


ϕ -informational measures: some results and interrelations.

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Abstract: In this paper we study extend well-know informational measure –Shannon entropy, Fisher information, statistical moments– and revisit the relationships they satisfy. Between usual information measures in a wider context. More precisely, we consider generalizations of these measure, generated by a convex function ϕ : ϕ -entropies, ϕ -fisher information and generalized moments via a ϕ -escort distribution. First, we study the maximum entropy problem, and more specially the inverse problem –search for the entropy so that a given distribution is its maximizer–. We then this paper focus on the interrelations between informational measures defined via a convex function ϕ . maximum entropy pconstraints. Contrary to the usual problem of finding the maximizer of a given entropy, or of selecting constraints such that a given distribution is a maximizer of the considered entropy, we consider here the problem of the determination of an entropy such that a given distribution is its maximizer. Our goal is to adapt the entropy to its maximizer, with potential application in entropy-based goodness-of-fit tests for instance. It allows us to consider distributions outside the exponential family – to which the maximizers of the Shannon entropy belong –, and also to consider simple moment constraints, estimated from the observed sample. Our approach also yields entropic functionals that are function of both probability density and state, allowing us to include skew-symmetric or multimodal distributions in the setting. Finally, extended informational quantities are introduced, such that generalized moments and generalized Fisher informations. With these extended quantities, we propose extended version of the Cramér-Rao inequality and of the de Bruijn identity, valid or saturated for the maximal entropy distribution corresponding to the generalized entropy previously studied.

Keywords: ϕ -entropies; state-dependent ϕ -entropies; (inverse) maximum ϕ -entropy distribution; ϕ -escort distributions; ϕ -Fisher information; ϕ -moments; generalized Cramér-Rao inequality; ϕ -heat equation; generalized de Bruijn's identity/

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1. Introduction

Since the pioneer works of von Neuman [1], Shannon [2], Boltzmann, Maxwell, Planck and Gibbs [3–9] on the entropy as a tool for uncertainty or information measure, many investigations were devoted to the generalization of the so-called Shannon entropy and its associated measures [10–22]. If the Shannon measures are compelling, especially in the communication domain, for compression purposes, many generalizations proposed later on has also showed promising interpretations and applications (Panter-Dite formula in quantification where the Rényi or Havdra-Charvát entropy emerges [23–25], codification penalizing long codewords where the Rényi entropy appears [26,27] for instance). The great majority of the extended entropies found in the literature belongs to a very general class of entropic measures called (h, ϕ) -entropies [13,19,20,28–30]. Such a general class (or more precisely the subclass of ϕ -entropies) traced back to the work of Burbea & Rao [28]. They offer not only a general framework to study general properties

shared by special entropies, but they also offer many potential applications as described for instance in [30]. Note that if a large amount of work deals with divergences, entropies occur as special cases when one takes a uniform reference measure.

In the settings of these generalized entropies, the so-called maximum entropy principle takes a special place. This principle, advocated by Jaynes, states that the statistical distribution that describes a system in equilibrium maximizes the entropy while satisfying the system's physical constraints (e.g., the center of mass, energy) [31–35]. In other words, it is the less informative law given the constraints of the system. In the Bayesian approach, dealing with the stochastic modelisation of a parameter, such a principle (or a minimum divergence principle) is often used to choose a prior distribution for the parameter [22,36–39]. It also finds its counterpart in communication, clustering, pattern recognition, problems, among many others [32,33,40–43]. In statistics, some goodness-of-fit tests are based on entropic criteria derived from the same idea of constrained maximal entropic law [44–49]. In a large number of works using the maximum entropy principle, the entropy used is the Shannon entropy, although several extensions exist in the literature. However, if for some reason a generalized entropy is considered, the approach used in the Shannon case does not fundamentally change [50–53].

One can consider the inverse problem which consists in finding the moment constraints leading to the observed distribution as a maximal entropy distribution [50]. Kesavan & Kapur also envisaged a second inverse problem, where both the distribution and the moments are given. The question is thus to determine the entropy so that the distribution is its maximizer. *As a matter of fact, dealing with the Shannon entropy, whatever the constraints considered, the maximum entropy distribution falls in the exponential family [33,34,52,54]. Considering more general entropies allows to escape from this limitation. Moreover, if the Shannon entropy (or the Gibbs entropy in physics) is well adapted to the study of systems in the equilibrium (or thermodynamic limit), extended entropies allow a finer description of systems out of equilibrium [17,55–59], exhibiting their importance.* While the problem was considered mainly in the discrete Shannon settings by Kesavan & Kapur in [50], we will recall it in the general framework of the (h, ϕ) -entropies, and make a further step considering an extended class of these entropies.

While the entropy is a widely used tool for quantifying information (or uncertainty) attached to a random variable or to a probability distribution, other quantities are used as well, such as the moments of the variable (giving information for instance on center of mass, dispersion, skewness, impulsive character), or the Fisher information. In particular, the Fisher information appears in the context of estimation [60,61], in Bayesian inference through the Jeffrey's prior [39,62], but also for complex physical systems descriptions [61,63–67].

Although coming from different worlds (information theory and communication, estimation, statistics, physics), these informational quantities are linked by well-known relations such the Cramér-Rao's inequality, the de Bruijn's identity, the Stam's inequality [34,68–70]. These relationships have been proved very useful in various areas, for instance in communications [34,68,69], in estimation [60] or in physics [71,72], among others. When generalized entropies are considered, it is natural to question the other informational measures' generalization and the associated identities or inequalities. This question gave birth to a large amount of work and is still an active field of research [28,73–84].

In this paper, we show that it is possible to build a whole framework, which associates a target maximum entropy distribution to generalized entropies, generalized moments and generalized Fisher information. In this setting, we derive generalized inequalities and identities relating these quantities, which are all linked in some sense to the maximum entropy distribution.

The paper is organized as follows. In section 2 we recall the definition of the generalized ϕ -entropy. Thus, we come back to the maximum entropy problem in this general settings. Following the sketch of [50], we present a sufficient condition linking the entropic functional and the maximizing distribution, allowing to both solve the direct and the inverse problems. When the sufficient conditions linking the entropic function and the distribution cannot be satisfied, the problem can be solved by introducing state-dependent generalized entropies, which is the purpose of section 3. In section 4, we introduce informational quantities associated to the generalized entropies of the previous sections, such that a generalized escort distribution, generalized moments and generalized Fisher informations. These generalized informational quantities allow to extend the usual informational relations such that the Cramér-Rao inequality, relations saturated (or valid) dealing precisely for the generalized maximum entropy distribution. Finally, in section 5, we show that the extended quantities allows to obtain an extended de Bruijn identity, provided the distribution follows a non-linear heat equation. Some example of determination of ϕ -entropies solving the inverse maximum entropy problem are provided in a short series of appendix, showing in other that the usual quantities are recovered in the well known cases (Gaussian distribution, Shannon entropy, Fisher information, variance).

In what follows, we will define a series of generalized information quantities relative to a probability density defined with respect to a given reference measure μ (e.g., the Lebesgue measure when dealing with continuous random variables, discrete measure for discrete-state random variables, ...). Therefore, rigorously, all these quantities depend on the particular choice of this reference measure. However, for simplicity, we will omit to mention this dependence in the notation along the paper.

2. ϕ -entropies – direct and inverse maximum entropy problems.

Let us first recall the definition of the generalized ϕ -entropies introduced by Csiszàr in terms of divergences, and by Burbea and Rao in terms of entropies:

Definition 1 (ϕ -entropy [28]). *Let $\phi : \mathcal{Y} \subseteq \mathbb{R}_+ \mapsto \mathbb{R}$ be a convex function defined on a convex set \mathcal{Y} . Then, if f is a probability distribution defined with respect to a general measure μ on a set $\mathcal{X} \subseteq \mathbb{R}^d$ such that $f(\mathcal{X}) \subseteq \mathcal{Y}$, when this quantity exists,*

$$H_\phi[f] = - \int_{\mathcal{X}} \phi(f(x)) d\mu(x) \quad (1)$$

is the ϕ -entropy of f .

The (h, ϕ) -entropy is defined by $H_{(h, \phi)}[f] = h(H_\phi[f])$ where h is a nondecreasing function. The definition is extended by allowing ϕ to be concave, together with h nonincreasing [13,19,20,29,30]. If additionally h is concave, then the entropy functional $H_{(h, \phi)}[f]$ is concave.

Since we are interested in the maximum entropy problem and because h is monotone, we can restrict our study to the ϕ -entropies. Additionally, we will assume that ϕ is strictly convex and differentiable.

