# Problem Set 5

Jules Faucher

McGill University Department of Physics

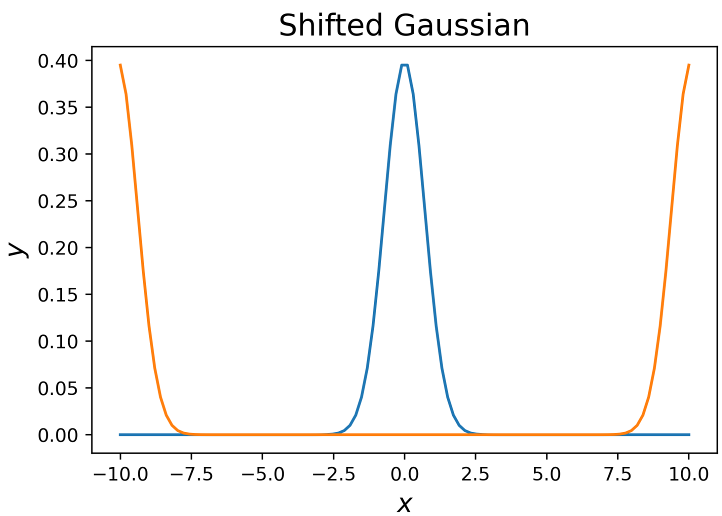
Instructor: Prof. Jonathan Sievers

October 30, 2021

# 1 Problem 1

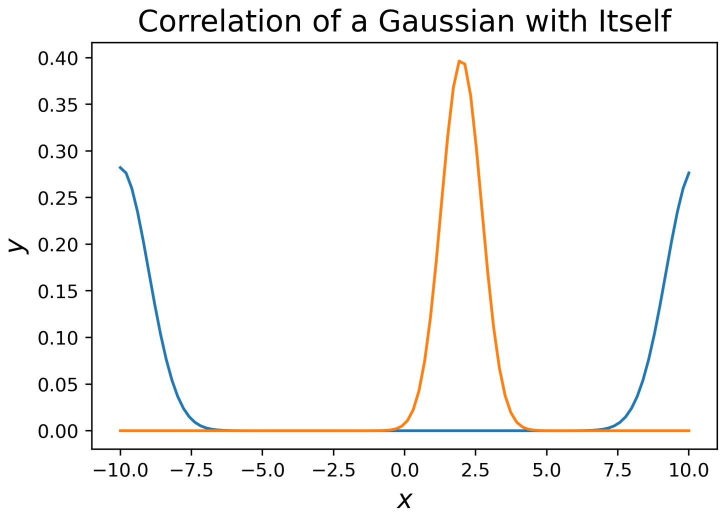
Any function can be shifted by doing applying a convolution with a Dirac delta function. We can show that using the sifting property of the Dirac delta function.

$$(f * \delta(a))(y) = \int_{-\infty}^{\infty} f(x)\delta(x - a - y)dx \tag{1}$$
$$= \int_{-\infty}^{\infty} f(x - y)\delta(x - a)dx \tag{2}$$
$$= f(y - a) \tag{3}$$



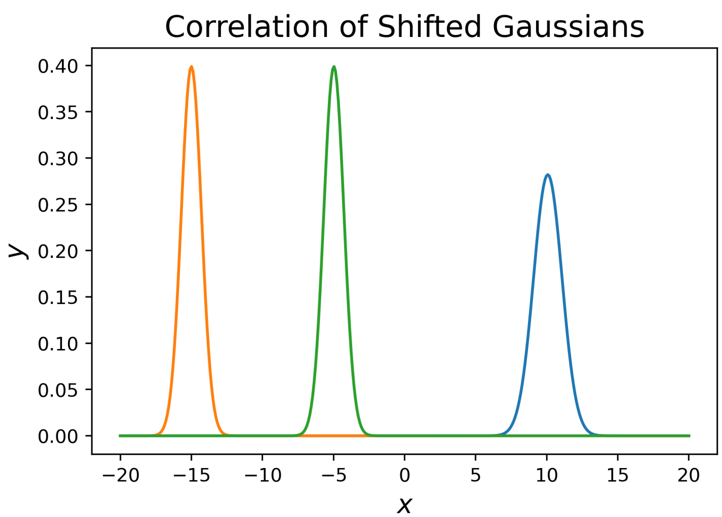

# 2 Problem 2

Here is the discrete convolution of a Gaussian with itself:



