Linearized Equations and BCs

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We're studying the equations

$$\begin{aligned}\frac{\partial J}{\partial t} + \frac{\partial}{\partial r}\left(\frac{J^2}{n} + v_s^2 n\right) - \frac{\eta}{m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\frac{J}{n} = \gamma(J_0 - J) - \frac{J^2}{n}\frac{1}{r},\\ \frac{\partial n}{\partial t} + \frac{\partial J}{\partial r} = -\frac{J}{r}.\end{aligned} \tag{1}$$

We are going to try to linearize these by writing $n = n_0 + n_1 e^{-\mathrm{i}\omega t}$ and $J = J_0 + J_1 e^{-\mathrm{i}\omega t}$. We'll also git rid of any terms quadratic in any of the parameters η, γ, and v_0. So we'll only get the frequency ω accurately to that same order.

1 Boundary Conditions

Remember that the steady state is $J_0(r) = n_0 v_0 R_2/r$. That makes the Neumann condition a little tricky. We want the BCs:

$$\begin{aligned}\frac{\partial J}{\partial r}(r = R_1) = 0\\ J(r = R_2) = n_0 v_0\end{aligned}$$

But if $J = J_0 + J_1 e^{-\mathrm{i}\omega t}$ and $n = n_0 + n_1 e^{-\mathrm{i}\omega t}$, then for the Neumann condition, you need, at $r = R_2$, the derivative to vanish:

$$\frac{\partial J_0}{\partial r} + \frac{\partial J_1}{\partial r} e^{i\omega t} = 0.$$

But the steady state solution J_0 does *not* obey this Neumann condition. So I think the boundary conditions are (in terms of the linearized things, n_1 and J_1):

$$\begin{aligned}\frac{\partial J}{\partial r}(r = R_1) = \frac{n_0 v_0 R_2}{R_1^2} e^{\mathrm{i}\omega t}\\ J(r = R_2) = 0\end{aligned} \tag{2}$$

(NB: if R_1 is very large compared to the length scale of the problem, which is L, the Neumann condition looks pretty much homogenous).

2 Equations

For the real part of the frequency, you get:

$$r^2 \frac{\partial^2 J_1}{\partial r^2} + r \frac{\partial J_1}{\partial r} + \left(r^2 \frac{\omega_1^2}{v_s^2} - 1 \right) J_1 = 0. \tag{3}$$

For the complex part of the frequency, you get:

$$r \frac{\partial^2 J_1}{\partial r^2} + \left(1 - \frac{2 v_0 R_2}{\nu} \right) \frac{\partial J_1}{\partial r} - \frac{1}{\nu} r (2\omega_2 + \gamma) J_1 = 0 \tag{4}$$