

# **Echoes**

**Extended Calculator of HOMogEnization Schemes**

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# Welcome



The library `echoes` allows to implement various mean-field homogenization schemes of random media involving different types of heterogeneities in the framework of elasticity, conductivity, viscoelasticity as well as nonlinear homogenization.

This book gathers tutorials presenting the main features of the library:

- elements of tensor calculus,
- Hill and Eshelby tensors and their derivatives with respect to reference medium moduli,
- concentration problems,
- RVEs and schemes in linear homogenization,
- extension to nonlinear homogenization,
- extension to linear time-dependent behaviors.

## Download

The core of `echoes` has been developed in C++ and wrapped by a Python interface. Hence its use requires first the installation of a Python environment including `pip` executable (for instance [Anaconda](#)).

Wheel packages can be downloaded for various versions of Python under Windows or Linux by choosing the appropriate file for your configuration under the link

<https://doi.org/10.5281/zenodo.10559657>

Once in possession of the relevant `.whl` file, the package can be installed in a console (`Anaconda` console or any console allowing to run `pip`) by

```
pip install -U echoes-XYZ.whl
# replacing echoes-XYZ.whl by the correct path to the whl file
```

## Citation

If you use echoes, please cite it as ([Barthélémy, 2022](#)) or in bibtex style

```
@software{echoes,  
  title = {Echoes: {{Extended Calculator}} of {{HOMogEnization Schemes}}},  
  shorttitle = {Echoes},  
  author = {Barthélémy, Jean-François},  
  date = {2022-11-22},  
  doi = {10.5281/ZENODO.10559657},  
  url = {https://zenodo.org/record/10559657},  
  organization = {Zenodo},  
  version = {v1.0.0},  
}
```

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## About the author

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# Introduction

This book is aimed at researchers, engineers and students, knowing the fundamentals of mean-field theory to help them learn how to use the echoes library with some brief theoretical recalls when relevant. For a more exhaustive presentation of the theory of random medium homogenization, see ([Bornert et al., 2001](#)), ([Milton, 2002](#)), ([Torquato, 2002](#)) or ([Kachanov and Sevostianov, 2018](#)) among others.

The objectives of the library can be summarized as follows:

- simple and quick implementation of Eshelby problems and homogenization schemes,
- multi-physics and multi-scale homogenization,
- effects of microstructure changes by chemical, physical or mechanical process.

## Features

- Eshelby problem solved at 2nd (conductivity) et 4th orders (elasticity)
- Isotropy and anisotropy
- Several types of inclusions including generic (user-defined) inclusion
- Large variety of schemes
- Derivatives of the macroscopic elasticity with respect to lower scale moduli
- Aging linear viscoelasticity
- Complex moduli

In this manual, some snippets of Python codes are presented. The echoes library can be imported as

```
from echoes import *
```

or, to avoid any ambiguity between libraries, as

```
import echoes as ec
```

A usual start of any tutorial could be the following

```
import numpy as np
from echoes import *
import matplotlib.pyplot as plt # if plots are needed

np.set_printoptions(precision=8, suppress=True)
# to display only 8 significant digits of array components
```

Whenever they are omitted, it is implicitly considered that these lines have previously been added.

## **Part I**

# **Elements of tensor calculus**

# 1 Kelvin-Mandel notation

## Objectives

Before introducing the specific objects of echoes devoted to tensor calculations in isotropic or anisotropic contexts, this tutorial aims at providing the syntax allowing to represent second-order or fourth-order tensors under the form of matrices in the Kelvin-Mandel notation as detailed in Section A.2.

A symmetric  $3 \times 3$  second-order matrix can be transformed in a vector of  $\mathbb{R}^6$  by the function `KM` consistently with A.14. The inverse is done by `invKM`.

```
import numpy as np
from echoes import *
import math
import random

np.set_printoptions(precision=8, suppress=True)
# to display only 8 significant digits of array components

α = np.random.rand(3, 3) ; ε = (α+α.T)/2
print("ε =\n",ε)
print("KM(ε) =\n",KM(ε))
assert np.allclose(invKM(KM(ε)), ε), "error"
```

```
ε =
[[0.87003209 0.66893188 0.30024916]
 [0.66893188 0.47709377 0.18807881]
 [0.30024916 0.18807881 0.33936939]]
KM(ε) =
[0.87003209 0.47709377 0.33936939 0.26598361 0.42461644 0.94601253]
```

Given a  $3 \times 3 \times 3 \times 3$  array `c` (of type `numpy.ndarray`) satisfying major and minor symmetries (see Section A.2), the corresponding  $6 \times 6$  matrix `C` obtained by Kelvin-Mandel transform is calculated by `C = KM(c)`. Conversely, if `C` is a positive definite matrix, `c` is calculated by `c = invKM(C)`.

```
A = np.random.rand(6,6)
C = A.T.dot(A) + np.eye(6) # generation of an arbitrary positive definite matrix
c = invKM(C)
```



```

print("C =\n",C)
print("c =\n",c)
assert np.allclose(KM(c), C), "error: KM(c) should be equal to C"

```

```

C =
[[3.74866352 2.11390671 1.97211173 1.9766162 2.54645325 1.61400264]
 [2.11390671 3.4320992 1.34798882 1.28679106 1.79307082 1.94895435]
 [1.97211173 1.34798882 2.91277253 1.39248291 1.95691486 1.34954307]
 [1.9766162 1.28679106 1.39248291 2.91018205 1.99972402 0.81517301]
 [2.54645325 1.79307082 1.95691486 1.99972402 3.57327274 1.53698396]
 [1.61400264 1.94895435 1.34954307 0.81517301 1.53698396 3.12105742]]

c =
[[[3.74866352 1.14127221 1.80061436]
 [1.14127221 2.11390671 1.39767872]
 [1.80061436 1.39767872 1.97211173]]

 [[1.14127221 1.56052871 0.76849198]
 [1.56052871 1.37811884 0.40758651]
 [0.76849198 0.40758651 0.95427105]]

 [[1.80061436 0.76849198 1.78663637]
 [0.76849198 1.26789254 0.99986201]
 [1.78663637 0.99986201 1.38374776]]]

[[[1.14127221 1.56052871 0.76849198]
 [1.56052871 1.37811884 0.40758651]
 [0.76849198 0.40758651 0.95427105]]

 [[2.11390671 1.37811884 1.26789254]
 [1.37811884 3.4320992 0.90989868]
 [1.26789254 0.90989868 1.34798882]]

 [[1.39767872 0.40758651 0.99986201]
 [0.40758651 0.90989868 1.45509103]
 [0.99986201 1.45509103 0.98463411]]]

[[[1.80061436 0.76849198 1.78663637]
 [0.76849198 1.26789254 0.99986201]
 [1.78663637 0.99986201 1.38374776]]

 [[1.39767872 0.40758651 0.99986201]
 [0.40758651 0.90989868 1.45509103]
 [0.99986201 1.45509103 0.98463411]]]

```

```
[[1.97211173 0.95427105 1.38374776]  
[0.95427105 1.34798882 0.98463411]  
[1.38374776 0.98463411 2.91277253]]]
```

## 2 Rotation matrices

### Objectives

This tutorial presents the construction of rotation matrices in  $\mathbb{R}^3$  in the convention proposed in Section A.3 for Euler angles.

