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ϕ-informational measures: some results and interrelations.

Steeve Zozor [1,*] and Jean-François Bercher [2]

[1] Univ. Grenoble Alpes, CNRS, Grenoble INP*, GIPSA-Lab, 38000 Grenoble, France
*Institute of Engineering Univ. Grenoble Alpes
steeve.zozor@cnrs.fr

[2] LIGM, Univ Gustave Eiffel, CNRS, F-77454 Marne-la-Vallée, France
jean-francois.bercher@esiee.fr

* Correspondence: steeve.zozor@cnrs.fr

Abstract: In this paper we focus on extended informational measures based on a convex function ϕ –entropies extended Fisher information, generalized moments–. Both the generalization of the Fisher's information and the moments are based on the definition of an escort distribution, precisely based on the (entropic) function ϕ. We thus revisit the usual maximum entropy principle –more especially its inverse problem, starting from the distribution and constraints–, which conduct to a wider extension of the ϕ-entropy with a state-dependent entropic functional. Then, we generalize some interrelations between the extended informationa measures –Cramer-Rao inequality, de Bruijn's identity– in this broader context. In this particular framework, the maximum entropy distributions play en central role. All the results derived in the paper include the usual ones as special cases.

Keywords: ϕ-entropies; state-dependent ϕ-entropies; (inverse) maximum ϕ-entropy problem; ϕ-escort distributions; ϕ-Fisher information; ϕ-moments; generalized Cramér-Rao inequality; ϕ-heat equation; generalized de Bruijn's identity.

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1. Introduction

Since the pionner works of von Neuman [1], Shannon [2], Boltzmann, Maxwell, Planck and Gibbs [3–9] on the entropy as a tool for uncertainty or information measure, many investigations were devoted to the generalization of the so-called Shannon entropy and its associated measures [10–22]. If the Shannon measures are compelling, especially in the communication domain, for compression purposes, many generalizations proposed later on has also showed promising interpretations and applications (Panter-Dite formula in quantification where the Rényi or Havdra-Charvát entropy emerges [23–25], codification penalizing long codewords where the Rényi entropy appears [26,27] for instance). The great majority of the extended entropies found in the literature belongs to a very general class of entropic measures called (h, ϕ)-entropies [13,19,20,28–30]. Such a general class (or more precisely the subclass of ϕ-entropies) traced back to the work of Burbea & Rao [28]. They offer not only a general framework to study general properties shared by special entropies, but they also offer many potential applications as described for instance in [30]. Note that if a large amount of work deals with divergences, entropies occur as special cases when one takes a unform reference measure.

In the settings of these generalized entropies, the so-called maximum entropy principle takes a special place. This principle, advocated by Jaynes, states that the statistical distribution that describes a system in equilibrium maximizes the entropy while satisfying the system's physical constraints (e.g., the center of mass, energy) [31–35]. In other words, it is the less informative law given the constraints of the system. In the Bayesian approach, dealing with the stochastic modelisation of a parameter, such a principle (or a minimum divergence principle) is often used to choose a prior distribution for the parameter [22,36–39]. It also finds its counterpart in communication, clustering, pattern recognition, problems, among many others [32,33,40–43]. In statistics, some goodness-of-fit tests are based on entropic criteria derived from the same idea of constrained maximal entropic law [44–49]. In a large number of works using the maximum entropy principle, the entropy used is the Shannon entropy, although several extensions exist in the literature. However, if for some reason a generalized entropy is considered, the approach used in the Shannon case does not fundamentally change [50–53].

One can consider the inverse problem which consists in finding the moment constraints leading to the observed distribution as a maximal entropy distribution [50]. Kesavan & Kapur also envisaged a second inverse problem, where both the distribution and the moments are given. The question is thus to determine the entropy so that the distribution is its maximizer. As a matter of fact, dealing with the Shannon entropy, whatever the constraints considered, the maximum entropy distribution falls in the exponential family [33,34,52,54]. Considering more general entropies allows to escape from this limitation. Moreover, if the Shannon entropy (or the Gibbs entropy in physics) is well adapted to the study of systems in the equilibrium (or thermodynamic limit), extended entropies allow a finer description of systems out of equilibrium [17,55–59], exhibiting their importance. While the problem was considered mainly in the discrete setting by Kesavan & Kapur in [50], we will recall it in the general framework of the *phi*-entropies probability densities with respect to any reference measure, and make a further step considering an extended class of these entropies.

While the entropy is a widely used tool for quantifying information (or uncertainty) attached to a random variable or to a probability distribution, other quantities are used as well, such as the moments of the variable (giving information for instance on center of mass, dispersion, skewness, impulsive character), or the Fisher information. In particular, the Fisher information appears in the context of estimation [60,61], in Bayesian inference through the Jeffrey's prior [39,62], but also for complex physical systems descriptions [61,63–67].

Although coming from different worlds (information theory and communication, estimation, statistics, physics), these informational quantities are linked by well-known relations such the Cramér-Rao's inequality, the de Bruijn's identity, the Stam's inequality [34,68–70]. These relationships have been proved very useful in various areas, for instance in communications [34,68,69], in estimation [60] or in physics [71,72], among others. When generalized entropies are considered, it is natural to question the other informational measures' generalization and the associated identities or inequalities. This question gave birth to a large amount of work and is still an active field of research [28,73–84].

In this paper, we show that it is possible to build a whole framework, which associates a target maximum entropy distribution to generalized entropies, generalized moments and generalized Fisher information. In this setting, we derive generalized inequalities and identities relating these quantities, which are all linked in some sense to the maximum entropy distribution.

The paper is organized as follows. In section 2 we recall the definition of the generalized ϕ-entropy. Thus, we come back to the maximum entropy problem in this general settings. Following the sketch of [50], we present a sufficient condition linking the entropic functional and the maximizing distribution, allowing to both solve the direct and the inverse problems. When the sufficient conditions linking the entropic function and the distribution cannot be satisfied, the problem can be solved by introducing state-dependent generalized entropies, which is the purpose of section 3. In section 4, we introduce informational quantities associated to the generalized entropies of the previous sections, such that a generalized escort distribution, generalized moments and generalized Fisher informations. These generalized informational quantities allow to extend the usual informational relations such that the Cramér-Rao inequality, relations saturated (or valid) dealing precisely for the generalized maximum entropy distribution. Finally, in section 5, we show that the extended quantities allows to obtain an extended de Bruijn identity, profided the distribution follows a non-linear heat equation. Some example of determination of ϕ-entropies solving the inverse maximum entropy problem are provided in a short series of appendix, showing in other that the usual quantities are recovered in the well known cases (Gausian distribution, Shannon entropy, Fisher information, variance).

In what follows, we will define a series of generalized information quantities relative to a probability density defined with respect to a given reference measure μ (e.g., the Lebesgue measure when dealing with continuous random variables, discrete measure for discrete-state random variables,...). Therefore, rigorously, all these quantities depend on the particular choice of this reference measure. However, for simplicity, we will omit to mention this dependence in the notation along the paper.

2. ϕ-entropies – direct and inverse maximum entropy problems.

Let us first recall the definition of the generalized ϕ-entropies introduced by Csiszàr in terms of divergences, and by Burbea and Rao in terms of entropies:

Definition 1 (ϕ-entropy [28])**.** *Let $\phi : \mathcal{Y} \subseteq \mathbb{R}_+ \mapsto \mathbb{R}$ be a convex function defined on a convex set $\mathcal{Y}$. Then, if f is a probability distribution defined with respect to a general measure μ on a set $\mathcal{X} \subseteq \mathbb{R}^d$ such that $f(\mathcal{X}) \subseteq \mathcal{Y}$, when this quantity exists,*

$$H_\phi[f] = - \int_{\mathcal{X}} \phi(f(x)) \, \mathrm{d}\mu(x) \tag{1}$$

is the ϕ-entropy of f.

The (h, ϕ)-entropy is defined by $H_{(h,\phi)}[f] = h\big(H_\phi[f]\big)$ where h is a nondecreasing function. The definition is extended by allowing ϕ to be concave, together with h nonincreasing [13,19,20,29,30]. If additionnaly h is concave, then the entropy functional $H_{(h,\phi)}[f]$ is concave.

Since we are interested in the maximum entropy problem and because h is monotone, we can restrict our study to the ϕ-entropies. Additionally, we will assume that ϕ is *strictly* convex and *differentiable*.

A related quantity is the Bregman divergence associated with convex function ϕ:

Definition 2 (Bregman divergence and functional Bregman divergence [22,85])**.** *With the same assumptions as in 1, the Bregman divergence associated with ϕ defined on a convex set $\mathcal{Y}$ is given by the function defined on $\mathcal{Y} \times \mathcal{Y}$,*

$$D_\phi(y_1, y_2) = \phi(y_1) - \phi(y_2) - \phi'(y_2)(y_1 - y_2). \tag{2}$$

Applied to two functions $f_i : \mathcal{X} \mapsto \mathcal{Y}$, $i = 1, 2$, the functional Bregman divergence writes

$$\mathcal{D}_\phi(f_1, f_2) = \int_{\mathcal{X}} \phi(f_1(x)) \, \mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi(f_2(x)) \, \mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi'(f_2(x))(f_1(x) - f_2(x)) \, \mathrm{d}\mu(x). \tag{3}$$

A direct consequence of the strict convexity of ϕ is the nonnegativity of the (functional) Bregman divergence: $D_\phi(y_1, y_2) \geq 0$ and $\mathcal{D}_\phi(f_1, f_2) \geq 0$, with equality if and only if $y_1 = y_2$ and $f_1 = f_2$ almost everywhere respectively.

More generally, the Bregman divergence is defined for multivariate convex functions, where the derivative is replaced by gradient operator [85]. Extensions for convex function of functions also exist, where the derivative is in the sense of Gâteau [86]. Such general extensions are not useful for our purposes; thus, we restrict ourselves to the above definition where $\mathcal{Y} \subseteq \mathbb{R}_+$.

