ϕ-informational measures: some results in a generalized settings

by jfb & sz

Abstract

To be modified in the light of the new direction of the paper This paper focus on maximum entropy problems under moment constraints. Contrary to the usual problem of finding the maximizer of a given entropy, or of selecting constraints such that a given distribution is a maximizer of the considered entropy, we consider here the problem of the determination of an entropy such that a given distribution is its maximizer. Our goal is to adapt the entropy to its maximizer, with potential application in entropy-based goodness-of-fit tests for instance. It allows us to consider distributions outside the exponential family – to which the maximizers of the Shannon entropy belong -, and also to consider simple moment constraints, estimated from the observed sample. Out approach also yields entropic functionals that are function of both probability density and state, allowing us to include skew-symmetric or multimodal distributions in the setting. Finally, extended informational quantities are introduced, such that generalized moments and generalized Fisher informations. With thes extended quantities, we propose extended version of the Cramér-Rao inequality and of the de Bruijn identity, valid or saturated for the maximal entropy distribution corresponding to the generalized entropy previsouly studied.

1. Introduction

Since the pionner works of von Neuman [1], Shannon [2], Boltzmann, Maxwell, Planck and Gibbs [3–7] on the entropy as a tool for uncertainty or information measure, many investigations were devoted to the generalization of the so-called Shannon entropy and its associated measures [8–20]. If the Shannon measures are compelling, especially in the communication domain, for compression purposes, many generalizations proposed later on has also showed promising interpretations and applications (Panter-Dite formula in quantification where the Rényi or Havdra-Charvát entropy emerges [21–23], codification penalizing long codewords where the Rényi entropy appears [24, 25] for instance). The great majority of the extended entropies found in the literature belongs to a very general class of entropic measures called (h, ϕ)-entropies [11, 17, 18, 26–28]. Such a general class (or more precisely the subclass of ϕ-entropies) traced back to the work of Burbea & Rao [26]. They offer not only a general framework to study general properties shared by special entropies, but they also offer many potential applications as described for instance in [28]. Note that if a large amount of work deals with divergences, entropies occur as special cases when one takes a unform reference measure.

In the settings of these generalized entropies, the so-called maximum entropy principle takes a special place. This principle, advocated by Jaynes, states that the statistical distribution that describes a system in equilibrium maximizes the entropy while satisfying the system's physical constraints (e.g., the center of mass, energy) [29–32]. In other words, it is the less informative law given the constraints of the system. In the Bayesian approach, dealing with the stochastic modelisation of a parameter, such a principle (or a minimum divergence principle) is often used to choose a prior distribution for the parameter [20, 33–36]. It also finds its counterpart in communication, clustering, pattern recognition, problems, among many others [30, 31, 37–39]. In statistics, some goodness-of-fit tests are based on entropic criteria derived from the same idea of constrained maximal entropic law [40–45]. In a large number of works using the maximum entropy principle, the entropy used is the Shannon entropy, although several extensions exist in the literature. However, if for some reason a generalized entropy is considered, the approach used in the Shannon case does not fundamentally change [46–49].

One can consider the inverse problem which consists in finding the moment constraints leading to the observed distribution as a maximal entropy distribution [46]. Kesavan & Kapur also envisaged a second inverse problem, where both the distribution and the moments are given. The question is thus to determine the entropy so that the distribution is its maximizer. jfbThis problem has also a physical interpretation, in the sense that the Shannon entropy described system in the thermodynamic limit, i.e., in equilibrium, whereas other entropies are often evoked to describe systems out of equilibrium [15, 50–54]. While the problem was considered mainly in the discrete Shannon settings by Kesavan & Kapur in [46], we will recall it in the general framework of the (h, ϕ)-entropies, and make a further step considering an extended class of these entropies.

While the entropy is a widely used tool for quantifying information (or uncertainty) attached to a random variable or to a probability distribution, other quantities are used as well, such as the moments of the variable (giving information for instance on center of mass, dispersion, skewness, impulsive character), or the Fisher information. In particular, the Fisher information appears in the context of estimation [55, 56], in Bayesian inference through the Jeffrey's prior [36, 57], but also for complex physical systems descriptions [56, 58–62].

Although coming from different worlds (information theory and communication, estimation, statistics, physics), these informational quantities are linked by well-known relations such the Cramér-Rao's inequality, the de Bruijn's identity, the Stam's inequality [32, 63–65]. These relationships have been proved very useful in various areas, for instance in communications [32, 63, 64], in estimation [55] or in physics [66, 67], among others. When generalized entropies are considered, it is natural to question the other informational measures' generalization and the associated identities or inequalities. This question gave birth to a large amount of work and is still an active field of research [26, 68–79].

In this paper, we show that it is possible to build a whole framework, which associates a target maximum entropy distribution to generalized entropies, generalized moments and generalized Fisher information. In this setting, we derive generalized inequalities and identities relating these quantities, which are all linked in some sense to the maximum entropy distribution.

The paper is organized as follows. In section 2 we recall the definition of the generalized ϕ-entropy. Thus, we come back to the maximum entropy problem in this general settings. Following the sketch of [46], we present a sufficient condition linking the entropic functional and the maximizing distribution, allowing to both solve the direct and the inverse problems. When the sufficient conditions linking the entropic function and the distribution cannot be satisfied, the problem can be solved by introducing state-dependent generalized entropies, which is the purpose of section 3. In section 4, we introduce informational quantities associated to the generalized entropies of the previous sections, such that a generalized escort distribution, generalized moments and generalized Fisher informations. These generalized informational quantities allow to extend the usual informational relations such that the Cramér-Rao inequality, relations saturated (or valid) dealing precisely for the generalized maximum entropy distribution. Finally, in section 5, we show that the extended quantities allows to obtain an extended de Bruijn identity, profided the distribution follows a non-linear heat equation. Some exemple of determination of ϕ-entropies solving the inverse maximum entropy problem are provided in a short series of appendix, showing in other that the usual quantities are recovered in the well known cases (Gausian distribution, Shannon entropy, Fisher information, variance).

In what follows we will define a series of generalized informational quantities of a probability density that is defined with respect to a given reference measure μ (e.g., the Lebesgue measure when dealing with continuous random variables, discrete measure for discrete-state random variabes,...). Therefore, rigorously, all this quantities depend on the particular choice of this reference measure. However, for sake of simplicity we will omit to mention this dependence in the notation along the paper.

2. ϕ-entropies – direct and inverse maximum entropy problems.

Let us first recall the definition of the generalized ϕ-entropies introduced by Csiszàr in terms of divergences, and by Burbea and Rao in terms of entropies:

Definition 1 (ϕ-entropy [26])**.** Let $\phi : \mathcal{Y} \subseteq \mathbb{R}_+ \mapsto \mathbb{R}$ be a convex function defined on a convex set $\mathcal{Y}$. Then, if f is a probability distribution defined with respect to a general measure μ on a set $\mathcal{X} \subseteq \mathbb{R}^d$ such that $f(\mathcal{X}) \subseteq \mathcal{Y}$, when this quantity exists,

$$H_\phi[f] = -\int_{\mathcal{X}} \phi(f(x))\,\mathrm{d}\mu(x) \tag{1}$$

is the ϕ-entropy of f.

The (h,ϕ)-entropy is defined by $H_{(h,\phi)}[f] = h\left(H_\phi[f]\right)$ where h is a nondecreasing function. The definition is extended by allowing ϕ to be concave, together with h nonincreasing [11, 17, 18, 27, 28]. If additionnaly h is concave, then the entropy functional $H_{(h,\phi)}[f]$ is concave.

In the following, since we are interested by the maximum entropy problem and because h is monotone, we can restrict our study to the ϕ-entropies. Additionnaly, we will assume that ϕ is *strictly* convex and *differentiable*.