The rotation matrix (A.19) can be built with `rot3( $\theta$ ,  $\phi$ ,  $\psi$ )`. See Section A.3 and more particularly Fig. A.1 for more details about Euler angles  $\theta$ ,  $\phi$ ,  $\psi$ .

```
import numpy as np
from echoes import *
import math

np.set_printoptions(precision=8, suppress=True)
# to display only 8 significant digits of array components

 $\pi$  = math.pi
 $\theta$ ,  $\phi$ ,  $\psi$  =  $\pi/3$ ,  $\pi/4$ ,  $\pi/5$ 
R = rot3( $\theta$ ,  $\phi$ ,  $\psi$ )
print(R)
```

```
[[-0.12959624 -0.77987487  0.61237244]
 [ 0.70165764  0.36424793  0.61237244]
 [-0.70062927  0.50903696  0.5         ]]
```

The vectors of the spherical basis correspond to the column of the rotation matrix for  $\psi = 0$ . They can individually be obtained by the function `es( $i$ ,  $\theta$ ,  $\phi$ )`

### Warning

Python numbering starts at 0 so  $i$  takes the values 0, 1 and 2.  
Besides vectors of the spherical basis are ordered as  $\underline{e}_\theta$ ,  $\underline{e}_\phi$ ,  $\underline{e}_r$ .

```
 $\theta$ ,  $\phi$  =  $\pi/8$ ,  $\pi/5$ 
R = rot3( $\theta$ ,  $\phi$ )
print("R =\n",R)
for i in range(3): print(f"es({i},  $\theta$ ,  $\phi$ ) = ", es(i,  $\theta$ ,  $\phi$ ), f"  $\rightarrow \underline{e}$  {[ $\theta$ ',' $\phi$ ','r']}[i]}")
```

```

R =
[[ 0.74743424 -0.58778525  0.3095974 ]
 [ 0.54304276  0.80901699  0.22493568]
 [-0.38268343  0.          0.92387953]]
es(0,  $\theta$ ,  $\phi$ ) = [ 0.74743424  0.54304276 -0.38268343]  $\rightarrow e_\theta$ 
es(1,  $\theta$ ,  $\phi$ ) = [-0.58778525  0.80901699  0.          ]  $\rightarrow e_\phi$ 
es(2,  $\theta$ ,  $\phi$ ) = [0.3095974  0.22493568  0.92387953]  $\rightarrow e_r$ 

```

As shown in Section A.3 the rotation matrix applying on fourth-order tensors in Kelvin-Mandel notation can be deduced from the  $3 \times 3$  rotation matrix from A.24. It can be directly calculated by `rot6( $\theta$ ,  $\phi$ ,  $\psi$ )`.

```

 $\theta$ ,  $\phi$ ,  $\psi$  =  $\pi/3$ ,  $\pi/4$ ,  $\pi/5$ 
R = rot3( $\theta$ ,  $\phi$ ,  $\psi$ )
sboxtimes=lambda a,b:0.5*(np.einsum('ik,jl',a,b)+np.einsum('il,jk',a,b))
print("R⊗R =\n",KM(sboxtimes(R,R)))

R6 = rot6( $\theta$ ,  $\phi$ ,  $\psi$ )
print("R6 =\n",R6)

```

```

R⊗R =
[[ 0.01679518  0.60820482  0.375      -0.67539145 -0.11223363  0.14293294]
 [ 0.49232344  0.13267656  0.375      0.31544796  0.60765334  0.36144095]
 [ 0.49088137  0.25911863  0.25       0.35994349 -0.49541971 -0.50437388]
 [-0.69523004  0.26221734  0.4330127   0.49384417 -0.07821723  0.10196691]
 [ 0.12840906 -0.56142176  0.4330127  -0.07821723 -0.49384417  0.48043389]
 [-0.12859754 -0.40173255  0.53033009 -0.25451848  0.35031463 -0.59441032]]
R6 =
[[ 0.01679518  0.60820482  0.375      -0.67539145 -0.11223363  0.14293294]
 [ 0.49232344  0.13267656  0.375      0.31544796  0.60765334  0.36144095]
 [ 0.49088137  0.25911863  0.25       0.35994349 -0.49541971 -0.50437388]
 [-0.69523004  0.26221734  0.4330127   0.49384417 -0.07821723  0.10196691]
 [ 0.12840906 -0.56142176  0.4330127  -0.07821723 -0.49384417  0.48043389]
 [-0.12859754 -0.40173255  0.53033009 -0.25451848  0.35031463 -0.59441032]]

```

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# A Tensor algebra

## A.1 Conventions of tensor algebra

This appendix presents some conventions regarding tensor algebra in the usual three-dimensional euclidean space  $E = \mathbb{R}^3$ . In the sequel, tensor components are associated to an orthonormal frame  $(\underline{e}_i)_{i=1,2,3}$  so that introducing the notion of tensor variance is useless here. The following presentation relies on the prior knowledge of the definition of tensors as multilinear operators and the classical isomorphism between the euclidean space and its dual through the scalar product

$$\begin{aligned} \phi : E &\longrightarrow E^* \\ \underline{v} &\longmapsto \underline{v} \cdot \bullet \end{aligned} \quad (\text{A.1})$$

which allows to identify vectors and linear forms.

Consider two tensors  $\mathcal{T}$  and  $\mathcal{T}'$  of respective orders  $p$  and  $q$ . The tensor product  $\mathcal{T} \otimes \mathcal{T}'$  is the  $(p + q)$  order tensor decomposed as

$$\mathcal{T} \otimes \mathcal{T}' = \mathcal{T}_{i_1, \dots, i_p} \mathcal{T}'_{i_{p+1}, \dots, i_{p+q}} \underline{e}_{i_1} \otimes \dots \otimes \underline{e}_{i_{p+q}} \quad (\text{A.2})$$

where Einstein convention of implicit summation over repeated indices is adopted and  $\underline{e}_{i_1} \otimes \dots \otimes \underline{e}_{i_{p+q}}$  is the multilinear form such that<sup>1</sup>

$$(\underline{e}_{i_1} \otimes \dots \otimes \underline{e}_{i_{p+q}})(\underline{e}_{j_1}, \dots, \underline{e}_{j_{p+q}}) = \delta_{i_1, j_1} \dots \delta_{i_{p+q}, j_{p+q}} \quad (\text{A.3})$$

The notation  $\mathcal{T} \overset{s}{\otimes} \mathcal{T}'$  indicates a tensor product followed by a symmetrization over the last index of  $\mathcal{T}$  and the first of  $\mathcal{T}'$ , i.e.

$$\mathcal{T} \overset{s}{\otimes} \mathcal{T}' = \frac{\mathcal{T}_{i_1, \dots, i_p} \mathcal{T}'_{i_{p+1}, \dots, i_{p+q}} + \mathcal{T}_{i_1, \dots, i_{p+1}} \mathcal{T}'_{i_p, \dots, i_{p+q}}}{2} \underline{e}_{i_1} \otimes \dots \otimes \underline{e}_{i_{p+q}} \quad (\text{A.4})$$

It follows that

$$\underline{u} \overset{s}{\otimes} \underline{v} = \frac{\underline{u} \otimes \underline{v} + \underline{v} \otimes \underline{u}}{2} \quad (\text{A.5})$$

and an example of generalization involving a second-order tensor  $\mathbf{a}$  and vectors  $\underline{u}$  and  $\underline{v}$

$$\underline{u} \overset{s}{\otimes} \mathbf{a} \overset{s}{\otimes} \underline{v} = \frac{u_i a_{jk} v_l + u_i a_{jl} v_k + u_j a_{ik} v_l + u_j a_{il} v_k}{4} \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l \quad (\text{A.6})$$

The simple dot product or contracted product between  $\mathcal{T}$  and  $\mathcal{T}'$  involves by convention a contraction between the last index of  $\mathcal{T}$  and the first of  $\mathcal{T}'$ , which leads to the  $(p + q - 2)$  order tensor

$$\mathcal{T} \cdot \mathcal{T}' = \mathcal{T}_{i_1, \dots, i_{p-1}, \textcolor{red}{k}} \mathcal{T}'_{\textcolor{red}{k}, i_p, \dots, i_{p+q-2}} \underline{e}_{i_1} \otimes \dots \otimes \underline{e}_{i_{p+q-2}} \quad (\text{A.7})$$

---

<sup>1</sup> $\delta_{ij} = 1$  if  $i = j$  and 0 if  $i \neq j$  (Kronecker symbol)