2.1. Maximum entropy principle: the direct problem

Let us first recall the maximum entropy problem that consists in searching for the distribution maximizing the ϕ-entropy (1) subject to constraints on some moments $\mathbb{E}[T_i(X)]$ with $T_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \ldots, n$. This direct problem writes

$$f^\star = \underset{f \in C_t}{\operatorname{argmax}} \left(- \int_{\mathcal{X}} \phi(f(x)) \, \mathrm{d}\mu(x) \right) \tag{4}$$

with

$$C_t = \{ f \geq 0 : \ \mathbb{E}[T_i(X)] = t_i, i = 0, \ldots, n \}, \tag{5}$$

where $T_0(x) = 1$ and $t_0 = 1$ (normalization constraint). The maximization problem being strictly concave, the solution exists and is unique. A technique to solve the problem can be to use the classical Lagrange multipliers technique and solving the Euler-Lagrange equation from the variational problem, but this approach requires mild conditions [50,51,53,87–89]. In the following proposition, we recall a sufficient condition relating f and ϕ so that f is the problem's solution. Below, we prove the result without the use of the Lagrange technique.

Proposition 1 (Maximal ϕ-entropy solution [50])**.** *Suppose that there exists a probability distribution $f \in C_t$ satisfying*

$$\phi'(f(x)) = \sum_{i=0}^{n} \lambda_i \, T_i(x), \tag{6}$$

for some $(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$. Then, f is the unique solution of the maximal entropy problem (4).

Proof. Suppose that distribution f satisfies (6) and consider any distribution $g \in C_t$. The functional Bregman divergence between f and g writes

$$\begin{aligned}
\mathcal{D}_\phi(g, f) &= \int_{\mathcal{X}} \phi(g(x)) \, \mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi(f(x)) \, \mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi'(f(x))(g(x) - f(x)) \, \mathrm{d}\mu(x) \\
&= -H_\phi[g] + H_\phi[f] - \sum_{i=0}^{n} \lambda_i \int_{\mathcal{X}} T_i(x)(g(x) - f(x)) \, \mathrm{d}\mu(x) \\
&= H_\phi[f] - H_\phi[g]
\end{aligned}$$

where we used the fact that g and f are both probability distributions with the same moments $\mathbb{E}[T_i(X)] = t_i$. By nonnegativity of the Bregman functional divergence, we finally get that

$$H_\phi[f] \geq H_\phi[g]$$

for all distribution g with the same moments as f, with equality if and only if $g = f$ almost everywhere. In other words, this shows that if f satisfies (6), then it is the desired solution. □

Hence, given an entropic functional ϕ and moments constraints T_i, eq. (6) leads the the maximum entropy distribution $f^\star$. This distribution is parametrized by the λ_i or, equivalently, by the moments t_i.

Note that the reciprocal is not necesarilly true, as shown for instance in [53]. However, the reciprocal is true when $\mathcal{X}$ is a compact [89] or for any $\mathcal{X}$ provided that ϕ is locally bounded on $\mathcal{X}$ [90].

2.2. Maximum entropy principle: the inverse problems

As stated in the introduction, two inverse problems can be considered starting from a given distribution f. These problems were considered by Kesavan & Kapur in [50] in the discrete framework.

The first inverse problem consists in searching for the adequate moments so that a desired distribution f is the maximum entropy distribution of a given ϕ-entropy. This amounts to find functions T_i and coefficients λ_i satisfying eq. (6). This is not always an easy task, and even not always possible. For instance, it is well known that the maximum Shannon entropy distribution given moment constraints fall in the exponential family [33,34,52,54]. Therefore, if f does not belong to this family, the problem has no solution.

The second inverse problem consists in designing the entropy itself, given a target distribution f and given the T_i. In other words, given a distribution f, eq. (6) may allow to determine the entropic functional ϕ so that f is its maximizer.

As for the direct problem, in the second inverse problem, the solution is parametrized by the λ_i. Here, required properties on ϕ will shape the domain the λ_i live in. In particular, ϕ must satisfy

- the domain of definition of ϕ' must include $f(\mathcal{X})$; this will be satisfied by construction.
- from the strict convexity property of ϕ, ϕ' must be strictly increasing.

Hence, because ϕ' must be stricly increasing, its clear that solving eq. (6) requires the following two conditions:

(C1) $f(x)$ and $\displaystyle\sum_{i=1}^{n} \lambda_i\, T_i(x)$ must have the same variations, i.e., $\displaystyle\sum_{i=0}^{n} \lambda_i\, T_i(x)$ is increasing (resp. decreasing, constant) where f is increasing (resp. decreasing, constant).

(C2) $f(x)$ and $\displaystyle\sum_{i=1}^{n} \lambda_i\, T_i(x)$ must have the same level sets,

$$f(x_1) = f(x_2) \;\Leftrightarrow\; \sum_{i=0}^{n} \lambda_i\, T_i(x_1) \;=\; \sum_{i=0}^{n} \lambda_i\, T_i(x_2)$$

For instance, in the univariate case, for one moment constraint,

- for $\mathcal{X} = \mathbb{R}_+$, $T_1(x) = x$, λ_1 must be negative and $f(x)$ must be decreasing,
- for $\mathcal{X} = \mathbb{R}$, $T_1(x) = x^2$ or $T_1(x) = |x|$, λ_1 must be negative and $f(x)$ must be even and unimodal.

Under conditions (C1) and (C2), the solutions of eq. (6) are given by

$$\phi'(y) = \sum_{i=0}^{n} \lambda_i\, T_i\Big(f^{-1}(y)\Big) \tag{7}$$

where f^{-1} can be multivalued.

Eq. (6) provides an effective way to solve the inverse problem. However, there exist situations where there do not exist any set of λ_i such that conditions (C1)-(C2) are satisfied (e.g., $T_1(x) = x^2$ with f not even). In such a case, a way to go is to extend the definition of the ϕ-entropy. This is precisely the purpose of section 3.

2.3. Second inverse maximum entropy problem: some examples

To illustrate the previous subsection, let us analyze briefly three examples: the famous Gaussian distribution (example 1), the q-Gaussian distribution also intensively studied (example 2) and the arcsine distribution (example 3), both three with a second-order moment constraint. The Gaussian, q-Gaussian, and arcsine distributions will serve as a guideline all along the paper. The details of the calculus, together with a deeper study related to the sequel of the paper, are rejected in the appendix. Other examples are also given in this appendix. In both three examples, except in the next section, we consider the second-order moment constraint $T_1(x) = x^2$.

Example 1. *Let us consider the well-known Gaussian distribution $f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$, defined over $\mathcal{X} = \mathbb{R}$, and let us search for the ϕ-entropy so that the Gaussian is its maximizer subject to the constraint $T_1(x) = x^2$. To satisfy condition (C1) we must have $\lambda_1 < 0$. Rapid calculus detailed in appendix A.1 gives the entropic functional, after a reparametrization of the λ_i's, of the form,*

$$\phi(y) = \alpha\, y \log(y) + \beta\, y + \gamma \quad \textit{with} \quad \alpha > 0,$$

that is nothing more than the Shannon entropy, up to the scaling factor α, and a shift (to avoid the divergence of the entropy when $\mathcal{X}$ is unbounded, one will take $\gamma = 0$). One thus recovers the long outstanding fact that the Gaussian is the maximum Shannon entropy distribution with the second order moment constraint.

Example 2. *Let us consider the q-Gaussian distribution, also known as Tsallis distribution, $f_X(x) = C_q\left(1-(q-1)\frac{x^2}{\sigma^2}\right)_+^{\frac{1}{1-q}}$, where $q > 0$, $q \neq 1$, $x_+ = \max(x,0)$ and C_q is the normalization coefficient, defined over $\mathcal{X} = \mathbb{R}$ when $q < 1$ or over $\mathcal{X} = \left(-\frac{\sigma}{\sqrt{q-1}}\,;\,\frac{\sigma}{\sqrt{q-1}}\right)$ when $q > 1$, and let search for the ϕ-entropy so that the q-Gaussian is its maximizer with the constraint $T_1(x) = x^2$. Here again, condition (C1) is satisfied if and only if $\lambda_1 < 0$. Rapid calculus detailed in appendix A.2 leads to the entropic functional, after a reparametrization of the λ_i's, as,*

$$\phi(y) = \alpha\, \frac{y^q - y}{q-1} + \beta\, y + \gamma \quad \textit{with} \quad \alpha > 0,$$

where q is thus a additional parameter of the family. This entropy is nothing more than the Havrdat-Charvát-Tsallis entropy [12,14,17,91], up to the scaling factor α, and a shift (here also, to avoid the divergence of the entropy when $\mathcal{X}$ is unbounded, one will take $\gamma = 0$). One recover the also well known fact that the q-Gaussian is the maximum Shannon entropy distribution with the second order moment constraint [91]. In the limit case $q \to 1$, the distribution f_X tends to the Gaussian, whereas the Havrdat-Charvát-Tsallis entropy tends to the Shannon entropy.

Example 3. *Consider the arcsine distribution, $f_X(x) = \frac{1}{\sqrt{s^2 - \pi^2 x^2}}$, defined over $\mathcal{X} = \left(-\frac{s}{\pi}\,;\,\frac{s}{\pi}\right)$ and let us determine the entropic functionals ϕ so that f_X is the maximum ϕ-entropy distribution subject to the constraint $T_1(x) = x^2$. Now, to fullfill condition (C1) we must impose $\lambda_1 > 0$. Some algebra detailed in appendix A.4.1 leads to the entropic functional, after a reparametrization of the λ_i's,*

$$\phi(y) = \frac{\alpha}{y} + \beta\, y + \gamma \quad \textit{with} \quad \alpha > 0$$

(again, to avoid the divergence of the entropy one can adjust parameter γ). This entropy is unusual and, due to its form, is potentially finite only for densities defined over a bounded support and that are divergent in its boundary (integrable divergence).

3. State-dependent entropic functionals and mimization revisited

In order to follow asymmetries of the distribution f and address the limitation raised above, an idea is to allow the entropic functional to also depend on the state variable x:

Definition 3 (State-dependent ϕ-entropy)**.** *Let $\phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ such that for any $x \in \mathcal{X} \subseteq \mathbb{R}^d$, function $\phi(x, \cdot)$ is a convex function on the closed convex set $\mathcal{Y} \subseteq \mathbb{R}_+$. Then, if f is a probability distribution defined with respect to a general measure μ on set $\mathcal{X}$ and such that $f(\mathcal{X}) \subseteq \mathcal{Y}$,*

$$H_\phi[f] = - \int_{\mathcal{X}} \phi(x, f(x))\, \mathrm{d}\mu(x) \tag{8}$$

will be called state-dependent ϕ-entropy of f. Since $\phi(x, \cdot)$ is convex, then the entropy functional $H_\phi[f]$ is concave. A particular case arises when, for a given partition $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ of $\mathcal{X}$, functional ϕ writes

$$\phi(x, y) = \sum_{l=1}^{k} \phi_l(y) \mathbb{1}_{\mathcal{X}_l}(x) \tag{9}$$

where $\mathbb{1}_A$ denotes the indicator of set A. This functional can be viewed as a "$(\mathcal{X}_1, \ldots, \mathcal{X}_k)$-extension" over $\mathcal{X} \times \mathcal{Y}$ of a multiform function defined on $\mathcal{Y}$, with k branches ϕ_l and the associated ϕ-entropy will be called $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$-multiform ϕ-entropy.

