Problem Set 5

Jules Faucher

McGill University Department of Physics

Instructor: Prof. Jonathan Sievers

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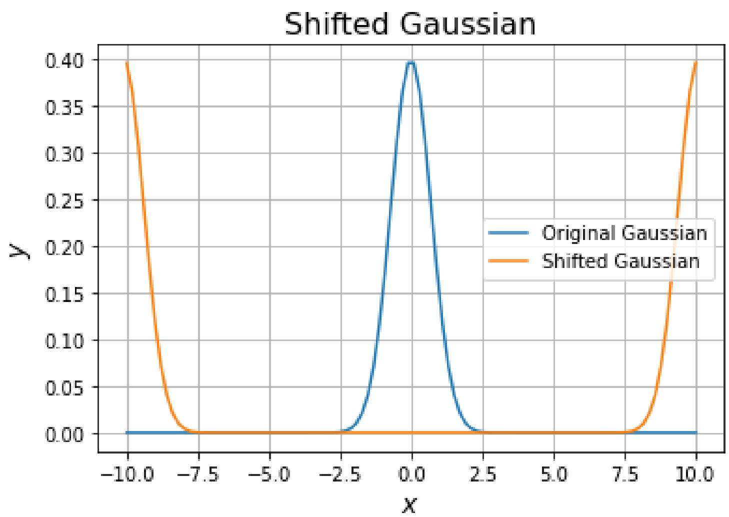
1 Problem 1

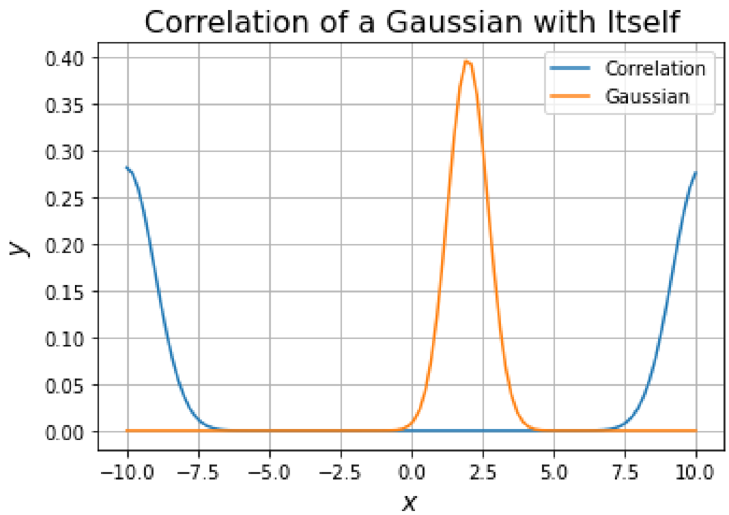
Any function can be shifted by doing applying a convolution with a Dirac delta function. We can show that using the sifting property of the Dirac delta function.

$$(f * \delta(a))(y) = \int_{-\infty}^{\infty} f(x)\delta(x-a-y)dx \tag{1}$$
$$= \int_{-\infty}^{\infty} f(x-y)\delta(x-a)dx \tag{2}$$
$$= f(y-a) \tag{3}$$



2 Problem 2

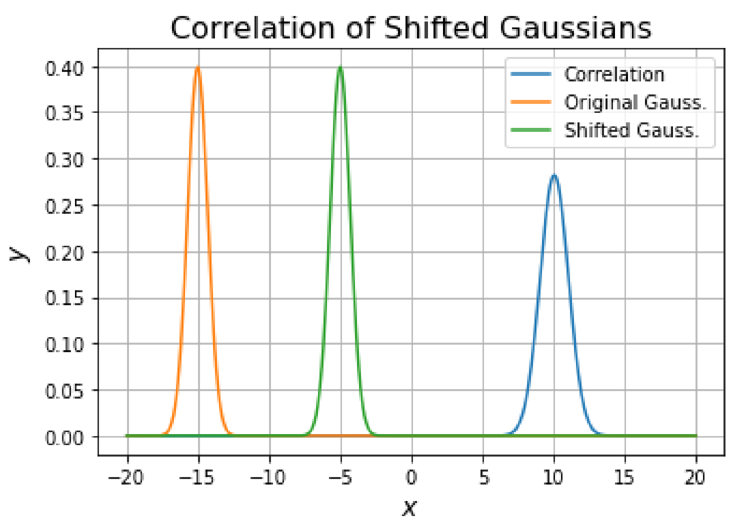
Here is the discrete convolution of a Gaussian with itself:



It is interesting to note that no matter where the Gaussians are centered at, there correlation will always be centered at 0 (Note that the correlation appears on the edges of the array due to rap around).