As regards the double dot product, the classical convention consists in consuming the indices going up from the extremities, which means that a first contraction acts as in the simple dot product then a second contraction is performed between the penultimate index of  $\mathcal{T}$  and the second one of  $\mathcal{T}'$ . However an alternate convention adopted here is proposed in (Brisard, 2014)<sup>2</sup>, which somehow consists in considering that the double contraction operates over the two last indices of  $\mathcal{T}$  as a pair and the two first indices of  $\mathcal{T}'$  as the corresponding pair. In other words, this operation is such that if  $\mathbf{a}$  and  $\mathbf{b}$  are two second-order tensors and  $\mathbb{T}$  is a fourth-order tensor

$$\mathbf{a} : \mathbf{b} = a_{ij} b_{ij} \quad \text{and} \quad \mathbb{T} : \mathbf{a} = T_{ijkl} a_{kl} \underline{e}_i \otimes \underline{e}_j \quad (\text{A.8})$$

and the transpose tensor  ${}^t\mathbb{T}$  is consistently defined by

$${}^t\mathbb{T} : \mathbf{a} = \mathbf{a} : \mathbb{T} \quad \Leftrightarrow \quad ({}^t\mathbb{T})_{ijkl} = (\mathbb{T})_{klij} \quad (\text{A.9})$$

The quadruple dot product is introduced as a scalar product between fourth-order tensors as

$$\mathbb{T} : \mathbb{T}' = T_{ijkl} T'_{ijkl} \quad (\text{A.10})$$

Another useful operator introduced in (Brisard, 2014)<sup>3</sup> is the modified tensor product denoted by  $\boxtimes$ . The fourth-order tensor  $\mathbf{a} \boxtimes \mathbf{b}$  (where  $\mathbf{a}$  and  $\mathbf{b}$  are two second-order tensors) is defined by its operation over any second-order tensor  $\mathbf{p}$  and by its components

$$\begin{aligned} (\mathbf{a} \boxtimes \mathbf{b}) : \mathbf{p} &= \mathbf{a} \cdot \mathbf{p} \cdot {}^t\mathbf{b} = a_{ik} p_{kl} b_{jl} \underline{e}_i \otimes \underline{e}_j \\ (\mathbf{a} \boxtimes \mathbf{b})_{ijkl} &= a_{ik} b_{jl} \end{aligned} \quad (\text{A.11})$$

A symmetrized version of  $\boxtimes$  denoted by  $\overset{s}{\boxtimes}$  can also be introduced. It operates as

$$\begin{aligned} (\mathbf{a} \overset{s}{\boxtimes} \mathbf{b}) : \mathbf{p} &= (\mathbf{a} \boxtimes \mathbf{b}) : \left( \frac{\mathbf{p} + {}^t\mathbf{p}}{2} \right) = \mathbf{a} \cdot \left( \frac{\mathbf{p} + {}^t\mathbf{p}}{2} \right) \cdot {}^t\mathbf{b} \\ (\mathbf{a} \overset{s}{\boxtimes} \mathbf{b})_{ijkl} &= \frac{a_{ik} b_{jl} + a_{il} b_{jk}}{2} \end{aligned} \quad (\text{A.12})$$

It follows from these definitions that the fourth-order identity, as an operator over second-order tensors, writes  $\mathbb{1} = \mathbf{1} \boxtimes \mathbf{1}$  where  $\mathbf{1}$  is the second-order identity. The fourth-order operator allowing to extract the symmetric part of a second-order tensor writes  $\mathbb{1} = \mathbf{1} \overset{s}{\boxtimes} \mathbf{1}$ . The latter tensor, which obviously complies with the conditions of minor symmetries, is classically used to play the role of fourth-order identity operating over symmetric second-order tensors.

<sup>2</sup>see [https://sbrisard.github.io/posts/20140219-on\\_the\\_double\\_dot\\_product.html](https://sbrisard.github.io/posts/20140219-on_the_double_dot_product.html)

<sup>3</sup>see [https://sbrisard.github.io/posts/20140226-decomposition\\_of\\_transverse\\_isotropic\\_fourth-rank\\_tensors.html](https://sbrisard.github.io/posts/20140226-decomposition_of_transverse_isotropic_fourth-rank_tensors.html)



Some remarkable relationships result from the previous definitions

$$\begin{aligned}
(\mathbf{a} \boxtimes \mathbf{b}) : (\mathbf{u} \otimes \mathbf{v}) &= (\mathbf{a} \cdot \mathbf{u}) \otimes (\mathbf{b} \cdot \mathbf{v}) & (a) \\
(\mathbf{a} \boxtimes \mathbf{b}) : (\mathbf{c} \boxtimes \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c}) \boxtimes (\mathbf{b} \cdot \mathbf{d}) & (b) \\
(\mathbf{a} \overset{s}{\boxtimes} \mathbf{b}) : (\mathbf{c} \overset{s}{\boxtimes} \mathbf{d}) &= \frac{(\mathbf{a} \cdot \mathbf{c}) \overset{s}{\boxtimes} (\mathbf{b} \cdot \mathbf{d}) + (\mathbf{a} \cdot \mathbf{d}) \overset{s}{\boxtimes} (\mathbf{b} \cdot \mathbf{c})}{2} & (c) \\
(\mathbf{a} \overset{s}{\boxtimes} \mathbf{a}) : (\mathbf{b} \overset{s}{\boxtimes} \mathbf{b}) &= (\mathbf{a} \cdot \mathbf{b}) \overset{s}{\boxtimes} (\mathbf{a} \cdot \mathbf{b}) & (d) \\
{}^t(\mathbf{a} \boxtimes \mathbf{b}) &= {}^t\mathbf{a} \boxtimes {}^t\mathbf{b} & (e) \\
{}^t(\mathbf{a} \overset{s}{\boxtimes} \mathbf{a}) &= {}^t\mathbf{a} \overset{s}{\boxtimes} {}^t\mathbf{a} \text{ but } {}^t(\mathbf{a} \overset{s}{\boxtimes} \mathbf{b}) \neq {}^t\mathbf{a} \overset{s}{\boxtimes} {}^t\mathbf{b} & (f) \\
(\mathbf{a} \boxtimes \mathbf{b})^{-1} &= \mathbf{a}^{-1} \boxtimes \mathbf{b}^{-1} & (g) \\
(\mathbf{a} \overset{s}{\boxtimes} \mathbf{a})^{-1} : \mathbf{p} &= (\mathbf{a}^{-1} \overset{s}{\boxtimes} \mathbf{a}^{-1}) : \mathbf{p} \text{ if } {}^t\mathbf{p} = \mathbf{p} \text{ but } (\mathbf{a} \overset{s}{\boxtimes} \mathbf{b})^{-1} \neq \mathbf{a}^{-1} \overset{s}{\boxtimes} \mathbf{b}^{-1} & (h)
\end{aligned} \tag{A.13}$$

## A.2 Kelvin-Mandel notation

The Kelvin-Mandel notation allows to write the matrix of a symmetric second-order tensor in a given orthonormal frame  $(\underline{e}_i)_{i=1,2,3}$  under the form of a vector of  $\mathbb{R}^6$

$$\text{Mat}(\varepsilon, (\underline{e}_i)) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{31} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2} \varepsilon_{23} \\ \sqrt{2} \varepsilon_{31} \\ \sqrt{2} \varepsilon_{12} \end{pmatrix} \tag{A.14}$$

The vector of  $\mathbb{R}^6$  in (A.14) corresponds to the components of the second-order tensor  $\varepsilon$  in the basis ordered as

$$\mathcal{B} = (\underline{e}_1 \otimes \underline{e}_1, \underline{e}_2 \otimes \underline{e}_2, \underline{e}_3 \otimes \underline{e}_3, \sqrt{2} \underline{e}_2 \otimes \underline{e}_3, \sqrt{2} \underline{e}_3 \otimes \underline{e}_1, \sqrt{2} \underline{e}_1 \otimes \underline{e}_2) \tag{A.15}$$

The tensors of the basis (A.15) form an orthonormal frame spanning the space of symmetric second-order tensors equipped with the double contraction “:” as scalar product. It follows that the double contraction between symmetric second-order tensors is no other than the classical scalar product of the corresponding vectors of  $\mathbb{R}^6$  written according to the convention (A.14).

### **i** Note

Note the effect of  $\sqrt{2}$  on the off-diagonal components. Unlike [Voigt convention](#), the choice of Kelvin-Mandel convention throughout all the library allows to address strain and stress tensors under the same synthetic representation.

