# Dyakonov Shur Instability - Finite Volume Method

May 21, 2020

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## 1 Preliminary

This notebook tries to reproduce the results of Fig. 2 in Mendl et al. 2019. They claim that hydrodynamic electrons satisfying certain boundary conditions act as a nonlinear oscillator in the Terahertz range.

### 1.1 Equations

The equations are the isothermal equations of gas dynamics, with a couple extra terms that I put on the RHS. In their non-dimensional form, they look like:

$$\partial_t J + \partial_x \left( n + \frac{J^2}{n} \right) = \eta \partial_x^2 \left( \frac{J}{n} \right) + \gamma (n v_0 - J),$$

$$\partial_t n + \partial_x J = 0.$$

So they are the Navier-Stokes equation and the continuity equation. The term $\partial_x n$ in the Navier-Stokes equation comes from assuming a simple form for the pressure. The weird boundary conditions are:

$$n(0) = 1,$$
$$\partial_x J(0) = 0,$$
$$J(1) = v_0.$$

We choose dimensionless parameters $\eta = 0.02$, $\gamma = 0.04$, $v_0 = 0.14$.

### 1.2 Numerical Method

#### 1.2.1 Conservation Law

If you forget about the terms on the RHS of the Navier-Stokes equation, we can write the remaining stuff as a *vector conservation law*. A conservation law has the form: $\partial_t \vec{u} + \partial_x \vec{f}(\vec{u}) = 0$, where $\vec{u}$ is a state vector and $\vec{f}$ is the "flux vector". In our case, the conservation law is is:

$$\partial_t \begin{pmatrix} n \\ J \end{pmatrix} + \partial_x \begin{pmatrix} J \\ \frac{J^2}{n} + n \end{pmatrix} = 0.$$

Conservation laws are handled well by finite volume methods, so we'll use one of those. But we have to still deal with the other stuff that is not part of the conservation law. For the dissipative term with $\eta$, we'll directly incorporate it in the time-stepping as a finite differences quotient. For the relaxation term with $\gamma$, we'll employ an approach called "operator splitting", specifically, "Strang Splitting".

**Slope Limiting** The conservation law step uses a high resolution method that approximates the function as piecewise - linear. This means we have to pick the slopes somehow - this code implements two "slope-limiting" techniques: One is "minmod" which is the most diffusive one, and the other is "superbee", which should be a little bit sharper [3].

### 1.2.2 Relaxation Step

Pretend the Navier-Stokes equation only has the term with $\gamma$. Then it's an ODE that reads:

$$\partial_t J = \gamma(nv_0 - J).$$

You can solve this one exactly, and the answer is $J(t) = nv_0 + (J_0 - nv_0)e^{-\gamma t}$. So this step we can perform exactly, just setting $J_0$ as the value one time step in the past!

## 2 Simulation

[1]:
```
#Imports
import numpy as np
import matplotlib.pyplot as plt
from scipy import linalg
```

[2]:
```
#Settings
imageLog = False #during the simulation, will plot a graph every 1.0 sec of sim␣
 ↪time so you can check on it
saveFigures = True #If true, will save pdfs of all figures
```

[3]:
```
#Global things
eps = np.finfo(float).eps #machine epsilon
k = 0.001 #Time Step
h = 1/50. #Mesh width
T = 152.0 #stopping time
```