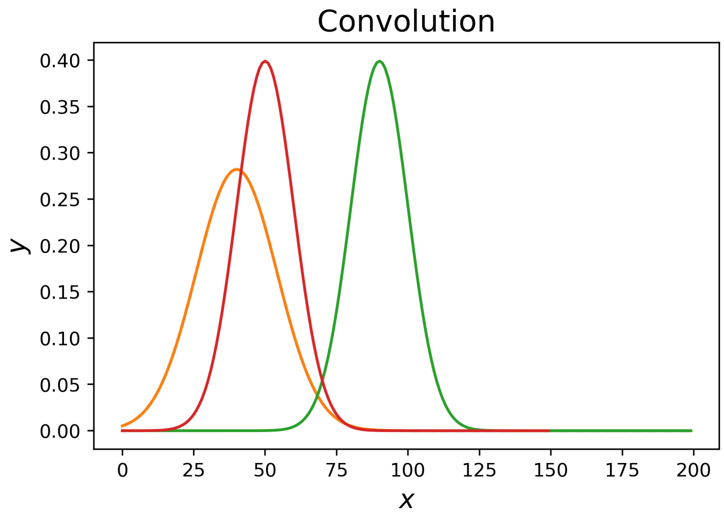

# 3 Problem 3

The closer both Gaussians are, the further away is the Gaussian generated from the correlation.




# 4 Problem 4

The resulting array as the same length as the one representing $f(x)$ since I return an array without the zeros that were generated because of the zeros that were first added to both arrays. I decided to do so to make plotting easier.




# 5 Problem 5

## 5.1 a)

Recall that for geometric series, we can use the following equation:

$$\sum_{i=1}^{n} a_i r^i = a_1 \frac{1 - r^n}{1 - r}. \tag{4}$$

With this in mind, we can see that

$$\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) = \sum_{x=0}^{N-1} (\exp(-2\pi i k/N))^x. \tag{5}$$

We can also rework the range of indices over which the sum is made. Let $y = x + 1$.

$$\sum_{x=0}^{N-1} (\exp(-2\pi i k/N))^x = 1 - 1 + \sum_{y=1}^{N} (\exp(-2\pi i k/N))^y \tag{6}$$

Finally, we can use the equation for geometric series.

$$\sum_{y=1}^{N} (\exp(-2\pi i k/N))^y = \frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)} \tag{7}$$

### 5.2 b)

We may take the limit of the expression above as k approaches 0.

$$\lim_{k \to 0} \frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)} \stackrel{H}{=} \lim_{k \to 0} \frac{(2\pi i) \exp(-2\pi i k)}{(2\pi i/N) \exp(-2\pi i k/N)} \tag{8}$$

$$= \lim_{k \to 0} N \frac{\exp(-2\pi i k)}{\exp(-2\pi i k/N)} \tag{9}$$

$$= N \tag{10}$$

We can expand the expression to reveal more about the behaviour of this equation.

$$\frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)} = \frac{1 - \cos(2\pi k) + i \sin(2\pi k)}{1 - \cos(2\pi k/N) + i \sin(2\pi k/N)} \tag{11}$$

Since $k$ is an integer, we can get rid of some trig terms.

$$\frac{1 - \cos(2\pi k) + i \sin(2\pi k)}{1 - \cos(2\pi k/N) + i \sin(2\pi k/N)} = \frac{0}{1 - \cos(2\pi k/N) + i \sin(2\pi k/N)} \tag{12}$$

Here, we can see that any integer $k$ will lead the numerator of this expression to go to 0. Meanwhile, in the denominator, if $k$ is not a multiple of $N$, $k/N$ is not an integer. Hence, the denominator will not reach 0. Therefore, the fraction as a whole will equal 0.