A fourth-order tensor with minor symmetries ( $C_{jikl} = C_{ijlk} = C_{ijkl}$ ), which can be seen as a linear operator acting over symmetric second-order tensors by double contraction, writes in the same convention under the form of a  $6 \times 6$  square matrix (the solid lines separate blocks

affected by different factors whereas the colored components highlight a central block playing a major role in the sequel)

$$\text{Mat}(\mathbb{C}, \mathcal{B}) = \left( \begin{array}{ccc|ccc} C_{1111} & C_{1122} & C_{1133} & \sqrt{2} C_{1123} & \sqrt{2} C_{1131} & \sqrt{2} C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & \sqrt{2} C_{2223} & \sqrt{2} C_{2231} & \sqrt{2} C_{2212} \\ C_{3311} & C_{3322} & \color{red}{C_{3333}} & \color{red}{\sqrt{2} C_{3323}} & \color{red}{\sqrt{2} C_{3331}} & \sqrt{2} C_{3312} \\ \hline \sqrt{2} C_{2311} & \sqrt{2} C_{2322} & \color{red}{\sqrt{2} C_{2333}} & \color{red}{2 C_{2323}} & \color{red}{2 C_{2331}} & 2 C_{2312} \\ \sqrt{2} C_{3111} & \sqrt{2} C_{3122} & \color{red}{\sqrt{2} C_{3133}} & \color{red}{2 C_{3123}} & \color{red}{2 C_{3131}} & 2 C_{3112} \\ \sqrt{2} C_{1211} & \sqrt{2} C_{1222} & \sqrt{2} C_{1233} & 2 C_{1223} & 2 C_{1231} & 2 C_{1212} \end{array} \right) \quad (\text{A.16})$$

The result of  $\mathbb{C} : \varepsilon$  writes as a classical matrix-vector product of (A.16) by (A.14).

However another way of ordering the tensors of (A.15) which proves useful for the calculation of crack compliance is based on a gathering of one set of three in-plane and another one of three out-of-plane tensors (the latter involving  $\underline{n} = \underline{e}_3$  assumed to be the normal of the crack and the former not)

$$\mathcal{B}^* = \left( \underbrace{\underline{e}_1 \otimes \underline{e}_1, \underline{e}_2 \otimes \underline{e}_2, \sqrt{2} \underline{e}_1 \overset{s}{\otimes} \underline{e}_2}_{\text{in-plane}}, \underbrace{\underline{e}_3 \otimes \underline{e}_3, \sqrt{2} \underline{e}_2 \overset{s}{\otimes} \underline{e}_3, \sqrt{2} \underline{e}_3 \overset{s}{\otimes} \underline{e}_1}_{\text{out-of-plane}} \right) \quad (\text{A.17})$$

such that the matrix of  $\mathbb{C}$  in  $\mathcal{B}^*$  is now obtained by permutations of lines and columns of (A.16) to give

$$\text{Mat}(\mathbb{C}, \mathcal{B}^*) = \left( \begin{array}{ccc|ccc} C_{1111} & C_{1122} & \sqrt{2} C_{1112} & C_{1133} & \sqrt{2} C_{1123} & \sqrt{2} C_{1131} \\ C_{2211} & C_{2222} & \sqrt{2} C_{2212} & C_{2233} & \sqrt{2} C_{2223} & \sqrt{2} C_{2231} \\ \sqrt{2} C_{1211} & \sqrt{2} C_{1222} & 2 C_{1212} & \sqrt{2} C_{1233} & 2 C_{1223} & 2 C_{1231} \\ \hline C_{3311} & C_{3322} & \sqrt{2} C_{3312} & C_{3333} & \sqrt{2} C_{3323} & \sqrt{2} C_{3331} \\ \sqrt{2} C_{2311} & \sqrt{2} C_{2322} & 2 C_{2312} & \sqrt{2} C_{2333} & 2 C_{2323} & 2 C_{2331} \\ \sqrt{2} C_{3111} & \sqrt{2} C_{3122} & 2 C_{3112} & \sqrt{2} C_{3133} & 2 C_{3123} & 2 C_{3131} \end{array} \right) \quad (\text{A.18})$$

One may notice that the bottom right  $3 \times 3$  block of (A.18) exactly corresponds to the colored block in (A.16).

### A.3 Rotation of tensors

Recalling that the set of second-order tensors is isomorphic to the set of endomorphism in an euclidean space, it is natural to define a rotation tensor as an element of the special orthogonal group, i.e. tensors  $\mathbf{R}$  such that  ${}^t\mathbf{R} \cdot \mathbf{R} = 1$  and  $\det \mathbf{R} = 1$ . In  $\mathbb{R}^3$  equipped with an orthonormal frame  $(\underline{e}_i)_{i=1,2,3}$ , these tensors can be defined by three Euler angles  $(\theta, \phi, \psi) \in [0, \pi] \times [0, 2\pi] \times [0, 2\pi]$  (see Fig. A.1) such that<sup>4</sup>

$$\text{Mat}(\mathbf{R}, (\underline{e}_i)) = \begin{pmatrix} c_\theta c_\psi c_\phi - s_\psi s_\phi & -c_\theta c_\phi s_\psi - c_\psi s_\phi & c_\phi s_\theta \\ c_\theta c_\psi s_\phi + c_\phi s_\psi & -c_\theta s_\psi s_\phi + c_\psi c_\phi & s_\theta s_\phi \\ -c_\psi s_\theta & s_\theta s_\psi & c_\theta \end{pmatrix} \quad (\text{A.19})$$

A rotated frame  $(\underline{e}'_i)_{i=1,2,3}$  is obtained by application of the rotation  $\mathbf{R}$  on the vectors  $\underline{e}_i$  (see Fig. A.1)

$$\forall i \in \{1, 2, 3\} \quad \underline{e}'_i = \mathbf{R} \cdot \underline{e}_i \quad (\text{A.20})$$

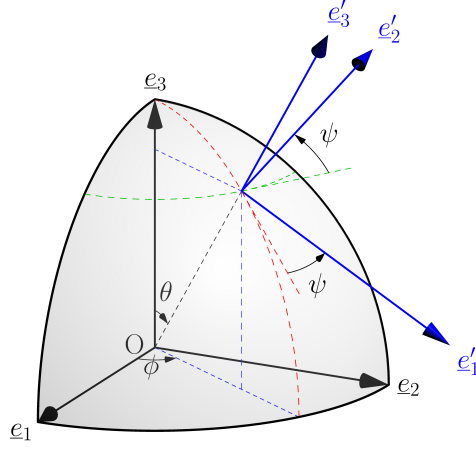


Figure A.1: Euler angles

The rotation of Euler angles  $(\theta, \phi, \psi)$  applies on a  $p$  order tensor  $\mathcal{T}$  as

$$\mathcal{T} = \mathcal{T}_{i_1, \dots, i_p} e_{i_1} \otimes \dots \otimes e_{i_p} \xrightarrow{\mathbf{R}} \mathbf{R}(\mathcal{T}) = \mathcal{T}_{i_1, \dots, i_p} (\mathbf{R} \cdot e_{i_1}) \otimes \dots \otimes (\mathbf{R} \cdot e_{i_p}) \quad (\text{A.21})$$

