MATH 263: Homework #5

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1. *Show that M, the Möbius band, is homeomorphic to the punctured real projective plane.*

   Recall that the Klein bottle K is $\mathbb{P}^2\#\mathbb{P}^2$. Write K as a square with labeling $aba^{-1}b$. Trisect this square into three parts of equal height. Then the middle section is a Möbius band, as are the upper and lower sections since the top and bottom edges are identified. The two lines trisecting the square form the boundary of the Möbius bands, and, moreover, constitute a circle by the identification alone the left and right edges of K. Hence K is constructed by taking two Möbius bands and identifying their boundaries with a circle. But recall that $K = \mathbb{P}^2\#\mathbb{P}^2$, which is formed by removing two contractible neighborhoods from each $\mathbb{P}^2$ and identifying their boundaries, which are homotopic to S^1. In terms of CW-complexes, it then follows that a punctured $\mathbb{P}^2$ is homeomorphic to M.

2. *For $n > 1$ show that the fundamental group of the n-torus is not abelian.*

   Lemma 1. *Let $f : G \to H$ be a homomorphism of groups. If $I \subseteq \ker f$ is normal in G then there exists a homomorphism $\tilde{f} : G/I \to H$ such that $\tilde{f} \circ p = f$, where $p : G \to G/I$ is the natural homomorphism.*

   Proof. Define $\tilde{f}(a + I) = f(a)$. It suffices to show that $\tilde{f}$ is well-defined, since if it were then it is immediately a homomorphism by the fact that f is a homomorphism. Let $a + I = b + I$, so that $a - b \in I$. Then since $I \subseteq \ker f$,

   $$0 = f(a - b) = f(a) - f(b) = \tilde{f}(a + I) - \tilde{f}(b + I)$$

   and therefore $\tilde{f}(a + I) = \tilde{f}(b + I)$. ■

   Let G be the free group on $2n$ generators $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ and H the free group on 2 generators, $\{\gamma, \delta\}$. Define a homomorphism $\varphi : G \to H$ by sending α_1, β_1 to γ and all other generators of G to δ. The fundamental group of the n-fold torus is isomorphic to G modulo the least normal subgroup N generated by $[\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ where $[\alpha_i, \beta_i] = \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$. However, the image of $[\alpha_i, \beta_i]$ under φ is the empty word since for fixed i, α_i, β_i map to the same element in F. Hence $N \subseteq \ker\varphi$ and by (1) there exists a well-defined homomorphism from G/N to H.

   But $G/N \cong \pi_1(T_n)$, where T_n denotes the n-fold torus. Clearly φ (and therefore $\tilde{\varphi}$) is surjective. If $\pi_1(T_n)$ were abelian then $\pi_1(T_n)/\ker\tilde{\varphi}$ would also be abelian and isomorphic to H, which is impossible since $Z(H) = 0$. Therefore $\pi_1(T_n)$ cannot be abelian.

3. *For $m > 1$ show that the fundamental group of the m-fold projective plane is not abelian.*

   Let $\mathbb{P}_m$ denote the m-fold torus. Let G be the free group on m generators, $\{\alpha_1, \ldots, \alpha_n\}$, and let $H = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, generated by $\{\gamma, \delta\}$. Define a homomorphism $\varphi : G \to H$ by sending α_1 to γ and α_i to δ for $i > 1$. Then $\pi_1(\mathbb{P}_m) \cong G/N$ where N is the least normal subgroup generated by $\alpha_1^2 \cdots \alpha_m^2$. But clearly φ sends this element to the empty word, as x^2 is trivial for all $x \in \mathbb{Z}/2\mathbb{Z}$. Again by (1), there exists a surjective homomorphism $\tilde{\varphi} : \pi_1(\mathbb{P}_m) \to H$. As above this is impossible, since the first isomorphism theorem would them imply that H is abelian. Therefore $\pi_1(\mathbb{P}_m)$ is not abelian.

4. *Calculate $H_1(\mathbb{P}^2 \# T)$. Assuming that the list of compact surfaces given in Theorem 75.5 is a complete list, to which of these surfaces if $\mathbb{P}^2 \# T$ homeomorphic?*

   $H_1(\mathbb{P}^2 \# T)$ is isomorphic to the free abelian group F with three generators, call them a, b, c, generated by the element $a^2bcb^{-1}c^{-1}$. Since F is abelian, the element $bcb^{-1}c^{-1}$ is the identity element, and $p(a^2) = p(a)^2$. Hence $H_1(\mathbb{P}^2 \# T) \cong \frac{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Assuming the list is complete, this would imply that $\mathbb{P}^2 \# T$ is homeomorphic to $\mathbb{P}_3$, the 3-fold projective plane.

5. *Let K be the Klein Bottle. Calculate $H_1(K)$ directly.*

   Recall that $K = \langle a, b \mid aba^{-1}b \rangle$. Hence $H_1(K)$ is isomorphic to the free abelian group on two generators generated by the element $aba^{-1}b$. Since the group is abelian, $aba^{-1}b$ becomes a^2, and it follows that $H_1(K) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

6. *Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labeling scheme $acadbcb^{-1}d$.*

   (a) *Check that all vertices of P are mapped to the same point of the quotient space X by the pasting map.*

   Since I'm not drawing this, you get to see it written out. In all instances "tip" and "end" mean relative to the direction in the labeling scheme. The tip of d is connected to the end of a, which is also connected to the tip of c and the end of b. But the tip of b is connected to both the tip and end of c, and this suffices since the other vertices belong to edges already considered.

   (b) *Calculate $H_1(X)$.*

   Under the projection map from a free group on 4 generators to its abelianization, $acadbcb^{-1}d$ becomes $a^2c^2d^2$. Hence $H_1(X)$ is isomorphic to the abelian group on 4 generators modulo the subgroup generated by $a^2c^2d^2$. Rewriting the basis as $a, b, c, (a + c + d)$ shows that this is, in fact, the free abelian group on 4 generators modulo the subgroup generated by $2(a + c + d)$. In other words, $H_1(X) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (c) *Assume X is homeomorphic to one of the surfaces given in Theorem 75.5, which surface is it?*

   X would seem to be homeomorphic to $\mathbb{P}_3$, the 3-fold projective plane, since it has a torsion subgroup of rank 2 and the quotient with this subgroup of rank 3.