[4]:
```
#Define Helper Functions
def minmod(a,b):
    """
    The minmod function gives a simple way of doing slope-limiting (used in the␣
 ↪high resolution correction)
    """
    if np.abs(a) < np.abs(b) and a * b > 0:
        return a
    elif np.abs(b) < np.abs(a) and a * b > 0:
        return b
    else:
```

```
        return 0

def maxmod(a,b):
    """
    Helper function for superbee limiter
    """
    if np.abs(a) > np.abs(b) and a * b > 0:
        return a
    elif np.abs(b) > np.abs(a) and a * b > 0:
        return b
    else:
        return 0

def superbee(a,b):
    """
    A less diffusive slope-limiter
    """
    s1 = minmod(b, 2*a)
    s2 = minmod(2 * b, a)

    return maxmod(s1, s2)

#Select which slope-limiter
slopeLimiter = superbee

def eigenExpand(uleft, uright):
    """
    Expands jump in state vector from cell to cell in terms of
    the eigenfunctions of the Roe matrix 'A' which is hard
    coded in.  uL and uR should be two-component state vectors,
    i.e., numpy arrays with 2 components.
    """
    nleft, Jleft = uleft
    nright, Jright = uright
    jump_in = uright - uleft
    v_in =(nleft**0.5*Jleft/(nleft) + nright**0.5*Jleft/(nright))/(nleft**0.5 +
 ↪nright**0.5)
    #Want to solve for "alpha1, alpha2", the coefficients of each eigenvector
    #in the expansion of the jump.  We do that here!
    alpha1_in = (jump_in[0] * (v_in + 1) - jump_in[1])/2
    alpha2_in = (-jump_in[0] * (v_in - 1) + jump_in[1])/2

    return alpha1_in, alpha2_in

def f(u):
    """
    flux term in equation.  In our case, the it is a vector [J, J**2/n + n].
```

```
    Just a convenience function that does this calcul
    """
    n_in, J_in = u
    return np.array([J_in,J_in**2/n_in + n_in])

def flux(uL, uR, UL, UR):
    """
    Finite volume methods approximate the average of the solution on a bunch of
    cells.  The averages are updated by the "flux" through the boundaries of␣
 ↪the
    cell.  This function computes the flux at the boundary between uL and uR.
    Because of the high resolution method used, the values at one previous␣
 ↪cell,
    UL and UR are also needed.
    """
    nL, JL = uL
    nR, JR = uR
    jump = uR - uL
    #rho averaged velocity
    v = (nL**0.5*JL/(nL+eps) + nR**0.5*JR/(nR))/(nL**0.5 + nR**0.5)
    #Hard-code eigenvectors and eigenvalues (from [3])
    r1 = [1, v - 1]
    w1 = v - 1
    r2 = [1, v + 1]
    w2 = v + 1
    alpha1, alpha2 = eigenExpand(uL, uR)

    #Get the gobunov flux
    fG = f(uL) + min(w1, 0) *  alpha1 * np.array(r1) + min(w2, 0) * alpha2 * np.
 ↪array(r2)

    #For next part, need the "j - 1" value of the alpha1, alpha2 expansion
    #coefficients.  I give these new variable names by switching the case
    #of everything
    NL, jL = UL
    NR, jR = UR
    Jump = UR - UL
    V = (NL**0.5*jL/NL + NR**0.5*jR/NR)/(NL**0.5 + NR**0.5)
    R1 = [1, V - 1]
    W1 = V - 1
    R2 = [1, V + 1]
    W2 = V + 1
    Alpha1, Alpha2 = eigenExpand(UL, UR)

    #Here's the slope limiting thing
    sigma1 = 1/h*np.array(
```

```
    [slopeLimiter(alpha1 * r1[0], Alpha1 * R1[0]), slopeLimiter(alpha1 * r1[1],␣
↪Alpha1 * R1[1])])

    sigma2 = 1/h*np.array(
    [slopeLimiter(alpha2 * r2[0], Alpha2 * R2[0]), slopeLimiter(alpha2 * r2[1],␣
↪Alpha2 * R2[1])])

    #The high resolution correction to the flux:
    additionalFlux = 1/2 * (w1 * (np.sign(k * w1 / h) - k * w1 / h) * h *␣
↪sigma1 + w2 * (np.sign(k * w2 / h) - k * w2 / h) * h * sigma2)

    F = fG + additionalFlux

    return F
```

[ ]:
```
#Set up the domain of the problem in space and time.
x = np.arange(0., 1.0, h)
tau = np.arange(0., T, k)

#like to do computations at midpoints for finite volume methods
xMid = x[:-1] + h / 2

#Parameters
v0 = 0.14 #dimensionless velocity
eta = 0.01 #dimensionless viscosity
gamma = 0.04 #dimensionless momentum relaxation rate

n0 = 1. + 0.1 * np.sin(xMid*np.pi)
J0 = v0 * (1. + 0.2 * np.cos(np.pi * xMid / 2))

u = np.vstack((n0, J0)).T

#Storage
J_list = []
n_list = []

J_list.append(np.copy(u[:,1]))
n_list.append(np.copy(u[:,0]))
N = 0

for t in np.arange(0, T, k):
     #First part of the Strang Splitting - integrate the relaxation term up to
    #dt / 2


    #Now impose boundary conditions on left and right!
    uLeft = np.array([[1.,u[2][1]], [1.,u[1][1]]])
```

