Ising Model

The source for these notes is Chapter (2) of Shankar's Book *Quantum Field Theory for Condensed Matter Physics*.

Ising Model in $d = 0$

Let's consider a system with only two degrees of freedom, which we'll call s_1 and s_2. They are *spins*, so they only take on the values $s = 1$ and $s = -1$. To specify a system you just need to give values for s_1 and s_2.

Imagine the spins have magnetic moments of $\mu_B = 1$ (just so we don't have to put it in the formulas), and also imagine that there is a uniform magnetic field of magnitude B along the same axis as the spins. Then the energy has two parts — an interaction between the two spins, and an interaction between each spin and the external field:

$$E(s) = -Js_1 s_2 - B(s_1 + s_2). \tag{1}$$

It's useful to define some additional constants:

$$\begin{aligned} K &= \beta J, \\ h &= \beta B. \end{aligned}$$

with $\beta = 1/k_b T$ as usual.

To solve the problem, we'll need to work out the partition function, which will be:

$$\begin{aligned} Z &= \mathrm{e}^{-\beta E(s_1,s_2)} \\ &= \sum_{s_1,s_2} \mathrm{e}^{Ks_1s_2 + h(s_1+s_2)}. \end{aligned}$$

In this case, the sums over s_1 and s_2 are just over two values , $s_1 = 1$ and $s_2 = -1$. So we get:

$$Z(K, h) = 2\cosh(2h)\mathrm{e}^{K} + 2\mathrm{e}^{-K}. \tag{2}$$

To get a sense of the physics, we'll use the partition function to calculate some averages of quantities (I mean *thermal* averages). For example, consider the (*spin*-averaged) magnetization, $M = (s_1 + s_2)/2$. We take thermal average by working out:

$$\begin{aligned}
\langle M \rangle &= \frac{1}{Z} \sum_{s_1,s_2} M \mathrm{e}^{-E(s)}, \\
&= \frac{1}{Z} \sum_{s_1,s_2} \frac{1}{2}(s_1 + s_2)\mathrm{e}^{Ks_1s_2 + h(s_1+s_2)}, \\
&= \frac{1}{2}\frac{1}{Z}\frac{\partial Z}{\partial h}, \\
&= \frac{1}{2}\frac{1}{Z}\frac{\partial \ln Z}{\partial h}.
\end{aligned} \tag{3}$$

Or, if we define free energy F by:

$$F(K,h) \equiv \mathrm{e}^{-\beta F}, \tag{4}$$

then we can write:

$$\langle M \rangle = \frac{1}{2}\frac{\partial}{\partial h}\left(-\beta F\right). \tag{5}$$

For us, $-\beta F = \ln\left(2\cosh(2h)\mathrm{e}^{K} + 2\mathrm{e}^{-K}\right)$, so we get:

$$\langle M \rangle \frac{\sinh(2h)}{\cosh(2h) + \mathrm{e}^{-2K}}. \tag{6}$$

If we wanted instead the *thermal* average of a *particular* spin (instead of the spin-average), we need to add a source term to the Boltzmann weight. The idea is to couple each spin to its own magnetic field — the source term is $h_1 s_1 + h_2 s_2$, and it gives the partition function:

$$Z = \sum \mathrm{e}^{Ks_1s_2 + h_1s_1 + h_2s_2} \equiv \mathrm{e}^{-\beta F}. \tag{7}$$

We can check that as $h \longrightarrow \infty$ and $K \longrightarrow \infty$, $\langle M \rangle \longrightarrow 1$. This makes sense, since in the limit of strong fields, the spins should be completely aligned.

Correlators

Taking single spatial derivatives of the free energy in (7) with respect to the h_i is the same as taking a thermal average of the corresponding spin, because basically, the derivative just brings down a factor of s_i from the exponential. *e.g.*

$$\langle s_1 \rangle = \frac{1}{Z} \sum_{s_1 1, s_2} s_1 \mathrm{e}^{K s_1 s_2 + h_1 s_1 + h_2 s_2} = \frac{\partial(-\beta F)}{\partial h_1} = \frac{\partial \ln Z}{\partial h_1} \tag{8}$$

And, similarly, taking two mixed derivatives with respect to, say, h_1 and h_2 is *kind of* like taking an average of the product $s_1 s_2$. But it's not quite — we actually get (using the product rule):

$$\begin{aligned} \frac{\partial^2}{\partial h_1 \partial h_2} &= \frac{\partial}{\partial h_1} \left(\frac{1}{Z} \frac{\partial Z}{\partial h_2} \right) \\ &= \frac{1}{Z} \frac{\partial^2}{\partial h_1 \partial h_2} - \frac{1}{Z^2} \frac{\partial Z}{\partial h_1} \frac{\partial Z}{\partial h_2} \\ &= \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle \end{aligned} \tag{9}$$

This kind of thing comes up often, motivating a definition:

Definition: Connected Correlation Function The connected correlation function $\langle s_1 s_2 \rangle_c$ is defined as:

$$\langle s_1 s_2 \rangle_c = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle \tag{10}$$