A useful related quantity to these entropies is the Bregman divergence associated with convex function ϕ:

Definition 2 (Bregman divergence and functional Bregman divergence [20, 80])**.** With the same assumptions as in definition 1, the Bregman divergence associated with ϕ defined on a convex set $\mathcal{Y}$, is given by the function defined on $\mathcal{Y} \times \mathcal{Y}$,

$$D_\phi(y_1,y_2) = \phi(y_1) - \phi(y_2) - \phi'(y_2)\,(y_1 - y_2)\,. \tag{2}$$

Applied to two functions $f_i : \mathcal{X} \mapsto \mathcal{Y}$, $i = 1,2$, the functional Bregman divergence writes

$$\mathcal{D}_\phi(f_1,f_2) = \int_{\mathcal{X}} \phi(f_1(x))\,\mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi(f_2(x))\,\mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi'(f_2(x))\,(f_1(x) - f_2(x))\;\mathrm{d}\mu(x). \tag{3}$$

A direct consequence of the strict convexity of ϕ is the nonnegativity of the (functional) Bregman divergence: $D_\phi(y_1,y_2) \geq 0$ and $\mathcal{D}_\phi(f_1,f_2) \geq 0$, with equality if and only if $y_1 = y_2$ and $f_1 = f_2$ almost everywhere respectively.

Note that, more generally, the Bregman divergence is defined for multivariate convex functions, where the derivative is replaced by gradient operator [80]. Extensions for convex function of functions also exist, where the derivative is in the sense of Gâteau [81]. Such general extensions are not useful for our purposes, thus, we restrict to the above definition where $\mathcal{Y} \subseteq \mathbb{R}_+$.

2.1. Maximum entropy principle: the direct problem

Let us here recall the maximum entropy problem that consists in searching for the distribution maximizing the ϕ-entropy (1) subject to constraints on some moments $\mathbb{E}\left[T_i(X)\right]$ with $T_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \ldots, n$. This direct problem writes

$$f^\star \ = \underset{f \in C_t}{\operatorname{argmax}} \left(- \int_{\mathcal{X}} \phi(f(x)) \, \mathrm{d}\mu(x) \right) \tag{4}$$

with

$$C_t = \left\{ f \geq 0 : \ \mathbb{E}\left[T_i(X)\right] = t_i, i = 0, \ldots, n \right\}, \tag{5}$$

where $T_0(x) = 1$ and $t_0 = 1$ (normalization constraint). The maximization problem being strictly concave, the solution exists and is unique. A technique to solve the problem can be to use the classical Lagrange multipliers technique, but this approach requires mild conditions [46, 47, 49, 82–84]. A sufficient condition relating f and ϕ so that f is the desired solution of the problem is then obtained, as recalled in the following proposition. Below, we prove the result without the use of the Lagrange technique.

Proposition 1 (Maximal ϕ-entropy solution [46])**.** *Suppose that there exists a probability distribution $f \in C_t$ satisfying*

$$\phi'\big(f(x)\big) = \sum_{i=0}^{n} \lambda_i \, T_i(x), \tag{6}$$

for some $(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$. Then, f is the unique solution of the maximal entropy problem (4).

Proof. Suppose that distribution f satisfies (6) and consider any distribution $g \in C_t$. The functional Bregman divergence between f and g writes

$$\begin{aligned}
\mathcal{D}_\phi(g,f) \quad &= \quad \int_{\mathcal{X}} \phi(g(x)) \, \mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi(f(x)) \, \mathrm{d}\mu(x) - \int_{\mathcal{X}} \phi'(f(x)) \left(g(x) - f(x)\right) \, \mathrm{d}\mu(x) \\
&= \quad -H_\phi[g] + H_\phi[f] - \sum_{i=0}^{n} \lambda_i \int_{\mathcal{X}} T_i(x) \left(g(x) - f(x)\right) \, \mathrm{d}\mu(x) \\
&= \quad H_\phi[f] - H_\phi[g]
\end{aligned}$$

where we used the fact that g and f are both probability distributions with the same moments $\mathbb{E}\left[T_i(X)\right] = t_i$. By nonnegativity of the Bregman functional divergence, we finally get that

$$H_\phi[f] \geq H_\phi[g]$$

for all distribution g with the same moments as f, with equality if and only if $g = f$ almost everywhere. In other words, this shows that if f satisfies (6), then it is the desired solution. □

Hence, given an entropic functional ϕ and moments constraints T_i, eq. (6) leads the the maximum entropy distribution $f^\star$. This distribution is parametrized by the λ_i or, equivalently, by the moments t_i.

Note that the reciprocal is not necesarilly true, as shown for instance in [49]. However, the reciprocal is true when $\mathcal{X}$ is a compact [84] or for any $\mathcal{X}$ provided that ϕ is locally bounded on $\mathcal{X}$ [85].

2.2. Maximum entropy principle: the inverse problems

As stated in the introdution, two inverse problems can be considered started from a given distribution f. These problems were considered by Kesavan & Kapur in [46] in the discrete framework.

The first inverse problem consists in searching for the adequate moments so that a desired distribution f is the maximum entropy distribution of a given ϕ-entropy. A solution can thus consist in identifying functions T_i and coefficients λ_i in order to satisfy eq. (6). Obviously, this is not always an easy task, and even not always possible. For instance, it is well known that the maximum Shannon entropy distribution given moment constraints fall in the exponential family [31, 32, 48]. Therefore, if f does not belong to this family, the problem has no solution.

The second inverse problem consists in designing the entropy itself, given a target distribution f and given the T_i. In other words, given a distribution f, eq. (6) may allow to determine the entropic functional ϕ so that f is its maximizer.

As for the direct problem, in the second inverse problem, the solution is parametrized by the λ_i. Here, required properties on ϕ will shape the domain the λ_i live in. In particular ϕ, must satisfy the following properties:

- the domain of definition of ϕ' must include $f(\mathcal{X})$; this will be satisfied by construction.
- from the strict convexity property of ϕ, ϕ' must be strictly increasing.

Hence, because ϕ' must be stricly increasing, its clear that solving eq. (6) requires the following two conditions:

(C1) $f(x)$ and $\displaystyle\sum_{i=1}^{n} \lambda_i\, T_i(x)$ must have the same variations, i.e., $\displaystyle\sum_{i=0}^{n} \lambda_i\, T_i(x)$ is increasing (resp. decreasing, resp. constant) where f is increasing (resp. decreasing, resp. constant).

(C2) $f(x)$ and $\displaystyle\sum_{i=1}^{n} \lambda_i\, T_i(x)$ must have the same level sets, $f(x_1) = f(x_2) \;\Leftrightarrow\; \displaystyle\sum_{i=0}^{n} \lambda_i\, T_i(x_1) = \sum_{i=0}^{n} \lambda_i\, T_i(x_2)$

For instance, in the univariate case, for one moment constraint,

- for $\mathcal{X} = \mathbb{R}_+$, $T_1(x) = x$, λ_1 must be negative and $f(x)$ must be decreasing,
- for $\mathcal{X} = \mathbb{R}$, $T_1(x) = x^2$ or $T_1(x) = |x|$, λ_1 must be negative and $f(x)$ must be even and unimodal.

Under conditions 3 and (C2), the solutions of eq. (6) are given by

$$\phi'(y) = \sum_{i=0}^{n} \lambda_i\, T_i\big(f^{-1}(y)\big) \tag{7}$$

where f^{-1} can be multivalued.

Eq. (6) provides an effective way to solve the inverse problem. However, there exist situations where there do not exist any set of λ_i such that conditions 3-(C2) are satisfied (e.g., $T_1(x) = x^2$ with f not even). In such a case, a way to go is to extent the definition of the ϕ-entropy, purpose of section 3.