A related quantity is the Bregman divergence associated with convex function ϕ :

Definition 2 (Bregman divergence and functional Bregman divergence [22,85]). *With the same assumptions as in 1, the Bregman divergence associated with ϕ defined on a convex set \mathcal{Y} is given by the function defined on $\mathcal{Y} \times \mathcal{Y}$,*

$$D_\phi(y_1, y_2) = \phi(y_1) - \phi(y_2) - \phi'(y_2)(y_1 - y_2). \quad (2)$$

Applied to two functions $f_i : \mathcal{X} \mapsto \mathcal{Y}$, $i = 1, 2$, the functional Bregman divergence writes

$$\mathcal{D}_\phi(f_1, f_2) = \int_{\mathcal{X}} \phi(f_1(x)) \, d\mu(x) - \int_{\mathcal{X}} \phi(f_2(x)) \, d\mu(x) - \int_{\mathcal{X}} \phi'(f_2(x))(f_1(x) - f_2(x)) \, d\mu(x). \quad (3)$$

131 A direct consequence of the strict convexity of ϕ is the nonnegativity of the (functional) Bregman
132 divergence: $\mathcal{D}_\phi(y_1, y_2) \geq 0$ and $\mathcal{D}_\phi(f_1, f_2) \geq 0$, with equality if and only if $y_1 = y_2$ and
133 $f_1 = f_2$ almost everywhere respectively.

134 More generally, the Bregman divergence is defined for multivariate convex func-
135 tions, where the derivative is replaced by gradient operator [85]. Extensions for convex
136 function of functions also exist, where the derivative is in the sense of Gâteaux [86]. Such
137 general extensions are not useful for our purposes; thus, we restrict ourselves to the
138 above definition where $\mathcal{Y} \subseteq \mathbb{R}_+$.

139 2.1. Maximum entropy principle: the direct problem

Let us first recall the maximum entropy problem that consists in searching for the distribution maximizing the ϕ -entropy (1) subject to constraints on some moments $\mathbb{E}[T_i(X)]$ with $T_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \dots, n$. This direct problem writes

$$f^* = \operatorname{argmax}_{f \in C_t} \left(- \int_{\mathcal{X}} \phi(f(x)) \, d\mu(x) \right) \quad (4)$$

with

$$C_t = \{f \geq 0 : \mathbb{E}[T_i(X)] = t_i, i = 0, \dots, n\}, \quad (5)$$

140 where $T_0(x) = 1$ and $t_0 = 1$ (normalization constraint). The maximization problem being
141 strictly concave, the solution exists and is unique. A technique to solve the problem can
142 be to use the classical Lagrange multipliers technique, but this approach requires mild
143 conditions [50,51,53,87–89]. In the following proposition, we recall a sufficient condition
144 relating f and ϕ so that f is the problem's solution. Below, we prove the result without
145 the use of the Lagrange technique.

Proposition 1 (Maximal ϕ -entropy solution [50]). *Suppose that there exists a probability distribution $f \in C_t$ satisfying*

$$\phi'(f(x)) = \sum_{i=0}^n \lambda_i T_i(x), \quad (6)$$

146 for some $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$. Then, f is the unique solution of the maximal entropy prob-
147 lem (4).

148 **Proof.** Suppose that distribution f satisfies (6) and consider any distribution $g \in C_t$.
149 The functional Bregman divergence between f and g writes

$$\begin{aligned} \mathcal{D}_\phi(g, f) &= \int_{\mathcal{X}} \phi(g(x)) \, d\mu(x) - \int_{\mathcal{X}} \phi(f(x)) \, d\mu(x) - \int_{\mathcal{X}} \phi'(f(x))(g(x) - f(x)) \, d\mu(x) \\ &= -H_\phi[g] + H_\phi[f] - \sum_{i=0}^n \lambda_i \int_{\mathcal{X}} T_i(x)(g(x) - f(x)) \, d\mu(x) \\ &= H_\phi[f] - H_\phi[g] \end{aligned}$$

where we used the fact that g and f are both probability distributions with the same moments $\mathbb{E}[T_i(X)] = t_i$. By nonnegativity of the Bregman functional divergence, we finally get that

$$H_\phi[f] \geq H_\phi[g]$$

for all distribution g with the same moments as f , with equality if and only if $g = f$ almost everywhere. In other words, this shows that if f satisfies (6), then it is the desired solution. \square

Hence, given an entropic functional ϕ and moments constraints T_i , eq. (6) leads the the maximum entropy distribution f^* . This distribution is parametrized by the λ_i or, equivalently, by the moments t_i .

Note that the reciprocal is not necessarily true, as shown for instance in [53]. However, the reciprocal is true when \mathcal{X} is a compact [89] or for any \mathcal{X} provided that ϕ is locally bounded on \mathcal{X} [90].

2.2. Maximum entropy principle: the inverse problems

As stated in the introduction, two inverse problems can be considered starting from a given distribution f . These problems were considered by Kesavan & Kapur in [50] in the discrete framework.

The first inverse problem consists in searching for the adequate moments so that a desired distribution f is the maximum entropy distribution of a given ϕ -entropy. This amounts to find functions T_i and coefficients λ_i satisfying eq. (6). This is not always an easy task, and even not always possible. For instance, it is well known that the maximum Shannon entropy distribution given moment constraints fall in the exponential family [33,34,52,54]. Therefore, if f does not belong to this family, the problem has no solution.

The second inverse problem consists in designing the entropy itself, given a target distribution f and given the T_i . In other words, given a distribution f , eq. (6) may allow to determine the entropic functional ϕ so that f is its maximizer.

As for the direct problem, in the second inverse problem, the solution is parametrized by the λ_i . Here, required properties on ϕ will shape the domain the λ_i live in. In particular, ϕ must satisfy

- the domain of definition of ϕ' must include $f(\mathcal{X})$; this will be satisfied by construction.
- from the strict convexity property of ϕ , ϕ' must be strictly increasing.

Hence, because ϕ' must be strictly increasing, it's clear that solving eq. (6) requires the following two conditions:

(C1) $f(x)$ and $\sum_{i=1}^n \lambda_i T_i(x)$ must have the same variations, i.e., $\sum_{i=0}^n \lambda_i T_i(x)$ is increasing (resp. decreasing, constant) where f is increasing (resp. decreasing, constant).

(C2) $f(x)$ and $\sum_{i=1}^n \lambda_i T_i(x)$ must have the same level sets,

$$f(x_1) = f(x_2) \Leftrightarrow \sum_{i=0}^n \lambda_i T_i(x_1) = \sum_{i=0}^n \lambda_i T_i(x_2)$$

For instance, in the univariate case, for one moment constraint,

- for $\mathcal{X} = \mathbb{R}_+$, $T_1(x) = x$, λ_1 must be negative and $f(x)$ must be decreasing,
- for $\mathcal{X} = \mathbb{R}$, $T_1(x) = x^2$ or $T_1(x) = |x|$, λ_1 must be negative and $f(x)$ must be even and unimodal.

Under conditions (C1) and (C2), the solutions of eq. (6) are given by

$$\phi'(y) = \sum_{i=0}^n \lambda_i T_i(f^{-1}(y)) \quad (7)$$

where f^{-1} can be multivalued.

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Eq. (6) provides an effective way to solve the inverse problem. However, there exist situations where there do not exist any set of λ_i such that conditions (C1)-(C2) are satisfied (e.g., $T_1(x) = x^2$ with f not even). In such a case, a way to go is to extend the definition of the ϕ -entropy. This is precisely the purpose of section 3.

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2.3. Second inverse maximum entropy problem: some examples

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To illustrate the previous subsection, let us analyze briefly three examples: the famous Gaussian distribution (example 1), the q -Gaussian distribution also intensively studied (example 2) and the arcsine distribution (example 3), both three with a second-order moment constraint. The Gaussian, q -Gaussian, and arcsine distributions will serve as a guideline all along the paper. The details of the calculus, together with a deeper study related to the sequel of the paper, are rejected in the appendix. Other examples are also given in this appendix. In both three examples, except in the next section, we consider the second-order moment constraint $T_1(x) = x^2$.

Example 1. *Let us consider the well-known Gaussian distribution $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$, defined over $\mathcal{X} = \mathbb{R}$, and let us search for the ϕ -entropy so that the Gaussian is its maximizer subject to the constraint $T_1(x) = x^2$. To satisfy condition (C1) we must have $\lambda_1 < 0$. Rapid calculus gives the entropic functional, after a reparametrization of the λ_i 's, of the form,*

$$\phi(y) = \alpha y \log(y) + \beta y + \gamma \quad \text{with} \quad \alpha > 0,$$

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that is nothing more than the Shannon entropy, up to the scaling factor α , and a shift (except for $\gamma = -\beta$). One thus recovers the long outstanding fact that the Gaussian is the maximum Shannon entropy distribution with the second order moment constraint.

Example 2. *Let us consider the q -Gaussian distribution, also known as Tsallis distribution, $f_X(x) = C_q \left(1 - (q-1) \frac{x^2}{\sigma^2}\right)_+^{\frac{1}{1-q}}$, where $q > 0$, $q \neq 1$, $x_+ = \max(x, 0)$ and C_q is the normalization coefficient, defined over $\mathcal{X} = \mathbb{R}$, and let search for the ϕ -entropy so that the q -Gaussian is its maximizer with the constraint $T_1(x) = x^2$. Here again, condition (C1) is satisfied if and only if $\lambda_1 < 0$. Rapid calculus leads to the entropic functional, after a reparametrization of the λ_i 's, as,*

$$\phi(y) = \alpha \frac{y^q - y}{q-1} + \beta y + \gamma \quad \text{with} \quad \alpha > 0,$$

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that is nothing more than the Havrdat-Charvát-Tsallis entropy [12,14,17,91], up to the scaling factor α , and a shift (except for $\gamma = -\beta$). One recover the also well known fact that the q -Gaussian is the maximum Shannon entropy distribution with the second order moment constraint [91]. In the limit case $q \rightarrow 1$, the distribution f_X tends to the Gaussian, whereas the Havrdat-Charvát-Tsallis entropy tends to the Shannon entropy.

