MATH 208: Homework #4

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1. *Prove that $\mathbb{Q}[x]$ is dense in $C[0,1]$.*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_n, \ldots, a_0 \in \mathbb{R}$. And

$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

with $b_n, \ldots, b_0 \in \mathbb{Q}$.

By the density of $\mathbb{Q}$ in $\mathbb{R}$, choose each b_i such that $|a_i - b_i| < \frac{\epsilon}{n}$. Since $x \in [0,1]$ we have

$$\begin{aligned}
|p(x) - q(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 - (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0)| \\
&= |(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_1 - b_1)x + a_0 - b_0| \\
&\leq |a_n - b_n|x^n + |a_{n-1} - b_{n-1}|x^{n-1} + \cdots + |a_1 - b_1|x + |a_0 - b_0| \\
&\leq |a_n - b_n| + |a_{n-1} - b_{n-1}| + \cdots + |a_1 - b_1| + |a_0 - b_0| \\
&< \epsilon
\end{aligned}$$

By Stone-Weierstrass we know that $\mathbb{R}[x]$ is dense in $C[0,1]$, so it follows that $\mathbb{Q}[x]$ is also dense in $C[0,1]$.

2. *Let $\{T_j : V \to W\}$ be a sequence of bounded linear maps such that $T_j \to T$ pointwise. Show that T is linear.*

We know that $T(v_1 + v_2) = lim_{n\to\infty} T_n(v_1 + v_2)$, so all that remains to be shown is that $lim_{n\to\infty} T_n(v_1 + v_2) = T(v_1) + T(v_2)$.

Pick N_1, N_2 such that for $n, m \geq N$

$$\begin{aligned}
|T_n(v_1) - T(v_1)| &< \frac{\epsilon}{2} \\
|T_m(v_1) - T(v_1)| &< \frac{\epsilon}{2}
\end{aligned}$$

If we pick $N = \max\{N_1, N_2\}$ it follows that for all $n \geq N$

$$\begin{aligned}
|T_n(v_1 + v_2) - (T(v_1) + T(v_2))| &= |T_n(v_1) + T(v_2) - (T(v_1) + T(v_2))| \\
&\leq |T_n(v_1) - T(v_1)| + |T_n(v_2) - T(v_1)| \\
&< \epsilon
\end{aligned}$$

We can simply pick N such that $|T_n(\alpha v) - \alpha T(v)| = |\alpha T_n(v) - \alpha T(v)| < \epsilon$. Therefore T is linear.

3. *Let A be a $n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ its corresponding linear map. Compute $\|T\|$ if $\mathbb{R}^n$ is endowed with the $p-$ norm.*

4. *Find a topological basis for $l^\infty(F)$ under the l^∞ norm.*

   Let $V = \{(a_n) \mid a_i \in \{0,1\}\}$. This should work, as we can approximate values arbitrarily far out in the sequence with just one basis element, but I'm too tired to concentrate.

5. *Find a topological basis for $C([0,1], F)$ under the sup norm.*

   Since we're taking $F = \mathbb{R}, \mathbb{C}$, and we have proved the first already, take $F = \mathbb{C}$.

   Every $f \in C([0,1], \mathbb{C})$ can be written as $f = u + iv$ for $u, v \in C([0,1], \mathbb{R})$. We can pick $p, q \in \mathbb{R}[x]$ such that $|p - u| < \frac{\epsilon}{2}$ and $|q - v| < \frac{\epsilon}{2}$ by the Stone-Weierstrass Then $|(u - p) - i(v - q)| \leq |u - p| + |v - q| < \epsilon$. Additionally, by the first problem, we can use $\mathbb{Q}[x]$ instead of $\mathbb{R}[x]$.

6. *Does Stone-Weierstrass provide a topological basis for some well-known normed linear spaces?*

7. *Show that $C([0,1], F)$ is not complete under the $\|\cdot\|_1$ norm, where $\|f\|_1 := \int_0^1 |f(x)|dx$.*

8. *Find the dual spaces of the following vector spaces.*

   (a) $l^p(F)$
   (b) $l^\infty(F)$
   (c) $C_0(F)$
   (d) $C(F)$

9. *Let X be a complete, separable metric space with $\varnothing \neq A \subseteq X$. Show that if A is perfect then A is uncountable.*

   A cannot be finite as every neighborhood around each point contains an infinite number of points. Assume that A is infinite and countable, and let $A = \{a_1, a_2, a_3, \ldots\}$. We will show that no sequence can cover all of A., and that since every point in A can be described by a Cauchy sequence A must be uncountable. As on Monday we will denote $S_r(x) := \overline{B_r(x)}$.