As in the previous section, we restrict our study to functionals $\phi(x, y)$ *strictly convex and differentiable* versus y.

Following the lines of section 2, a generalized Bregman divergence can be associated to ϕ under the form $D_\phi(x, y_1, y_2) = \phi(x, y_1) - \phi(x, y_2) - \frac{\partial \phi}{\partial y}(x, y_2)(y_1 - y_2)$, and a generalized functional Bregman divergence $\mathcal{D}_\phi(f_1, f_2) = \int_{\mathcal{X}} D_\phi(x, f_1(x), f_2(x))\, \mathrm{d}\mu(x)$.

With these extended quantities, the direct problem becomes

$$f^\star = \underset{f \in C_t}{\operatorname{argmax}} \left(- \int_{\mathcal{X}} \phi(x, f(x))\, \mathrm{d}\mu(x) \right) \tag{10}$$

Although the entropic functional is now state dependent, the approach adopted before can be applied here, leading to

Proposition 2 (Maximum state-dependent ϕ-entropy solution)**.** *Suppose that there exists a probability distribution f satisfying*

$$\frac{\partial \phi}{\partial y}\big(x, f(x)\big) = \sum_{i=0}^{n} \lambda_i\, T_i(x), \tag{11}$$

for some $(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$, then f is the unique solution of the extended maximum entropy problem (10).
If ϕ is chosen in the $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$-multiform ϕ-entropy class, this sufficient condition writes

$$\sum_{l=1}^{k} \phi_l'\big(f(x)\big)\, \mathbb{1}_{\mathcal{X}_l}(x) = \sum_{i=0}^{n} \lambda_i\, T_i(x), \tag{12}$$

Proof. The proof follows the steps of Proposition 1, using the generalized functional Bregman divergence instead of the usual one. □

Resolution eq. (11) is not possible in all generality. However the sufficient condition. (12) can be rewritten as

$$\sum_{l=1}^{k}\left(\phi_l'(f(x)) - \sum_{i=0}^{n}\lambda_i\,T_i(x)\right)\mathbb{1}_{\mathcal{X}_l}(x) = 0. \tag{13}$$

Thus, if there exists (at least) a set of λ_i such that condition (C1) is satisfied (but not necessarily (C2)), one can always

- design a partition $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$ so that (C2) is satisfied *in each* $\mathcal{X}_l$ (at least, such that f is either strictly monotonic, or constant, on $\mathcal{X}_l$)
- determine ϕ_l as in eq. (7) in each $\mathcal{X}_l$, that is

$$\phi_l'(y) = \sum_{i=0}^{n}\lambda_i\,T_i\Big(f_l^{-1}(y)\Big) \tag{14}$$

where f_l^{-1} is the (possibly multivalued) inverse of f on $\mathcal{X}_l$.

In short, in the case where only condition (C1) is satisfied, one can obtain an extended entropic functionnal of $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$-multiform class so that eq. (13) provides an effective way to solve the inverse problem in the state-dependent entropic functional context.

Note however that it still may happen that there is no set of λ_i allowing to satisfy (C1). In such an harder context, the problem remains solvable when the moments are defined as partial moments like $\mathbb{E}\big[T_{l,i}(X)\mathbb{1}_{\mathcal{X}_l}(X)\big] = t_{l,i}$, $l = 1,\ldots,k$ and $i = 1,\ldots,n_l$ and when there exist on $\mathcal{X}_l$ a set of $\lambda_{l,i}$ such that (C1) and (C2) holds. The solution still writes as in eq. (14), but where now n, the λ_i and the T_i are replaced by n_l, $\lambda_{l,i}$ and $T_{l,i}$ respectively,

$$\phi_l'(y) = \sum_{i=0}^{n_l}\lambda_{l,i}\,T_{l,i}\Big(f_l^{-1}(y)\Big) \tag{15}$$

Let us now come back to the arcsine example $f_X(x) = \frac{1}{s^2-\pi^2x^2}$, defined over $\mathcal{X} = \left(-\frac{s}{\pi}\,;\,\frac{s}{\pi}\right)$ (example 3) of the previous section, when now we constraint the first order moment or partial first order moments.

Example 3-1. *Let us now consider this arcsince distribution, constraint uniformely by $T_1(x) = x$. Clearly, neither condition (C1) nor condition (C2) can be satisfied. Note that the arcsince distribution is bijective on each set $\mathcal{X}_- = \left(-\frac{s}{\pi}\,;0\right)$ and $\mathcal{X}_+ = \left[0\,;\frac{s}{\pi}\right)$ that partitions $\mathcal{X}$. Therfore, considering multiform entropic functionals with this partition allow to overcome the issue on condition (C2), but that on condition (C1) remains. If we ignore this issue and apply eq. (14), after a reparametrization of the λ_i's, we obtain $\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy)$ with $\widetilde{\phi}_{\pm,\mathrm{u}}(y) = \pm\alpha\left(\sqrt{u^2-1} + \arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;+\infty)}(u) + \beta\,u + \gamma_\pm$ where s is thus an additional parameter of the family. It appears that whereas these functionals are defined for $u > 1$, one can extend them continuously and with a continuous derivative for any $u > 0$ imposing $\beta = 0$, which finally leads to the family*

$$\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy) \qquad \textit{with}$$

$$\widetilde{\phi}_{\pm,\mathrm{u}}(y) = \pm\alpha\left(\sqrt{u^2-1} + \arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;+\infty)}(u) \; + \; \gamma_\pm$$

However, the functional are no more convex. (see appendix A.4.2 for more details).

Example 3-2. *If now we impose the partial constraint $T_{\pm,1}(x) = x\mathbb{1}_{\mathcal{X}_\pm}(x)$, and search for the ϕ-entropy so that f_X is the maximizer subject to these constraints, condition (C1) (on each $\mathcal{X}_\pm$) requires that $\lambda_{\pm,1} > 0$. We then obtain the associated multiform entropic functional, after a reparametrization of the λ_i's, as $\phi_\pm(y) = \phi_{\pm,\mathrm{u}}(sy)$ with $\phi_{\pm,\mathrm{u}}(u) = \alpha_\pm\left(\sqrt{u^2-1}+\arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;+\infty)}(u)+\beta\,u+\gamma_\pm$ with $\alpha_\pm > 0$ and where s is thus an additional parameter of the family. In this case, the entropic functionals can be considered for any $u > 0$ by imposing $\beta = 0$ and one can check that the obtained functions are of class C^1. This finally leads to the family*

$$\phi_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy) \qquad \textit{with}$$

$$\phi_{\pm,\mathrm{u}}(y) = \alpha_\pm\left(\sqrt{u^2-1}+\arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;+\infty)}(u)\;+\;\gamma_\pm, \quad \alpha_\pm > 0$$

In addition, remarkably, the entropic functional can be made univalued by choosing $\alpha_+ = \alpha_-$ and $\gamma_+ = \gamma_-$. In fact, such a choice is equivalent than considering the constraint $T_1(x) = |x|$ which respects the symmetries of the distribution, allowing thus to recover a classical ϕ-entropy. (see appendix A.4.2 for more details).

At a first glance, the two solutions seem to be identical. In fact, they drastically differ. Indeed, let us emphasize that the problem has one constraint in the first case, whereas in the second case, there is two. The consequence is that 4 parameters parametrize the first solution $\beta, \gamma_\pm$ and, especially, α, while 5 parameters $\beta, \gamma_\pm$ and $\alpha_\pm$ parametrize the second solution. This difference is not insignificant: the first case cannot be viewed as a special case of the second one, because $\alpha_\pm$ must be positive, which cannot be possible with only parameter α since $\pm\alpha$ rule the $\widetilde{\phi}_\pm$. For the first example, the solution does not lead to a convex function, because this would contradict the required condition (C1) on the parts $\mathcal{X}_\pm$. Coming back to the direct problem, the "ϕ-like-entropy" defined with $\widetilde{\phi}$ is no more concave (indeed, it is no more an entropy in the sense of definition 1), so that the maximum ϕ-entropy problem is no more concave: one cannot garantee the unicity and even the existence of a maximum so that there is no garantee that the arcsine distribution would be a maximizer. Indeed, equation (6) coming from the Euler-Lagrange equation (see paragraph previous to prop. 1), one can just conclude that the arsine is a critical point (either extremal, or inflection point) of the obtained ϕ-like-entropy.

In section 2 and 3 we established general entropies with a given maximizer. In what follows, we will complete the information theoretical setting by introducing generalized escort distributions, generalized moments, and generalized Fisher information associated to the same entropic functional. We will then explore some of their relationships.

4. ϕ-escort distribution, (ϕ,α)-moments, (ϕ,β)-Fisher informations, generalized Cramér-Rao inequalities

In this section, we begin by introducing the above-mentioned informational quantities. We will then show that generalizations of the celebrated Cramér-Rao inequalities hold and link the generalized moments and Fisher information. Furthermore, the lower bound of the inequalities are saturated precisely by maximal ϕ-entropy distributions.

Escort distributions have been introduced as an operational tool in the context of multifractals [92,93], with interesting connections with the standard thermodynamics [94] and with source coding [26,27]. In our context, we also define (generalized) escort distributions associated with a particular ϕ-entropy, and show how they pop up naturally. It is then possible to define generalized moments with respect to these escort distributions.

Definition 4 (ϕ-escort)**.** *Let $\phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ such that for any $x \in \mathcal{X} \subseteq \mathbb{R}^d$ function $\phi(x,\cdot)$ is a strictly convex twice differentiable function defined on the closed convex set $\mathcal{Y} \subseteq \mathbb{R}_+$. Then, if f is a probability distribution defined with respect to a general measure μ on a set $\mathcal{X}$ such that $f(\mathcal{X}) \subseteq \mathcal{Y}$, such that*

$$C_\phi[f] = \int_{\mathcal{X}} \frac{\mathrm{d}\mu(x)}{\frac{\partial^2 \phi}{\partial y^2}(x, f(x))} < +\infty \tag{16}$$

we define by

$$E_{\phi,f}(x) = \frac{1}{C_\phi[f] \, \dfrac{\partial^2 \phi}{\partial y^2}(x, f(x))} \tag{17}$$

the ϕ-escort density with respect to measure μ, associated to density f.

