Tiling for Tots

A Study & Application of Golomb's Tromino Theorem

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ABSTRACT

In the mid-1950's, Harvard undergraduate Samuel Golomb published a theorem on tiling with trominoes. In short, a square board of any size - made up of 2^n by 2^n tiles - can be filled with unit-sized L-shaped blocks, so long as one square unit tile was removed. This paper - along with its companion web application - demonstrates Golomb's Tromino Theorem inductively and explores its applications. Furthermore, this project was developed to study theoretical applications of the Golomb's Tromino Theorem and to introduce prospective computer scientists and engineers to algorithmic programming.

KEYWORDS

Tromino, polyomino, Golomb, algorithmic programming, secondary education (computer science), tiling, invariants

1 Problem Statement

Given a square board composed entirely of square unit tiles, prove that the board can be filled with "L-shaped" blocks of tiles, called trominoes.

1.1 Problem Parameters

A tile is a single square unit of measurement that takes up equal lengths of space horizontally and vertically. It is the only unit of measurement referenced in the problem.

The square board is a theoretical set of tiles aligned contiguously to form a square of tiles. In application, this means that the square board is composed of ($2^n * 2^n$), where n is any natural integer greater than or equal to one.

Although Golomb originally intended for this board to be a chessboard, by using induction, it can be proven that Golomb's Tromino Theorem will hold true for a square board of any length of tiles [See Section 3].[5]

The basic tromino is an "L-shaped" object the size of three contiguous tiles. As seen in Figure 1, one tile-sized object is aligned immediately vertical to a second tile-sized object, and a third tile-sized object is aligned immediately horizontal to the second. Technically, Golomb would have considered any object of three contiguous tiles to be a tromino – so long as the proposed object did not have holes.[1,5] Golomb originally referred to these as right and left trominoes, but – for our purposes – this paper will refer to it as the basic tromino. When "trominoes" are referenced in the following discussion, assume that basic trominoes are being discussed, unless the researcher indicates otherwise.

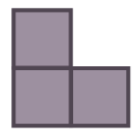

Figure 1. A basic tromino. Defined as a right tromino by Golomb.

When a tromino is placed onto the square board, it selects the three tiles immediately beneath it and 'removes' them from play, meaning that – so long as the tromino is unmoved – no other object (i.e.: tromino) can be placed in a position such that the object overlaps with a 'removed' tile.

2 Relevance and Application of Problem

Golomb defined and studied trominoes in the context of polyominoes. According to Golomb, a polyomino is composed of some natural integer number of contiguous, square unit blocks (i.e.: three), and is generally composed with empty blocks (or holes) inside of its structure.[1,5]

Polyominoes are considered to be integral to the study of graph theory, and consequently, familiarity with Golomb's Tromino Theorem is essential to understanding any other algorithm relevant to the field.

Additionally, Golomb's Tromino Theorem is considered to be an excellent example to use when introducing future computer scientists to the principles of invariant-based programming, where absolute conditionals relative to the problem are defined before developing an algorithm (i.e.: How many trominoes fit in a square board of *n* tiles?).

Lastly, studying this theorem bridges the gap between purely visual and pure algorithmic problem-solving strategies, encouraging students to utilize both when approaching a difficult algorithmic problem.

3 Theorem Proofs

3.1.1 "Visual" Demonstrations – Demonstration A

[See Figure 2.]

Approach the presented problem with a square board made of 4 tiles. Remove from play – or color in – one tile on the board and decide if and where a tromino will need to be placed. In this case, the answer is evident: in the only position where the tromino can be placed without overlapping onto the removed tile. This effectively proves that an answer is possible when the square board is made of 4 tiles.

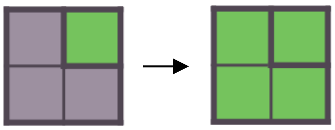

Figure 2. Demonstration A: The base-case solution for the problem is visually represented.

3.1.2 "Visual" Demonstrations – Demonstration B

[See Figure 3.]

Now approach the presented problem with a square board made of 64 tiles (with a length and width of 8 tiles). Remove one tile on the board from play, and we are left with 63 tiles. Since 63 is divisible by 3, we can be certain that an answer is possible.