The application of (A.21) to a second-order tensor  $\mathbf{a}$  therefore gives

$$\mathbf{R}(\mathbf{a}) = \mathbf{R} \cdot \mathbf{a} \cdot {}^t\mathbf{R} \quad (\text{A.22})$$

and to a fourth-order tensor  $\mathbb{T}$

$$\mathbf{R}(\mathbb{T}) = (\mathbf{R} \boxtimes \mathbf{R}) : \mathbb{T} : {}^s(\mathbf{R} \boxtimes \mathbf{R}) \quad (\text{A.23})$$

where the fourth-order rotation tensor  $\mathbb{R} = \mathbf{R} \boxtimes \mathbf{R}$  can be expressed in Kelvin-Mandel notation (A.16) by means of the components of  $\mathbf{R}$  defined in (A.19)

$$\text{Mat}(\mathbb{R}, \mathcal{B}) = \begin{pmatrix} R_{11}^2 & R_{12}^2 & R_{13}^2 & \sqrt{2}R_{12}R_{13} & \sqrt{2}R_{11}R_{13} & \sqrt{2}R_{11}R_{12} \\ R_{21}^2 & R_{22}^2 & R_{23}^2 & \sqrt{2}R_{22}R_{23} & \sqrt{2}R_{21}R_{23} & \sqrt{2}R_{21}R_{22} \\ R_{31}^2 & R_{32}^2 & R_{33}^2 & \sqrt{2}R_{32}R_{33} & \sqrt{2}R_{31}R_{33} & \sqrt{2}R_{31}R_{32} \\ \sqrt{2}R_{21}R_{31} & \sqrt{2}R_{22}R_{32} & \sqrt{2}R_{23}R_{33} & R_{22}R_{33} + R_{23}R_{32} & R_{21}R_{33} + R_{23}R_{31} & R_{21}R_{32} + R_{22}R_{31} \\ \sqrt{2}R_{11}R_{31} & \sqrt{2}R_{12}R_{32} & \sqrt{2}R_{13}R_{33} & R_{12}R_{33} + R_{13}R_{32} & R_{11}R_{33} + R_{13}R_{31} & R_{11}R_{32} + R_{12}R_{31} \\ \sqrt{2}R_{11}R_{21} & \sqrt{2}R_{12}R_{22} & \sqrt{2}R_{13}R_{23} & R_{12}R_{23} + R_{13}R_{22} & R_{11}R_{23} + R_{13}R_{21} & R_{11}R_{22} + R_{12}R_{21} \end{pmatrix} \quad (\text{A.24})$$

It results that (A.23) can be seen as a classical rotation operation involving matrix multiplications in  $\mathbb{R}^6$ .

## A.4 Fourth-order isotropic tensors

This paragraph concerns fourth-order tensors operating over symmetrical second-order tensors, which allows to impose that they satisfy the minor symmetries ( $C_{jikl} = C_{ijlk} = C_{ijkl}$ ). The identity operator is given by

$$\mathbb{I} = \mathbf{1} \boxtimes \mathbf{1} = \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} e_i \otimes e_j \otimes e_k \otimes e_l \quad (\text{A.25})$$

<sup>4</sup>with the simplifying writing convention  $c_\theta = \cos \theta$ ,  $s_\theta = \sin \theta$ ,  $c_\phi = \cos \phi$ ,  $s_\phi = \sin \phi$ ,  $c_\psi = \cos \psi$ ,  $s_\psi = \sin \psi$

of identity matrix in Kelvin-Mandel convention

$$\text{Mat}(\mathbb{I}, \mathcal{B}) = \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad (\text{A.26})$$

It is classically proven that any minor-symmetrical fourth-order tensor invariant by rotation (A.23) write as a linear combination on the two projectors  $\mathbb{J}$  and  $\mathbb{K}$  which respectively extract the spherical and deviatoric part of any symmetrical second-order tensor

$$\mathbb{J} : \mathbf{a} = \frac{1}{3} \text{tr} \mathbf{a} \mathbf{1} \quad \text{and} \quad \mathbb{K} : \mathbf{a} = \mathbf{a} - \frac{1}{3} \text{tr} \mathbf{a} \mathbf{1} \quad (\text{A.27})$$

In other words,  $\mathbb{J}$  and  $\mathbb{K}$  are defined by

$$\mathbb{J} = \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \quad \text{and} \quad \mathbb{K} = \mathbb{I} - \mathbb{J} = \mathbf{1} \boxtimes \mathbf{1} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \quad (\text{A.28})$$

their components by

$$J_{ijkl} = \frac{1}{3} \delta_{ij} \delta_{kl} \quad \text{and} \quad K_{ijkl} = \frac{\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}}{2} - \frac{1}{3} \delta_{ij} \delta_{kl} \quad (\text{A.29})$$

and their matrices in Kelvin-Mandel notation relatively to any orthonormal frame by

$$\text{Mat}(\mathbb{J}, \mathcal{B}) = \left( \begin{array}{ccc|ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (\text{A.30})$$

$$\text{Mat}(\mathbb{K}, \mathcal{B}) = \left( \begin{array}{ccc|ccc} \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} & 0 & 0 & 0 \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} & 0 & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad (\text{A.31})$$

The following relationships are easily obtained

$$\mathbb{J} : \mathbb{J} = \mathbb{J} \quad ; \quad \mathbb{K} : \mathbb{K} = \mathbb{K} \quad ; \quad \mathbb{J} : \mathbb{K} = 0 \quad ; \quad \mathbb{J} :: \mathbb{J} = 1 \quad ; \quad \mathbb{K} :: \mathbb{K} = 5 \quad (\text{A.32})$$

The isotropisation of any fourth-order tensor  $\mathbb{T}$  is defined by (Bornert et al., 2001)

$$\text{ISO}(\mathbb{T}) = (\mathbb{T} :: \mathbb{J}) \mathbb{J} + \left( \frac{\mathbb{T} :: \mathbb{K}}{5} \right) \mathbb{K} \quad (\text{A.33})$$

It is easy to show that (A.33) is no other than the closest isotropic tensor to  $\mathbb{T}$  if the distance is chosen as the euclidean one i.e. associated to the scalar product (A.10). Note however that this isotropisation relying on euclidean distance to the set of isotropic tensors does not lead to the same result if  $\mathbb{T}'$  is considered instead of  $\mathbb{T}$ . Other distance definitions such as log-Euclidean distance fullfilling invariance by inversion are analyzed in (Morin et al., 2020).

## A.5 Fourth-order transversely isotropic tensors and Walpole basis

The Walpole basis ((Walpole, 1984), (Brisard, 2014)<sup>5</sup>) allowing to write any fourth-order transversely isotropic relatively to a an axis oriented by the unit vector  $\underline{n}$  is composed of the six following tensors built from  $\mathbf{1}_n = \underline{n} \otimes \underline{n}$  and  $\mathbf{1}_T = \mathbf{1} - \mathbf{1}_n$

$$\begin{aligned} \mathbb{E}_1 &= \mathbf{1}_n \otimes \mathbf{1}_n \quad ; \quad \mathbb{E}_2 = \frac{\mathbf{1}_T \otimes \mathbf{1}_T}{2} \quad ; \quad \mathbb{E}_3 = \frac{\mathbf{1}_n \otimes \mathbf{1}_T}{\sqrt{2}} \quad ; \quad \mathbb{E}_4 = \frac{\mathbf{1}_T \otimes \mathbf{1}_n}{\sqrt{2}} \quad (a) \\ \mathbb{E}_5 &= \mathbf{1}_T \boxtimes \mathbf{1}_T - \frac{\mathbf{1}_T \otimes \mathbf{1}_T}{2} \quad ; \quad \mathbb{E}_6 = \mathbf{1}_T \boxtimes \mathbf{1}_n + \mathbf{1}_n \boxtimes \mathbf{1}_T \quad (b) \end{aligned} \quad (\text{A.34})$$

Any transversely isotropic fourth-order tensor can be decomposed as

$$\mathbb{L} = \ell_1 \mathbb{E}_1 + \ell_2 \mathbb{E}_2 + \ell_3 \mathbb{E}_3 + \ell_4 \mathbb{E}_4 + \ell_5 \mathbb{E}_5 + \ell_6 \mathbb{E}_6 \quad (\text{A.35})$$

The six parameters can be conveniently gathered in a triplet composed of a  $2 \times 2$  matrix containing the four first parameters  $\ell_i$  ( $1 \leq i \leq 4$ ) and the two last parameters  $\ell_5$  and  $\ell_6$

$$\mathbb{L} \equiv (L, \ell_5, \ell_6), \quad L = \begin{pmatrix} \ell_1 & \ell_3 \\ \ell_4 & \ell_2 \end{pmatrix} \quad (\text{A.36})$$

Such a synthetic notation allows simple calculations of products and inverses which consist in classical matrix or scalar products and inverses

$$\begin{aligned} \mathbb{L} : \mathbb{M} &\equiv (LM, \ell_5 m_5, \ell_6 m_6) \quad (a) \\ \mathbb{L}^{-1} &\equiv \left( L^{-1}, \frac{1}{\ell_5}, \frac{1}{\ell_6} \right) \quad (b) \end{aligned} \quad (\text{A.37})$$