Example 3. *Consider the arcsine distribution, $f_X(x) = \frac{1}{\pi\sqrt{2\sigma^2-x^2}}$, defined over $\mathcal{X} = (-\sigma\sqrt{2}; \sigma\sqrt{2})$, and let us determine the entropic functionals ϕ so that f_X is the maximum ϕ -entropy distribution subject to the constraint $T_1(x) = x^2$. Now, to fulfill condition (C1) we must impose $\lambda_1 > 0$. Some algebra leads to the entropic functional, after a reparametrization of the λ_i 's,*

$$\phi(y) = \frac{\alpha}{y} + \beta y + \gamma \quad \text{with} \quad \alpha > 0.$$

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This entropy is unusual and, due to its form, is potentially finite only for densities defined over a bounded support and that are divergent in its boundary (integrable divergence).

217 3. State-dependent entropic functionals and mimization revisited

218 In order to follow asymmetries of the distribution f and address the limitation
219 raised above, an idea is to allow the entropic functional to also depend on the state
220 variable x :

Definition 3 (State-dependent ϕ -entropy). *Let $\phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ such that for any $x \in \mathcal{X} \subseteq \mathbb{R}^d$, function $\phi(x, \cdot)$ is a convex function on the closed convex set $\mathcal{Y} \subseteq \mathbb{R}_+$. Then, if f is a probability distribution defined with respect to a general measure μ on set \mathcal{X} and such that $f(\mathcal{X}) \subseteq \mathcal{Y}$,*

$$H_\phi[f] = - \int_{\mathcal{X}} \phi(x, f(x)) \, d\mu(x) \quad (8)$$

will be called state-dependent ϕ -entropy of f . Since $\phi(x, \cdot)$ is convex, then the entropy functional $H_\phi[f]$ is concave. A particular case arises when, for a given partition $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of \mathcal{X} , functional ϕ writes

$$\phi(x, y) = \sum_{l=1}^k \phi_l(y) \mathbb{1}_{\mathcal{X}_l}(x) \quad (9)$$

221 *where $\mathbb{1}_A$ denotes the indicator of set A . This functional can be viewed as a “ $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ -*
222 *extension” over $\mathcal{X} \times \mathcal{Y}$ of a multiform function defined on \mathcal{Y} , with k branches ϕ_l and the*
223 *associated ϕ -entropy will be called $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ -multiform ϕ -entropy.*

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225 As in the previous section, we restrict our study to functionals $\phi(x, y)$ *strictly convex*
226 *and differentiable versus y .*

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228 Following the lines of section 2, a generalized Bregman divergence can be associ-
229 ated to ϕ under the form $D_\phi(x, y_1, y_2) = \phi(x, y_1) - \phi(x, y_2) - \frac{\partial \phi}{\partial y}(x, y_2)(y_1 - y_2)$, and a
230 generalized functional Bregman divergence $\mathcal{D}_\phi(f_1, f_2) = \int_{\mathcal{X}} D_\phi(x, f_1(x), f_2(x)) \, d\mu(x)$.
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With these extended quantities, the direct problem becomes

$$f^* = \operatorname{argmax}_{f \in \mathcal{C}_t} \left(- \int_{\mathcal{X}} \phi(x, f(x)) \, d\mu(x) \right) \quad (10)$$

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233 Although the entropic functional is now state dependent, the approach adopted
before can be applied here, leading to

Proposition 2 (Maximum state-dependent ϕ -entropy solution). *Suppose that there exists a probability distribution f satisfying*

$$\frac{\partial \phi}{\partial y}(x, f(x)) = \sum_{i=0}^n \lambda_i T_i(x), \quad (11)$$

for some $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$, then f is the unique solution of the extended maximum entropy problem (10).

If ϕ is chosen in the $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ -multiform ϕ -entropy class, this sufficient condition writes

$$\sum_{l=1}^k \phi'_l(f(x)) \mathbb{1}_{\mathcal{X}_l}(x) = \sum_{i=0}^n \lambda_i T_i(x), \quad (12)$$

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235 **Proof.** The proof follows the steps of Proposition 1, using the generalized functional Bregman divergence instead of the usual one. \square

Resolution eq. (11) is not possible in all generality. However the sufficient condition. (12) can be rewritten as

$$\sum_{l=1}^k \left(\phi'_l(f(x)) - \sum_{i=0}^n \lambda_i T_i(x) \right) \mathbb{1}_{\mathcal{X}_l}(x) = 0. \quad (13)$$

Thus, if there exists (at least) a set of λ_i such that condition (C1) is satisfied (but not necessarily (C2)), one can always

- design a partition $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ so that (C2) is satisfied in each \mathcal{X}_l (at least, such that f is either strictly monotonic, or constant, on \mathcal{X}_l)
- determine ϕ_l as in eq. (7) in each \mathcal{X}_l , that is

$$\phi'_l(y) = \sum_{i=0}^n \lambda_i T_i(f_l^{-1}(y)) \quad (14)$$

where f_l^{-1} is the (possibly multivalued) inverse of f on \mathcal{X}_l .

In short, in the case where only condition (C1) is satisfied, one can obtain an extended entropic functional of $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ -multiform class so that eq. (13) provides an effective way to solve the inverse problem in the state-dependent entropic functional context.

Note however that it still may happen that there is no set of λ_i allowing to satisfy (C1). In such an harder context, the problem remains solvable when the moments are defined as partial moments like $\mathbb{E}[T_{l,i}(X) \mathbb{1}_{\mathcal{X}_l}(X)] = t_{l,i}$, $l = 1, \dots, k$ and $i = 1, \dots, n_l$ and when there exist on \mathcal{X}_l a set of $\lambda_{l,i}$ such that (C1) and (C2) holds. The solution still writes as in eq. (14), but where now n , the λ_i and the T_i are replaced by n_l , $\lambda_{l,i}$ and $T_{l,i}$ respectively.

Let us now come back to the arcsine example $f_X(x) = \frac{1}{\pi\sqrt{2\sigma^2-x^2}}$, defined over $\mathcal{X} = (-\sigma\sqrt{2}; \sigma\sqrt{2})$ (example 3) of the previous section, when now we constraint the first order moment or partial first order moments.

Example 3-1. The arcsine distribution is bijective on each set $\mathcal{X}_- = (-\sigma\sqrt{2}; 0)$ and $\mathcal{X}_+ = [0; \sigma\sqrt{2})$ that partitions \mathcal{X} . If now we impose the partial constraint $T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_{\pm}}(x)$, and search for the ϕ -entropy so that f_X is the maximizer subject to these constraints, condition (C1) requires that $\lambda_{\pm,1} > 0$. We then obtain the associated multiform entropic functional, after a reparametrization of the λ_i 's, as

$$\phi_{\pm}(y) = \alpha_{\pm} \left(\sqrt{u^2 - 1} + \arctan\left(\frac{1}{\sqrt{u^2 - 1}}\right) \right) \mathbb{1}_{(1; +\infty)}(u) + \beta u + \gamma_{\pm}$$

$$\text{with } u = \sqrt{2}\pi\sigma y \text{ and } \alpha_{\pm} > 0$$

(see appendix for more details).

Example 3-2. This arcsine distribution, now constraint uniformly by $T_1(x) = x$, will be an extremal ϕ -entropy provided that $\lambda_1 > 0$ and therefore with the associated multiform entropic functional, after a reparametrization of the λ_i 's, is given by

$$\tilde{\phi}_{\pm}(y) = \pm\alpha \left(\sqrt{u^2 - 1} + \arctan\left(\frac{1}{\sqrt{u^2 - 1}}\right) \right) \mathbb{1}_{(1; +\infty)}(u) + \beta u + \gamma$$

$$\text{with } u = \sqrt{2}\pi\sigma y \text{ and } \alpha > 0$$

(see appendix for more details). The entropic functional is no more convex.

At a first glance, the two solutions seem to be identical. In fact, they drastically differ. Indeed, let us emphasize that the problem has two constraints in the first case, whereas in the second case, there is only one. The consequence is that 5 parameters parametrize the first solution β, γ_{\pm} and, especially, α_{\pm} , while only 3 parameters β, γ and α parametrize the second solution. This difference is not insignificant: the second case cannot be viewed as a special case of the first one, because α_{\pm} must be positive, which cannot be possible with only one parameter since $\pm\alpha$ rule the $\tilde{\phi}_{\pm}$. For the second example, the solution does not lead to a convex function, because this would contradict the required condition (C1) on the parts \mathcal{X}_{\pm} . The “ ϕ -entropy” defined with $\tilde{\phi}$ is thus no more concave, so that the maximum ϕ -entropy problem is no more concave: one cannot guarantee the unicity and even the existence of a maximum.