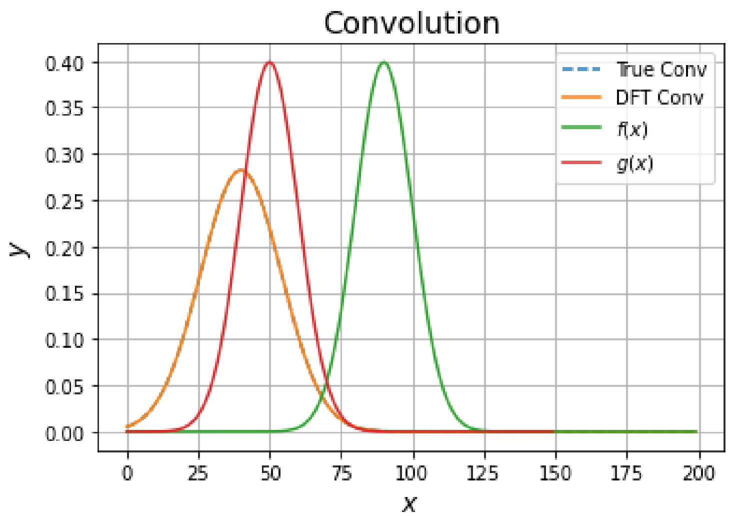
3 Problem 3

The closer both Gaussians are, the further away is the Gaussian generated from the correlation. This is not surprising since we established that a Gaussian correlated with itself is centered at 0, which is the edges of the array because of the wrap around.



4 Problem 4

The resulting array as the same length as the one representing $f(x)$ since I return an array without the zeros that were generated because of the zeros that were first added to both arrays. I decided to do so to make plotting easier.



5 Problem 5

5.1 a)

Recall that for geometric series, we can use the following equation:

$$\sum_{i=1}^{n} a_i r^i = a_1 \frac{1-r^n}{1-r}. \tag{4}$$

With this in mind, we can see that

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = \sum_{x=0}^{N-1} (\exp(-2\pi ik/N))^x. \tag{5}$$

We can also rework the range of indices over which the sum is made. Let $y = x + 1$.

$$\sum_{x=0}^{N-1} (\exp(-2\pi ik/N))^x = 1 - 1 + \sum_{y=1}^{N} (\exp(-2\pi ik/N))^y \tag{6}$$

Finally, we can use the equation for geometric series.

$$\sum_{y=1}^{N} (\exp(-2\pi ik/N))^y = \frac{1 - \exp(-2\pi ik)}{1 - \exp(-2\pi ik/N)} \tag{7}$$

5.2 b)

We may take the limit of the expression above as k approaches 0.

$$\begin{aligned}
\lim_{k\to 0} \frac{1 - \exp(-2\pi ik)}{1 - \exp(-2\pi ik/N)} &\overset{H}{=} \lim_{k\to 0} \frac{(2\pi i)\exp(-2\pi ik)}{(2\pi i/N)\exp(-2\pi ik/N)} &(8)\\
&= \lim_{k\to 0} N \frac{\exp(-2\pi ik)}{\exp(-2\pi ik/N)} &(9)\\
&= N &(10)
\end{aligned}$$

We can expand the expression to reveal more about the behaviour of this equation.

$$\frac{1 - \exp(-2\pi ik)}{1 - \exp(-2\pi ik/N)} = \frac{1 - \cos(2\pi k) + i\sin(2\pi k)}{1 - \cos(2\pi k/N) + i\sin(2\pi k/N)} \tag{11}$$

Since k is an integer, we can get rid of some trig terms.