2.3. Second inverse maximum entropy problem: some examples

To illustrate the previous subsection, let us analyze very briefly three examples: the very famous Gaussian distribution (example 1), the q-Gaussian distribution also intensively studied (example 2) and the arcsine distribution (example 3), both three with a second order moment constrainst. The Gaussian, q-Gaussian, and arcsine distributions will serve as a a guideline all along the paper. The details of the calculus, together with a deeper study related to the sequel of the paper, are rejected in the appendix. Other examples are also given in this appendix. In both three examples, except in the next section, we consider the second order moment constraint $T_1(x) = x^2$.

Example 1 The Gaussian distribution $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\,\sigma^2}\right)$ defined over $\mathcal{X} = \mathbb{R}$, with the constraint $T_1(x) = x^2$ viewed as a maximum ϕ-entropy imposes that $\lambda_1 < 0$. Rapid calculus leads to the entropic functional, after a reparametrization of the λ_i's, of the form,

$$\phi(y) = \alpha\, y \log(y) + \beta\, y + \gamma \quad \text{with} \quad \alpha > 0,$$

that is nothing more than the Shannon entropy, up to the scaling factor α, and a shift (except for $\gamma = -\beta$). One thus recovers the long outstanding fact that the Gaussian is the maximum Shannon entropy distribution with the second order moment constraint.

Example 2 The q-Gaussian distribution, also known as Tsallis distribution, $f_X(x) = C_q \left(1 - (q-1)\frac{x^2}{\sigma^2}\right)_+^{\frac{1}{1-q}}$, where $q > 0$, $q \neq 1$, $x_+ = \max(x, 0)$ and C_q is the normalization coefficient, defined over $\mathcal{X} = \mathbb{R}$, with the constraint $T_1(x) = x^2$ viewed as a maximum ϕ-entropy imposes also that $\lambda_1 < 0$. Rapid calculus leads to the entropic functional, after a reparametrization of the λ_i's, as,

$$\phi(y) = \alpha\, \frac{y^q - y}{q - 1} + \beta\, y + \gamma \quad \text{with} \quad \alpha > 0,$$

that is nothing more than the Havrdat-Charvát-Tsallis entropy [10, 12, 15, 86], up to the scaling factor α, and a shift (except for $\gamma = -\beta$). One recover the also well known fact that the q-Gaussian is the maximum Shannon entropy distribution with the second order moment constraint [86]. In the limit case $q \to 1$, the distribution f_X tends to the Gaussian, whereas the Havrdat-Charvát-Tsallis entropy tends to the Shannon entropy.

Example 3 The arcsince distribution, $f_X(x) = \frac{1}{\pi\sqrt{2\,\sigma^2 - x^2}}$, defined over $\mathcal{X} = \left(-\sigma\sqrt{2}\,;\,\sigma\sqrt{2}\right)$, with the constraint $T_1(x) = x^2$ viewed as a maximum ϕ-entropy imposes now that $\lambda_1 > 0$. Short algebra leads to the entropic functional, after a reparametrization of the λ_i's,

$$\phi(y) = \frac{\alpha}{y} + \beta\, y + \gamma \quad \text{with} \quad \alpha > 0.$$

This entropy is non usual and, due to its form, is potentially finite only for densities defined over a bounded support and that are divergent in its boundary (integrable divergence).

3. State-dependent entropic functionals and mimization revisited

In order to follow asymmetries of the distribution f and adress the limiation raised above, an idea is to allow the entropic functional to be depend also on the state variable x:

Definition 3 (State-dependent ϕ-entropy)**.** Let $\phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ such that for any $x \in \mathcal{X} \subseteq \mathbb{R}^d$, function $\phi(x,\cdot)$ is a convex function on the closed convex set $\mathcal{Y} \subseteq \mathbb{R}_+$. Then, if f is a probability distribution defined with respect to a general measure μ on set $\mathcal{X}$ and such that $f(\mathcal{X}) \subseteq \mathcal{Y}$,

$$H_\phi[f] = -\int_{\mathcal{X}} \phi(x, f(x))\,\mathrm{d}\mu(x) \tag{8}$$

will be called state-dependent ϕ-entropy of f. Since $\phi(x,\cdot)$ is convex, then the entropy functional $H_\phi[f]$ is concave. A particular case arises when, for a given partition $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$ of $\mathcal{X}$, functional ϕ writes

$$\phi(x,y) = \sum_{l=1}^{k} \phi_l(y) \mathbb{1}_{\mathcal{X}_l}(x) \tag{9}$$

where $\mathbb{1}_A$ denotes the indicator of set A. This functional can be viewed as a "$(\mathcal{X}_1,\ldots,\mathcal{X}_k)$-extension" over $\mathcal{X} \times \mathcal{Y}$ of a multiform function defined on $\mathcal{Y}$, with k branches ϕ_l and the associated ϕ-entropy will be called $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$-multiform ϕ-entropy.

As in the previous section, we restrict our study to functionals $\phi(x,y)$ *strictly convex and differentiable* versus y.

Following the lines of section 2, a generalized Bregman divergence can be associated to ϕ under the form $D_\phi(x,y_1,y_2) = \phi(x,y_1) - \phi(x,y_2) - \frac{\partial \phi}{\partial y}(x,y_2)\,(y_1-y_2)$, and a generalized functional Bregman divergence $\mathcal{D}_\phi(f_1,f_2) = \displaystyle\int_{\mathcal{X}} D_\phi(x, f_1(x), f_2(x))\,\mathrm{d}\mu(x)$.

With these extended quantities, the direct problem becomes

$$f^\star = \underset{f \in C_t}{\operatorname{argmax}} \left(-\int_{\mathcal{X}} \phi(x, f(x))\,\mathrm{d}\mu(x) \right) \tag{10}$$

Although the entropic functional is now state dependent, the approach adopted before can be applied here, leading to

Proposition 2 (Maximum state-dependent ϕ-entropy solution)**.** *Suppose that there exists a probability distribution f satisfying*

$$\frac{\partial \phi}{\partial y}\big(x, f(x)\big) = \sum_{i=0}^{n} \lambda_i\, T_i(x), \tag{11}$$

for some $(\lambda_0,\ldots,\lambda_n) \in \mathbb{R}^{n+1}$, then f is the unique solution of the extended maximum entropy problem (10).
If ϕ is chosen in the $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$-multiform ϕ-entropy class, this sufficient condition writes

$$\sum_{l=1}^{k} \phi_l'\big(f(x)\big)\, \mathbb{1}_{\mathcal{X}_l}(x) = \sum_{i=0}^{n} \lambda_i\, T_i(x), \tag{12}$$

Proof. The proof is the very same as that of Proposition 1, using the generalized functional Bregman divergence instead of the usual one. □

Resolution eq. (11) is not possible in all generality. However the sufficient condition. (12) can be rewritten as

$$\sum_{l=1}^{k} \left(\phi_l'\big(f(x)\big) - \sum_{i=0}^{n} \lambda_i\, T_i(x) \right) \mathbb{1}_{\mathcal{X}_l}(x) = 0. \tag{13}$$

Thus, if there exists (at least) a set of λ_i such that condition 3 is satisfied (but not necessarily (C2)), one can always

- design a partition $(\mathcal{X}_1,\ldots,\mathcal{X}_k)$ so that (C2) is satisfied *in each* $\mathcal{X}_l$ (at least, such that f is either strictly monotonic, or constant, on $\mathcal{X}_l$)
- determine ϕ_l as in eq. (7) in each $\mathcal{X}_l$, that is

$$\phi_l'(y) = \sum_{i=0}^{n} \lambda_i\, T_i\big(f_l^{-1}(y)\big) \tag{14}$$

where f_l^{-1} is the (possibly multivalued) inverse of f on $\mathcal{X}_l$.

In a conclusion, in the case where only condition 3 is satisfied, one can obtain an extended entropic functionnal of $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$-multiform class so that eq. (13) provides an effective way to solve the inverse problem in the state-dependent entropic functional context.

