MATH 262: Homework #8

Jesse Farmer

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1. *Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.*

   Every compact metrizable space (X, ρ) is second-countable. To see this, consider

   $$\mathcal{O}_n = \left\{ B_{\frac{1}{n}}(x) \mid x \in X \right\}$$

   where $n \in \mathbb{Z}_+$ is fixed. Since X is compact there exists a finite subcover $\mathcal{A}_n$. Let $\mathcal{B} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{A}_n$. $\mathcal{B}$ is countable. Since the usual basis for X, $\{B_\epsilon(x)\}$, contains $\mathcal{B}$, it is sufficient to show that for every ϵ-ball there exists some $B \in \mathcal{B}$ such that B is contains in that ϵ-ball. Simply let n be such that $\frac{1}{n} < \frac{\epsilon}{2}$, then there is some $B \in \mathcal{B}$ of radius $\frac{1}{n}$ containing x and $B \subset B_\epsilon(x)$. Therefore $\mathcal{B}$ is a basis for the metric topology on X.

   The other direction is easier. Every compact Hausdorff space is normal, and any second-countable normal space is metrizable by Urysohn.

2. *Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis then X is metrizable? What about the converse?*

   Let X be an uncountable discrete space. Then X is locally compact, Hausdorff, and metrizable, but not second-countable.

   Let X be a second-countable, locally compact Hausdorff space. Let S be the one-point compactification of X. By 29.4 it follows that X is homeomorphic to an open subspace of S. Since S is compact Hausdorff it is normal, and hence completely regular. X inherits this property as a subspace of S, and therefore X is also regular. As X is also second-countable, it follows that X is metrizable (since second-countable and regular implies metrizable).

3. *Let (X, ρ) be a metric space.*

   (a) *Fix $\epsilon > 0$. Show that if every ball of radius ϵ in X has a compact closure then X is complete.*

   Let $\{x_k\}$ be a Cauchy sequence in X, and fix $\epsilon > 0$ such that every ball of radius ϵ has compact closure. Then there exists some $N \in \mathbb{N}$ such that $\rho(x_k, x_n) < \epsilon$ for all $k \in \mathbb{N}$. Hence the sequence $\{x_{k+N}\}$ is contained entirely in $\overline{B_\epsilon(x_k)}$. As a subspace this is compact since it is closed, and therefore sequentially compact. But this means that it contains a convergence subsequence, and therefore that $\{x_k\}$ converges.

   (b) *Show that if for every x there exists an $\epsilon > 0$ such that $B_\epsilon(x)$ has compact closure then X need not be complete.*

   Let $X = (0, 1)$. For every $x \in X$ choose ϵ such that $0 < x - \epsilon < x < x + \epsilon < 1$. Then $\overline{B_\epsilon(x)} = [x - \epsilon, x + \epsilon]$, which is compact. However, X is not complete. In particular the sequence $\{1/n\}$ is Cauchy but does not converge in X.

4. *Let (X, ρ) be a complete metric space. Show that if $f : X \to X$ is a contraction mapping then there is a unique point $x \in X$ such that $f(x) = x$.*

   Define a sequence in X as follows. Let $x_0 \in X$ be arbitrary, and let $x_n = f(x_{n-1})$. Let $0 < \alpha < 1$ be the coefficient of contraction, then it follows by induction that

$$\rho(x_n, x_0) \leq \frac{1-\alpha^n}{1-\alpha}\rho(x_1, x_0$$

   Since this is obvious for $n = 0$, assume it to be true for $n - 1$. Then

$$\begin{aligned}
\rho(x_n, x_0) &\leq \rho(x_n, x_{n-1}) + \rho(x_{n-1}, x_0) \\
&\leq \alpha^{n-1}\rho(x_1, x_0) + \frac{1-\alpha^{n-1}}{1-\alpha}\rho(x_1, x_0) \\
&= \left(\frac{\alpha^{n-1}-\alpha^n}{1-\alpha} + \frac{1-\alpha^{n-1}}{1-\alpha}\right)\rho(x_1, x_0) \\
&= \frac{1-\alpha^n}{1-\alpha}\rho(x_1, x_0)
\end{aligned}$$

   And therefore it is true for every $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$, and, without loss of generality, assume $m \geq n$. Then by the above

$$\begin{aligned}
\rho(x_m, x_n) &\leq \alpha^n \rho(x_{m-n}, x_0) \\
&\leq \alpha^n \frac{1-\alpha^{m-n}}{1-\alpha}\rho(x_1, x_0) \\
&= \frac{\alpha^n - \alpha^m}{1-\alpha}\rho(x_1, x_0) \\
&< \frac{\alpha^n}{1-\alpha}\rho(x_1, x_0)
\end{aligned}$$

   However, as $\frac{\alpha^n}{1-\alpha} \to 0$ as $n \to \infty$, we can choose N such that if $n \geq N$ then $\frac{\alpha^n}{1-\alpha} < \epsilon$ for all $\epsilon > 0$. Therefore $\{x_n\}$ is a Cauchy sequence, and by the completeness of X it has some limit $x \in X$. Assume for contradiction that this is not a fixed point of f, i.e., $\rho(f(x), x) > 0$. Then choose N such that for $n \geq N$, $\rho(x, x_n) < \frac{\rho(f(x),x)}{2}$. Then

$$\begin{aligned}
\rho(f(x), x) &\leq \rho(f(x), x_{N+1}) + \rho(x_{N+1}, x) \\
&\leq \alpha\rho(x, x_N) + \rho(x_{N+1}, x) \\
&< \frac{\rho(f(x), x)}{2} + \frac{\rho(f(x), x)}{2} < \rho(f(x), x)
\end{aligned}$$

   which is absurd. Therefore $\rho(f(x), x) = 0$ and hence $f(x) = x$. To see that x is unique, let y be such that $f(y) = y$. Then

$$\rho(x, y) = \rho(f(x), f(y)) \leq \alpha\rho(x, y)$$

   which is true if and only if $\rho(x, y) = 0$, i.e., $x = y$.

5. *Let X be a compact metric space. Is $C(X, \mathbb{R})$ necessarily second-countable? What about $C(X, Y)$ if Y is also a compact metric space?*

6. *If X and Y are compact metric spaces such that there are isometric embeddings of X into Y and Y into X, must X and Y be isometric?*