In section 2 and 3 we established general entropies with a given maximizer. In what follows, we will complete the information theoretical setting by introducing generalized escort distributions, generalized moments, and generalized Fisher information associated to the same entropic functional. We will then explore some of their relationships.

4. ϕ -escort distribution, (ϕ, α) -moments, (ϕ, β) -Fisher informations, generalized Cramér-Rao inequalities

In this section, we begin by introducing the above-mentioned informational quantities. We will then show that generalizations of the celebrated Cramér-Rao inequalities hold and link the generalized moments and Fisher information. Furthermore, the lower bound of the inequalities are saturated precisely by maximal ϕ -entropy distributions.

Escort distributions have been introduced as an operational tool in the context of multifractals [92,93], with interesting connections with the standard thermodynamics [94] and with source coding [26,27]. In our context, we also define (generalized) escort distributions associated with a particular ϕ -entropy, and show how they pop up naturally. It is then possible to define generalized moments with respect to these escort distributions.

Definition 4 (ϕ -escort). Let $\phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ such that for any $x \in \mathcal{X} \subseteq \mathbb{R}^d$ function $\phi(x, \cdot)$ is a strictly convex twice differentiable function defined on the closed convex set $\mathcal{Y} \subseteq \mathbb{R}_+$. Then, if f is a probability distribution defined with respect to a general measure μ on a set \mathcal{X} such that $f(\mathcal{X}) \subseteq \mathcal{Y}$, such that

$$C_{\phi}[f] = \int_{\mathcal{X}} \frac{d\mu(x)}{\frac{\partial^2 \phi}{\partial y^2}(x, f(x))} < +\infty \quad (15)$$

we define by

$$E_{\phi, f}(x) = \frac{1}{C_{\phi}[f] \frac{\partial^2 \phi}{\partial y^2}(x, f(x))} \quad (16)$$

the ϕ -escort density with respect to measure μ , associated to density f .

Note that from the strict convexity of ϕ with respect to its second argument, this probability density is well defined and is strictly positive. Moreover, coming back to the previous examples, one can see that:

Example 1. In the context of the Shannon entropy, entropy for which the Gaussian is the maximal entropy law for the second order moment constraint, $\phi(x, y) = \phi(y) = y \log y$, the ϕ -escort density associated to f restricts to density f itself.

Example 2. In the Rényi-Tsallis context, entropy for which the q -Gaussian is the maximal entropy law for the second order moment constraint $\phi(x, y) = \phi(y) = \frac{y^q - y}{q-1}$, and $E_{\phi, f} \propto f^{2-q}$ which recovers the escort distributions used in the Rényi-Tsallis context up to a duality transformation [94].

Example 3. For the entropy that is maximal for the arcsine distribution under the second order moment constraint, $\phi(x, y) = \phi(y) = \frac{1}{y}$, and $E_{\phi, f} \propto f^3$ which is nothing more than an escort distributions used in the Rényi-Tsallis context. Indeed, although the arcsine distribution does not fall in the q -Gaussian family, its form is very similar to a q -distribution where the “scaling” would not be related to the exponent q . It is thus not suprising to recover an escort distribution associated to this family.

Definition 5 ((α, ϕ) -moments). Under the assumptions of definition 4, with \mathcal{X} equipped with a norm $\|\cdot\|_{\mathcal{X}}$, we define by

$$M_{\alpha, \phi}[f; X] = \int_{\mathcal{X}} \|x\|_{\mathcal{X}}^{\alpha} E_{\phi, f}(x) d\mu(x) \quad (17)$$

if this quantity exists, as the (α, ϕ) -moment of X associated to distribution f .

For our three examples, we have:

Example 1. In the context of the Shannon entropy, the (α, ϕ) -moments are the usual moments of $\|X\|_{\mathcal{X}}^{\alpha}$.

Example 2. In the Rényi-Tsallis context the generalized moments introduced in [55,95] are recovered.

Example 3. For $\phi(x, y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalized moments introduced in [55,95] (see the items related to the escort distributions).

The Fisher information’s importance is well known in estimation theory: the estimation error of a parameter is bounded by the inverse of the Fisher information associated with this distribution [34,60]. The Fisher information is also used as a method of inference and understanding in statistical physics and biology, as promoted by Frieden [61] and has been generalized in the Rényi-Tsallis context in a series of papers [75,78,80–83,96,97]. In what follows, we generalize these definitions a step further in our ϕ -entropy context.

Definition 6 (Nonparametric (β, ϕ) -Fisher information). With the same assumption as in definition 5, denoting by $\|\cdot\|_{\mathcal{X}^*}$ the dual norm, for any differentiable density f , we define the quantity

$$I_{\beta, \phi}[f] = \int_{\mathcal{X}} \left\| \frac{\nabla_x f(x)}{E_{\phi, f}(x)} \right\|_{\mathcal{X}^*}^{\beta} E_{\phi, f}(x) d\mu(x) \quad (18)$$

if this quantity exists, as the nonparametric (β, ϕ) -Fisher information of f .

Note that when ϕ is state-independent, $\phi(x, y) = \phi(y)$, as for the usual Fisher information, this quantity is shift-invariant, i.e., for $g(x) = f(x - x_0)$ one have $I_{\beta, \phi}[g] = I_{\beta, \phi}[f]$. This property is unfortunately lost in the state-dependent context.

Definition 7 (Parametric (β, ϕ) -Fisher information). Let consider the same assumption as in definition 5, such that density f is parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$. The set Θ is

equipped with a norm $\|\cdot\|_{\Theta}$ and the corresponding dual norm is denoted $\|\cdot\|_{\Theta^*}$. Assume that f is differentiable with respect to θ . We define by

$$I_{\beta,\phi}[f;\theta] = \int_{\mathcal{X}} \left\| \frac{\nabla_{\theta} f(x)}{E_{\phi,f}(x)} \right\|_{\Theta^*}^{\beta} E_{\phi,f}(x) d\mu(x) \quad (19)$$

as the parametric (β, ϕ) -Fisher information of f .

Note that, as for the usual Fisher information, when the norm on \mathcal{X} and on Θ are the same, the nonparametric and parametric information coincide when θ is a location parameter. For our three examples, we have:

Example 1. In the Shannon entropy context, when the norm is the euclidean norm and $\beta = 2$, the nonparametric and parametric informations (β, ϕ) -Fisher give the usual nonparametric and parametric Fisher informations respectively.

Example 2. Similarly, in the Rényi-Tsallis context, the generalizations proposed in [81–83] are recovered.

Example 3. For $\phi(x, y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalizations proposed in [81–83] (see the items related to the escort distributions).

We have now the quantities that allow to generalize the Cramér-Rao inequalities as follows.

Proposition 3 (Nonparametric (α, ϕ) -Cramér-Rao inequality). Assume that a differentiable probability density function with respect to a measure μ , defined on a domain \mathcal{X} , admits an (α, ϕ) -moment and a (α^*, ϕ) -Fisher information with $\alpha \geq 1$ and α^* Holder-conjugated $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$, and that $xf(x)$ vanishes at the boundary of \mathcal{X} . Thus, density f satisfies the (α, ϕ) extended Cramér-Rao inequality

$$M_{\alpha,\phi}[f; X]^{\frac{1}{\alpha}} I_{\alpha^*,\phi}[f]^{\frac{1}{\alpha^*}} \geq d \quad (20)$$

When ϕ is state independent, $\phi(x, y) = \phi(y)$, the equality occurs when f is the maximal ϕ entropy distribution subject to the moment constraint $T(x) = \|x\|_{\mathcal{X}}^{\alpha}$.

Proof. The approach follows [83], starting from the differentiable probability density f (derivative denoted $\nabla_x f$), since $xf(x)$ vanishes in the boundaries of X from the divergence theorem one has

$$0 = \int_{\mathcal{X}} \nabla_x^t (xf(x)) d\mu(x) = \int_{\mathcal{X}} (\nabla_x^t x) f(x) d\mu(x) + \int_{\mathcal{X}} x^t (\nabla_x f(x)) d\mu(x)$$

Now, for the first term, we use the fact that $\nabla_x x = d$ and that f is a density to achieve

$$d = - \int_{\mathcal{X}} x^t \frac{\nabla_x f(x)}{g(x)} g(x) d\mu(x)$$

for any function g non-zero on \mathcal{X} . Now, noting that $d > 0$, we obtain from [83, Lemma 2]

$$\begin{aligned} d &= \left| \int_{\mathcal{X}} x^t \left(\frac{\nabla_x f(x)}{g(x)} \right) g(x) d\mu(x) \right| \\ &\leq \left(\int_{\mathcal{X}} \|x\|_{\mathcal{X}}^{\alpha} g(x) d\mu(x) \right)^{\frac{1}{\alpha}} \left(\int_{\mathcal{X}} \left\| \frac{\nabla_x f(x)}{g(x)} \right\|_{\Theta^*}^{\alpha^*} g(x) d\mu(x) \right)^{\frac{1}{\alpha^*}} \end{aligned}$$

The proof ends by choosing $g = E_{\phi,f}$ the ϕ -escort density associated to density f . Note now that, again from [83, Lemma 2] the equality is obtained when

$$\nabla_x f(x) \frac{\partial^2 \phi}{\partial y^2}(x, f(x)) = \lambda_1 \nabla_x \|x\|_{\mathcal{X}}^{\alpha}$$

where λ_1 is a negative constant. Consider now the case where $\phi(x, y) = \phi(y)$ is state-independent. Thus, $\nabla_x f(x) \frac{\partial^2 \phi}{\partial y^2}(x, f(x)) = \nabla_x \phi'(f(x))$, that gives

$$\phi'(f(x)) = \lambda_0 + \lambda_1 \|x\|_{\mathcal{X}}^{\alpha}$$

349 This last equation has precisely the form eq. (6) of proposition 1. \square

350 An obvious consequence of the proposition is that the probability density that
351 minimizes the (α^*, ϕ) -Fisher information subject to the moment constraint $T(x) =$
352 $\|x\|_{\mathcal{X}}^{\alpha}$ coincides with the maximal ϕ -entropy distribution subject to the same moment
353 constraint.