Note however that it still may happen that there is no set of λ_i allowing to satisfy 3. In such an harder context, the problem remains solvable when then moments are defined as partial moments like $\mathbb{E}\left[T_{l,i}(X) \mathbb{1}_{\mathcal{X}_l}(X)\right] = t_{l,i}$, $l = 1, \ldots, k$ and $i = 1, \ldots, n_l$ and when there exist on $\mathcal{X}_l$ a set of $\lambda_{l,i}$ such that 3 and (C2) holds. The solution still writes as in eq. (14), but where now n, the λ_i and the T_i are replaced by n_l, $\lambda_{l,i}$ and $T_{l,i}$ respectively.

Let us now come back to the arcsine example (example 3) of the previous section, when now we constraint the first order moment or partial first order moments.

Example 3-1 The arcsince distribution, $f_X(x) = \frac{1}{\pi\sqrt{2\,\sigma^2 - x^2}}$, defined over $\mathcal{X} = \left(-\sigma\sqrt{2}\,;\,\sigma\sqrt{2}\right)$, is bijective on each set $\mathcal{X}_- = (-\sigma\sqrt{2}\,;\,0)$ and $\mathcal{X}_+ = [0\,;\,\sigma\sqrt{2})$ that partitions $\mathcal{X}$. With the partial constraint $T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_\pm}(x)$, this distribution viewed as a maximum ϕ-entropy imposes that $\lambda_{\pm,1} > 0$ and the associated multiform entropic functional, after a reparametrization of the λ_i's, writes

$$\phi_\pm(y) = \alpha_\pm \left(\sqrt{2\pi^2\sigma^2y^2 - 1} + \arctan\left(\frac{1}{\sqrt{2\pi^2\sigma^2y^2 - 1}} \right) \right) \mathbb{1}_{(1\,;\,+\infty)}(\sqrt{2}\,\pi\sigma\,y) + \beta\,y + \gamma_\pm \quad \text{with} \quad \alpha_\pm > 0$$

(see appendix for more details).

Example 3-2 This arcsince distribution, now constraint uniformely by $T_1(x) = x$, viewed as an extremal ϕ-entropy imposes again that $\lambda_1 > 0$ and the associated multiform entropic functional, after a reparametrization of the λ_i's, is given by

$$\widetilde{\phi}_\pm(y) = \pm\alpha \left(\sqrt{2\pi^2\sigma^2y^2 - 1} + \arctan\left(\frac{1}{\sqrt{2\pi^2\sigma^2y^2 - 1}} \right) \right) \mathbb{1}_{(1\,;\,+\infty)}(\sqrt{2}\,\pi\sigma\,y) + \beta\,y + \gamma \quad \text{with} \quad \alpha > 0$$

(see appendix for more details). The entropic functional is no more convex.

At a first glance, the two solutions seem to be identical. In fact, they drastically differ. Indeed, let us insist on the fact that in the first case, the problem has two constraints whereas in the second case there is only one. The consequence is that the first solution is parametrized by 5 parameters $\beta, \gamma_\pm$ and, especially, $\alpha_\pm$, while only 3 parametrize the second solution β, γ and α. This difference is not insignificant: the second case cannot be viewed as a special case of the first one because $\alpha_\pm$ must be positive, which cannot be possible with only one parameter because $\pm\alpha$ rule the $\widetilde{\phi}_\pm$. The consequcence is that for the second example, the solution does not lead to a convex function, otherwise, it would have contradict the required condition on the parts $\mathcal{X}_\pm$.

In section 2 and 3 we established general entropies with a given maximizer. In what follows, we will complete the information theoretical settings by introducing generalized escort distributions, generalized moments, and generalized Fisher information associated to the same entropic functional, and study their relationships.

4. ϕ-escort distribution, (ϕ, α)-moments, (ϕ, β)-Fisher informations, generalized Cramér-Rao inequalities

In this section, after introducing the above mentioned informational quantities, we will show that generalizations of the celebrated Cramér-Rao inequalities hold. The lower bound of the inequalities are saturated precisely by maximal ϕ-entropy distributions.

Escort distributions have been introduced as an operational tool in the context of multifractals [87, 88], with interesting connections with the standard thermodynamics [89] and with source coding [24, 25]. In our context, we also define (generalized) escort distributions associated with a particular ϕ-entropy, and show how they pop up naturally. It is then possible to define generalized moments with respect these escort distributions.

Definition 4 (ϕ-escort)**.** Let $\phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ such that for any $x \in \mathcal{X} \subseteq \mathbb{R}^d$ function $\phi(x, \cdot)$ is a strictly convex twice differentiable function defined on the closed convex set $\mathcal{Y} \subseteq \mathbb{R}_+$. Then, if f is a probability distribution defined with respect to a general measure μ on a set $\mathcal{X}$ such that $f(\mathcal{X}) \subseteq \mathcal{Y}$, such that

$$C_\phi[f] = \int_{\mathcal{X}} \frac{\mathrm{d}\mu(x)}{\frac{\partial^2\phi}{\partial y^2}(x, f(x))} < +\infty \tag{15}$$

we define by

$$E_{\phi,f}(x) = \frac{1}{C_\phi[f]\, \dfrac{\partial^2\phi}{\partial y^2}(x, f(x))} \tag{16}$$

the ϕ-escort density with respect to measure μ, associated to density f .

Note that from the strict convexity of ϕ with respect to its second argument, this probability density is well defined and is strictly positive. Moreover, coming back to the previous examples, one can see that:

Example 1 In the context of the Shannon entropy, entropy for which the Gaussian is the maximal entropy law for the second order moment constraint, $\phi(x,y) = \phi(y) = y \log y$, the ϕ-escort density associated to f restricts to density f itself.

Example 2 In the Rényi-Tsallis context, entropy for which the q-Gaussian is the maximal entropy law for the second order moment constraint $\phi(x,y) = \phi(y) = \frac{y^q - y}{q-1}$, and $E_{\phi,f} \propto f^{2-q}$ which recovers the escort distributions used in the Rényi-Tsallis context up to a duality transformation [89].

Example 3 For the entropy that is maximal for the arcsine distribution under the second order moment constraint, $\phi(x,y) = \phi(y) = \frac{1}{y}$, and $E_{\phi,f} \propto f^3$ which is nothing more than an escort distributions used in the Rényi-Tsallis context. Indeed, although the arcsine distribution does not fall in the q-Gaussian family, its form is very similar to a q-distribution where the "scaling" would not be related to the exponent q. It is thus not suprising to recover an escort distribution associated to this family.

Definition 5 ((α,ϕ)-moments)**.** Under the assumptions of definition 4, with $\mathcal{X}$ equiped with a norm $\|\cdot\|_{\mathcal{X}}$, we define by

$$M_{\alpha,\phi}[f;X] = \int_{\mathcal{X}} \|x\|_{\mathcal{X}}^{\alpha} \, E_{\phi,f}(x) \, \mathrm{d}\mu(x) \tag{17}$$

if this quantity exists, as the (α,ϕ)-moment of X associated to distribution f.

Note that:

Example 1 In the context of the Shannon entropy, the (α,ϕ)-moments are the usual moments of $\|X\|_{\mathcal{X}}^{\alpha}$.

Example 2 In the Rény-Tsallis context the generalized moments introduced in [50, 90] are recovered.

Example 3 For $\phi(x,y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalized moments introduced in [50, 90] (see the items related to the escort distributions).

The importance of the Fisher information is well known in estimation theory: the estimation error of a parameter is bounded by the inverse of the Fisher information associated with this distribution [32, 55]. The Fisher information is also used as a method of inference and understanding in statistical physics and biology, as promoted by Frieden [56] and has been generalized in the Rényi-Tsallis context in a series of papers [70, 73, 75–78, 91, 92]. In what follows, we generalize these definitions a step further in our ϕ-entropy context by using the above defined ϕ-escort distribution.