Note that from the strict convexity of ϕ with respect to its second argument, this probability density is well defined and is strictly positive. Moreover, coming back to the previous examples, one can see that:

Example 1. *In the context of the Shannon entropy, entropy for which the Gaussian is the maximal entropy law for the second order moment constraint, $\phi(x,y) = \phi(y) = y \log y$, the ϕ-escort density associated to f restricts to density f itself.*

Example 2. *In the Rényi-Tsallis context, entropy for which the q-Gaussian is the maximal entropy law for the second order moment constraint $\phi(x,y) = \phi(y) = \frac{y^q - y}{q-1}$, and $E_{\phi,f} \propto f^{2-q}$ which recovers the escort distributions used in the Rényi-Tsallis context up to a duality transformation [94].*

Example 3. *For the entropy that is maximal for the arcsine distribution under the second order moment constraint, $\phi(x,y) = \phi(y) = \frac{1}{y}$, and $E_{\phi,f} \propto f^3$ which is nothing more than an escort distributions used in the Rényi-Tsallis context. Indeed, although the arcsine distribution does not fall in the q-Gaussian family, its form is very similar to a q-distribution (with $q = -1$) where the "scaling" would not be related to the exponent q. It is thus not suprising to recover an escort distribution associated to this family.*

Definition 5 ((α, ϕ)-moments)**.** *Under the assumptions of definition 4, with $\mathcal{X}$ equiped with a norm $\|\cdot\|_{\mathcal{X}}$, we define by*

$$M_{\alpha,\phi}[f; X] = \int_{\mathcal{X}} \|x\|_{\mathcal{X}}^{\alpha} \, E_{\phi,f}(x) \, \mathrm{d}\mu(x) \tag{18}$$

if this quantity exists, as the (α, ϕ)-moment of X associated to distribution f.

For our three examples, we have:

Example 1. *In the context of the Shannon entropy, the (α, ϕ)-moments are the usual moments of $\|X\|_{\mathcal{X}}^{\alpha}$.*

Example 2. *In the Rény-Tsallis context the generalized moments introduced in [55,95] are recovered.*

Example 3. *For $\phi(x,y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalized moments introduced in [55,95] (see the items related to the escort distributions).*

The Fisher information's importance is well known in estimation theory: the estimation error of a parameter is bounded by the inverse of the Fisher information associated with this distribution [34,60]. The Fisher information is also used as a method of inference and understanding in statistical physics and biology, as promoted by Frieden [61] and has been generalized in the Rényi-Tsallis context in a series of papers [75,78,80–83,96,97]. In what follows, we generalize these definitions a step further in our ϕ-entropy context.

Definition 6 (Nonparametric (β,ϕ)-Fisher information)**.** *With the same assumption as in definition 5, denoting by $\|\cdot\|_{\mathcal{X}*}$ the dual norm, for any differentiable density f, we define the quantity*

$$I_{\beta,\phi}[f] = \int_{\mathcal{X}} \left\| \frac{\nabla_x f(x)}{E_{\phi,f}(x)} \right\|_{\mathcal{X}*}^{\beta} E_{\phi,f}(x)\,\mathrm{d}\mu(x) \tag{19}$$

if this quantity exists, as the nonparametric (β,ϕ)-Fisher information of f.

Note that when ϕ is state-independent, $\phi(x,y) = \phi(y)$, as for the usual Fisher information, this quantity is shift-invariant, i.e., for $g(x) = f(x-x_0)$ one have $I_{\beta,\phi}[g] = I_{\beta,\phi}[f]$. This property is unfortunately lost in the state-dependent context.

Definition 7 (Parametric (β,ϕ)-Fisher information)**.** *Let consider the same assumption as in definition 5, such that density f is parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$. The set Θ is equipped with a norm $\|\cdot\|_\Theta$ and the corresponding dual norm is denoted $\|\cdot\|_{\Theta*}$. Assume that f is differentiable with respect to θ. We define by*

$$I_{\beta,\phi}[f;\theta] = \int_{\mathcal{X}} \left\| \frac{\nabla_\theta f(x)}{E_{\phi,f}(x)} \right\|_{\Theta*}^{\beta} E_{\phi,f}(x)\,\mathrm{d}\mu(x) \tag{20}$$

as the parametric (β,ϕ)-Fisher information of f.

Note that, as for the usual Fisher information, when the norm on $\mathcal{X}$ and on Θ are the same, the nonparametric and parametric information coincide when θ is a location parameter. For our three examples, we have:

Example 1. *In the Shannon entropy context, when the norm is the euclidean norm and $\beta = 2$, the nonparametric and parametric informations (β,ϕ)-Fisher give the usual nonparametric and parametric Fisher informations respectively.*

Example 2. *Similarly, in the Rényi-Tsallis context, the generalizations proposed in [81–83] are recovered.*

Example 3. *For $\phi(x,y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalizations proposed in [81–83] (see the items related to the escort distributions).*

We have now the quantities that allow to generalize the Cramér-Rao inequalities as follows.

Proposition 3 (Nonparametric (α,ϕ)-Cramér-Rao inequality)**.** *Assume that a differentiable probability density function with respect to a measure μ, defined on a domain $\mathcal{X}$, admits an (α,ϕ)-moment and a (α^*,ϕ)-Fisher information with $\alpha \geq 1$ and α^* Holder-conjugated $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$, and that $x f(x)$ vanishes at the boundary of $\mathcal{X}$. Thus, density f satisfies the (α,ϕ) extended Cramér-Rao inequality*

$$M_{\alpha,\phi}[f;X]^{\frac{1}{\alpha}}\, I_{\alpha^*,\phi}[f]^{\frac{1}{\alpha^*}} \;\geq\; d \tag{21}$$

When ϕ is state independent, $\phi(x,y) = \phi(y)$, the equality occurs when f is the maximal ϕ entropy distribution subject to the moment constraint $T(x) = \|x\|_{\mathcal{X}}^{\alpha}$.

Proof. The approach follows [83], starting from the differentiable probability density f (derivative denoted $\nabla_x f$), since $xf(x)$ vanishes in the boundaries of X from the divergence theorem one has

$$0 = \int_{\mathcal{X}} \nabla_x^t (x f(x)) \,\mathrm{d}\mu(x) = \int_{\mathcal{X}} (\nabla_x^t\, x) f(x) \,\mathrm{d}\mu(x) + \int_{\mathcal{X}} x^t (\nabla_x f(x)) \,\mathrm{d}\mu(x)$$

Now, for the first term, we use the fact that $\nabla_x x = d$ and that f is a density to achieve

$$d = -\int_{\mathcal{X}} x^t \frac{\nabla_x f(x)}{g(x)}\, g(x) \,\mathrm{d}\mu(x)$$

for any function g non-zero on $\mathcal{X}$. Now, noting that $d > 0$, we obtain from [83, Lemma 2]

$$\begin{aligned} d &= \left| \int_{\mathcal{X}} x^t \left(\frac{\nabla_x f(x)}{g(x)} \right) g(x) \,\mathrm{d}\mu(x) \right| \\ &\le \left(\int_{\mathcal{X}} \|x\|_{\mathcal{X}}^{\alpha}\, g(x) \,\mathrm{d}\mu(x) \right)^{\frac{1}{\alpha}} \left(\int_{\mathcal{X}} \left\| \frac{\nabla_x f(x)}{g(x)} \right\|_{\mathcal{X}^*}^{\alpha^*} g(x) \,\mathrm{d}\mu(x) \right)^{\frac{1}{\alpha^*}} \end{aligned}$$

The proof ends by choosing $g = E_{\phi,f}$ the ϕ-escort density associated to density f. Note now that, again from [83, Lemma 2] the equality is obtained when

$$\nabla_x f(x) \frac{\partial^2 \phi}{\partial y^2}(x, f(x)) = \lambda_1 \nabla_x \|x\|_{\mathcal{X}}^{\alpha}$$

where λ_1 is a negative constant. Consider now the case where $\phi(x,y) = \phi(y)$ is state-independent. Thus, $\nabla_x f(x) \frac{\partial^2 \phi}{\partial y^2}(x, f(x)) = \nabla_x \phi'(f(x))$, that gives

$$\phi'(f(x)) = \lambda_0 + \lambda_1 \|x\|_{\mathcal{X}}^{\alpha}$$

This last equation has precisely the form eq. (6) of proposition 1. □

An obvious consequence of the proposition is that the probabiliy density that minimizes the (α^*, ϕ)-Fisher information subject to the moment constraint $T(x) = \|x\|_{\mathcal{X}}^{\alpha}$ coincides with the maximal ϕ-entropy distribution subject to the same moment constraint.

In the problem of estimation, the purpose is to determine a function $\hat{\theta}(x)$ in order to estimate an unknown parameter θ. In such a context, the Cramér-Rao inequality allows to lowerbound the variance of the estimator thanks to the parametric Fisher information. The idea is thus to extend this to bound any α order mean error using our generalized Fisher information.

Proposition 4 (Parametric (α, ϕ)-Cramér-Rao inequality)**.** *Let f be a probability density function with respect to a general measure μ defined over a set $\mathcal{X}$, where f is parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, and satisfies the conditions of definition 7. Assume that both μ and $\mathcal{X}$ do not depend on θ neither, that f is a jointly measurable function of x and θ, is integrable with respect to x, is absolutely continuous with respect to θ and that the derivatives with respect to each component of θ are locally integrable. Thus, for any estimator $\widehat{\theta}(X)$ of θ that does not depend on θ, we have*

$$M_{\alpha,\phi}\Big[f; \widehat{\theta}(X) - \theta\Big]^{\frac{1}{\alpha}}\, I_{\alpha^*,\phi}[f;\theta]^{\frac{1}{\alpha^*}} \;\ge\; \left| \, m \,+\, \nabla_\theta^t\, b(\theta) \, \right| \tag{22}$$

where

$$b(\theta) = \mathbb{E}\Big[\widehat{\theta}(X) - \theta\Big] \tag{23}$$

is the bias of the estimator and α and α^* are Holder conjugated. When ϕ is state independent, $\phi(x,y) = \phi(y)$, the equality occurs when f is the maximal ϕ entropy distribution subject to the moment constraint $T(x) = \|\Theta(x) - \theta\|_{\Theta}^{\alpha}$.