## 5.3 c)

$$F(k') = 4\sum_{x=0}^{N-1} \sin(2\pi kx/N) e^{-i2\pi k'x/N} \tag{13}$$

$$= \sum_{x=0}^{N-1} \frac{e^{-i2\pi kx/N} - e^{i2\pi kx/N}}{2i} e^{-i2\pi k'x/N} \tag{14}$$

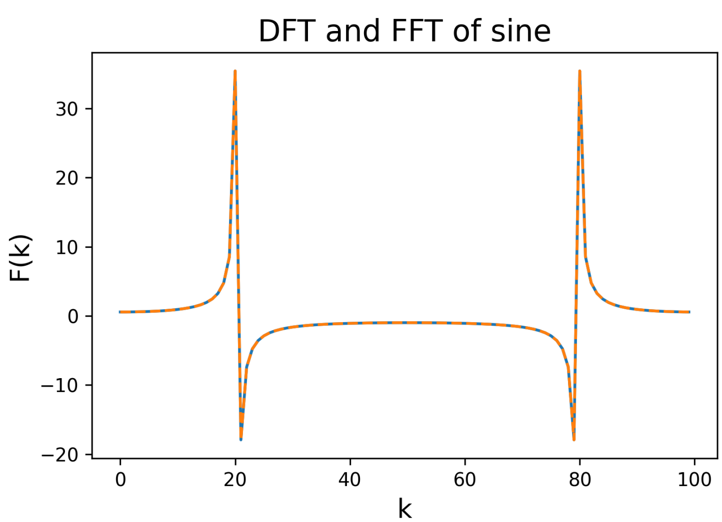
$$= \frac{1}{2i} \sum_{x=0}^{N-1} e^{2\pi i(k'-k)x} - e^{2\pi i(k'+k)x} \tag{15}$$

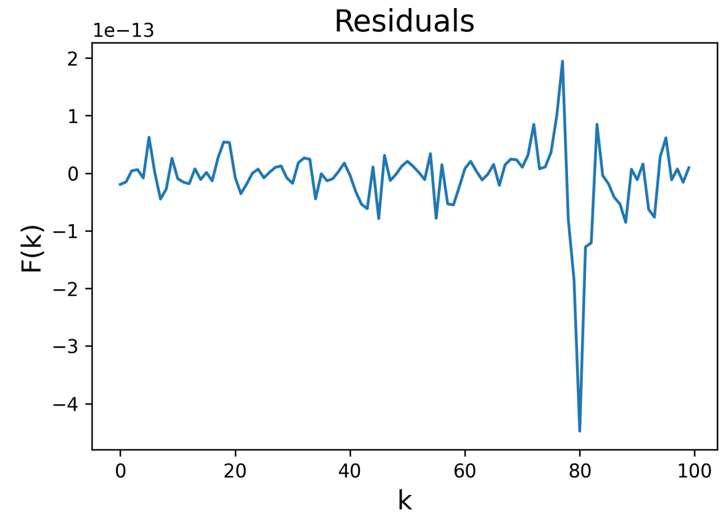
$$= \frac{1}{2i} \sum_{x=0}^{N-1} e^{2\pi i(k'-k)x} - \frac{1}{2i} \sum_{x=0}^{N-1} e^{2\pi i(k'+k)x} \tag{16}$$

$$= \frac{1}{2i} \left( \frac{1 - e^{-2\pi i(k'-k)}}{1 - e^{-2\pi i(k'-k)/N}} - \frac{1 - e^{-2\pi i(k'+k)}}{1 - e^{-2\pi i(k'+k)/N}} \right) \tag{17}$$

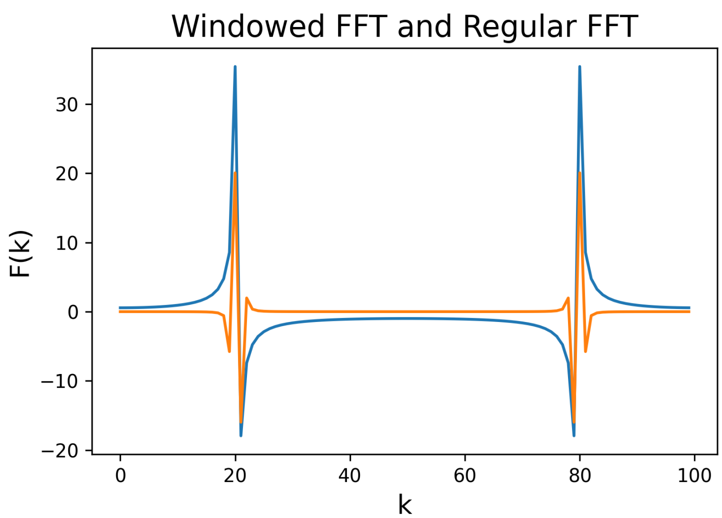
$$\tag{18}$$

If $k$ and $k'$ are integers, this expression will equal 0 whenever $k \neq k'$ or $k \neq -k'$, and $N$ when $k = k'$ or $k = -k'$. This means we should have something that looks like two Dirac Delta functions. Using non-integer values for $k$ will make the Diracs imperfect.