Definition 6 (Nonparametric (β,ϕ)-Fisher information)**.** With the same assumption as in definition 5, denoting by $\|\cdot\|_{\mathcal{X}*}$ the dual norm, for any differentiable density f, we define the quantity

$$I_{\beta,\phi}[f] = \int_{\mathcal{X}} \left\| \frac{\nabla_x f(x)}{E_{\phi,f}(x)} \right\|_{\mathcal{X}*}^{\beta} E_{\phi,f}(x) \, \mathrm{d}\mu(x) \tag{18}$$

if this quantity exists, as the nonparametric (β,ϕ)-Fisher information of f.

Note that when ϕ is state-independent, $\phi(x,y) = \phi(y)$, as for the usual Fisher information, this quantity is shift-invariant, i.e., for $g(x) = f(x - x_0)$ one have $I_{\beta,\phi}[g] = I_{\beta,\phi}[f]$. This property is unfortunately lost in the state-dependent context.

Definition 7 (Parametric (β,ϕ)-Fisher information)**.** Let consider the same assumption as in definition 5, such that density f is parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$. The set Θ is equipped with a norm $\|\cdot\|_{\Theta}$ and the corresponding dual norm is denoted $\|\cdot\|_{\Theta*}$. Assume that f is differentiable with respect to θ. We define by

$$I_{\beta,\phi}[f;\theta] = \int_{\mathcal{X}} \left\| \frac{\nabla_\theta f(x)}{E_{\phi,f}(x)} \right\|_{\Theta*}^{\beta} E_{\phi,f}(x) \, \mathrm{d}\mu(x) \tag{19}$$

as the parametric (β,ϕ)-Fisher information of f.

Note that, as for the usual Fisher information, when the norm on $\mathcal{X}$ and on Θ are the same, the nonparametric and parametric information coincide when θ is a location parameter. Note also that:

Example 1 In the Shannon entropy context, when the norm is the euclidean norm and $\beta = 2$, the nonparametric and parametric informations (β,ϕ)-Fisher give the usual nonparametric and parametric Fisher informations respectively.

Example 2 Similarly, in the Rényi-Tsallis context, the generalizations proposed in [76–78] are recovered.

Example 3 For $\phi(x,y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalized moments introduced in [50, 90] (see the items related to the escort distributions).

We have now the quantities that allow to generalize the Cramér-Rao inequalities as follows.

Proposition 3 (Nonparametric (α,ϕ)-Cramér-Rao inequality). *Assume that a differentiable probability density function with respect to a measure μ, defined on a domain $\mathcal{X}$, admits an (α,ϕ)-moment and a (α^*,ϕ)-Fisher information with $\alpha \geq 1$ and α^* Holder-conjugated $\frac{1}{\alpha}+\frac{1}{\alpha^*}=1$, and that $xf(x)$ vanishes at the boundary of $\mathcal{X}$. Thus, density f satisfies the (α,ϕ) extended Cramér-Rao inequality*

$$M_{\alpha,\phi}[f;X]^{\frac{1}{\alpha}}\, I_{\alpha^*,\phi}[f]^{\frac{1}{\alpha^*}} \;\geq\; d \tag{20}$$

When ϕ is state independent, $\phi(x,y)=\phi(y)$, the equality occurs when f is the maximal ϕ entropy distribution subject to the moment constraint $T(x)=\|x\|_{\mathcal{X}}^{\alpha}$.

Proof. The approach follows [78], starting from the differentiable probability density f (derivative denoted $\nabla_x f$), since $xf(x)$ vanishes in the boundaries of X from the divergence theorem one has

$$0=\int_{\mathcal{X}} \nabla_x^t\left(xf(x)\right)\,\mathrm{d}\mu(x)=\int_{\mathcal{X}}\left(\nabla_x^t\, x\right) f(x)\,\mathrm{d}\mu(x)+\int_{\mathcal{X}} x^t\left(\nabla_x f(x)\right)\,\mathrm{d}\mu(x)$$

Now, for the first term, we use the fact that $\nabla_x x = d$ and that f is a density to achieve

$$d=-\int_{\mathcal{X}} x^t\,\frac{\nabla_x f(x)}{g(x)}\,g(x)\,\mathrm{d}\mu(x)$$

for any function g non-zero on $\mathcal{X}$. Now, noting that $d>0$, we obtain from [78, Lemma 2]

$$d=\left|\int_{\mathcal{X}} x^t\left(\frac{\nabla_x f(x)}{g(x)}\right) g(x)\,\mathrm{d}\mu(x)\right| \leq \left(\int_{\mathcal{X}}\|x\|_{\mathcal{X}}^{\alpha}\, g(x)\,\mathrm{d}\mu(x)\right)^{\frac{1}{\alpha}}\left(\int_{\mathcal{X}}\left\|\frac{\nabla_x f(x)}{g(x)}\right\|_{\mathcal{X}^*}^{\alpha^*} g(x)\,\mathrm{d}\mu(x)\right)^{\frac{1}{\alpha^*}}$$

The proof ends by choosing $g=E_{\phi,f}$ the ϕ-escort density associated to density f. Note now that, again from [78, Lemma 2] the equality is obtained when

$$\nabla_x f(x)\,\frac{\partial^2\phi}{\partial y^2}(x,f(x))=\lambda_1\nabla_x\|x\|_{\mathcal{X}}^{\alpha}$$

where λ_1 is a negative constant. Consider now the case where $\phi(x,y)=\phi(y)$ is state-independent. Thus, $\nabla_x f(x)\,\frac{\partial^2\phi}{\partial y^2}(x,f(x))=\nabla_x\phi'(f(x))$, that gives

$$\phi'(f(x))=\lambda_0+\lambda_1\|x\|_{\mathcal{X}}^{\alpha}$$

This last equation has precisely the form eq. (6) of proposition 1. □

An obvious consequence of the proposition is that the probabiliy density that minimizes the (α^*,ϕ)-Fisher information subject to the moment constraint $T(x)=\|x\|_{\mathcal{X}}^{\alpha}$ coincides with the maximal ϕ-entropy distribution subject to the same moment constraint.

In the problem of estimation, the purpose is to determine a function $\hat{\theta}(x)$ in order to estimate an unknown parameter θ. In such a context, the Cramér-Rao inequality allows to lowerbound the variance of the estimator thanks to the parametric Fisher information. The spirit is thus to extend such an inequality to bound any α order mean error thanks to generalized Fisher information.

Proposition 4 (Parametric (α,ϕ)-Cramér-Rao inequality). *Let f be a probability density function with respect to a general measure μ define over a set $\mathcal{X}$, where f is parametrized by a parameter $\theta\in\Theta\subseteq\mathbb{R}^m$ but μ does not depend on θ and satisfying the conditions of definition 7. Assume that domain $\mathcal{X}$ does not depend on θ neither, that f is a jointly measurable function of x and θ, is integrable with respect to x, is absolutely continuous with respect to θ and that the derivatives with respect to each component of θ are locally integrable. Thus, for any estimator $\widehat{\theta}(X)$ of θ that does not depend on θ, we have*

$$M_{\alpha,\phi}\left[f;\widehat{\theta}(X)-\theta\right]^{\frac{1}{\alpha}}\; I_{\alpha^*,\phi}[f;\theta]^{\frac{1}{\alpha^*}} \;\geq\; \left|\,m\,+\,\nabla_\theta^t\, b(\theta)\,\right| \tag{21}$$

where

$$b(\theta)=\mathbb{E}\left[\widehat{\theta}(X)-\theta\right] \tag{22}$$

is the bias of the estimator and α and α^* are Holder conjugated. When ϕ is state independent, $\phi(x,y)=\phi(y)$, the equality occurs when f is the maximal ϕ entropy distribution subject to the moment constraint $T(x)=\|\Theta(x)-\theta\|_{\Theta}^{\alpha}$.