Proof. The proof follows again that of [83], and starts by evaluating the divergence of the bias. The regularity conditions in the statement of the theorem enable to interchange integration with respect to x and differentiation with respect to θ, so that

$$\nabla_\theta^t\, b(\theta) \; = \; \int_{\mathcal{X}} \Big(\nabla_\theta^t\, \widehat{\theta}(x) \; - \; \nabla_\theta^t\, \theta \Big) f(x) \, \mathrm{d}\mu(x) \; + \; \int_{\mathcal{X}} \Big(\widehat{\theta}(x) - \theta \Big)^t \nabla_\theta f(x) \, \mathrm{d}\mu(x)$$

Note then that $\nabla_\theta^t\, \theta \; = \; m$ and that $\widehat{\theta}$ being independent on θ, one has $\nabla_\theta^t\, \widehat{\theta}(x) \; = \; 0$. Thus, f being a probability density, the equality becomes

$$m + \nabla_\theta^t\, b(\theta) = \int_{\mathcal{X}} \Big(\widehat{\theta}(x) - \theta \Big)^t \frac{\nabla_\theta f(x)}{g(x)}\, g(x) \, \mathrm{d}\mu(x)$$

for any density g non-zero on $\mathcal{X}$. The proof ends with the very same steps that in proposition 4 using [83, Lemma 2]. □

For our three examples, this leads to what follows.

Example 1. *The usual parametric and nonparametric Cramér-Rao inequality are recovered in the usual Shannon context $\phi(x,y) = y \log y$, using the euclidean norm and $\alpha = 2$. The bound in the nonparametric context is saturated for the maximal entropy law, namely the Gaussian.*

Example 2. *In the Rényi-Tsallis context, the generalizations proposed in [81–83] are recovered and, again, when $\alpha = 2$, the bound is saturated in the nonparametric context for the q-Gaussian, maximal entropy law under the second order moment constraint.*

Example 3. *For $\phi(x,y) = \phi(y) = \frac{1}{y}$, aagain, the generalizations proposed in [81–83] are recovered (see the items related to the escort distributions).*

Beyond the mathematical aspect of these relations, they may have great interest to assess an estimator when the usual variance/mean square error does not exist. Moreover, the escort distribution is also a way to emphasis some part of a distribution. For instance, in the Rényi-Tsallis context, one can see that in f^q either the tails or the head of the distribution is emphasized. Playing with q is a way to penalize either the tails, or the head of the distribution in the estimation process.

5. ϕ-heat equation and extended de Bruijn identity

An important relation connecting the Shannon entropy H, coming from the "information world", with the Fisher information I, living in the "estimation world", is given by the de Bruijn identity and is closely linked to the Gaussian distribution. Considering a noisy random variable $Y_t = X + \sqrt{t}N$ where N is a zero-mean d-dimensional standard Gaussian random vector and X a d-dimensional random vector independant of N, and of support independent on parameter t, then

$$\frac{d}{dt} H[f_{Y_t}] = \frac{1}{2} I[f_{Y_t}]$$

where f_{Y_t} stands for the probability distribution of Y_t. This identity is a critical ingredient in proving the entropy power and Stam's inequalities [34]. The starting point to establish the de Bruijn identity is the heat equation satisfied by the probability distribution f_{Y_t}, $\frac{\partial f}{\partial t} = \frac{1}{2} \Delta f$, where Δ stands for the Laplacian operator [98].

Let us consider probability distributions f parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, satisfying what we will call *generalized ϕ-heat equation*,

$$\nabla_\theta f = K \operatorname{div}\Big(\|\nabla_x \phi'(f)\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f \Big) \tag{24}$$

for some $K \in \mathbb{R}^m$ (possibly dependent on θ) and where ϕ is a convex twice differentiable function defined over a set $\mathcal{X} \in \mathbb{R}_+$.

When θ is scalar, this equation is an instance of what is known as quasilinear parabolic equations [99, § 8.8] and arise in various physical problems.

Proposition 5 (Extended de Bruijn identity). *Let f be a probability distribution, parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, defined over a set $\mathcal{X} \subset \mathbb{R}^d$ that do not depend on θ, and satisfying the nonlinear ϕ-heat equation eq. (24) for a twice differentiable convex function ϕ. Assume that $\nabla_\theta \phi(f)$ is absolutely integrable and locally integrable with respect to θ, and that the function $\|\nabla_x \phi'(f)\|_{\mathcal{X}^*}^{\beta-2} \nabla_x \phi(f)$ vanishes at the boundary of $\mathcal{X}$. Thus, distribution f satisfies the extended de Bruijn identity, relating the ϕ-entropy of f and its nonparametric (β, ϕ)-Fisher information as follows,*

$$\nabla_\theta H_\phi[f] = K C_\phi^{1-\beta} \, I_{\beta,\phi}[f] \tag{25}$$

with C_ϕ is the normalisation constant given eq. (16).

Proof. From the definition of the ϕ-entropy, the smoothness of the assumption enabling to use the Leibnitz' rule and differentiate under the integral,

$$\begin{aligned}
\nabla_\theta H_\phi[f] &= -\int_{\mathcal{X}} \phi'(f(x)) \, \nabla_\theta f(x) \, \mathrm{d}\mu(x) \\
&= -K \int_{\mathcal{X}} \phi'(f(x)) \operatorname{div}\Big(\|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f(x) \Big) \, \mathrm{d}\mu(x) \\
&= -K \int_{\mathcal{X}} \operatorname{div}\Big(\phi'(f(x)) \|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f(x) \Big) \, \mathrm{d}\mu(x) \\
&\quad + K \int_{\mathcal{X}} \nabla_x^t \phi'(f(x)) \|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f(x) \, \mathrm{d}\mu(x) \\
&= -K \int_{\mathcal{X}} \operatorname{div}\Big(\|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x \phi(f(x)) \Big) \, \mathrm{d}\mu(x) \\
&\quad + K \int_{\mathcal{X}} \big(\phi''(f(x))\big)^{\beta-1} \|\nabla_x f(x)\|_{\mathcal{X}^*}^{\beta} \, \mathrm{d}\mu(x)
\end{aligned}$$

where the second line comes from the ϕ-heat equation and where the third line comes from the product derivation rule.

Now, from the divergence theorem, the first term of the right handside reduces to the integral of $\|\nabla_x \phi'(f)\|_{\mathcal{X}^*}^{\beta-2} \nabla_x \phi(f)$ on the boundary of $\mathcal{X}$, that vanishes from the assumption of the proposition, while the second term of the right handside related to C_ϕ and the (β, ϕ)-Fisher information from eqs. (16), (17) and definition 6. $\square$

Coming back to the special examples we presented all along the paper:

Example 1. *In the Shannon entropy context, for $K = \frac{1}{2}$ and $\beta = 2$, the standard heat-equation is recovered and the usual de Bruijn identity is recovered.*

Example 2. *The case where $\phi(y) = y^q$ was intensively studied in [84] and the results of the paper are naturally recovered. In particular, the generalized ϕ-heat equation appears in anomalous diffusion in porous medium [84,100,101].*

Example 3. *For $\phi(x,y) = \phi(y) = \frac{1}{y}$, once again one find the same form for the generalized heat equation than in [84,100,101], and therefore the same forme of the generalized de Bruijn's identity of [84] (see the items related to the escort distributions).*

Note that various physical non linear diffusions equation are encompassed in the generalized ϕ-heat equation [101,102].

6. Concluding remarks

In this paper, we extended as far as possible the identities and inequalities which link the classical informational quantities – Shannon entropy, Fisher information, moments, ..., in the framework of the ϕ-entropies. Our first result concerns the inverse maximum entropy problem, starting with a probability distribution and constraints and searching for which entropy the distribution is the maximizer. If such a study was already tackled, it is extended here in a much more general context. We used general reference measures — not necessarily discrete or of Lebesgue. We also considered the case where the distribution and constraints do not share the same symmetries, which leads to state-dependent entropic functionals. Our second result is the generalization of the Cramér-Rao inequality in the same setting: to this end, a generalized Fisher information and generalized moments are introduced, both based on a convex function ϕ (and a so-called ϕ-escort distribution). The Cramér-Rao inequality is saturated precisely for the maximum ϕ-entropy distribution with the same moment constraints, linking all information quantities together. Finally, our third result is the statement of a generalized de Bruijn identity, linking the ϕ-entropy rate and the ϕ-Fisher information of a distribution satisfying an extended heat equation, called ϕ-heat equation. Moreover, dealing with usual distributions (Gaussian, q-Gaussian, exponential) and usual moments (mean, second order), all classical results are recovered as limit cases. Beyond these results, remind that the Shannon (Gibbs) entropy is well adapted to the study of systems in the equilibrium and the maximal entropy distributions or Boltzmann distributions fall to the exponential family [33,34,52,54]. As an alternative, extended entropies are often considered better suited to analyse systems out of equilibrium, where the observed distributions do not belong to the exponential (Boltzman-type) family [17,55–59], like the maximal entropy distributions in the general setting of this paper.

In this panel, two important inequalities still miss. The first one is entropy power inequality (EPI), which states that the entropy power (exponential of twice the entropy) of the sum of two continuous independent random variables is higher than the sum of the individual entropy powers [1]. The second one is the Stam's inequality which lowerbounds the product of the entropy power and the Fisher information. For the former, despite many efforts, the literature on extended version only treat special cases. For instance, some extensions in the classical settings exist for discrete variables but are somewhat limited [103–105]. In the continuous framework, the EPI was also extended to the class of the Rényi entropies (log of a ϕ-entropy with $\phi(u) = u^\alpha$) [106]. Important properties that play a key role in the inequality is that the Rényi's entropies are invariant to an afine transform of unit determinant and monotonic under convolution, properties that seem lost in the very general setting considered here. This fact leaves little room to extend the EPI in our general settings. About the Stam inequality, at a first glance, the fact that the proof is based on the EPI seems to close any hope to extend it to the ϕ-entropy framework. However, it was remarkably extended to the Rényi's entropies, base on the Gagliardo-Nirenberg inequality [78,80,81,107]. Nevertheless, a key property is that both the entropy power and the extended Fisher information have scaling properties, that are lost in the general setting of the ϕ-entropies. A possible way to overcome the (apparent) limits just evoked could be to mimic alternative proofs such that based on optimal transport [108]. This approach precisely drops off any use of Young or Sobolev-like

[1] In fact, there exist other equivalent versions which can be found e.g., in [34,69].

inequalities. As far as we feel, there is thus a little room for extensions in the settings of the paper. Both the extension of the EPI and Stam inequalities are left as a perspective.