   First, it is obvious that any perfect set P in a complete metric space X is complete since every Cauchy sequence converges and no limit point could be in $X \setminus P$ by the definition of a perfect set.

   For any sequence $\{x_1, x_2, \ldots\}$ we do the following. Take $x_1 \in A$ and for $r_1 > 0$ consider $B_{r_1}(x_1)$. We can pick $x_2 \in B_{r_1}(x_1)$ and $r_2 > 0$ such that $S_{r_2}(x_2) \subsetneq S_{r_1}(x_1)$ with $x_1 \notin S_{r_2}(x_2)$. Continue inductively, choosing $r_{k+1} > 0$ such that $x_{k+1} \in S_{r_{k+1}}(x_{k+1}) \subsetneq S_{r_k}(x_k)$ and $x_k \notin S_{r_{k+1}}(x_{k+1})$.

   We then have a sequence of $S_{r_1}(x_1) \supsetneq S_{r_2}(x_2) \supsetneq S_{r_3}(x_3) \supsetneq \cdots$. By the Lemma from Monday's problem session, $\bigcap_{k=1}^\infty S_{r_k} x_k \neq \varnothing$. But whatever point is in this set could not be one of the points of the sequence, since $x_k \notin S_{r_{k+1}}(x_{k+1})$. Therefore A is uncountable.

10. *Prove the following statements about the Cantor set, denoted C.*

(a) *$x \in C$ if and only if the ternary expansion of x contains only 0 and 2.*

Define the following

$$\begin{aligned} A_0 &= [0,1] & (1) \\ A_n &= A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) & (2) \end{aligned}$$

Then $C = \bigcap_{n=0}^{\infty} A_n$. We see that at the n^{th} step the left-most interval removed is always $\left(\frac{1}{3^n}, \frac{2}{3^n}\right)$, and that the ternary expansion for this is $(0.00\ldots01, 0.00\ldots02)$ where the digit is in the n^{th} place. Note also that, for example, 0.1 can be rewritten as $0.0\overline{2}$, so number of this form can be said to "not contain any ones." Since every other interval is an integral multiple of the first, we get that at the i^{th} step all digits whose ternary expansion contains a 1 in the i^{th} place is removed. Therefore this construction of the Cantor set is equivalent to removing all points from $[0,1]$ whose ternary expansion contains a 1.

(b) *C is uncountable.*

Since C is closed and perfect (and $\mathbb{R}$ is a complete, separable metric space), C is uncountable.

(c) *C is closed.*

Since each A_n is closed, the Cantor set, which is the arbitrary intersection of the A_n, is also closed.

(d) *C is perfect.*

Let $x \in C$ and look at its ternary expansion $0.a_1a_2a_3\ldots$. If x has an infinite number of 2s then we can easily find a sequence which approximates $x = 0.a_1a_2a_3\ldots$ by defining x_n as follows

$$x_n = 0.b_1b_2b_3b_4\ldots = \begin{cases} b_i = a_i & i \leq n \\ b_i = 0 & i > n \end{cases}$$

Then this is clearly a ternary approximation for x.

If x has a finite ternary expansion then let k denote the place of the final 2 in that expansion and define x_n as follows

$$x_n = 0.b_1b_2b_3b_4\ldots = \begin{cases} b_i = a_i & i \neq k+n \\ b_i = 2 & i = k+n \end{cases}$$

Then $|x - x_n| \leq \frac{2}{3^{k+n}}$.

Therefore C is a perfect set.

(e) *C is nowhere dense.*

We will show that no open set is contained in C, immediately implying that C is nowhere dense.

Assume there is an open set in C, then since $C \subset \mathbb{R}$ there is some open interval (a,b) in C. But this means that (a,b) must be contained in all of the closed intervals which comprise each A_i. Pick $k \in \mathbb{N}$ such that $\frac{1}{3^k} < b - a$. Then (a,b) is not contained in A_k since each interval has a length of $\frac{1}{3^k}$. Therefore no open set is contained in C.

(f) *Define $D(C) = \{x - y \mid x, y \in C\}$. Show that $D(C) = [-1,1]$.*