Divide the board into four quadrants and take note of the quadrant which the removed tile is placed in. Given that there are an equal number of tiles in each quadrant, the next objective will be to remove a single tile from the remaining three quadrants. There will always exist exactly one location and position where this is possible. [For the first tromino placement, this will always be at the center of the board.] Place the first tromino in this position. Now, all four quadrants have one tile removed.

Next, isolate a single quadrant and note the location of the removed tile. We will now treat this quadrant as a smaller square board made up of 16 tiles (with a length and width of 4 tiles). Break this smaller board into 4 smaller quadrants and take note of the quadrant which the removed tile is placed in. Place the next tromino in the only position through which the 3 empty quadrants – and consequently, all four quadrants – will have one tile removed.

Isolate another quadrant and note the location of the removed tile. This quadrant now represents a square board of 4 tiles (with a length and width of 2 tiles). Place the tromino in the only legal position that fills the three unfilled tiles – or quadrants – of the board. This fills a board of 4 tiles, which is actually a quadrant in a board of made up of 16 tiles.

Repeat this process with the three other quadrants of 4 tile boards. This fills a board of 16 tiles, which is actually a quadrant in a board made up of 64 tiles.

Repeat this process with the remaining three quadrants of 16 tiles. This fills a board of 64 tiles, which proves that a solution to the problem is possible when the square board is made up of 64 tiles.

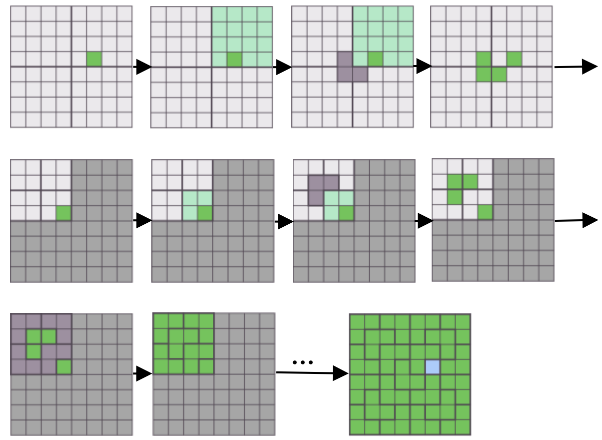

Figure 3. Demonstration B: An inventive, inductive solution for the problem is represented when the number of tiles equals 64.

3.2 *Inductive Proof*

Ultimately, the process highlighted in Section 3.1 proves that an answer to the guding problem exists and that it hasas to do with breaking up the square board – made up of ($2^n * 2^n$) tiles – into smaller quadrants – each made up of ($2^{n-1} * 2^{n-1}$) tiles.

By removing one tile, we are left with ($2^n * 2^n$) – 1] tiles to fill or remove, or ($4^n – 1$) tiles. The binomial theorem can be applied to this equation, such that [$3(4^{n-1}+\ldots+1)$] tiles need to be filled. From this point on, the problem continues recursively, where one tile is removed from each quadrant either by the base condition or a tromino placement overlapping contiguous borders in its position.

3.3 *Algebraic Proof*

The math for the inductive case is shown below. It is assumed that, if the number of tiles in the square at the start of the problem is divisible by 3, then a solution is possible.

- Base Case: ($n = 1$); 4 tiles, Demonstration A.

  Known to be possible. $((4^1 – 1) \% 3 = 0)$
- Inductive Case: ($n = k$); 4^k tiles.

  Assumed to be possible. $((4^k – 1) \% 3 = 0)$
- Next Case: ($n = k+1$); 4^{k+1} tiles.

  $= ((4^{k+1} – 1) \% 3) = ([4(4^k) – 1] \% 3) = ([4(4^k) – 1 – 3] \% 3)$

  $= ([4(4^k) – 4] \% 3) = ([4(4^k – 1)] \% 3) = ((4^k – 1) \% 3 = 0)$.

Therefore, it is proven by induction that trominoes of tiles can fill any square board of tiles, so long as a single tile is removed from the board.