Another perspective lies in the estimation of the generalized moments from data (or from estimates). Such a possibility would confer an operational role to our Cramér-Rao inequality, i.e., by computing the estimator's generalized moments and comparing them to the bound. A difficulty resides in the presence of the ϕ-escort distribution which forbids empirical or Monte-Carlo approaches. The escort distribution needs to be estimated. This problem seems not far from the estimation of entropies from data and plug-in approaches used in such problems can thus be considered, like kernel approaches [109–111], nearest neighbor approaches [111,112], or minimal spanning tree approaches [42]. Of course this perspective goes far beyond the scope of this paper.

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Appendix A. Inverse maximum entropy problem and associated inequalities: some examples

In this appendix, we will now derive in detail several case of inverse problem of the maximal entropy problem. In each case, we will thus provide the quantities and inequalities associated with the entropic functional ϕ, as derived in the text. In the sequel, for sake of simplicity, we restricts our example to the univariate context $d = 1$.

Appendix A.1. Normal distribution and second-order moment

For a normal distribution, and second order moment constraint

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\,\sigma^2}\right) \qquad \text{and} \qquad T_1(x) = x^2 \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}.$$

We begin by computing the inverse of $y = f_X(x)$, that gives $T_1(x) = x^2 = -\sigma^2 \ln\left(2\pi\sigma^2 y^2\right)$. Note that f_X^{-1} is multivalued, but $T_1\left(f_X^{-1}(\cdot)\right)$ is univalued. Injecting the expression of $T_1\left(f_X^{-1}(y)\right)$ into eq. (7) we obtain

$$\phi'(y) = \left(\lambda_0 - \sigma^2 \log(2\pi\sigma^2)\,\lambda_1\right) - 2\sigma^2\lambda_1 \log y \qquad \text{with} \quad \lambda_1 < 0$$

where the requisit $\lambda_1 < 0$ is necessary to satisfy condition (C1), being condition (C2) satisfied because f_X and T_1 share the same symmetries. This gives, after a reparametrization of the λ_is,

$$\phi(y) = \alpha\, y \log(y) + \beta\, y + \gamma \qquad \text{with} \quad \alpha > 0$$

The judicious choice $\alpha = 1,\ \beta = \gamma = 0$ leads to function

$$\phi(y) = y \log y$$

that gives nothing more than the Shannon entropy as expected,

$$H_\phi[f] = -\int_{\mathcal{X}} f(x) \ln f(x)\, \mathrm{d}\mu(x)$$

where $\mathcal{X}$ is now the support of f (overall, the obtained family of entropy is the Shannon's one up to a scaling and a shift).

Now, $\phi''(y) \propto \frac{1}{y}$ leading to the escort distribution def. 4 as $E_{\phi,f} = f$ so that, as expected, the (α, ϕ) moments def. 5 are the usual moments of order α. When $\beta = 2$ and the usual euclidean norm is considered, the (β, ϕ)-Fisher informations def. 6 & 7 are the usual Fisher informations and the usual Cramér-Rao inequalities prop. 3 & 4 are recovered for $\alpha = 2$. Finally, for $\beta = 2$, the usual euclidean norm, the ϕ-heat equation eq. (24) turns to be the heat equation, satisfied by the gaussian, so that the usual de Bruijn identity is naturally recovered from prop. 5.

Appendix A.2. q-Normal distribution and second-order moment

For q-normal distribution, also known as Tsallis distributions, Student-t and -r, and a second order moment constraint,

$$f_X(x) = C_q \left(1 - (q-1)\frac{x^2}{\sigma^2}\right)_+^{\frac{1}{(q-1)}} \qquad \text{and} \qquad T_1(x) = x^2,$$

where $q > 0$, $q \neq 1$, $x_+ = \max(x, 0)$ and C_q is a normalization coefficient. The support of f_X is $\mathcal{X} = \mathbb{R}$ when $q < 1$ and $\mathcal{X} = \left(-\frac{\sigma}{\sqrt{q-1}}\,;\,\frac{\sigma}{\sqrt{q-1}}\right)$ when $q > 1$.

The inverse of $y = f_X(x)$ gives $T_1(x) = x^2 = \frac{\sigma^2}{q-1}\left(1 - \left(\frac{y}{C_q}\right)^{q-1}\right)$. Note that, again, f_X^{-1} is multivalued, but $T_1\left(f_X^{-1}(\cdot)\right)$ is univalued. Injecting the expression of $T_1\left(f_X^{-1}(y)\right)$ into eq. (7) we get

$$\phi'(y) = \left(\lambda_0 + \frac{\lambda_1 \sigma^2}{q-1}\right) - \frac{\lambda_1 \sigma^2}{(q-1) C_q^{q-1}}\, y^{q-1} \qquad \text{with} \quad \lambda_1 < 0$$

where the requisit $\lambda_1 < 0$ is necessary to satisfy condition (C1), being condition (C2) satisfied because f_X and T_1 share the same symmetries. This gives, after a reparametrization of the λ_is,

$$\phi(y) = \alpha\, \frac{y^q - y}{q-1} + \beta y + \gamma \qquad \text{with} \quad \alpha > 0$$

Note that the inverse of f_X is defined over $\left(0\,;\,C_q\right)$ but, without contradiction, the domain of definition of the entropic functional can be extended to $\mathbb{R}_+$.

Then, a judicious choice of parameters is $\alpha = 1, \beta = \gamma = 0$ that yields

$$\phi(y) = \frac{y^q - y}{q-1}.$$

and an associated entropy is then

$$H_\phi[f] = \frac{1}{1-q}\left(\int_{\mathcal{X}} f(x)^q \,\mathrm{d}\mu(x) - 1\right) :$$

where $\mathcal{X}$ is now the support of f. This entropy is nothing but the Havrdat-Charvát-Tsallis entropy [12,14,17,91] (overall, we obtain this entropy up to a scaling and a shift).

Then, $\phi''(y) = q y^{q-2}$: so that, from def. 4, and then from def. 5, def. 6 & 7 respectively, we achieve to $M_{\phi,\alpha}[f]$ and $I_{\phi,\alpha}[f]$ as respectively the q-moment of order α and the (q, β)-Fisher information defined previously in [78–83] (with the symmetric q index given here by $2 - q$). The extended Cramér-Rao inequality proved in [78,82,83] is then recovered from prop. 3 & 4, and the generalized de Bruijn's identity of [84] is also recovered from eq. (24) & prop. 5.

Note that when $q \to 1$: f_X tends to the gaussian distribution. It appears that H_ϕ tends to the Shannon's entropy, $I_{\phi,2}$ to the usual Fisher's information and $M_{\phi,\alpha}$ to the usual moments (both considering the euclidean norm): all the settings related to the Gaussian distribution is naturally recovered.

Appendix A.3. q-exponential distribution and first-order moment

The same entropy functional can readily be obtained for the so-called q-exponential

$$f_X(x) = C_q(1-(q-1)\beta x)_+^{\frac{1}{(q-1)}} \qquad \text{and} \qquad T_1(x) = x \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}_+.$$

It suffices to follow the very same steps as above, leading again to the Havrdat-Charvát-Tsallis entropy, the q-moments of order α and the (q, β)-Fisher information defined previously in [78–83] (with the symmetric q index given here by $2 - q$) as for the q-Gaussian distribution and to the extended Cramér-Rao inequality proved in [82,83] as well.

Now when $q \to 1$: f_X tends to the exponential distribution, known to be of maximum Shannon's entropy on $\mathbb{R}_+$ under the first order moment constraint. Again H_ϕ tends to the Shannon's entropy, $I_{\phi,2}$ to the usual Fisher's information and $M_{\phi,\alpha}$ to the usual moments (both considering the euclidean norm): all the settings related to the exponential distribution is naturally recovered.

Appendix A.4. The arcsine distribution

The arcsine distribution is a special case of the beta distribution with $\alpha = \beta = \frac{1}{2}$. We consider here the centered and scaled version of this distribution which writes

$$f_X(x) = \frac{1}{\sqrt{s^2 - \pi^2 x^2}} \qquad \text{on} \qquad \mathcal{X} = \left(-\frac{s}{\pi}\,;\,\frac{s}{\pi}\right).$$

The inverse distributions $f_{X,\pm}^{-1}$ on $\mathcal{X}_- = \left(-\frac{s}{\pi}\,;\,0\right)$ and $\mathcal{X}_+ = \left(0\,;\,\frac{s}{\pi}\right)$ write then

$$f_{X,\pm}^{-1}(y) = \pm\frac{\sqrt{s^2y^2-1}}{\pi y}, \qquad y \ge \frac{1}{s}$$

Let us now consider again either a second order moment as the constraint, or (partial) first order moment(s).

Appendix A.4.1. Second order moment

When the second order moment $T_1(x) = x^2$ is constrained, conditions(C2) is satisfied, so that, injecting the expression of $T_1\left(f_X^{-1}(y)\right)$ into eq. (7) one immediately obtains

$$\phi'(y) = \lambda_0 + \lambda_1\left(\frac{s^2}{\pi^2} - \frac{1}{\pi^2 y^2}\right) \qquad \text{with} \quad \lambda_1 > 0$$

where the requisit $\lambda_1 < 0$ is necessary to satisfy condition (C1). After a reparametrization of the λ_is, the family of entropy functional is then

$$\phi(y) = \frac{\alpha}{y} + \beta y + \gamma \quad \text{with} \quad \alpha > 0$$

Note that this entropy can be viewed as Havrdat-Charvát-Tsallis entropy for $q = -1$, so that all the generalizations (escort, moments, Cramer-Rao inequality, de Bruijn identity) set out appendix A.2 are recovered taking the limit $q \to -1$.