354 In the problem of estimation, the purpose is to determine a function $\hat{\theta}(x)$ in order to
355 estimate an unknown parameter θ . In such a context, the Cramér-Rao inequality allows
356 to lowerbound the variance of the estimator thanks to the parametric Fisher information.
357 The idea is thus to extend this to bound any α order mean error using our generalized
358 Fisher information.

Proposition 4 (Parametric (α, ϕ) -Cramér-Rao inequality). *Let f be a probability density function with respect to a general measure μ defined over a set \mathcal{X} , where f is parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, and satisfies the conditions of definition 7. Assume that both μ and \mathcal{X} do not depend on θ neither, that f is a jointly measurable function of x and θ , is integrable with respect to x , is absolutely continuous with respect to θ and that the derivatives with respect to each component of θ are locally integrable. Thus, for any estimator $\hat{\theta}(X)$ of θ that does not depend on θ , we have*

$$M_{\alpha, \phi}[f; \hat{\theta}(X) - \theta]^{\frac{1}{\alpha}} I_{\alpha^*, \phi}[f; \theta]^{\frac{1}{\alpha^*}} \geq |m + \nabla_{\theta}^t b(\theta)| \quad (21)$$

where

$$b(\theta) = \mathbb{E}[\hat{\theta}(X) - \theta] \quad (22)$$

359 is the bias of the estimator and α and α^* are Holder conjugated. When ϕ is state independent,
360 $\phi(x, y) = \phi(y)$, the equality occurs when f is the maximal ϕ entropy distribution subject to the
361 moment constraint $T(x) = \|\Theta(x) - \theta\|_{\Theta}^{\alpha}$.

Proof. The proof follows again that of [83], and starts by evaluating the divergence of the bias. The regularity conditions in the statement of the theorem enable to interchange integration with respect to x and differentiation with respect to θ , so that

$$\nabla_{\theta}^t b(\theta) = \int_{\mathcal{X}} (\nabla_{\theta}^t \hat{\theta}(x) - \nabla_{\theta}^t \theta) f(x) d\mu(x) + \int_{\mathcal{X}} (\hat{\theta}(x) - \theta)^t \nabla_{\theta} f(x) d\mu(x)$$

Note then that $\nabla_{\theta}^t \theta = m$ and that $\hat{\theta}$ being independent on θ , one has $\nabla_{\theta}^t \hat{\theta}(x) = 0$. Thus, f being a probability density, the equality becomes

$$m + \nabla_{\theta}^t b(\theta) = \int_{\mathcal{X}} (\hat{\theta}(x) - \theta)^t \frac{\nabla_{\theta} f(x)}{g(x)} g(x) d\mu(x)$$

362 for any density g non-zero on \mathcal{X} . The proof ends with the very same steps that in
363 proposition 4 using [83, Lemma 2]. \square

364 For our three examples, this leads to what follows.

365 **Example 1.** *The usual parametric and nonparametric Cramér-Rao inequality are recovered in*
 366 *the usual Shannon context $\phi(x, y) = y \log y$, using the euclidean norm and $\alpha = 2$. The bound*
 367 *in the nonparametric context is saturated for the maximal entropy law, namely the Gaussian.*

368 **Example 2.** *In the Rényi-Tsallis context, the generalizations proposed in [81–83] are recovered*
 369 *and, again, when $\alpha = 2$, the bound is saturated in the nonparametric context for the q -Gaussian,*
 370 *maximal entropy law under the second order moment constraint.*

371 **Example 3.** *For $\phi(x, y) = \phi(y) = \frac{1}{y}$ again, the generalizations proposed in [81–83] are*
 372 *recovered* (see the items related to the escort distributions).

373
 374 Beyond the mathematical aspect of these relations, they may have great interest to
 375 assess an estimator when the usual variance/mean square error does not exist. Moreover,
 376 the escort distribution is also a way to emphasis some part of a distribution. For instance,
 377 in the Rényi-Tsallis context, one can see that in f^q either the tails or the head of the
 378 distribution is emphasized. Playing with q is a way to penalize either the tails, or the
 379 head of the distribution in the estimation process.

380 5. ϕ -heat equation and extended de Bruijn identity

An important relation connecting the Shannon entropy H , coming from the “information world”, with the Fisher information I , living in the “estimation world”, is given by the de Bruijn identity and is closely linked to the Gaussian distribution. Considering a noisy random variable $Y_t = X + \sqrt{t}N$ where N is a zero-mean d -dimensional standard Gaussian random vector and X a d -dimensional random vector independent of N , and of support independent on parameter t , then

$$\frac{d}{dt} H[f_{Y_t}] = \frac{1}{2} I[f_{Y_t}]$$

381 where f_{Y_t} stands for the probability distribution of Y_t . This identity is a critical ingredient
 382 in proving the entropy power and Stam’s inequalities [34]. The starting point to establish
 383 the de Bruijn identity is the heat equation satisfied by the probability distribution f_{Y_t} ,
 384 $\frac{\partial f}{\partial t} = \frac{1}{2} \Delta f$, where Δ stands for the Laplacian operator [98].

Let us consider probability distributions f parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, satisfying what we will call *generalized ϕ -heat equation*,

$$\nabla_\theta f = K \operatorname{div} \left(\|\nabla_x \phi'(f)\|_{\chi^*}^{\beta-2} \nabla_x f \right) \quad (23)$$

385 for some $K \in \mathbb{R}^m$ (possibly dependent on θ) and where ϕ is a convex twice differentiable
 386 function defined over a set $\mathcal{X} \in \mathbb{R}_+$.

Proposition 5 (Extended de Bruijn identity). *Let f be a probability distribution, parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, defined over a set $\mathcal{X} \subset \mathbb{R}^d$ that do not depend on θ , and satisfying the nonlinear ϕ -heat equation eq. (23) for a twice differentiable convex function ϕ . Assume that $\nabla_\theta \phi(f)$ is absolutely integrable and locally integrable with respect to θ , and that the function $\|\nabla_x \phi'(f)\|_{\chi^*}^{\beta-2} \nabla_x \phi(f)$ vanishes at the boundary of \mathcal{X} . Thus, distribution f satisfies the extended de Bruijn identity, relating the ϕ -entropy of f and its nonparametric (β, ϕ) -Fisher information as follows,*

$$\nabla_\theta H_\phi[f] = K C_\phi^{1-\beta} I_{\beta, \phi}[f] \quad (24)$$

387 with C_ϕ is the normalisation constant given eq. (15).

Proof. From the definition of the ϕ -entropy, the smoothness of the assumption enabling to use the Leibnitz' rule and differentiate under the integral,

$$\begin{aligned}
 \nabla_{\theta} H_{\phi}[f] &= - \int_{\mathcal{X}} \phi'(f(x)) \nabla_{\theta} f(x) d\mu(x) \\
 &= -K \int_{\mathcal{X}} \phi'(f(x)) \operatorname{div} \left(\|\nabla_x \phi'(f(x))\|_{\chi^*}^{\beta-2} \nabla_x f(x) \right) d\mu(x) \\
 &= -K \int_{\mathcal{X}} \operatorname{div} \left(\phi'(f(x)) \|\nabla_x \phi'(f(x))\|_{\chi^*}^{\beta-2} \nabla_x f(x) \right) d\mu(x) \\
 &\quad + K \int_{\mathcal{X}} \nabla_x^t \phi'(f(x)) \|\nabla_x \phi'(f(x))\|_{\chi^*}^{\beta-2} \nabla_x f(x) d\mu(x) \\
 &= -K \int_{\mathcal{X}} \operatorname{div} \left(\|\nabla_x \phi'(f(x))\|_{\chi^*}^{\beta-2} \nabla_x \phi(f(x)) \right) d\mu(x) \\
 &\quad + K \int_{\mathcal{X}} (\phi''(f(x)))^{\beta-1} \|\nabla_x f(x)\|_{\chi^*}^{\beta} d\mu(x)
 \end{aligned}$$

where the second line comes from the ϕ -heat equation and where the third line comes from the product derivation rule.