Proof. The proof follows again that of [78], and start first by evaluating the divergence of the bias. The regularity conditions in the statement of the theorem enable to interchange integration with respect to x and differentiation with respect to θ, thus

$$\nabla_\theta^t\, b(\theta)\;=\;\int_{\mathcal{X}}\left(\nabla_\theta^t\,\widehat{\theta}(x)\;-\;\nabla_\theta^t\,\theta\right) f(x)\,\mathrm{d}\mu(x)\;+\;\int_{\mathcal{X}}\left(\widehat{\theta}(x)-\theta\right)^t\nabla_\theta f(x)\,\mathrm{d}\mu(x)$$

Note then that $\nabla_\theta^t\,\theta\;=\;m$ and that $\widehat{\theta}$ being independent on θ one has $\nabla_\theta^t\,\widehat{\theta}(x)\;=\;0$. Thus, f being a probability density, the equality becomes

$$m+\nabla_\theta^t\, b(\theta)=\int_{\mathcal{X}}\left(\widehat{\theta}(x)-\theta\right)^t\,\frac{\nabla_\theta f(x)}{g(x)}\,g(x)\,\mathrm{d}\mu(x)$$

for any density g non-zero on $\mathcal{X}$. The proof ends with the very same steps that in proposition 4 using [78, Lemma 2]. □

Note that:

Example 1 The usual parametric and nonparametric Cramér-Rao inequality are recovered in the usual Shannon context $\phi(x,y) = y \log y$, using the euclidean norm and $\alpha = 2$. The bound in the nonparametric context is saturated for the maximal entropy law, namely the Gaussian.

Example 2 In the Rényi-Tsallis context, the generalizations proposed in [76–78] are recovered and, again, when $\alpha = 2$, the bound is saturated in the nonparametric context for the q-Gaussian, maximal entropy law under the second order moment constraint.

Example 3 For $\phi(x,y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalized moments introduced in [50, 90] (see the items related to the escort distributions).

Beyond the mathematical aspect of these relations, they may have great interest to asses for instance an estimator when the usual variance/mean square error does not exist. Moreover, the escort distribution is also a way to emphasis some part of a distribution. For instance, in the Rényi-Tsallis context, one can see that in f^q either the tails of the head of the distribution is emphasis. Playing with q is thus a way to penalize more either the tails, or the head of the distribution in the estimation (estimation using the escort and adequate Cramér-Rao bound to assess the estimator).

5. ϕ-heat equation and extended de Bruijn identity

An important relation connecting the Shannon entropy H, comming from the "information world", with the Fisher information I, living in the "estimation world", is given by the de Bruijn identity and is closely linked to the Gaussian distribution. Considering a noisy random variable $Y_t = X + \sqrt{t}N$ where N is a zero-mean d-dimensional standard Gaussian random vector and X a d-dimensional random vector independant of N, and of support independent on parameter t, then

$$\frac{d}{dt} H[f_{Y_t}] = \frac{1}{2} I[f_{Y_t}]$$

where f_{Y_t} stands for the probability distribution of Y_t. This identity is in the heart of the proof of the entropy power inequality, and then to a proof of the Stam's inequality [32]. The key point of the proof of this identity is the heat equation satisfied by the probability distribution f_{Y_t}, $\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f$ where Δ stands for the Laplacian operator [93].

Inspired by the work [79], we consider in the following, probability distributions f parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, satisfying what we will call *generalized ϕ-heat equation*,

$$\nabla_\theta f = K \ \mathrm{div} \left(\|\nabla_x \phi'(f)\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f \right) \tag{23}$$

for some $K \in \mathbb{R}^m$ (possibly dependent on θ) and where ϕ is a convex twice differentiable function defined over a set $\mathcal{X} \in \mathbb{R}_+$.

Proposition 5 (Extended de Bruijn identity). *Let f be a probability distribution, parametrized by a parameter $\theta \in \Theta \subseteq \mathbb{R}^m$, defined over a set $\mathcal{X} \subset \mathbb{R}^d$ that do not depend on θ, and satisfying the nonlinear ϕ-heat equation eq.* (23) *for a twice differentiable convex function ϕ. Assume that $\nabla_\theta \phi(f)$ is absolutely integrable and locally integrable with respect to θ, and that the function $\|\nabla_x \phi'(f)\|_{\mathcal{X}^*}^{\beta-2} \nabla_x \phi(f)$ vanishes at the boundary of $\mathcal{X}$. Thus, distribution f satisfies the extended de Bruijn identity, relating the ϕ-entropy of f and its nonparametric (β,ϕ)-Fisher information as follows,*

$$\nabla_\theta H_\phi[f] = K C_\phi^{1-\beta} \, I_{\beta,\phi}[f] \tag{24}$$

with C_ϕ is the normalisation constant given eq. (15).

Proof. From the definition of the ϕ-entropy, the smoothness of the assumption enabling to use the Leibnitz' rule and differentiate under the integral,

$$\begin{aligned}
\nabla_\theta H_\phi[f] &= -\int_{\mathcal{X}} \phi'(f(x)) \, \nabla_\theta f(x) \, \mathrm{d}\mu(x) \\
&= -K \int_{\mathcal{X}} \phi'(f(x)) \ \mathrm{div} \left(\|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f(x) \right) \ \mathrm{d}\mu(x) \\
&= -K \int_{\mathcal{X}} \mathrm{div} \left(\phi'(f(x)) \|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f(x) \right) \ \mathrm{d}\mu(x) + K \int_{\mathcal{X}} \nabla_x^t \phi'(f(x)) \|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x f(x) \, \mathrm{d}\mu(x) \\
&= -K \int_{\mathcal{X}} \mathrm{div} \left(\|\nabla_x \phi'(f(x))\|_{\mathcal{X}^*}^{\beta-2} \nabla_x \phi(f(x)) \right) \ \mathrm{d}\mu(x) + K \int_{\mathcal{X}} \left(\phi''(f(x))\right)^{\beta-1} \|\nabla_x f(x)\|_{\mathcal{X}^*}^{\beta} \, \mathrm{d}\mu(x)
\end{aligned}$$

where the second line comes from the ϕ-heat equation and where the third line comes from the product derivation rule.

Now, from the divergence theorem, the first term of the right handside reduces to the integral of $\|\nabla_x \phi'(f)\|_{\mathcal{X}^*}^{\beta-2} \nabla_x \phi(f)$ on the boundary of $\mathcal{X}$, that vanishes from the assumption of the proposition, while the second term of the right handside related to C_ϕ and the (β,ϕ)-Fisher information from eqs. (15), (16) and definition 6. □

Coming back to the special examples we presented all along the paper:

Example 1 In the Shannon entropy context, for $K = \frac{1}{2}$ and $\beta = 2$, the standard heat-equation is recovered and the usual de Bruijn identity is recovered.

Example 2 The case where $\phi(y) = y^q$ was intensively studied in [79] and the results of the paper are naturally recovered. In particular, the generalized ϕ-heat equation appears in anomalous diffusion in porous medium [79, 94, 95].

Example 3 For $\phi(x,y) = \phi(y) = \frac{1}{y}$ one also naturally find the generalized moments introduced in [50, 90] (see the items related to the escort distributions).

Note that various physical non linear diffusions equation are encompassed in the generalized ϕ-heat equation [95, 96]. **Anomalous diffusion, NL FK: [94?]**

6. Concluding remarks

Dans une telle construction, lorsque loi cible tends vers la Gaussienne/puissance par exemple, la phi va tendre vers la Shannon.