Appendix A.4.2. (Partial) first-order moment(s)

Since the distribution has not the same variation as $T_1(x) = x$, i.e., condition (C1) cannot be satisfied, either we turn out to consider the arcsine distribution a critical point (extremal, inflection point) of a non concave “entropy”, or as a maximum entropy when constraints are of the type

$$T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_\pm}(x)$$

Now, dealing respectively with the partial-moment constraints $T_{\pm,1}$ and with the uniform constraint T_1, we obtain from eq. (15) and eq. (14) respectively,

$$\phi_\pm'(y) = \lambda_0 + \lambda_{\pm,1} \frac{\sqrt{s^2y^2-1}}{\pi y} \qquad \text{and} \qquad \widetilde{\phi}_\pm'(y) = \lambda_0 \pm \lambda_1 \frac{\sqrt{s^2y^2-1}}{\pi y}$$

where the sign is absorbed in the factors $\lambda_{\pm,1}$ in the first case. Dealing with the partial moments, to satisfy condition (C1) one must impose

$$\lambda_{\pm,1} > 0$$

At the opposite, condition (C1) cannot be satisfied for the second case (one would has to impose $\pm\lambda_1 > 0$ on $\mathcal{X}_\pm$). After a reparametrization of the λ_is, one obtain the branches of the entropic functional under the form $\phi_\pm(y) = \phi_{\pm,\mathrm{u}}(sy)$ with $\phi_{\pm,\mathrm{u}}(u) = \alpha_\pm\left(\sqrt{u^2-1} + \arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;\,+\infty)}(u) + \beta\, u + \gamma_\pm$ and with $\alpha_\pm > 0$, and the branches for the non-convex case $\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy)$ with $\widetilde{\phi}_{\pm,\mathrm{u}}(u) = \pm\alpha\left(\sqrt{u^2-1} + \arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;\,+\infty)}(u) + \beta\, u + \gamma_\pm$.

In this case, s appears as an additional parameter of this family of the ϕ-entropy.

In both case, the entropic functionals are defined for $u > 1$ due to the domain where f_X is invertible. However, in the first case, one can extend the domain to $\mathbb{R}_+$ insuring both the continuity of the entropic functional and its derivative at $u = 1$ (and thus everywhere), by vanishing the derivative of the entropic functional at $u = 1$, which impose $\beta = 0$. This is also possible for the functionals $\widetilde{\phi}_{\pm,\mathrm{u}}$ by also imposing condition $\beta = 0$. This leads respectively to

$$\phi_\pm(y) = \phi_{\pm,\mathrm{u}}(sy) \qquad \text{with}$$

$$\phi_{\pm,\mathrm{u}}(u) = \alpha_\pm\left(\sqrt{u^2-1} + \arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;\,+\infty)}(u) + \gamma_\pm, \qquad \alpha_\pm > 0$$

and the branches for the non-convex case

$$\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy) \qquad \text{with}$$

$$\widetilde{\phi}_{\pm,\mathrm{u}}(u) = \pm\alpha\left(\sqrt{u^2-1} + \arctan\left(\frac{1}{\sqrt{u^2-1}}\right)\right)\mathbb{1}_{(1\,;\,+\infty)}(u) + \gamma_\pm$$

Remarkably, in the first case, an univalued entropic functional can be obtain imposing both $\alpha_+ = \alpha_-$, $\gamma_+ = \gamma_-$. Looking more attentively this choice, one can observe that it corresponds to the one obtained by the moment constraint $T_1(x) = |x|$, which have the same symmetries that f_X.

The uniform function ϕ_u is represented figure A1 for $\alpha_\pm = 1$, $\gamma_\pm = 0$.

On ne parle pas des moment, Fisher and so on... À faire ?

Appendix A.5. The logistic distribution

In this case,

$$f_X(x) = \frac{1-\tanh^2\left(\frac{2x}{s}\right)}{s} \qquad \text{and} \qquad T_1(x) = x^2 \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}.$$

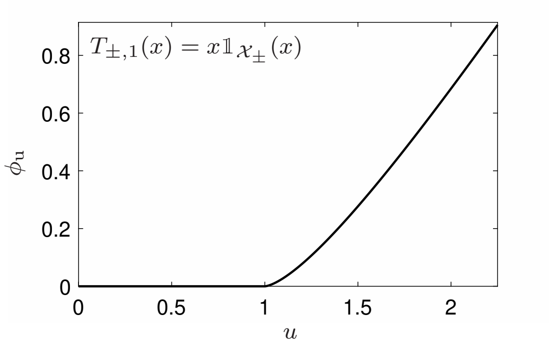

Figure A1. Univalued entropy functional ϕ_{u} derived from the arcsine distribution with partial constraints $T_{\pm,1}(x) = x\mathbb{1}_{\mathcal{X}_\pm}(x)$.

This distribution, which resembles the normal distribution but has heavier tails, has been used in many applications **[?] des references?**. One can then check that over each interval

$$\mathcal{X}_\pm = \mathbb{R}_\pm$$

the inverse distribution writes

$$f_{X,\pm}^{-1}(y) = \pm\frac{s}{2} \operatorname{argtanh} \sqrt{1-sy}, \qquad y \in \left(0\,;\,\frac{1}{s}\right]$$

We concentrate now on a second order constraint, that respect the symmetry of the distribution, and on first order constrain(s) that does not respect the symmetry.

Appendix A.5.1. Second order moment constraint

In this case, injecting the expression of $T_1\left(f_X^{-1}(y)\right)$ into eq. (7), we immediately obtain

$$\phi'(y) = \lambda_0 + \frac{\lambda_1 s^2}{4}\left(\operatorname{argtanh}\sqrt{1-sy}\right)^2 \qquad \text{with} \qquad \lambda_1 < 0$$

where $\lambda_1 < 0$ is required to satisfy condition (C1). After a reparametrization, we thus achieve the family of entropy functionals $\phi(y) = \phi_{\mathrm{u}}(sy)$ with $\phi_{\mathrm{u}}(u) = -\alpha\left[u\left(\operatorname{argtanh}\sqrt{1-u}\right)^2 - 2\sqrt{1-u}\operatorname{argtanh}\sqrt{1-u} - \log u\right]\mathbb{1}_{(0\,;1]}(u) + \beta u + \gamma$ with $\alpha > 0$.

Here again, s is an additional parameter for this family of ϕ-entropies.

The entropic functional is defined for $u \le 1$ due to the domain f_X is invertible. To evaluate the ϕ-entropy for a given distribution f, one can play on parameter s so as to restrain sf to $[0\,;1]$. But one can also extend the functional to $\mathbb{R}_+$ while remaining of class C^1 by vanishing the derivative at $u = 1$: this imposes $\beta = 0$ and leads to the entropic functional

$$\phi(y) = \phi_{\mathrm{u}}(sy) \qquad \text{with}$$

$$\phi_{\mathrm{u}}(u) = \gamma - \alpha\left[u\left(\operatorname{argtanh}\sqrt{1-u}\right)^2 - 2\sqrt{1-u}\operatorname{argtanh}\sqrt{1-u} - \log u\right]\mathbb{1}_{(0\,;1]}(u), \;\; \alpha > 0$$

depicted figure A2(a) for $\alpha = 1$, $\gamma = 0$.

On ne parle pas des moment, Fisher and so on... À faire ?

Appendix A.5.2. (Partial) first-order moment(s) constraint(s)

Since f_X and $T(x) = x$ do no share the same symmetries, one cannot interpret the logistic distribution as a maximum entropy constraint by the first order moment. However, constraining the partial means over $\mathcal{X}_\pm = \mathbb{R}_\pm$ allows such an interpretation, using then multiform entropies, while the alternative is to relax the concavity property of the entropy (but again) one can only insure that the distribution from which it comes from is a critical point. To be more precise, one chooses

$$T_{\pm,1}(x) = x\, \mathbb{1}_{\mathcal{X}_\pm}(x) \qquad \text{or} \qquad T_1(x) = x$$

We thus obtain from eq. (15) and eq. (14) respectively, over each set $\mathcal{X}_\pm$, the branches

$$\phi_\pm'(y) = \lambda_0 + \frac{\lambda_{\pm,1} s}{2} \operatorname{argtanh} \sqrt{1-sy} \qquad \& \qquad \widetilde{\phi}_\pm'(y) = \lambda_0 \pm \frac{\lambda_1 s}{2} \operatorname{argtanh} \sqrt{1-sy}$$

where the sign is absorbed on $\lambda_\pm$ for the first case. Dealing with the partial moments, to satisfy condition (C1) one must impose

$$\lambda_\pm < 0$$

At the opposite, condition (C1) cannot be satisfied for the second case (one would has to impose $\pm\lambda_1 < 0$ on $\mathcal{X}_\pm$). After a reparametrization of the λ_is, one obtain the branches of the entropic functional under the form $\phi_\pm(y) = \phi_{\pm,\mathrm{u}}(sy)$ with $\phi_{\pm,u}(u) = -\alpha_\pm\big(u \operatorname{argtanh}\sqrt{1-u} - \sqrt{1-u}\big)\mathbb{1}_{(0\,;1]}(u) + \beta\, u + \gamma_\pm$ where $\alpha_\pm > 0$ and the branches for the non-convex case $\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy)$ with $\widetilde{\phi}_{\pm,u}(u) = \pm\alpha\big(u \operatorname{argtanh}\sqrt{1-u} - \sqrt{1-u}\big)\mathbb{1}_{(0\,;1]}(u) + \beta\, u + \gamma_\pm$.

Once again, appear an additional parameter, s, for these families of entropies.

In both cases, even if the inverse of f_X restricts u to be lower than 1, one can either play on parameter s to allow to compute the ϕ-entropy of a distribution f, or to extend the entropic functionals to $\mathbb{R}_+$ by vanishing the derivative at $u = 1$. This impose $\beta = 0$ and thus the entropic functional,

$$\phi_\pm(y) = \phi_{\pm,\mathrm{u}}(sy) \qquad \text{with}$$
$$\phi_{\pm,u}(u) = \gamma_\pm - \alpha_\pm\Big(u \operatorname{argtanh}\sqrt{1-u} - \sqrt{1-u}\Big)\mathbb{1}_{(0\,;1]}(u), \qquad \alpha_\pm > 0$$

and the branches for the non-convex case

$$\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(sy) \qquad \text{with}$$
$$\widetilde{\phi}_{\pm,u}(u) = \gamma_\pm \pm \alpha\Big(u \operatorname{argtanh}\sqrt{1-u} - \sqrt{1-u}\Big)\mathbb{1}_{(0\,;1]}(u)$$

Remarkably, in the first case, an univalued entropic functional can be obtain imposing both $\alpha_+ = \alpha_-$, $\gamma_+ = \gamma_-$. Here also, such a choice is equivalent than considering the constraint $T_1(x) = |x|$, and thus allows to respect the symmetries of the distribution, allowing thus to recover a classical ϕ-entropy.