Now, from the divergence theorem, the first term of the right handside reduces to the integral of $\|\nabla_x \phi'(f)\|_{\chi^*}^{\beta-2} \nabla_x \phi(f)$ on the boundary of \mathcal{X} , that vanishes from the assumption of the proposition, while the second term of the right handside related to C_{ϕ} and the (β, ϕ) -Fisher information from eqs. (15), (16) and definition 6. \square

Coming back to the special examples we presented all along the paper:

Example 1. In the Shannon entropy context, for $K = \frac{1}{2}$ and $\beta = 2$, the standard heat-equation is recovered and the usual de Bruijn identity is recovered.

Example 2. The case where $\phi(y) = y^q$ was intensively studied in [84] and the results of the paper are naturally recovered. In particular, the generalized ϕ -heat equation appears in anomalous diffusion in porous medium [84,99,100].

Example 3. For $\phi(x, y) = \phi(y) = \frac{1}{y}$ once again one find the same form for the generalized heat equation than in [84,99,100], and therefore the same forme of the generalized de Bruijn's identity of [84] (see the items related to the escort distributions).

Note that various physical non linear diffusions equation are encompassed in the generalized ϕ -heat equation [100,101].

Anomalous diffusion, NL FK: [99?]

6. Concluding remarks

In this paper, we extended as far as possible the identities and inequalities which link the classical informational quantities – Shannon entropy, Fisher information, moments, ..., in the framework of the ϕ -entropies. Our first result concerns the inverse maximum entropy problem, starting with a probability distribution and constraints and searching for which entropy the distribution is the maximizer. If such a study was already tackled, it is extended here in a much more general context. We used general reference measures — not necessarily discrete or of Lebesgue. We also considered the case where the distribution and constraints do not share the same symmetries, which leads to state-dependent entropic functionals. Our second result is the generalization of the Cramér-Rao inequality in the same setting: to this end, a generalized Fisher information and generalized moments are introduced, both based on a convex function ϕ (and a so-called ϕ -escort distribution). The Cramér-Rao inequality is saturated precisely

for the maximum ϕ -entropy distribution with the same moment constraints, linking all information quantities together. Finally, our third result is the statement of a generalized de Bruijn identity, linking the ϕ -entropy rate and the ϕ -Fisher information of a distribution satisfying an extended heat equation, called ϕ -heat equation. Moreover, dealing with usual distributions (Gaussian, q -Gaussian, exponential) and usual moments (mean, second order), all classical results are recovered as limit cases. Beyond these results, remind that the Shannon (Gibbs) entropy is well adapted to the study of systems in the equilibrium and the maximal entropy distributions or Boltzmann distributions fall to the exponential family [33,34,52,54]. As an alternative, extended entropies are often considered better suited to analyse systems out of equilibrium, where the observed distributions do not belong to the exponential (Boltzmann-type) family [17,55–59], like the maximal entropy distributions in the general setting of this paper.

In this panel, two important inequalities still miss. The first one is entropy power inequality (EPI), which states that the entropy power (exponential of twice the entropy) of the sum of two continuous independent random variables is higher than the sum of the individual entropy powers¹. The second one is the Stam's inequality which lowerbounds the product of the entropy power and the Fisher information. For the former, despite many efforts, the literature on extended version only treat special cases. For instance, some extensions in the classical settings exist for discrete variables but are somewhat limited [102–104]. In the continuous framework, the EPI was also extended to the class of the Rényi entropies (log of a ϕ -entropy with $\phi(u) = u^\alpha$) [105]. Important properties that play a key role in the inequality is that the Rényi's entropies are invariant to an affine transform of unit determinant and monotonic under convolution, properties that seem lost in the very general setting considered here. This fact leaves little room to extend the EPI in our general settings. About the Stam inequality, at a first glance, the fact that the proof is based on the EPI seems to close any hope to extend it to the ϕ -entropy framework. However, it was remarkably extended to the Rényi's entropies, base on the Gagliardo-Nirenberg inequality [78,80,81,106]. Nevertheless, a key property is that both the entropy power and the extended Fisher information have scaling properties, that are lost in the general setting of the ϕ -entropies. A possible way to overcome the (apparent) limits just evoked could be to mimic alternative proofs such that based on optimal transport [107]. This approach precisely drops off any use of Young or Sobolev-like inequalities. As far as we feel, there is thus a little room for extensions in the settings of the paper. Both the extension of the EPI and Stam inequalities are left as a perspective.

Another perspective lies in the estimation of the generalized moments from data (or from estimates). Such a possibility would confer an operational role to our Cramér-Rao inequality, i.e., by computing the estimator's generalized moments and comparing them to the bound. A difficulty resides in the presence of the ϕ -escort distribution which forbids empirical or Monte-Carlo approaches. The escort distribution needs to be estimated. This problem seems not far from the estimation of entropies from data and plug-in approaches used in such problems can thus be considered, like kernel approaches [108–110], nearest neighbor approaches [110,111], or minimal spanning tree approaches [42]. Of course this perspective goes far beyond the scope of this paper.

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¹ In fact, there exist other equivalent versions which can be found e.g., in [34,69].

469 **Appendix A. Inverse maximum entropy problem and associated inequalities: some** 470 **examples**

471 In this appendix, we will now derive in detail several case of inverse problem of
472 the maximal entropy problem. In each case, we will thus provide the quantities and
473 inequalities associated with the entropic functional ϕ , as derived in the text. In the sequel,
474 for sake of simplicity, we restricts our example to the univariate context $d = 1$.

475 *Appendix A.1. Normal distribution and second-order moment*

For a normal distribution, and second order moment constraint

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \text{and} \quad T_1(x) = x^2 \quad \text{on} \quad \mathcal{X} = \mathbb{R}.$$

We begin by computing the inverse of $y = f_X(x)$ where $x \in \mathbb{R}_+$ for instance, which gives

$$\phi'(y) = \left(\lambda_0 - \sigma^2 \log(2\pi\sigma^2)\lambda_1\right) - 2\sigma^2\lambda_1 \log y \quad \text{with } \lambda_1 < 0.$$

The judicious choice

$$\lambda_0 = 1 - \log(\sqrt{2\pi}\sigma) \quad \text{and} \quad \lambda_1 = -\frac{1}{2\sigma^2}$$

leads to function

$$\phi(y) = y \log y$$

476 that gives nothing more than the Shannon entropy as expected.

477 Now, $\phi''(y) \propto \frac{1}{y}$ leading to the escort distribution $E_{\phi,f} = f$ so that, as expected, the
478 (α, ϕ) moments are the usual moments of order α . When $\beta = 2$ and the usual euclidean
479 norm is considered, the (β, ϕ) -Fisher informations are the usual Fisher information
480 and the usual Cramér-Rao inequalities are recovered for $\alpha = 2$. Finally, for $\beta = 2$, the
481 usual euclidean norm, the ϕ -heat equation turns to be the heat equation, satisfied by the
482 gaussian, so that the usual de Bruijn identity is naturally recovered.

483 *Appendix A.2. q -Normal distribution and second-order moment*

For q -normal distribution, also known as Tsallis distributions, Student-t and -r, and
a second order moment constraint,

$$f_X(x) = C_q \left(1 - (q-1)\beta x^2\right)_+^{\frac{1}{q-1}} \quad \text{and} \quad T_1(x) = x^2 \quad \text{on} \quad \mathcal{X} = \mathbb{R},$$

where $q > 0$, $x_+ = \max(x, 0)$ and C_q is a normalization coefficient, we get

$$\phi'(y) = \left(\lambda_0 + \frac{\lambda_1}{(q-1)\beta}\right) - \frac{\lambda_1 y^{q-1}}{C_q^{q-1}(q-1)\beta}.$$

In this case, a judicious choice of parameters is

$$\lambda_0 = \frac{q C_q^{q-1} - 1}{q-1} \quad \text{and} \quad \lambda_1 = -q C_q^{q-1} \beta$$

that yields to

$$\phi(y) = \frac{y^q - y}{q-1}.$$

and an associated entropy is then

$$H_\phi[f] = \frac{1}{1-q} \left(\int_{\mathcal{X}} f(x)^q d\mu(x) - 1 \right) :$$

484 It is nothing but Havrdat-Charvát-Tsallis entropy [12,14,17,91].

485 Then, $\phi''(y) = qy^{q-2}$: $M_{\phi,\alpha}[f]$ and $I_{\phi,\alpha}[f]$ are respectively the q -moment of order α
 486 and the (q, β) -Fisher information defined previously in [78–83] (with the symmetric q
 487 index given here by $2 - q$). The extended Cramér-Rao inequality proved in [78,82,83] is
 488 then recovered.

489 Note that when $q \rightarrow 1$: f_X tends to the gaussian distribution. It appears that H_ϕ
 490 tends to the Shannon's entropy, $I_{\phi,2}$ to the usual Fisher's information and $M_{\phi,\alpha}$ to the
 491 usual moments (both considering the euclidean norm): all the settings related to the
 492 Gaussian distribution is naturally recovered.

493 Appendix A.3. q -exponential distribution and first-order moment

The same entropy functional can readily be obtained for the so-called q -exponential

$$f_X(x) = C_q(1 - (q-1)\beta x)_{+}^{\frac{1}{q-1}} \quad \text{and} \quad T_1(x) = x \quad \text{on} \quad \mathcal{X} = \mathbb{R}_+.$$

494 It suffices to follow the very same steps as above, leading again to the Havrdat-Charvát-
 495 Tsallis entropy, the q -moments of order α and the (q, β) -Fisher information defined
 496 previously in [78–83] (with the symmetric q index given here by $2 - q$) as for the q -
 497 Gaussian distribution and to the extended Cramér-Rao inequality proved in [82,83] as
 498 well.