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<sup>5</sup>see [https://sbrisard.github.io/posts/20140226-decomposition\\_of\\_transverse\\_isotropic\\_fourth-rank\\_tensors.html](https://sbrisard.github.io/posts/20140226-decomposition_of_transverse_isotropic_fourth-rank_tensors.html)

## B Hill polarization tensor in elasticity

This section recalls some results about the calculation of the Hill polarization tensors related to a matrix of stiffness  $\mathbb{C}$  and an ellipsoid  $\mathcal{E}_{\mathbf{A}}$  of equation

$$\underline{x} \in \mathcal{E}_{\mathbf{A}} \Leftrightarrow \underline{x} \cdot ({}^t\mathbf{A} \cdot \mathbf{A})^{-1} \cdot \underline{x} \leq 1$$

where  $\mathbf{A}$  is an invertible second-order tensor so that  ${}^t\mathbf{A} \cdot \mathbf{A}$  is a positive definite symmetric tensor associated to 3 radii (eigenvalues  $a \geq b \geq c$  possibly written  $\rho_1 \geq \rho_2 \geq \rho_3$  for convenience) and 3 angles (orientation of the frame of eigenvectors  $\underline{e}_1, \underline{e}_2, \underline{e}_3$ )

$${}^t\mathbf{A} \cdot \mathbf{A} = a^2 \underline{e}_1 \otimes \underline{e}_1 + b^2 \underline{e}_2 \otimes \underline{e}_2 + c^2 \underline{e}_3 \otimes \underline{e}_3 = \sum_{i=1}^3 \rho_i \underline{e}_i \otimes \underline{e}_i \quad (\text{B.1})$$

### B.1 General expression

A general expression of the elastic polarization tensor is derived in (Willis, 1977) (see also (Mura, 1987))

$$\begin{aligned} \mathbb{P}(\mathbf{A}, \mathbb{C}) &= \frac{1}{4\pi} \int_{\|\underline{\zeta}\|=1} (\mathbf{A}^{-1} \cdot \underline{\zeta})^s \otimes \left( (\mathbf{A}^{-1} \cdot \underline{\zeta}) \cdot \mathbb{C} \cdot (\mathbf{A}^{-1} \cdot \underline{\zeta}) \right)^{-1} \otimes (\mathbf{A}^{-1} \cdot \underline{\zeta}) dS_{\underline{\zeta}} \\ &= \frac{\det \mathbf{A}}{4\pi} \int_{\|\underline{\xi}\|=1} \frac{\underline{\xi}^s \otimes (\underline{\xi} \cdot \mathbb{C} \cdot \underline{\xi})^{-1} \otimes \underline{\xi}}{\|\mathbf{A} \cdot \underline{\xi}\|^3} dS_{\underline{\xi}} \end{aligned} \quad (\text{B.2})$$

When  $\mathbb{C}$  is arbitrarily anisotropic, it is necessary to resort to numerical cubature to estimate  $\mathbb{P}$  as proposed in (Ghahremani, 1977), (Gavazzi and Lagoudas, 1990) or (Masson, 2008). However in some cases of anisotropy, analytical solutions are available ((Withers, 1989), (Barthélémy, 2020)). The case of isotropic matrix is particularly developed in the next section.

### B.2 Isotropic matrix

In this section, the matrix is assumed isotropic so that its stiffness tensor writes by means of a bulk  $k$  and shear  $\mu$  or Lamé  $\lambda$  and  $\mu$  moduli or even Young modulus  $E$  and Poisson ratio  $\nu$  with  $k = \frac{E}{3(1-2\nu)}$  and  $\mu = \frac{E}{2(1+\nu)}$ .

$$\begin{aligned} \mathbb{C} &= 3k\mathbb{J} + 2\mu\mathbb{K} = 3\lambda\mathbb{I} + 2\mu\mathbb{K} \\ \text{with } J_{ijkl} &= \frac{\delta_{ij}\delta_{kl}}{3}, I_{ijkl} = \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} \text{ and } \mathbb{K} = \mathbb{I} - \mathbb{J} \end{aligned} \quad (\text{B.3})$$

Introducing (B.3) in (B.2) leads to after some algebra

$$\mathbb{P} = \frac{1}{\lambda + 2\mu} \mathbb{U} + \frac{1}{\mu} (\mathbb{V} - \mathbb{U})$$

where the tensors  $\mathbb{U}$  and  $\mathbb{V}$ , depending only on the ellipsoidal tensor  $\mathbf{A}$  of (B.1), are given by (see (Barthélémy, 2020))

$$\begin{aligned} \mathbb{U} &= \frac{\det \mathbf{A}}{4\pi} \int_{\|\underline{\xi}\|=1} \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{\|\mathbf{A} \cdot \underline{\xi}\|^3} dS_{\underline{\xi}} \\ &= \frac{1}{4\pi} \int_{\|\underline{\zeta}\|=1} \frac{(\mathbf{A}^{-1} \cdot \underline{\zeta}) \otimes (\mathbf{A}^{-1} \cdot \underline{\zeta}) \otimes (\mathbf{A}^{-1} \cdot \underline{\zeta}) \otimes (\mathbf{A}^{-1} \cdot \underline{\zeta})}{\|\mathbf{A}^{-1} \cdot \underline{\zeta}\|^4} dS_{\underline{\zeta}} \end{aligned}$$

and

$$\begin{aligned} \mathbb{V} &= \frac{\det \mathbf{A}}{4\pi} \int_{\|\underline{\xi}\|=1} \frac{\underline{\xi} \otimes \mathbf{1} \otimes \underline{\xi}}{\|\mathbf{A} \cdot \underline{\xi}\|^3} dS_{\underline{\xi}} \\ &= \frac{1}{4\pi} \int_{\|\underline{\zeta}\|=1} \frac{(\mathbf{A}^{-1} \cdot \underline{\zeta}) \otimes \mathbf{1} \otimes (\mathbf{A}^{-1} \cdot \underline{\zeta})}{\|\mathbf{A}^{-1} \cdot \underline{\zeta}\|^2} dS_{\underline{\zeta}} \end{aligned}$$

For an arbitrary ellipsoid defined by (B.1), the components of  $\mathbb{U}$  and  $\mathbb{V}$  write

$$\begin{aligned} U_{iiii} &= \frac{3(I_i - \rho_i^2 I_{ii})}{2} \quad \forall i \in \{1, 2, 3\} \\ U_{ijjj} &= U_{ijji} = U_{ijji} = \frac{I_j - \rho_j^2 I_{ij}}{2} = \frac{I_i - \rho_j^2 I_{ij}}{2} \quad \forall i \neq j \in \{1, 2, 3\} \end{aligned}$$

and

$$\begin{aligned} V_{iiii} &= I_i \quad \forall i \in \{1, 2, 3\} \\ V_{ijij} &= V_{ijji} = \frac{I_i + I_j}{4} \quad \forall i \neq j \in \{1, 2, 3\} \end{aligned}$$

where the coefficients  $I_i$  and  $I_{ij}$  are given by (note that  $I_i$  and  $I_{ij}$  are adapted from those provided in (Kellogg, 1929) and (Eshelby, 1957): they differ by a factor of  $4\pi/3$  for  $I_{ij}$  with  $i \neq j$  and by  $4\pi$  for the others)

- if  $a > b > c$

$$\begin{aligned} I_1 &= \frac{a b c}{(a^2 - b^2) \sqrt{a^2 - c^2}} (F - E) \\ I_3 &= \frac{a b c}{(b^2 - c^2) \sqrt{a^2 - c^2}} \left( \frac{b \sqrt{a^2 - c^2}}{a c} - E \right) \\ I_2 &= 1 - I_1 - I_3 \\ I_{ij} &= \frac{I_j - I_i}{\rho_i^2 - \rho_j^2} \quad \forall i \neq j \in \{1, 2, 3\} \\ I_{ii} &= \frac{1}{3} \left( \frac{1}{\rho_i^2} - \sum_{j \neq i} I_{ij} \right) \quad \forall i \in \{1, 2, 3\} \end{aligned}$$

where  $F = F(\theta, \kappa)$  and  $E = E(\theta, \kappa)$  are respectively the elliptic integrals of the first and second kinds (see (Abramowitz and Stegun, 1972)) of amplitude and parameter

$$\theta = \arcsin \sqrt{1 - \frac{c^2}{a^2}} \quad ; \quad \kappa = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$$