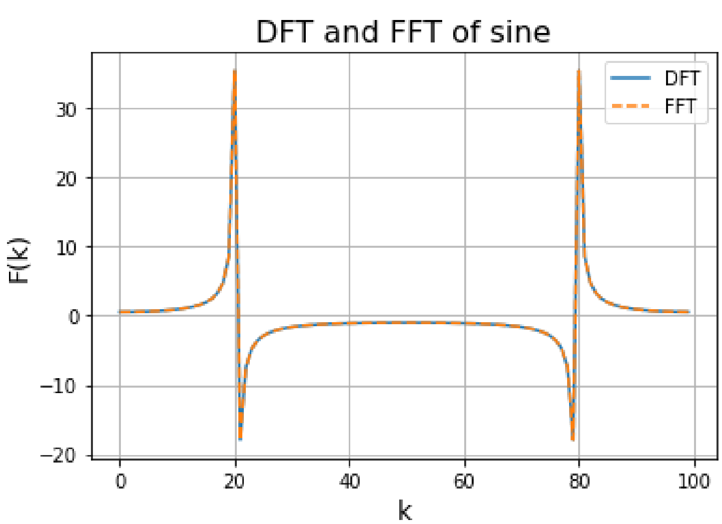
$$\frac{1 - \cos(2\pi k) + i\sin(2\pi k)}{1 - \cos(2\pi k/N) + i\sin(2\pi k/N)} = \frac{0}{1 - \cos(2\pi k/N) + i\sin(2\pi k/N)} \tag{12}$$

Here, we can see that any integer k will lead the numerator of this expression to go to 0. Meanwhile, in the denominator, if k is not a multiple of N, k/N is not an integer. Hence, the denominator will not reach 0. Therefore, the fraction as a whole will equal 0.

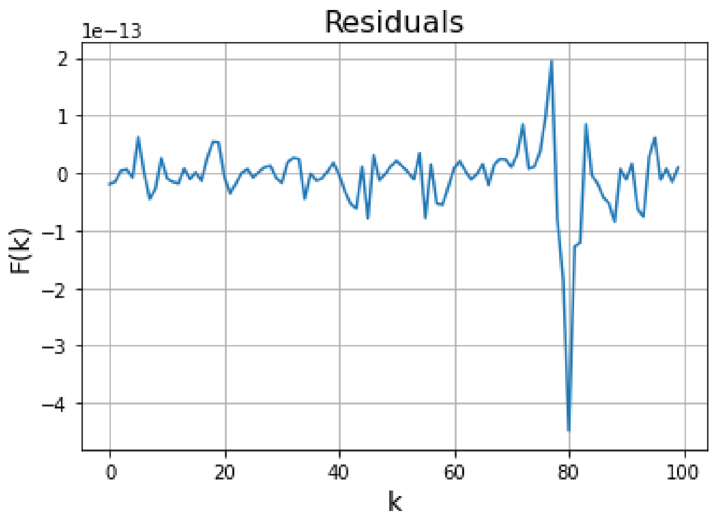
5.3 c)

$$
\begin{aligned}
F(k') &= 4\sum_{x=0}^{N-1} \sin(2\pi kx/N)e^{-i2\pi k'x/N} &(13)\\
&= \sum_{x=0}^{N-1} \frac{e^{-i2\pi kx/N} - e^{i2\pi kx/N}}{2i} e^{-i2\pi k'x/N} &(14)\\
&= \frac{1}{2i}\sum_{x=0}^{N-1} e^{2\pi i(k'-k)x} - e^{2\pi i(k'+k)x} &(15)\\
&= \frac{1}{2i}\sum_{x=0}^{N-1} e^{2\pi i(k'-k)x} - \frac{1}{2i}\sum_{x=0}^{N-1} e^{2\pi i(k'+k)x} &(16)\\
&= \frac{1}{2i}\left(\frac{1-e^{-2\pi i(k'-k)}}{1-e^{-2\pi i(k'-k)/N}} - \frac{1-e^{-2\pi i(k'+k)}}{1-e^{-2\pi i(k'+k)/N}}\right) &(17)\\
& &(18)
\end{aligned}
$$

If k and k' are integers, this expression will equal 0 whenever $k \neq k'$ or $k \neq -k'$, and N when $k = k'$ or $k = -k'$. This means we should have something that looks like two Dirac Delta functions. Using non-integer values for k will make the Diracs imperfect.