499 Now when $q \rightarrow 1$: f_X tends to the exponential distribution, known to be of max-
 500 imum Shannon's entropy on \mathbb{R}_+ under the first order moment constraint. Again H_ϕ
 501 tends to the Shannon's entropy, $I_{\phi,2}$ to the usual Fisher's information and $M_{\phi,\alpha}$ to the
 502 usual moments (both considering the euclidean norm): all the settings related to the
 503 exponential distribution is naturally recovered.

504 Appendix A.4. The logistic distribution

In this case,

$$f_X(x) = \frac{1 - \tanh^2\left(\frac{x}{2s}\right)}{4s} \quad \text{and} \quad T_1(x) = x^2 \quad \text{on} \quad \mathcal{X} = \mathbb{R}.$$

This distribution, which resembles the normal distribution but has heavier tails, has been used in many applications. One can then check that over each interval

$$\mathcal{X}_\pm = \mathbb{R}_\pm$$

the inverse distribution writes

$$f_{X,\pm}^{-1}(y) = \pm 2s \operatorname{arctanh} \sqrt{1 - 4sy}, \quad y \in \left[0; \frac{1}{4s}\right]$$

505 We concentrate now on a second order constraint, that respect the symmetry of the
 506 distribution, and on first order constrain(s) that does not respect the symmetry.

507 Appendix A.4.1. Second order moment constraint

In this case, immediately

$$\phi'(y) = 4s \left(\lambda_0 + \lambda_1 \left(\operatorname{arctanh} \sqrt{1 - 4sy} \right)^2 \right)$$

for $y \in \left[0; \frac{1}{4s}\right]$ and where the positive factors $\frac{1}{4s}$ and s are absorbed in λ_0 and λ_1 respectively. To impose the convexity of ϕ , one must impose

$$\lambda_1 < 0$$

that gives the family of entropy functionals $\phi(y) = \phi_u(4sy)$ with

$$\phi_u(u) = c + \lambda_0 u + \lambda_1 \left[u \left(\operatorname{argtanh} \sqrt{1-u} \right)^2 - 2\sqrt{1-u} \operatorname{argtanh} \sqrt{1-u} - \log u \right] \mathbb{1}_{[0;1]}(u).$$

508 where c is an integration constant. Figure A1(a) depicts function ϕ_u for the special choice
 509 $\lambda_0 = 0, \lambda_1 = -1$ and \mathcal{X} being unbounded, c is chosen to be zero.

510 Appendix A.4.2. (Partial) first-order moment(s) constraint(s)

Since f_X and $T(x) = x$ do not share the same symmetries, one cannot interpret the logistic distribution as a maximum entropy constraint by the first order moment. However, constraining the partial means over $\mathcal{X}_\pm = \mathbb{R}_\pm$ allows such an interpretation, using then multiform entropies, while the alternative is to relax the concavity property of the entropy. To be more precise, one chooses either functions $T_{-,1}(x)$ and $T_{+,1}$, or function T_1 under the form

$$T_{\pm,1}(x) = x, \quad x \in \mathcal{X}_\pm = \mathbb{R}_\pm \quad \text{or} \quad T_1(x) = x, \quad x \in \mathcal{X} = \mathbb{R}.$$

Over each set \mathcal{X}_\pm we immediately get

$$\phi'_\pm(y) = 4s \left(\lambda_0 + \lambda_{\pm,1} \operatorname{argtanh} \sqrt{1-4sy} \right)$$

or

$$\tilde{\phi}'_\pm(y) = 4s \left(\lambda_0 \pm \lambda_1 \operatorname{argtanh} \sqrt{1-4sy} \right)$$

where the sign and the factors are absorbed on λ_0 and $\lambda_{\pm,1}$. A judicious choice is then to impose

$$\lambda_{-,1} = \lambda_{+,1} = \bar{\lambda}_1 < 0 \quad (\lambda_1 < 0)$$

and the same integration constant c for each branch leading either to the family of (convex) uniform functions $\phi(y) = \phi_u(4sy)$ with,

$$\phi_u(u) = c + \lambda_0 u + \bar{\lambda}_1 \left(u \operatorname{argtanh} \sqrt{1-u} - \sqrt{1-u} \right) \mathbb{1}_{[0;1]}(u)$$

or to the family of multiform function $\tilde{\phi}$, with branches $\tilde{\phi}_{\pm,u}(4sy)$,

$$\tilde{\phi}_{\pm,u}(u) = c + \lambda_0 u \pm \lambda_1 \left(u \operatorname{argtanh} \sqrt{1-u} - \sqrt{1-u} \right) \mathbb{1}_{[0;1]}(u)$$

511 Function ϕ_u is represented figure A1(b) for the special choice $c = \lambda_0 = 0, \bar{\lambda}_1 = -1$
 512 (here, for $c = \lambda_0 = 0, \lambda_1 = -1, \tilde{\phi}_\pm = \pm\phi$). The choice of equal $\lambda_{\pm,1}$ is equivalent than
 513 considering the constraint $T_1(x) = |x|$, and thus allows to respect the symmetries of the
 514 distribution, allowing thus to recover a classical ϕ -entropy.

515 Appendix A.5. The arcsine distribution

The arcsine distribution is a special case of the beta distribution with $\alpha = \beta = \frac{1}{2}$. We consider here the centered and scaled version of this distribution which writes

$$f_X(x) = \frac{1}{\pi\sqrt{2\sigma^2 - x^2}} \quad \text{on} \quad \mathcal{X} = \left(-\sigma\sqrt{2}; \sigma\sqrt{2} \right).$$

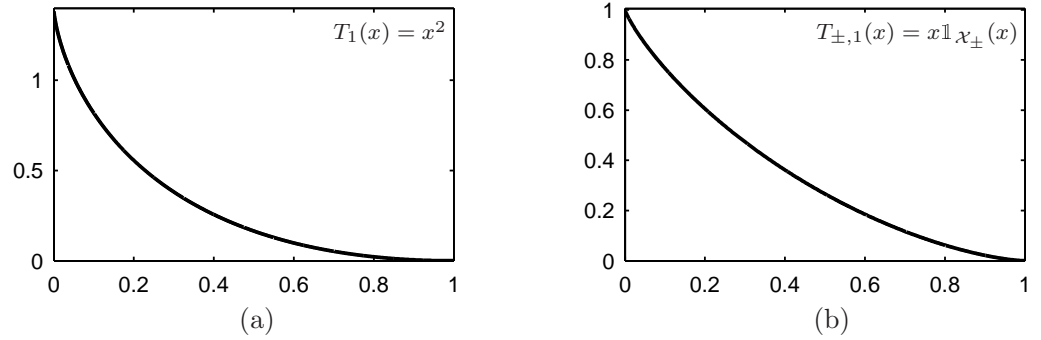


Figure A1. Entropy functional ϕ_u derived from the logistic distribution: (a) with $T_1(x) = x^2$ and (b) with $T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_{\pm}}(x)$.

The inverse distributions $f_{X,\pm}^{-1}$ on $\mathcal{X}_- = (-\sigma\sqrt{2}; 0)$ and $\mathcal{X}_+ = [0; \sigma\sqrt{2})$ write then

$$f_{X,\pm}^{-1}(y) = \pm \frac{\sqrt{2\pi^2\sigma^2y^2 - 1}}{\pi y}, \quad y \geq \frac{1}{\pi\sigma\sqrt{2}}$$

516 Let us now consider again either a second order moment as the constraint, or
517 (partial) first order moment(s).

518 Appendix A.5.1. Second order moment

When the second order moment $T_1(x) = x^2$ is constrained, one immediately obtains

$$\phi'(y) = \lambda_0 + \lambda_1 \left(2\sigma^2 - \frac{1}{\pi^2 y^2} \right)$$

The family of entropy functional is then

$$\phi(y) = c + \left(\lambda_0 + 2\sigma^2\lambda_1 \right) y + \frac{\lambda_1}{\pi^2 y}$$

which drastically simplifies with the special choice

$$c = 0, \quad \lambda_0 = -\frac{\sigma^2}{\pi^2} \quad \text{and} \quad \lambda_1 = \pi^2 \quad \text{to} \quad \phi(y) = \frac{1}{y}$$