- if  $a > b = c$  (prolate spheroid)

$$I_2 = I_3 = a \frac{a\sqrt{a^2 - c^2} - c^2 \operatorname{arccosh}(a/c)}{2(a^2 - c^2)^{3/2}}$$

$$I_1 = 1 - 2I_3$$

$$I_{1i} = I_{i1} = \frac{I_i - I_1}{a^2 - \rho_i^2} \quad \forall i \in \{2, 3\}$$

$$I_{ij} = \frac{1}{4} \left( \frac{1}{c^2} - I_{31} \right) \quad \forall i, j \in \{2, 3\}$$

$$I_{11} = \frac{1}{3} \left( \frac{1}{a^2} - 2I_{31} \right)$$

- if  $a = b > c$  (oblate spheroid)

$$I_1 = I_2 = c \frac{a^2 \arccos(c/a) - c\sqrt{a^2 - c^2}}{2(a^2 - c^2)^{3/2}}$$

$$I_3 = 1 - 2I_1$$

$$I_{3i} = I_{i3} = \frac{I_3 - I_i}{\rho_i^2 - c^2} \quad \forall i \in \{1, 2\}$$

$$I_{ij} = \frac{1}{4} \left( \frac{1}{a^2} - I_{31} \right) \quad \forall i, j \in \{1, 2\}$$

$$I_{33} = \frac{1}{3} \left( \frac{1}{c^2} - 2I_{31} \right)$$

- if  $a = b = c$  (sphere)

$$I_1 = I_2 = I_3 = \frac{1}{3}$$

$$I_{ij} = \frac{1}{5a^2} \quad \forall i, j \in \{1, 2, 3\}$$

In this last case of spherical inclusion ( $\mathbf{A} = \mathbf{1}$ ),  $\mathbb{U}$  and  $\mathbb{V}$  are simply decomposed as

$$\mathbb{U} = \frac{1}{3}\mathbb{J} + \frac{2}{15}\mathbb{K} \quad \text{and} \quad \mathbb{V} = \frac{1}{3}\mathbb{I}$$

### B.3 Case of cracks

The case of cracks corresponds to ellipsoids for which the smallest radius is very small compared to the two others, in other words the characteristic tensor  $\mathbf{A}$  (B.1) can be written here

$$\mathbf{A} = \underline{\ell} \otimes \underline{\ell} + \eta \underline{m} \otimes \underline{m} + \omega \underline{n} \otimes \underline{n} \quad \text{with} \quad \eta = \frac{b}{a} \quad \text{and} \quad \omega = \frac{c}{a}$$



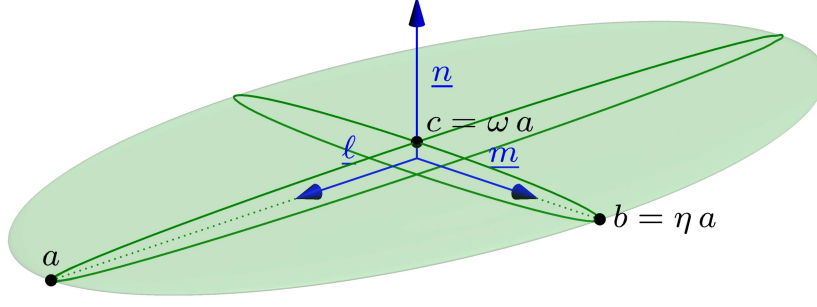


Figure B.1: Ellipsoidal crack

In the case of cracks, it is useful to introduce the second Hill polarization tensor defined as

$$\mathbb{Q} = \mathbb{C} - \mathbb{C} : \mathbb{P} : \mathbb{C}$$

and in particular  $\lim_{\omega \rightarrow 0} \omega \mathbb{Q}^{-1}$  in which it is recalled that  $\mathbb{P}$  and thus  $\mathbb{Q}$  depend on  $\omega$  such that the components  $(\mathbb{Q}^{-1})_{nijk}$  (with  $n$  corresponding to the crack normal) behave as  $1/\omega$  when  $\omega$  tends towards 0. The analytical expressions of this limit are fully detailed in (Barthélémy et al., 2021) which recalls in particular that  $\mathbb{L}$  actually derives from a symmetric second-order tensor  $\mathbf{B}$  as

$$\mathbb{L} = \lim_{\omega \rightarrow 0} \omega \mathbb{Q}^{-1} = \frac{3}{4} \underline{n}^s \otimes \mathbf{B} \otimes \underline{n}^s \quad (\text{B.4})$$

For an arbitrarily anisotropic matrix, an algorithm allowing to estimate the limit (B.4) is proposed in (Barthélémy, 2009) whereas in the isotropic case  $\mathbf{B}$  writes

$$\mathbf{B} = B_{nn} \underline{n} \otimes \underline{n} + B_{mm} \underline{m} \otimes \underline{m} + B_{\ell\ell} \underline{\ell} \otimes \underline{\ell}$$

with

$$\begin{aligned} B_{nn} &= \frac{8\eta(1-\nu^2)}{3E} \frac{1}{\mathcal{E}_\eta} \\ B_{mm} &= \frac{8\eta(1-\nu^2)}{3E} \frac{1-\eta^2}{(1-(1-\nu)\eta^2)\mathcal{E}_\eta - \nu\eta^2\mathcal{K}_\eta} \\ B_{\ell\ell} &= \frac{8\eta(1-\nu^2)}{3E} \frac{1-\eta^2}{(1-\nu-\eta^2)\mathcal{E}_\eta + \nu\eta^2\mathcal{K}_\eta} \end{aligned}$$

where  $\mathcal{K}_\eta = \mathcal{K}(\sqrt{1-\eta^2})$  and  $\mathcal{E}_\eta = \mathcal{E}(\sqrt{1-\eta^2})$  are the complete elliptic integrals of respectively the first and second kind (see (Abramowitz and Stegun, 1972)). If the crack is circular, the components of  $\mathbf{B}$  become

$$B_{nn} = \frac{16(1-\nu^2)}{3\pi E} \quad ; \quad B_{mm} = B_{\ell\ell} = \frac{B_{nn}}{1-\nu/2}$$

## B.4 Application of Hill calculation

### B.4.1 Definition of the matrix tensor

```
C = stiff_Enu(1.,0.2) ; print(C)
```

```
Order 4 ISO tensor | Param(size=2)=[ 1.66667 0.833333 ] | Angles(size=0)=[ ]
[ 1.11111 0.277778 0.277778 0 0 0
  0.277778 1.11111 0.277778 0 0 0
  0.277778 0.277778 1.11111 0 0 0
  0 0 0 0.833333 0 0
  0 0 0 0 0.833333 0
  0 0 0 0 0 0.833333 ]
```

### B.4.2 Calculation of the crack compliance $\mathbb{L} = \lim_{\omega \rightarrow 0} \omega \mathbb{Q}^{-1}$

Note that in *Echoes* it is necessary to provide an aspect ratio  $\omega$  for the crack even if the crack compliance is actually calculated as a limit (not depending on  $\omega$ )

```
 $\omega$  = 1.e-4
L = crack_compliance(spheroidal( $\omega$ ), C) ; print(L)
```

```
[[0.      0.      0.      0.      0.      0.      ]
 [0.      0.      0.      0.      0.      0.      ]
 [0.      0.      1.22230996 0.      0.      0.      ]
 [0.      0.      0.      0.67906109 0.      0.      ]
 [0.      0.      0.      0.      0.67906109 0.      ]
 [0.      0.      0.      0.      0.      0.      ]]
```