The uniform function ϕ_u is represented figure A2(b) for $\alpha_\pm = 1$, $\gamma_\pm = 0$.

On ne parle pas des moment, Fisher and so on... À faire ?

Appendix A.6. The gamma distribution and (partial) p-order moment(s)

As a very special case, consider here the gamma distribution expressed as

$$f_X(x) = \frac{(\Gamma(q)x)^{q-1}\exp\left(-\frac{\Gamma(q)}{r}\, x\right)}{r^q} \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}_+.$$

Parameter $q > 0$ is known as shape parameter of the law, while $\sigma = \frac{r}{\Gamma(q)} > 0$ is a scaling parameter.

Let us concentrate on the case $q > 1$ for which the distribution is non-monotonous, unimodal, where the mode is located at $x = \frac{r,(q-1)}{\Gamma(q)}$, and $f_X(\mathbb{R}_+) = \left[0\,;\, \frac{(q-1)^{q-1}e^{1-q}}{r}\right]$

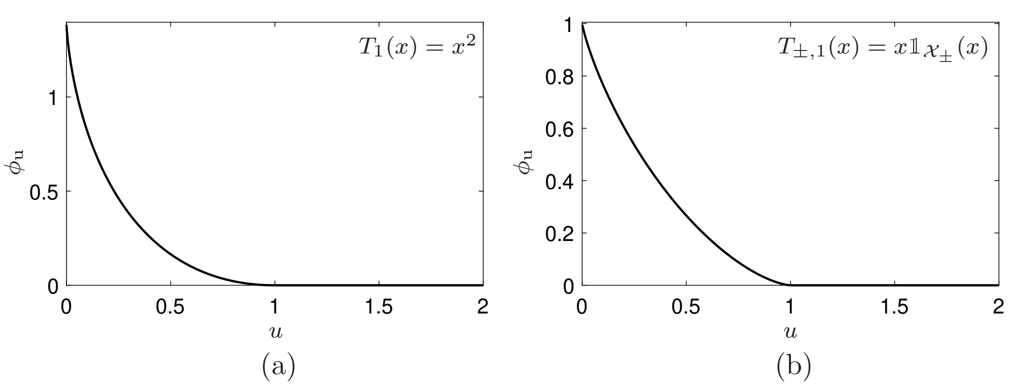

Figure A2. Entropy functional ϕ_u derived from the logistic distribution: (a) with $T_1(x) = x^2$ and (b) with $T_{\pm,1}(x) = x\mathbb{1}_{\mathcal{X}_\pm}(x)$.

Here again it cannot be as maximizer of a ϕ-entropy constraint subject to a moment of order $p > 0$. Here, we can again consider partial moments as constraints,

$$T_{k,1}(x) = x^p \, \mathbb{1}_{\mathcal{X}_k}(x), \qquad k \in \{0, -1\} \qquad \text{where}$$

$$\mathcal{X}_0 = \left[0\,;\, \frac{r\,(p-1)}{\Gamma(q)}\right) \qquad \text{and} \qquad \mathcal{X}_{-1} = \left[\frac{r\,(q-1)}{\Gamma(q)}\,;\, +\infty\right),$$

or as a critical point of an ϕ-like entropy by constraining the moment

$$T_1(x) = x^p \quad \text{over} \quad \mathcal{X} = \mathbb{R}_+$$

Inverting $y = f_X(x)$ leads to the equation

$$-\frac{\Gamma(q)\,x}{r\,(q-1)} \exp\left(-\frac{\Gamma(q)\,x}{r\,(q-1)}\right) = -\frac{(ry)^{\frac{1}{q-1}}}{q-1}$$

to be solved. As expected, this equation has two solutions. These solutions can be expressed via the multivalued Lambert-W function W defined by $z = \mathrm{W}(z)\exp(\mathrm{W}(z))$, i.e., W is the inverse function of $u \mapsto u\exp(u)$[113, § 1], leading to the inverse functions

$$f_{X,k}^{-1}(y) = -\frac{r\,(q-1)}{\Gamma(q)}\,\mathrm{W}_k\left(-\frac{(ry)^{\frac{1}{q-1}}}{q-1}\right), \qquad ry \in \left[0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right],$$

where k denotes the branch of the Lambert-W function. $k = 0$ gives the principal branch and here it is related to the entropy part on $\mathcal{X}_0$, while $k = -1$ gives the secondary branch, related to $\mathcal{X}_{-1}$ here.

Applying (15) to obtain the branches of the functionals of the multiform entropy, one has thus to integrate the functions

$$\phi_k'(y) = \lambda_0 + \lambda_{k,1}\left[-\frac{r\,(q-1)}{\Gamma(q)}\,\mathrm{W}_k\left(-\frac{(ry)^{\frac{1}{q-1}}}{q-1}\right)\right]^p$$

where, to insure the convexity of the ϕ_k,

$$(-1)^k \lambda_{k,1} > 0$$

The same approach allows to design $\widetilde{\phi}_k$, with a unique λ_1 instead of the $\lambda_{k,1}$ and without restriction on λ_1.

Integrating the previous expression is not an easy task. Relation $u\,(1 + \mathrm{W}_k(u))\,\mathrm{W}_k'(u) = \mathrm{W}_k(u)$ [113, Eq. 3.2] suggests that a way to make the integration is to search for

$$\phi_k(y) = \phi_{k,\mathrm{u}}(ry)$$

where the primitive of the term with the Lambert function in $\phi_{k,\mathrm{u}}(u)$ is searched under the form $u \sum_{l \ge 0} a_l \left[-\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^{l+p}$, identifying the coefficients a_l. Such an approach, after a reparametrization of the λ_is, leads to the family of entropic functional given by

$$\begin{aligned}\phi_{k,\mathrm{u}}(u) \;&=\; \beta\, u \,+\, \gamma_k \,+\, \alpha_k\, u \left[-\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^p \times\\ &\left[1 - \frac{p}{p+q-1}\, {}_1F_1\left(1\,;\, p+q\,;\, (1-q)\,\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right) \right] \mathbb{1}_{\left(0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right)}(u)\end{aligned}$$

with

$$(-1)^k \alpha_k > 0$$

and where ${}_1F_1$ is the confluent hypergeometric (or Kummer's) function [114, § 13] or [115, § 9.2]. One can verify a posteriori that these functions are the ones we search for.

Again, p, q, r are additional parameters for this famili of entropies.

Then, from the domaine of definition of the inverse of f_X, u is restricted to $\left(0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right)$, whcih can be compensated for by playing with parameter r. At the opposite, noting that $\mathrm{W}_k(-e^{-1}) = -1$, to extend the entropic functionals to C^1 functions on $\mathbb{R}_+$, one would have to impose $\beta + \alpha_k = 0$ to vanish the derivatives at $u = e^{1-a}$. This is impossible because from $(-1)\alpha_k > 0$ one cannot impose $\alpha_k = -\beta$. One can choose to impose

$$\beta = -\alpha_{-1}$$

to vanish the derivative for ϕ_{-1}, that is given for the semi-infinite domain $\mathcal{X}_{-1}$. Moreover, even a convex extension is impossible since we would have to impose $\beta + \alpha_k \le \beta$ to insure the increaseness of the ϕ_k. We can however choose the γ_k such that the ϕ_k coincide at $u = 0$ for instance (e.g., to vanish them at 0 to insure the existence of the ϕ-entropy), that gives

$$\gamma_0 = \gamma_{-1} \,-\, \frac{p\,\Gamma(p+q-1)}{(q-1)^p}\,\alpha_{-1}$$

using successively [113, Eq. 3.1] and [114, Eq. 13.1.2] for W_0, and successively [114, Eq. 13.1.4] (W_{-1} tending to $-\infty$ in 0^-), $\mathrm{W}_{-1}(u)\exp(\mathrm{W}_{-1}(u)) = u$, and [113, Eq. 4.6 & lines that follow] for W_{-1}.

The same algebra leads to the same expression for the $\widetilde{\phi}_k$, except that $\lambda_{k,1}$ are replaced by a unique λ_1.

Interestingly, when $a \to 1$, the gamma law tends to the exponential distribution and, at the same time, $\mathcal{X}_0 \to \emptyset$, $\mathcal{X}_{-1} \to \mathbb{R}_+$. to finish

The multivalued function ϕ_u in the concave context is represented figure A3 for $p = 2, q = 2$ and $q = 5$, and with the choice $\alpha_0 = 1$, $\alpha_{-1} = -0.05$, $\beta = -\alpha_{-1}$, $\gamma_0 = 0$, $\gamma_{-1} = \frac{p\Gamma(p+q-1)}{(q-1)^p}$.

On ne parle pas des moment, Fisher and so on... À faire ?

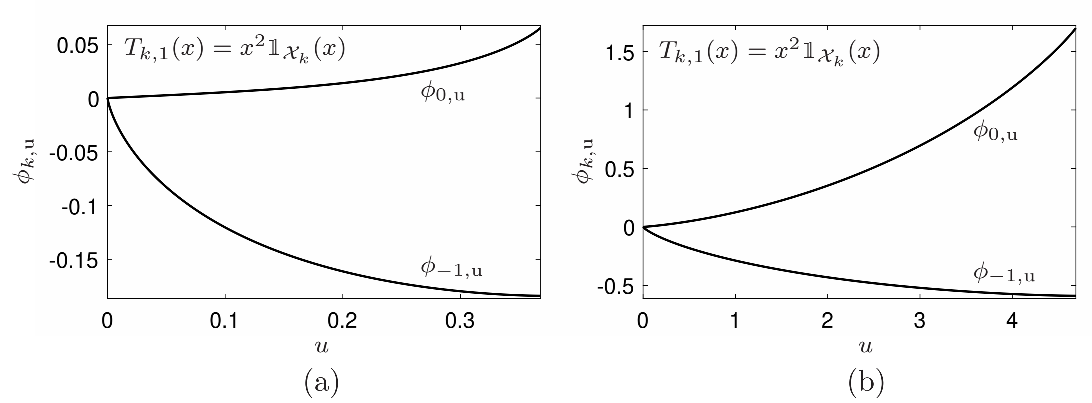

Figure A3. Multiform entropy functional ϕ_u derived from the gamma distribution with the partial moment constraints $T_{k,1}(x) = x^2 \mathbb{1}_{\mathcal{X}_k}(x)$, $k \in \{0, -1\}$. (a): $q = 2$; (b): $q = 5$.

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