519 Appendix A.5.2. (Partial) first-order moment(s)

Since the distribution does not share the sense of variation of $T_1(x) = x$, either we turn out to consider it as an extremal distribution of an entropy that is not concave, or as a maximum entropy when constraints are of the type

$$T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_{\pm}}(x)$$

now

$$\phi'_{\pm}(y) = \sqrt{2}\pi\sigma\lambda_0 + \lambda_{\pm,1} \frac{\sqrt{2\pi^2\sigma^2y^2 - 1}}{y} \quad \text{or} \quad \tilde{\phi}'_{\pm}(y) = \lambda_0 \pm \lambda_1 \frac{\sqrt{2\pi^2\sigma^2y^2 - 1}}{y}$$

where the different factors and the sign are absorbed in the factors $\lambda_0, \lambda_{\pm,1}$. A judicious choice can be to impose

$$\lambda_{-,1} = \lambda_{+,1} = \bar{\lambda}_1 > 0$$

and the same integration constant c for each branch, leading then either to the family of (convex) uniform of functions $\phi(y) = \phi_u(\sqrt{2\pi\sigma}y)$ with

$$\phi_u(u) = c + \lambda_0 u + \bar{\lambda}_1 \left(\sqrt{u^2 - 1} + \arctan\left(\frac{1}{\sqrt{u^2 - 1}}\right) \right) \mathbb{1}_{(1; +\infty)}(u)$$

or, in the non-convex case, to the family of functions with branches $\tilde{\phi}_{\pm}(y) = \tilde{\phi}_{\pm,u}(\sqrt{2\pi\sigma}y)$,

$$\tilde{\phi}_{\pm,u}(u) = c + \lambda_0 u \pm \lambda \left(\sqrt{u^2 - 1} + \arctan\left(\frac{1}{\sqrt{u^2 - 1}}\right) \right) \mathbb{1}_{(1; +\infty)}(u)$$

520 The uniform function ϕ_u is represented figure A2 for the special choice $c = \lambda_0 =$
 521 $0, \bar{\lambda}_1 = 1$ (here again, for $c = \lambda_0 = 0, \lambda_1 = 1, \tilde{\phi}_{\pm} = \pm\phi$). In this case again, the
 522 symmetrical choice for $\lambda_{\pm,1}$ allows to recover the symmetries of the probability density,
 523 and thus to a uniform convex entropy functional in the first context.

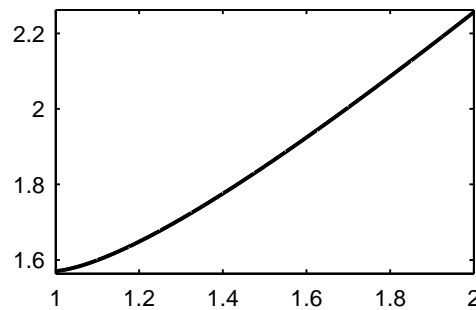


Figure A2. Entropy functional ϕ_u derived from the arcsine distribution with partial constraints $T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_{\pm}}(x)$.

524 Appendix A.6. The gamma distribution and (partial) p -order moment(s)

As a very special case, consider here this distribution, expressed as

$$f_X(x) = \frac{\beta^\alpha x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)} \quad \text{on} \quad \mathcal{X} = \mathbb{R}_+.$$

Let us concentrate on the case $\alpha > 1$ for which the distribution is non-monotonous, unimodal, where the mode is located at $x = x_m$, and $f_X(\mathbb{R}_+) = \left[0; \frac{1}{\tau e^{\alpha-1}}\right]$ with

$$x_m = \frac{\alpha - 1}{\beta} \quad \text{and} \quad \tau = \frac{\Gamma(\alpha)}{\beta (\alpha - 1)^{\alpha-1}}$$

Thus, here again it cannot be viewed as a maximum entropy constraint neither by any p -order moment. Here, we can again interpret it as a maximum entropy constrained by partial moments

$$T_{k,1}(x) = x^p, \quad k \in \{0, -1\} \quad \text{over} \quad \mathcal{X}_0 = [0; x_m) \quad \text{and} \quad \mathcal{X}_{-1} = [x_m; +\infty).$$

or as an extremal entropy constrained by the moment

$$T_1(x) = x^p \quad \text{over} \quad \mathcal{X} = \mathbb{R}_+$$

where $p > 0$. Inverting $y = f_X(x)$ leads to the equation

$$-\frac{x}{x_m} \exp\left(-\frac{x}{x_m}\right) = -(\tau y)^{\frac{1}{\alpha-1}}$$

to be solved. As expected, this equation has two solutions. These solutions can be expressed via the multivalued Lambert-W function W defined by $z = W(z) \exp(W(z))$, i.e., W is the inverse function of $u \rightsquigarrow u \exp(u)$ [112, § 1], leading to the inverse functions

$$f_{X,k}^{-1}(y) = -x_m W_k\left(-(\tau y)^{\frac{1}{\alpha-1}}\right), \quad y \in \left[0; \frac{1}{\tau e^{\alpha-1}}\right],$$

525 where k denotes the branch of the Lambert-W function. $k = 0$ gives the principal branch
 526 and here it is related to the entropy part on \mathcal{X}_0 , while $k = -1$ gives the secondary branch,
 527 related to \mathcal{X}_{-1} here.

One has thus to solve the equation

$$\phi'_k(y) = \lambda_0 \tau + \lambda_{k,1} \tau \left[-W_k\left(-(\tau y)^{\frac{1}{\alpha-1}}\right) \right]^p$$

where the positive factor are absorbed in the $\lambda_0, \lambda_{k,1}$ and where to insure the convexity of the ϕ_k ,

$$(-1)^k \lambda_{k,1} > 0$$

The same approach allows to design $\tilde{\phi}_k$, with a unique λ_1 instead of the $\lambda_{k,1}$. Integrating the previous expression is not an easy task. Relation $u(1 + W_k(u)) W'_k(x) = W_k(u)$ [112, Eq. 3.2] suggests that a way to make the integration is to search for $\phi_k(y) = \phi_{k,u}(\tau y)$ where $\phi_{k,u}(u)$ is searched as the product of $u \left[-W_k\left(-u^{\frac{1}{\alpha-1}}\right) \right]^p$ and a series of $\left[-W_k\left(-u^{\frac{1}{\alpha-1}}\right) \right]$ and then to recognize the coefficients of the series. Such an approach leads to the family of entropic functional $\phi_k(y) = \phi_{k,u}(\tau y)$ with

$$\begin{aligned} \phi_{k,u}(u) &= c_k + \lambda_0 u \\ &+ \lambda_{k,1} u \left[-W_k\left(-u^{\frac{1}{\alpha-1}}\right) \right]^p \left[1 - \frac{p}{p+\alpha-1} {}_1F_1\left(1; p+\alpha; (1-\alpha) W_k\left(-u^{\frac{1}{\alpha-1}}\right)\right) \right] \mathbb{1}_{(0; e^{1-\alpha})}(u) \end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric (or Kummer's) function [113, § 13] and c_k are integration constants. One can verify a posteriori that these functions are the ones we search for. The integration constant can be chosen such that ϕ_k coincide in 0 for instance, that gives

$$c_{-1} - c_0 = \frac{p \Gamma(p+\alpha-1)}{(\alpha-1)^{p+\alpha-1}} \lambda_{-1,1}$$

528 using successively [112, Eq. 3.1] and [113, Eq. 13.1.2] for W_0 , and successively [113,
 529 Eq. 13.1.4] (W_{-1} tending to $-\infty$ in 0^-), $W_{-1}(u) \exp(W_{-1}(u)) = u$, and [112, Eq. 4.6 &
 530 lines that follow] for W_{-1} . The same algebra leads to the same expression for the $\tilde{\phi}_k$,
 531 except that $\lambda_{k,1}$ are replaced by a unique λ_1 .

532 The multivalued function ϕ_u in the concave context is represented figure A3 for
 533 $p = 2, \alpha = 2$ and $\alpha = 5$, and with the choices $c_{-1} = \lambda_0 = 0, \lambda_{0,1} = 1, \lambda_{-1,1} = -0.1$.

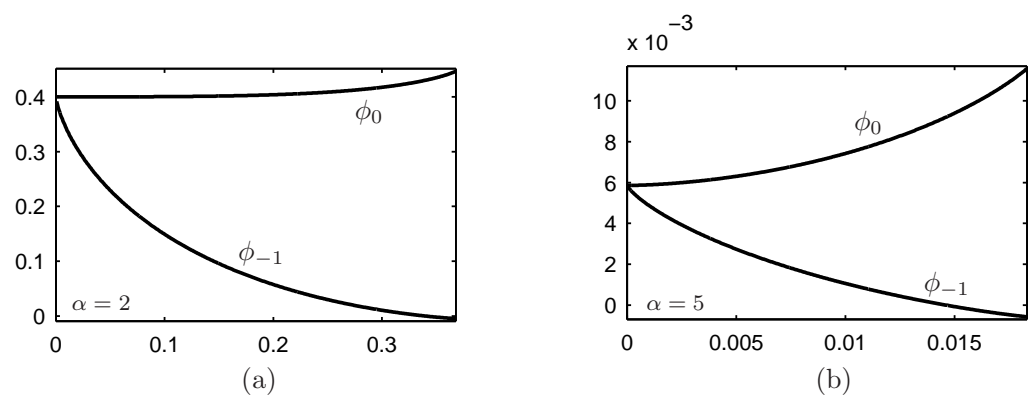


Figure A3. Multiform entropy functional ϕ_u derived from the gamma distribution with the partial moment constraints $T_{k,1}(x) = x^2 \mathbb{1}_{\mathcal{X}_k}(x)$, $k \in \{0, -1\}$. (a): $\alpha = 2$; (b): $\alpha = 5$.

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