The Dyakonov-Shur Instability in a Corbino Geometry

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1 Introduction

2 Equations

We model electrons in the disk $a < r < b$ subject to a steady radial electric field $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Also let η be the shear viscosity, γ be the momentum relaxation rate, $-e$ be the charge of the electron, and m be its effective mass. We assume radially symmetric $n = n(r,t)$ and $\mathbf{J} = J(r,t)\hat{\mathbf{r}} \equiv n(r,t)u(r,t)\hat{\mathbf{r}}$, and we use the equation of state for an idea gas, giving a pressure $p = v_s n$ where v_s is a density-independent speed of sound. These assumptions lead to the hydrodynamic equations:

$$\frac{\partial J}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{J^2}{n}\right) + v_s\frac{\partial n}{\partial r} - \eta\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\frac{J}{n} = \frac{-ne}{m}E(r) - \gamma J, \tag{1a}$$

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rJ) = 0. \tag{1b}$$

These are subject to the boundary conditions:

$$n(a) = n_a, \tag{2a}$$

$$v(b) = v_b, \tag{2b}$$

with n_a and v_b both constants.

Now we ask the question: What functional form of external $E(r)$ would we need to apply in order to admit a reasonable steady-state solution $\left(\frac{\partial n_0(r)}{\partial t} = \frac{\partial J_0(r)}{\partial t} = 0\right)$? In free electron theory, the equation of motion is the simpler:

$$\frac{\partial J}{\partial t} = \frac{-ne}{m}E(r) - \gamma J.$$

This gives the steady state balance $\gamma J = -neE$. Then nE and J will need to have the same spatial dependence, which means E and u should have the same spatial dependence. If the E field is going to be produced by an external source, it should have $\nabla \cdot \mathbf{E} = 0$, so $E \propto 1/r$. Then $u_0(r) \propto 1/r$, and from (1b) when $\frac{\partial n}{\partial t} = 0$, we must have that $J_0(r) \propto 1/r$, making $n_0 =$ constant. To satisfy the boundary conditions (2), we should have the steady state distribution for velocity $u_0(r) = v_b b/r$ and for the density $n_0(r) = n_a$. Then for $E(r)$, we get:

$$E(r) = -\frac{\gamma m b v_b}{er} \tag{3}$$

For the more complicated hydrodynamic theory of (1a), the steady state solution for $u_0(r)$ does not quite work. While $u_0(r) \propto 1/r$ does solve the viscous term, the convection term $\frac{\partial}{\partial r}(rJ^2/n)$ is nonzero. To account for this leftover, the electric field would have to have another component:

$$E = \frac{-m v_0 b}{er}\left(\gamma + \frac{v_0 b^2}{r^2}\right) \tag{4}$$

b

$$n_0 u_0 \frac{\partial u_0}{\partial r} - \eta\left(\frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} - \frac{u_0}{r^2}\right) = -\frac{n_0 e}{m}E - \gamma(n_0 u_0). \tag{5}$$

One way to guess a solution is to try and The viscous term admits two independent solutions: $u_0 \propto 1/r$ or $u_0 \propto r$. In the latter case, if $u_0 = c_2 r$, we see that:

$$E = \frac{-m}{e}\left(c_2^2 + \gamma c_2\right) r.$$

Applying a field with $E \propto r$ like this is not easy practically given the other features our device must satisfy — the whole apparatus would have to be immersed in a long cylinder of constant volume charge density. Further, $u_0 \propto r$ would require $n_0 \propto 1/r^2$ by Equation (1b). If, however, we take the other possibility, $u_0 = c_1/r$, we obtain for the electric field:

and n_0 just a constant. The boundary conditions (**??**) gives $c_1 = v_0 b$. Then, defining $L \equiv b - a$ at and letting $L/a \to 0$ with L fixed, we find that the second term of Equation (4) is the larger contribution. An electric field $E \propto 1/r$ is more reasonable practically since it does not require any set charge density in the region, as it satisfies $\nabla \cdot (E\hat{\mathbf{r}}) = 0$. With the motivation that an electric field of this form comes close to creating a steady state, we study the equations (1) with the simplified electric field:

$$E = \frac{-m\gamma b v_0}{e}\frac{1}{r}. \tag{6}$$

Finally, we write the equations in "conservation form", using as a variable $J \equiv nu$ instead of u. We also bring the equations into a non-dimensional form, scaling the spatial variables with $b \to b/L$, $a \to a/L$, $r \to r/L$. We also introduce dimensionless versions of the other parameters, $\eta \to$ the equations (1) become:

$$n\frac{\partial u}{\partial t} + nu\frac{\partial u}{\partial r} + v_s\frac{\partial n}{\partial r} - \eta\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) = \gamma\left(n\frac{v_0 b}{r} - nu\right), \tag{7a}$$

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rnu) = 0, \tag{7b}$$

subject again to the boundary conditions (2). Note that using the expression (6) is the same as assuming that the balance in (5) comes only from the momentum relaxation and the external electromagnetic force, in other words, neglecting convection.

3 Linear Theory