)}$, and hence $B_r(0) \subset \overline{T(B_1)}$. In general, $\|w\| < r2^{-n}$ implies $w \in \overline{T(B_{2^{-n}})}$.

To reduce this to the case of $T(B_1)$ instead of the closure it is sufficient to show that there exists $v \in B_1$ such that $Tv = w$, for $\|w\| < \frac{r}{2}$. We will do so by the completeness of V.

There exists a $v_1 \in B_{\frac{1}{2}}$ such that $\|w - Tv_1\| < \frac{r}{4}$. And, in general, there exists $v_n \in B_{2^{-n}}$ such that $\|w - \sum_{j=1}^{n} Tv_j\| < r2^{-n-1}$. Because V is a Banach space it follows that $\sum_{j=1}^{\infty} Tv_j = v \in V$, where $Tv = w$. Note that $\|v\| < \sum_{n=1}^{\infty} 2^{-n}$, so $B_{\frac{r}{2}}(0) \subset T(B_1)$.

3. *Prove the Hahn-Banach Theorem*

Let V ve a normed linear space, $V_0 \subset V$ a subspace, $f \in V_0^*$, and $v_0 \in V \setminus V_0$. We will show that it is possible to extend V_0 by v_0 and retain the desired properties, and then apply Zorn's Lemma to conclude for all $\mathbb{R}$ in general.

Take $F(v + \lambda v_0) = f(v) + \lambda F(v_0)$. Denote $F(v_0) = \alpha$. We need $|f(v) + \lambda\alpha| \leq \|v + \lambda v_0\|$.

This is equivalent to

$$-\|v + \lambda v_0\| \leq f(v) + \lambda\alpha \leq \|v + \lambda v_0\|$$

or

$$-f(v) - \|v + \lambda v_0\| \leq \lambda\alpha \leq -f(v) + \|v + \lambda v_0\|$$

It follows immediately that for arbitrary $v_1, v_1 \in V$ we have the inequality

$$-f(v_1) - \|v_1 + v_0\| \leq \alpha \leq -f(v_2) + \|v_2 + v_0\|$$

or

$$f(v_2 - v_1) = f(v_2) - f(v_1) \leq \|v_1 + v_0\| + \|v_2 + v_0\| \leq \|v_2 - v_1\|$$

Letting $a = \sup\{-f(v_1) - \|v_1 + v_0\|\}$ and $b = \inf\{-f(v_2) + \|v_2 + v_0\|\}$, we see that choosing $\alpha \in [a, b]$ allows us to extend f in such a way that the norm is preserved. We now must deal with arbitrary extensions.

Let $\mathcal{F}$ be the set of all extensions of f satisfying the conditions of the hypothesis. This set is partially ordered by set inclusion and each *totally ordered* subset $\mathcal{F}_0 \subset \mathcal{F}$ has an upper bound, namely the functional defined on the union of the domains of all functionals. By Zorn's Lemma $\mathcal{F}$ has a maximal element, $\tilde{f}$. This function is exactly the function which satisfies the conclusions of the Hahn-Banach Theorem, since, if it were not, we could extend $\tilde{f}$ from the proper subspace on which it is defined to a larger subspace – a contradiction of the maximality of $\tilde{f}$.

We can now consider the Hahn-Banach Theorem over $\mathbb{C}$. Let V_{0R} and V_R denote the spaces V_0, V as *real* linear spaces. Clearly $f_R(v) = \Re f(v) \leq \|v\|$. By the previous part there exists F such that $|F(v)| \leq \|v\|$ on all V_{0R}.

Define $\tilde{f}(v) := F(v) - iF(iv)$. It is clear that $\tilde{f}(v) = F(v)$ for $v \in V_0$ and that $\Re\tilde{f}(v) = F(v)$. Write $\tilde{f}(v) = \rho e^{i\theta}$ and $w = e^{-i\theta}v$ and assume for contradiction that $|\tilde{f}(v)| \geq \|v\|$. Then

$$F(w) = \Re\tilde{f}(w) = \Re[e^{-i\theta}\tilde{f}(v)] = \rho > \|v\| = \|w\|$$

which contradicts the properties of F given to us by the Hahn-Banach Theorem on $\mathbb{R}$.