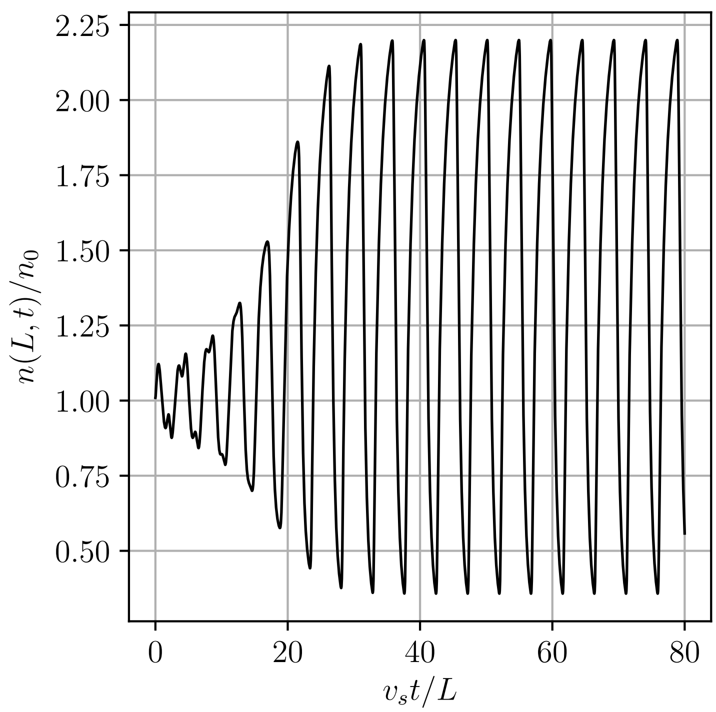
```
    uRight = np.array([[2*u[-1][0] - u[-2][0], v0], [u[-1,0], v0]])
    u[:,1] = u[:,0] * v0 + (u[:,1] - u[:,0] * v0) * np.exp(-gamma * k / 2)

    uBC = np.vstack((uLeft, u, uRight))
    U = np.copy(u)
    #the uBC array has the real physical domain and also some extra "ghost"␣
↪cells used to
    #do the boundary conditions.
    n = uBC[:,0] #just useful
    J = uBC[:,1]
    q = J / n

    for j in range(2, uBC.shape[0] - 2): #iterate through the *physical* domain
        #Call the flux() function at the left and right boundary of each cell
        FMinus = flux(uBC[j - 1], uBC[j], uBC[j - 2], uBC[j - 1])
        FPlus = flux(uBC[j], uBC[j + 1], uBC[j - 1], uBC[j])

        #Approximate the dissipative term by a finite differences quotient (2nd␣
↪order)
        dissipative = k * np.array([0, eta * 1/h**2 * (q[j + 1] - 2 * q[j] +␣
↪q[j - 1])])


        #Update each element of physical domain
        U[j-2] = u[j-2] - k / h * (FPlus - FMinus) + dissipative # + 0.0075 * k␣
↪* 1/h**2 * (uBC[j + 1] - 2*uBC[j] + uBC[j - 1])
        #Note the last term in the above is the-made up viscous term!

    #Second step of Strang Splitting, same integration
    U[:,1] = U[:,0] * v0 + (U[:,1] - U[:,0] * v0) * np.exp(-gamma * k / 2)

    #send to the storage lists
    J_list.append(np.copy(U[:,1]))
    n_list.append(np.copy(U[:,0]))
    u = np.copy(U)

    #Log Progress
    if N % 500 == 0:
        print("Time is {:.3f} out of {:.3f} - - - - Iteration {}".format(t, T,␣
↪N))
    if N % 1000 == 0 and imageLog:
        fig, axes = plt.subplots(1,2)
        ax1, ax2 = axes
        ax1.plot([list[-1] for list in n_list])
        ax1.set_title("n(L,t)")
        ax2.plot(J_list[-1])
        ax2.set_title("J")
```

