Ising Model

Ising Model in $d = 0$

Let's consider a system with only two degrees of freedom, which we'll call s_1 and s_2. They are *spins*, so they only take on the values $s = 1$ and $s = -1$. To specify a system you just need to give values for s_1 and s_2.

Imagine the spins have magnetic moments of $\mu_B = 1$ (just so we don't have to put it in the formulas), and also imagine that there is a uniform magnetic field of magnitude B along the same axis as the spins. Then the energy has two parts — an interaction between the two spins, and an interaction between each spin and the external field:

$$E(s) = -Js_1s_2 - B(s_1 + s_2). \tag{1}$$

It's useful to define some additional constants:

$$\begin{aligned} K &= \beta J, \\ h &= \beta B. \end{aligned}$$

with $\beta = 1/k_bT$ as usual.

To solve the problem, we'll need to work out the partition function, which will be:

$$\begin{aligned} Z &= \mathrm{e}^{-\beta E(s_1,s_2)} \\ &= \sum_{s_1,s_2} \mathrm{e}^{Ks_1s_2+h(s_1+s_2)}. \end{aligned}$$

In this case, the sums over s_1 and s_2 are just over two values , $s_1 = 1$ and $s_2 = -1$. So we get:

$$Z(K, h) = 2\cosh(2h)\mathrm{e}^{K} + 2\mathrm{e}^{-K}. \tag{2}$$

To get a sense of the physics, we'll use the partition function to calculate some averages of quantities (I mean *thermal* averages). For example, consider the (*spin*-averaged) magnetization, $M = (s_1 + s_2)/2$. We take thermal average by working out:

$$
\begin{aligned}
\langle M \rangle &= \frac{1}{Z} \sum_{s_1, s_2} M \mathrm{e}^{-E(s)} \\
&= \frac{1}{Z} \sum_{s_1, s_2} \frac{1}{2}(s_1 + s_2) \mathrm{e}^{K s_1 s_2 + h(s_1 + s_2)}
\end{aligned}
$$