### B.4.3 Checking the aspect ratio for which $\omega \mathbb{Q}^{-1} \approx \lim_{\omega \rightarrow 0} \omega \mathbb{Q}^{-1}$ is acceptable

```
tw = np.logspace(-5,1,20)
tab $\delta$  = []
for  $\omega$  in tw:
    Q = hill_dual(spheroidal( $\omega$ ), C)
    Lw =  $\omega$ *np.linalg.inv(Q)
     $\delta$ L = np.linalg.norm(Lw-L)/np.linalg.norm(L)
    tab $\delta$ .append( $\delta$ L)
plt.figure(figsize=(8,3))
plt.loglog(tw,tab $\delta$ ,'+-')
plt.xlabel(r"$\omega$")
```

```
plt.ylabel(r"$\frac{||\mathbb{L}-\omega\mathbb{Q}^{-1}||}{||\mathbb{L}||}$")
plt.grid(True,which='both')
plt.show()
```

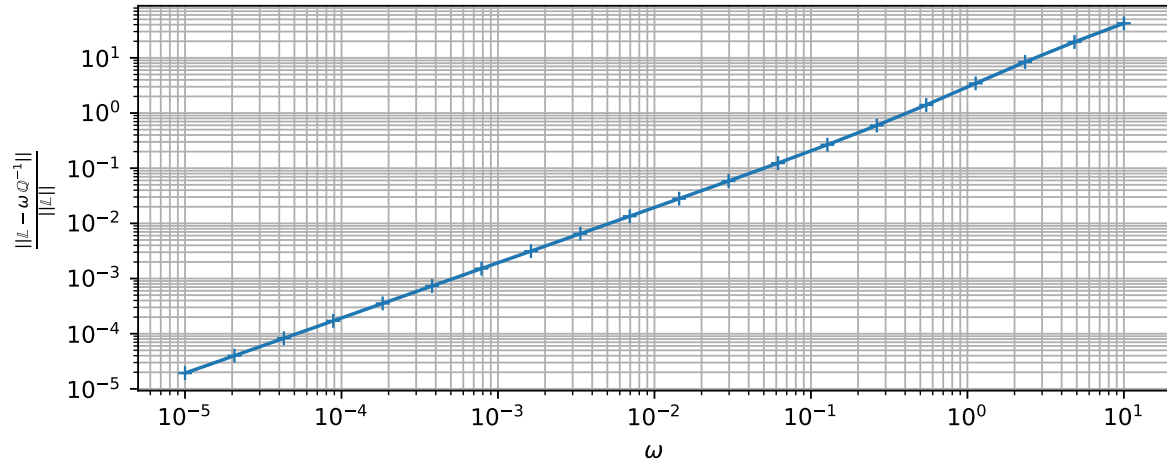


Figure B.2: Influence of the aspect ratio on the contribution tensor

## C Basic problem of elasticity homogenization

### C.1 System of equations

Consider a representative volume element (RVE)  $\Omega$  composed of a heterogeneous material. Neglecting body forces in a problem posed at the scale of a RVE is consistent with the fact that the order of magnitude of mechanical effects induced by body forces is in general much lower than that of the macroscopic strain  $\mathbf{E}$  or stress  $\Sigma$  effects accounting for interactions with particles surrounding the RVE (see (Dormieux et al., 2006)). The hypothesis of quasi-static equilibrium is also invoked here to write the balance law involving the Cauchy stress field  $\sigma$

$$\text{div } \sigma = \underline{0} \quad (\Omega) \quad (\text{C.1})$$

In the sequel, the small perturbation hypothesis is adopted so that the strain field  $\varepsilon$  derives from the displacement one  $\underline{u}$  as the symmetrical part of its gradient

$$\varepsilon = \frac{\text{grad } \underline{u} + {}^t\text{grad } \underline{u}}{2} \quad (\Omega) \quad (\text{C.2})$$

In the framework of random media homogenization, two types of conditions applied at the boundary  $\partial\Omega$  of a RVE  $\Omega$  are usually considered:

- *homogeneous strain boundary conditions* corresponding to prescribed displacements  $\underline{u}^g$  at  $\partial\Omega$

$$\underline{u}^g = \mathbf{E} \cdot \underline{x} \quad (\partial\Omega) \quad (\text{C.3})$$

It is noticeable that in this case the divergence theorem implies the following relationship between the microscopic and macroscopic strain tensors

$$\langle \varepsilon \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \varepsilon \, d\Omega = \frac{1}{|\Omega|} \int_{\partial\Omega} \underline{u} \otimes \underline{n} \, dS = \mathbf{E} \quad (\text{C.4})$$

where the spatial average over a domain  $\omega$  is denoted by  $\langle \bullet \rangle_\omega$  and  $\underline{n}$  is the unit outward normal at the boundary. The macroscopic stress tensor is then simply defined as the average

$$\Sigma = \langle \sigma \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \sigma \, d\Omega \quad (\text{C.5})$$

- *homogeneous stress boundary conditions* corresponding to prescribed surface tractions  $\underline{T}^g$  at  $\partial\Omega$

$$\underline{T}^g = \Sigma \cdot \underline{n} \quad (\partial\Omega) \quad (\text{C.6})$$

Now owing to the remarkable identity  $(x_i \sigma_{jk})_{,k} = \sigma_{ij}$  resulting from (C.1) and the symmetry of  $\sigma$ , the relationship between the microscopic and macroscopic stress is ensured by the divergence theorem

$$\langle \sigma \rangle_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \sigma \, d\Omega = \frac{1}{|\Omega|} \int_{\partial\Omega} \underline{x} \otimes (\sigma \cdot \underline{n}) \, dS = \Sigma \quad (\text{C.7})$$

The macroscopic stress tensor is then simply defined as the average

$$\mathbf{E} = \langle \varepsilon \rangle_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \varepsilon \, d\Omega \quad (\text{C.8})$$

### **i** Hill lemma

Note that whatever the choice of boundary conditions between (C.3) and (C.6), the consistency between the microscopic and macroscopic works is ensured by

$$\langle \sigma : \varepsilon \rangle_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \sigma : \varepsilon \, d\Omega = \frac{1}{|\Omega|} \int_{\partial\Omega} \underline{u} \cdot \sigma \cdot \underline{n} \, dS = \Sigma : \mathbf{E} \quad (\text{C.9})$$

which results from the application of the divergence theorem to  $(u_i \sigma_{ij})_{,k} = u_{i,j} \sigma_{ij} = \varepsilon_{ij} \sigma_{ij}$ .

The set of equations defining the problem posed on the RVE is finally completed by the local constitutive law relating the strain and stress fields. The hypothesis of linear elasticity is adopted in this part so that

$$\sigma = \mathbb{c} : \varepsilon \quad (\Omega) \quad (\text{C.10})$$

where  $\mathbb{c}(\underline{x})$  denotes the heterogeneous (positive definite fourth-order) stiffness tensor field satisfying the conditions of minor ( $c_{jikl} = c_{ijlk} = c_{ijkl}$ ) and major ( $c_{kl ij} = c_{ij kl}$ ) symmetries. The compliance tensor field is introduced as the inverse  $\mathbb{s} = \mathbb{c}^{-1}$  in the sense of fourth-order tensors operating over symmetrical second-order tensors.

In short, the system of equations posed on the RVE is given by (C.1), (C.2), (C.3) or (C.6) and (C.10).

## **C.2 Macroscopic stiffness or compliance tensors**

Whatever the boundary condition of homogeneous strain or stress type (C.3) or (C.6), the linearity of the problem allows to invoke the existence of concentration tensors relating the microscopic strain  $\varepsilon$  and stress  $\sigma$  fields to the macroscopic strain  $\mathbf{E}$  or stress  $\Sigma$  tensors

$$\begin{aligned} \varepsilon &= \mathbb{A}_E : \mathbf{E} \\ \sigma &= \mathbb{B}_E : \mathbf{E} \quad \text{with} \quad \mathbb{B}_E = \mathbb{c} : \mathbb{A}_E \\ \sigma &= \mathbb{B}_{\Sigma} : \Sigma \\ \varepsilon &= \mathbb{A}_{\Sigma} : \Sigma \quad \text{with} \quad \mathbb{A}_{\Sigma} = \mathbb{s} : \mathbb{B}_{\Sigma} \end{aligned} \quad (\text{C.11})$$