4 Properties of Golomb's Tromino Theorem

In the same publication that Dr. Golomb introduced his Tromino Theorem, he also identified possible limitations to its implementation. He asserted that the tromino used on the square board must be basic for the theorem to hold true for the problem regardless of the position of the first tile removed.[1]

However, if the tromino were straight, then the theorem would not only hold true for 4 out of the 64 possible problems on a square board with a length and width of 8 tiles [See Figure 4].[5] Furthermore, given a straight tromino, no solution would be possible for a square board with a length and width of 2 tiles.[5] This property emphasizes the importance of invariants in an algorithmic program, and how changing an independent conditional can destabilize the entire solution.

Furthermore, research conducted by a computer scientist and a high schooler have indicated that altered versions of Golomb's Tromino Theorem can be applied to boards and hypercubes of higher dimensions [i.e.: ($3^n * 3^n$) or ($4^n * 4^n$) tiled structures].[1] This further emphasizes the importance of invariants, as even drastic changes to dependent variables cause minimal impact to the format of the proof for the theorem.

5 Researcher Implementation

5.1 Added Value: Companion Application

A simple educational game, available here, was developed to demonstrate Golomb's Tromino Theorem and to test student understanding of the algorithm.[3] For the game to be easily accessible to high schoolers and middle schoolers, '*Trominoes! Trominominoes!*' was developed using Scratch.

The companion software has already been used to teach non-engineering students about Golomb's Tromino Theorem. Students have reportedly found the game engaging and fun.

5.2 Application Limitations

Controls are reportedly counterintuitive, making the game difficult to learn and play without additional instructions. A system to provide hints – or differentiate quadrants and sub-quadrants of the board – would make the learning and playing experience of the user more agreeable.

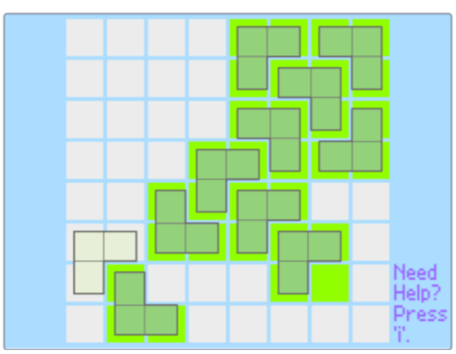

Figure 4. A screenshot of a game of *Trominoes! Trominominoes!* in-progress. Victory is now impossible. Press 'X' to restart.

6 Concluding Remarks

Golomb's Tromino Theorem largely serves two purposes in the modern study of algorithms:

1) To engage students in deconstructing visual problem through inductive and algorithmic strategies.
2) And to introduce future engineers to the fundamentals of Graph Theory.

If given the opportunity to share this research with students, the researcher plans to add more intuitive instructions and controls to the software upon initializing gameplay. Future work may include the creation of a simple program that can determine and display inductive solutions for a given, unfilled board.

REFERENCES

[1] Befumo, Arthur, and Jonathan Lenchner. "Extensions of Golomb's Tromino Theorem." *The 24th Fall Workshop on Computational Geometry*, University of Connecticut, 2014, https://fwcg14.cse.uconn.edu/program/wp-content/uploads/sites/863/2014/10/fwcg2014_submission_10.pdf.

[2] Bogomolny, A. (n.d.). Golomb's inductive proof of a tromino theorem. Retrieved from http://www.cut-the-knot.org/Curriculum/Geometry/Tromino.shtml.

[3] Fathi, J. (2019, December 14). Retrieved from https://scratch.mit.edu/projects/351295957/

[4] GeeksForGeeks. (2018, December 1). Tiling Problem using Divide and Conquer algorithm. Retrieved from https://www.geeksforgeeks.org/tiling-problem-using-divide-and-conquer-algorithm/.

[5] Golumb, Samuel W. "Checker Boards and Polyominoes." *The American Mathematical Monthly*, vol. 61, no. 10, Dec. 1954, pp. 675–682.

[6] Jain, S. (2018, February 7). Divide and Conquer. Retrieved from https://www.comp.nus.edu.sg/~sanjay/cs3230/dandc.pdf.

[7] Midlandman. (2008, November 8). Retrieved from https://scratch.mit.edu/projects/751577/

[8] Tromino. (2019, November 17). Retrieved from https://en.wikipedia.org/wiki/Tromino.

[9] Valentine, M., Wallmark, L., & Anderson, A. (2007). Retrieved from https://aanderson.people.amherst.edu/tromino/