Problem Set 3

October 8th, 2021
PHYS-512

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1. We need to cancel out 5th order terms. First, we can write out how exact solutions are composed of approximations and corrections.

$$y(x+h) = y_1 + (h)^5 \frac{y^{(5)}(x)}{5!} + O(h^6) \quad \text{for steps of lenght } h$$

$$y(x+h) = y_2 + 2(h/2)^5 \frac{y^{(5)}(x)}{5!} + O(h^6) \quad \text{for two half steps } h.$$

Let us define the truncation difference Δ.

$$\Delta \equiv y_2 - y_1$$

$$\Delta = y(x+h) - 2(h/2)^5 \frac{y^{(5)}(x)}{5!} + O_1(h^6) - y(x+h) + (h)^5 \frac{y^{(5)}(x)}{5!} - O_2(h^6)$$

$$\approx \left[-2(h/2)^5 + (h)^5\right] \frac{y^{(5)}(x)}{5!}$$

$$= \left(h^5 - \frac{h^5}{16}\right) \cdot \frac{1}{120} y^{(5)}(x) = \frac{15}{16} h^5 \cdot \frac{y^{(5)}(x)}{120}$$

$$\approx 15 \cdot \left(y(x+h) - y_2\right)$$

$$\therefore y(x+h) = y_2 - \frac{\Delta}{15} + O(h^6)$$

Using this equation will require 11 function evaluations per step instead of 4. However, given a certain number of function evaluation, it is still more accurate to use the 2 half-steps. (See figures)

2. a) I used the Radau method since it was way faster than Runge-Kutta. Runtime was important in this case since we had to keep track of 15 elements.

b) By looking at the Pb/U ratio plot, we can see that the ratio reaches 1 when we reach the half-life of U-238. This makes sense as there should be as much Uranium as Lead.

We can also see that the plot is somewhat exponential. This also makes sense given these relationships:

$$\frac{\text{Quantity of Pb210}}{\text{Quantity of U238}} = \frac{1 - e^{-\frac{\ln 2 \cdot t}{T}}}{e^{-\frac{\ln 2 \cdot t}{T}}} = \frac{1 - 2^{-t/T}}{2^{-t/T}} = 2^{t/T} - 1$$

With Th230 and U234, it is interesting to see that we reach an equilibrium for several million of years. This is due to the fact that the rate dN/dt is the same for both elements until no more Thorium is produced.

3.a) To make our equation linear, we can try to expand it.

$$z = a\left((x-x_0)^2 + (y-y_0)^2\right) + z_0$$

$$= a\left(x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2\right) + z_0$$

$$= a(x^2+y^2) \underbrace{-2ax_0}_{B}\cdot x \underbrace{-2ay_0}_{C}\cdot y + \underbrace{ax_0^2 + ay_0^2 + z_0}_{D}$$

$$= a(x^2+y^2) + Bx + Cy + D$$ This is linear

Note that $x_0 = \frac{-B}{2a}$, $y_0 = \frac{-C}{2a}$, $z_0 = D - \frac{B^2}{4a} - \frac{C^2}{4a}$

b) We obtain $a = 1{,}67 \times 10^{-4}$
$B = 4{,}54 \times 10^{-4}$
$C = -1{,}94 \times 10^{-2}$
$D = -1512$

This means that the old values are: $x_0 = -1{,}36$ mm
$y_0 = 58{,}22$ mm
$z_0 = -1512{,}88$ mm

c) Plotting the difference between the measured values of z versus the values computed from the best-fit parameters revealed a somewhat Gaussian distribution. This suggests that that the noise has no correlation. We can also take the standard deviation of the aforementioned difference as the error on z.

If we denote the error on z as σ_z, then we can write the noise matrix as $\sigma_z I$, where I is the identity matrix. From then on, we can find the error on our parameters by taking the square root of the diagonal elements of the matrix

$$Cov = A^T N^{-1} A$$

As such, we can see that $\sigma_a = 6{,}45 \times 10^{-8}$ mm

Since $f = \frac{1}{4a}$, we can propagate the uncertainty σ_a to obtain σ_f.

$$\frac{\sigma_f}{f} = \frac{\sigma_a}{a} \rightarrow \sigma_f = f \cdot \frac{\sigma_a}{a} = \frac{\sigma_a}{4a^2}$$

We can conclude that $f = 1499{,}7(6)$ mm, which is what was desired.