Appendix A. Inverse maximum entropy problem and associated inequalities: some examples

In this appendix, we will now derive in detail several case of inverse problem of the maximal entropy problem. In each case, we will thus provide the quantities and inequalities associated with the entropic functional ϕ, as derived in the text. In the sequel, for sake of simplicity, we restricts our example to the univariate context $d = 1$.

Appendix A.1. Normal distribution and second-order moment

For a normal distribution, and second order moment constraint

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\,\sigma^2}\right) \qquad \text{and} \qquad T_1(x) = x^2 \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}.$$

We begin by computing the inverse of $y = f_X(x)$ where $x \in \mathbb{R}_+$ for instance, which gives

$$\phi'(y) = \left(\lambda_0 - \sigma^2 \log(2\pi\sigma^2)\,\lambda_1\right) - 2\sigma^2\lambda_1 \log y \quad \text{with } \lambda_1 < 0.$$

The judicious choice

$$\lambda_0 = 1 - \log(\sqrt{2\pi}\sigma) \qquad \text{and} \qquad \lambda_1 = -\frac{1}{2\,\sigma^2}$$

leads to function

$$\phi(y) = y \log y$$

that gives nothing more than the Shannon entropy as expected.

Now, $\phi''(y) \propto \frac{1}{y}$ leading to the escort distribution $E_{\phi,f} = f$ so that, as expected, the (α,ϕ) moments are the usual moments of order α. When $\beta = 2$ and the usual euclidean norm is considered, the (β,ϕ)-Fisher informations are the usual Fisher information and the usual Cramér-Rao inequalities are recovered for $\alpha = 2$. Finally, for $\beta = 2$, the usual euclidean norm, the ϕ-heat equation turns to be the heat equation, satisfied by the gaussian, so that the usual de Bruijn identity is naturally recovered.

Appendix A.2. q-Normal distribution and second-order moment

For q-normal distribution, also known as Tsallis distributions, Student-t and -r, and a second order moment constraint,

$$f_X(x) = C_q\Big(1 - (q-1)\beta x^2\Big)_+^{\frac{1}{(q-1)}} \qquad \text{and} \qquad T_1(x) = x^2 \qquad \text{on} \qquad \mathcal{X} = \mathbb{R},$$

where $q > 0$, $x_+ = \max(x, 0)$ and C_q is a normalization coefficient, we get

$$\phi'(y) = \left(\lambda_0 + \frac{\lambda_1}{(q-1)\beta}\right) - \frac{\lambda_1\, y^{q-1}}{C_q^{q-1}(q-1)\beta}.$$

In this case, a judicious choice of parameters is

$$\lambda_0 = \frac{q\, C_q^{q-1} - 1}{q-1} \qquad \text{and} \qquad \lambda_1 = -q\, C_q^{q-1}\beta$$

that yields to

$$\phi(y) = \frac{y^q - y}{q-1}.$$

and an associated entropy is then

$$H_\phi[f] = \frac{1}{1-q} \left(\int_{\mathcal{X}} f(x)^q \, \mathrm{d}\mu(x) - 1 \right):$$

It is nothing but Havrdat-Charvát-Tsallis entropy [10, 12, 15, 86].

Then, $\phi''(y) = q y^{q-2}$: $M_{\phi,\alpha}[f]$ and $I_{\phi,\alpha}[f]$ are respectively the q-moment of order α and the (q, β)-Fisher information defined previously in [73–78] (with the symmetric q index given here by $2-q$). The extended Cramér-Rao inequality proved in [73, 77, 78] is then recovered.

Note that when $q \to 1$: f_X tends to the gaussian distribution. It appears that H_ϕ tends to the Shannon's entropy, $I_{\phi,2}$ to the usual Fisher's information and $M_{\phi,\alpha}$ to the usual moments (both considering the euclidean norm): all the settings related to the Gaussian distribution is naturally recovered.

Appendix A.3. q-exponential distribution and first-order moment

The same entropy functional can readily be obtained for the so-called q-exponential

$$f_X(x) = C_q \left(1 - (q-1)\beta x\right)_+^{\frac{1}{(q-1)}} \qquad \text{and} \qquad T_1(x) = x \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}_+.$$

It suffices to follow the very same steps as above, leading again to the Havrdat-Charvát-Tsallis entropy, the q-moments of order α and the (q, β)-Fisher information defined previously in [73–78] (with the symmetric q index given here by $2-q$) as for the q-Gaussian distribution and to the extended Cramér-Rao inequality proved in [77, 78] as well.

Now when $q \to 1$: f_X tends to the exponential distribution, known to be of maximum Shannon's entropy on $\mathbb{R}_+$ under the first order moment constraint. Again H_ϕ tends to the Shannon's entropy, $I_{\phi,2}$ to the usual Fisher's information and $M_{\phi,\alpha}$ to the usual moments (both considering the euclidean norm): all the settings related to the exponential distribution is naturally recovered.

Appendix A.4. The logistic distribution

In this case,

$$f_X(x) = \frac{1 - \tanh^2\left(\frac{x}{2s}\right)}{4s} \qquad \text{and} \qquad T_1(x) = x^2 \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}.$$

This distribution, which resembles the normal distribution but has heavier tails, has been used in many applications. One can then check that over each interval

$$\mathcal{X}_\pm = \mathbb{R}_\pm$$

the inverse distribution writes

$$f_{X,\pm}^{-1}(y) = \pm 2s \operatorname{argtanh} \sqrt{1 - 4sy}, \qquad y \in \left[0\,;\, \frac{1}{4s}\right]$$

We concentrate now on a second order constraint, that respect the symmetry of the distribution, and on first order constrain(s) that does not respect the symmetry.

Appendix A.4.1. Second order moment constraint

In this case, immediately

$$\phi'(y) = 4s \left(\lambda_0 + \lambda_1 \left(\operatorname{argtanh} \sqrt{1-4sy} \right)^2 \right)$$

for $y \in \left[0\,;\, \frac{1}{4s}\right]$ and where the positive factors $\frac{1}{4s}$ and s are absorbed in λ_0 and λ_1 respectively. To impose the convexity of ϕ, one must impose

$$\lambda_1 < 0$$

that gives the family of entropy functionals $\phi(y) = \phi_{\mathrm{u}}(4sy)$ with

$$\phi_{\mathrm{u}}(u) = c \,+\, \lambda_0 \, u \,+\, \lambda_1 \left[u \left(\operatorname{argtanh} \sqrt{1-u} \right)^2 - 2 \sqrt{1-u} \, \operatorname{argtanh} \sqrt{1-u} - \log u \right] \mathbb{1}_{[0\,;\,1]}(u).$$

where c is an integration constant. Figure A.1(a) depicts function ϕ_{u} for the special choice $\lambda_0 = 0, \lambda_1 = -1$ and $\mathcal{X}$ being unbounded, c is chosen to be zero.