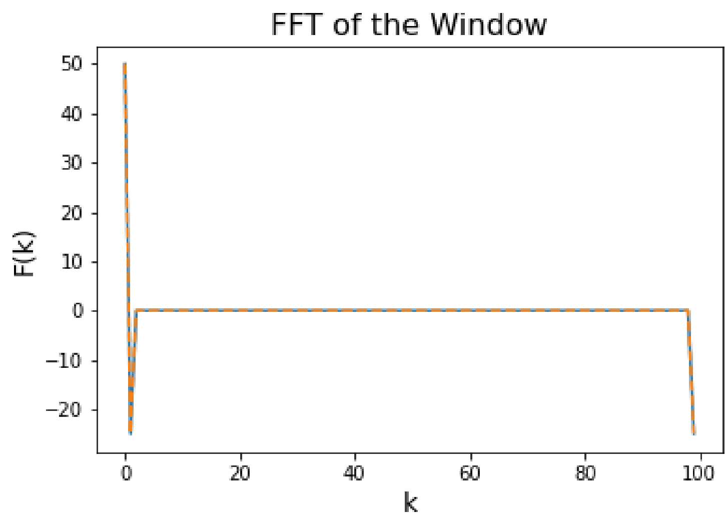


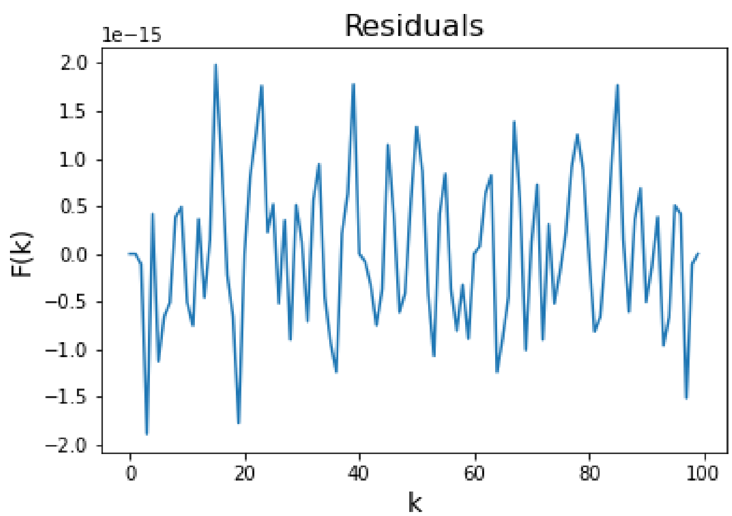

Residuals
1e−13
2
1
0
−1
−2
−3
−4
F(k)
0
20
40
60
80
100
k

## 5.4 d)


Windowed FFT and Regular FFT
30
20
10
0
−10
−20
F(k)
0
20
40
60
80
100
k

### 5.5 e)




# 6 Problem 6

### 6.1 a)

Let $y$ be the position of the particle undergoing a random walk. Each individual steps can be denoted as $\Delta y$. These steps are randomly selected from a Gaussian distribution centered at 0 ($\langle y \rangle = 0$).

The expected value of the position can be written as such:

$$\langle y^2 \rangle = \langle (\sum_i^x \Delta y_i)^2 \rangle \,. \tag{19}$$

Given that each individual steps $\Delta y$ are independent, we can rewrite the sum like so:

$$\langle (\sum_i^x \Delta y_i)^2 \rangle = \sum_i^x \langle (\Delta y_i)^2 \rangle = x\sigma^2 \tag{20}$$

We can conclude that correlations grow linearly.

$$F_{PS}(k) = | \int_0^\infty \sigma^2 x e^{ikx} dx | = \frac{\sigma^2}{k^2} \tag{21}$$