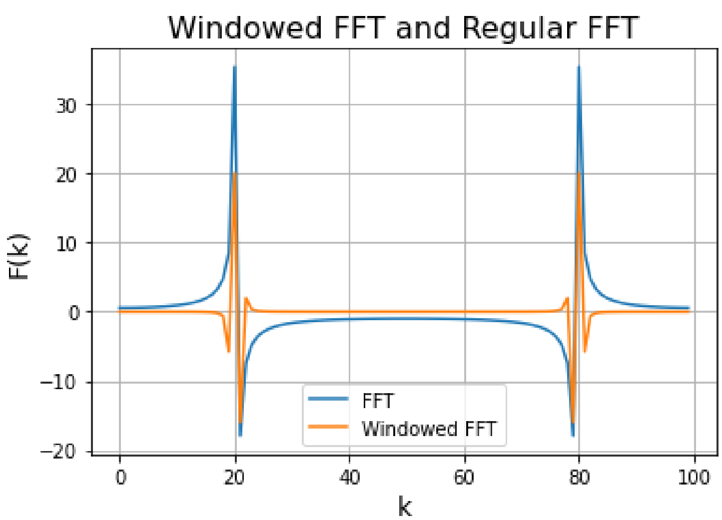


As explained earlier, I was not expecting perfect Diracs but we can see relatively sharp peaks around $k = 20.33$. Nonetheless, as illustrated by the residuals below, our derived DFT is close (up to machine precision) to the FFT of the same sine wave.



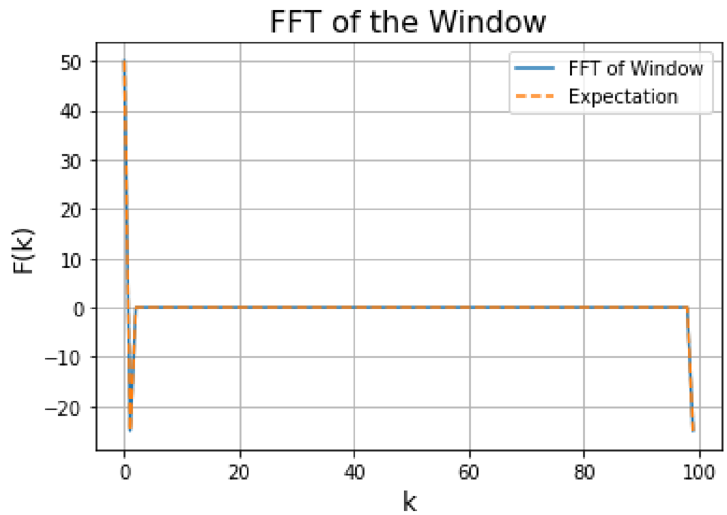
5.4 d)

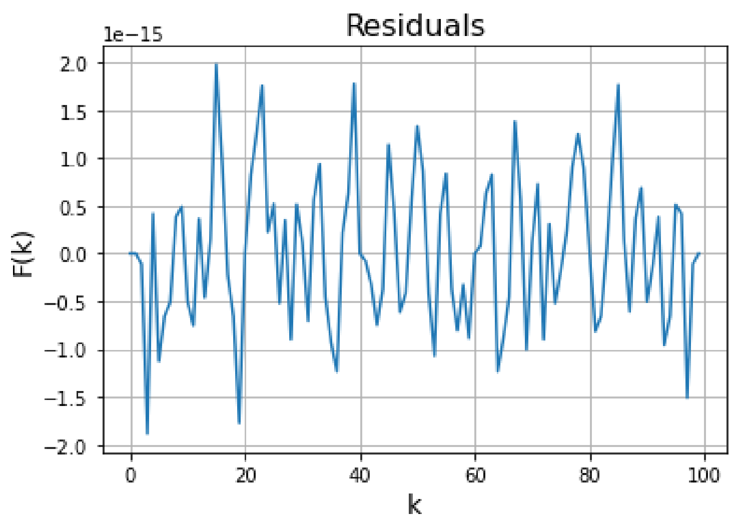
Using a window made the previously observed peaks a lot narrower.



5.5 e)

I decided to show that clever combinations of the unwindowed FFT points will lead to the same result as a windowed transformation numerically. As depicted below, both techniques yield the same result up to machine precision.





6 Problem 6

6.1 a)

Let y be the position of the particle undergoing a random walk. Each individual steps can be denoted as Δy. These steps are randomly selected from a Gaussian distribution centered at 0 ($\langle y \rangle = 0$).

The expected value of the position can be written as such:

$$\langle y^2 \rangle = \langle (\sum_i^x \Delta y_i)^2 \rangle \,. \tag{19}$$

Given that each individual steps Δy are independent, we can rewrite the sum like so:

$$\langle (\sum_i^x \Delta y_i)^2 \rangle = \sum_i^x \langle (\Delta y_i)^2 \rangle = x\sigma^2 \tag{20}$$

We can conclude that correlations grow linearly.

$$F_{PS}(k) = | \int_0^\infty \sigma^2 x e^{ikx} dx | = \frac{\sigma^2}{k^2} \tag{21}$$