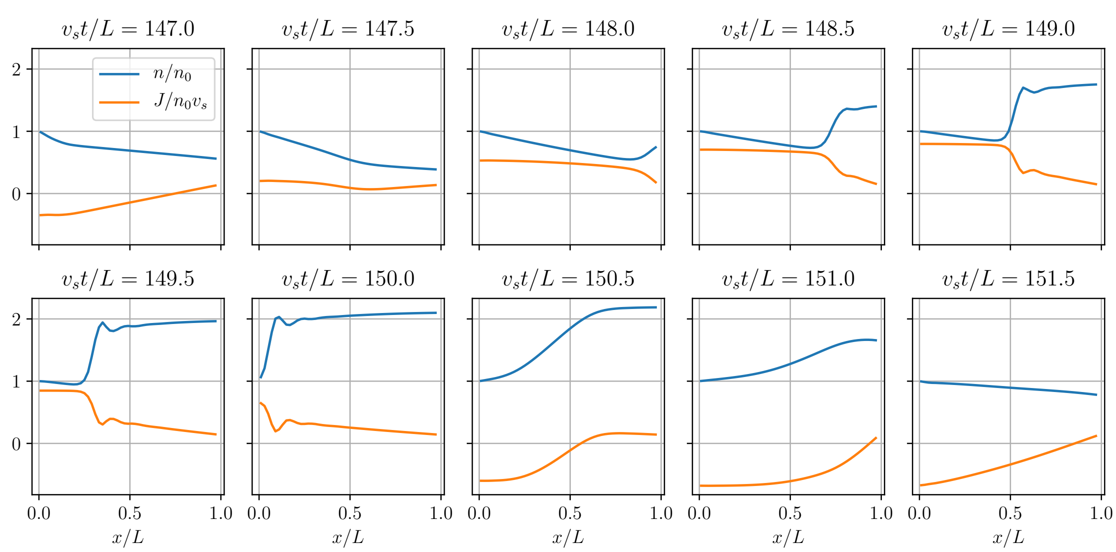
```
        plt.show()
    N += 1
```

# 3 Results

[10]:
```
#Figures
from matplotlib import rc
import matplotlib.ticker as tckr
from IPython.display import set_matplotlib_formats
set_matplotlib_formats('png', 'pdf')

#Matplotlib Parameters
plt.rc('text', usetex=True)
plt.rc('font', family='serif', size=12)

#Mendl et al. Fig. 2
fig2, axes2 = plt.subplots(1,1, figsize = (4,4))
ax21 = axes2
ax21.grid()
ax21.plot(tau[:int(80./k)], [list[-1] for list in n_list][:int(80./k)],lw = 1,␣
 ↪color = "Black")
ax21.set_xlabel("$v_s t / L$")
ax21.set_ylabel("$n(L,t)/n_0$")
fig2.tight_layout()
if saveFigures:
    plt.savefig("Figures/resonance_superbee.pdf")
plt.show()

#Mendl et. al Fig. 3 (Used different time for snapshots b.c. I only ran 80.0␣
 ↪seconds of sim time)
fig3, axes3 = plt.subplots(2, 5, figsize = (10,5), sharex = True, sharey = True)
J_snapshots = []
n_snapshots = []
delta = 0.5 #time difference between snapshots in seconds
start = 147. #time to start snapshots in seconds
for number in range(10):
    index = int((start + delta * number) / k)
    J_snapshots.append(J_list[index])
    n_snapshots.append(n_list[index])
index = 0
for i in axes3:
    for axis in i:
        axis.grid()
        axis.plot(xMid, n_snapshots[index], label = "$n/n_0$")
        axis.plot(xMid, J_snapshots[index], label = "$J/n_0v_s$")
        axis.set_title("$v_st/L = {:.1f}$".format(start + delta * index))
```

```
        index += 1
for axis in axes3[-1]:
    axis.set_xlabel("$x/L$") #Set label only on the bottom axes
axes3[0,0].legend()
fig3.tight_layout()
if saveFigures:
    plt.savefig("Figures/snapshots_superbee.pdf")
plt.show()
```



# 4 References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3. LeVeque, R. J. (1992). Numerical methods for conservation laws. Basel: Birkhauser. doi: DOI https://doi.org/10.1007/978-3- 0348-8629-1