Appendix A.4.2. (Partial) first-order moment(s) constraint(s)

Since f_X and $T(x) = x$ do no share the same symmetries, one cannot interpret the logistic distribution as a maximum entropy constraint by the first order moment. However, constraining the partial means over $\mathcal{X}_\pm = \mathbb{R}_\pm$ allows such an interpretation, using then multiform entropies, while the alternative is to relax the concavity property of the entropy. To be more precise, one chooses either functions $T_{-,1}(x)$ and $T_{+,1}$, or function T_1 under the form

$$T_{\pm,1}(x) = x, \quad x \in \mathcal{X}_\pm = \mathbb{R}_\pm \qquad \text{or} \qquad T_1(x) = x, \quad x \in \mathcal{X} = \mathbb{R}.$$

Over each set $\mathcal{X}_\pm$ we immediately get

$$\phi'_\pm(y) = 4s\left(\lambda_0 + \lambda_{\pm,1} \operatorname{argtanh}\sqrt{1-4sy}\right) \qquad \text{or} \qquad \widetilde{\phi}'_\pm(y) = 4s\left(\lambda_0 \pm \lambda_1 \operatorname{argtanh}\sqrt{1-4sy}\right)$$

where the sign and the factors are absorbed on λ_0 and $\lambda_{\pm,1}$. A judicious choice is then to impose

$$\lambda_{-,1} = \lambda_{+,1} = \overline{\lambda}_1 < 0 \qquad (\lambda_1 < 0)$$

and the same integration constant c for each branch leading either to the family of (convex) uniform functions $\phi(y) = \phi_\mathrm{u}(4sy)$ with,

$$\phi_\mathrm{u}(u) = c + \lambda_0\, u + \overline{\lambda}_1 \left(u \operatorname{argtanh}\sqrt{1-u} - \sqrt{1-u}\right) \mathbb{1}_{[0\,;\,1]}(u)$$

or to the family of multiform function $\widetilde{\phi}$, with branches $\widetilde{\phi}_{\pm,\mathrm{u}}(4sy)$,

$$\widetilde{\phi}_{\pm,\mathrm{u}}(u) = c + \lambda_0\, u \pm \lambda_1 \left(u \operatorname{argtanh}\sqrt{1-u} - \sqrt{1-u}\right) \mathbb{1}_{[0\,;\,1]}(u)$$

Function ϕ_u is represented figure A.1(b) for the special choice $c = \lambda_0 = 0, \overline{\lambda}_1 = -1$ (here, for $c = \lambda_0 = 0, \lambda_1 = -1$, $\widetilde{\phi}_\pm = \pm\phi$). The choice of equal $\lambda_{\pm,1}$ is equivalent than considering the constraint $T_1(x) = |x|$, and thus allows to respect the symmetries of the distribution, allowing thus to recover a classical ϕ-entropy.

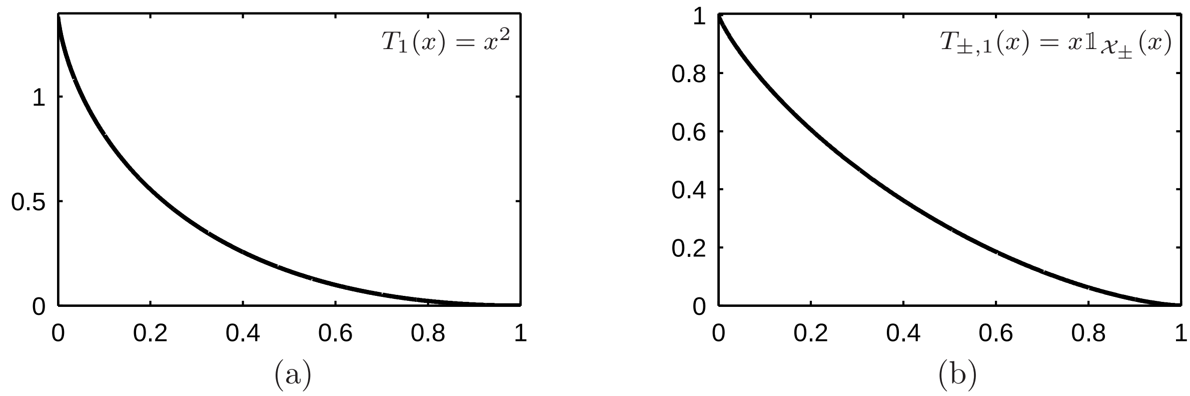

Figure A.1: Entropy functional ϕ_u derived from the logistic distribution: (a) with $T_1(x) = x^2$ and (b) with $T_{\pm,1}(x) = x\mathbb{1}_{\mathcal{X}_\pm}(x)$.

Appendix A.5. The arcsine distribution

The arcsine distribution is a special case of the beta distribution with $\alpha = \beta = \frac{1}{2}$. We consider here the centered and scaled version of this distribution which writes

$$f_X(x) = \frac{1}{\pi\sqrt{2\,\sigma^2 - x^2}} \qquad \text{on} \qquad \mathcal{X} = \left(-\sigma\sqrt{2}\,;\,\sigma\sqrt{2}\right).$$

The inverse distributions $f_{X,\pm}^{-1}$ on $\mathcal{X}_- = \left(-\sigma\sqrt{2}\,;\,0\right)$ and $X_+ = \left[0\,;\,\sigma\sqrt{2}\right)$ write then

$$f_{X,\pm}^{-1}(y) = \pm\frac{\sqrt{2\pi^2\sigma^2y^2 - 1}}{\pi y}, \qquad y \geq \frac{1}{\pi\sigma\sqrt{2}}$$

Let us now consider again either a second order moment as the constraint, or (partial) first order moment(s).

Appendix A.5.1. Second order moment

When the second order moment $T_1(x) = x^2$ is constrained, one immediately obtains

$$\phi'(y) = \lambda_0 + \lambda_1 \left(2\sigma^2 - \frac{1}{\pi^2 y^2} \right)$$

The family of entropy functional is then

$$\phi(y) = c + \left(\lambda_0 + 2\sigma^2 \lambda_1\right) y \, + \, \frac{\lambda_1}{\pi^2 y}$$

which drastically simplifies with the special choice

$$c = 0, \qquad \lambda_0 = -\frac{\alpha^2}{\pi^2} \qquad \text{and} \qquad \lambda_1 = \pi^2 \qquad \text{to} \qquad \phi(y) = \frac{1}{y}$$

Appendix A.5.2. (Partial) first-order moment(s)

Since the distribution does not share the sense of variation of $T_1(x) = x$, either we turn out to consider it as an extremal distribution of an entropy that is not concave, or as a maximum entropy when constraints are of the type

$$T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_\pm}(x)$$

now

$$\phi_\pm'(y) = \sqrt{2}\pi\sigma\lambda_0 + \lambda_{\pm,1} \frac{\sqrt{2\pi^2\sigma^2 y^2 - 1}}{y} \qquad \text{or} \qquad \widetilde{\phi}_\pm'(y) = \lambda_0 \pm \lambda_1 \frac{\sqrt{2\pi^2\sigma^2 y^2 - 1}}{y}$$

where the different factors and the sign are absorbed in the factors $\lambda_0, \lambda_{\pm,1}$. A judicious choice can be to impose

$$\lambda_{-,1} = \lambda_{+,1} = \overline{\lambda}_1 > 0$$

and the same integration constant c for each branch, leading then either to the family of (convex) uniform of functions $\phi(y) = \phi_{\mathrm{u}}(\sqrt{2}\pi\sigma y)$ with

$$\phi_{\mathrm{u}}(u) = c \, + \, \lambda_0 \, u \, + \, \overline{\lambda}_1 \, \left(\sqrt{u^2 - 1} \, + \, \arctan\left(\frac{1}{\sqrt{u^2-1}} \right) \right) \mathbb{1}_{(1\,;\,+\infty)}(u)$$

or, in the non-convex case, to the family of functions with branches $\widetilde{\phi}_\pm(y) = \widetilde{\phi}_{\pm,\mathrm{u}}(\sqrt{2}\pi\sigma y)$,

$$\widetilde{\phi}_{\pm,\mathrm{u}}(u) = c \, + \, \lambda_0 \, u \, \pm \, \lambda \, \left(\sqrt{u^2 - 1} \, + \, \arctan\left(\frac{1}{\sqrt{u^2-1}} \right) \right) \mathbb{1}_{(1\,;\,+\infty)}(u)$$

The uniform function ϕ_{u} is represented figure A.2 for the special choice $c = \lambda_0 = 0, \overline{\lambda}_1 = 1$ (here again, for $c = \lambda_0 = 0, \lambda_1 = 1$, $\widetilde{\phi}_\pm = \pm\phi$). In this case again, the symmetrical choice for $\lambda_{\pm,1}$ allows to recover the symmetries of the probability density, and thus to a uniform convex entropy functional in the first context.

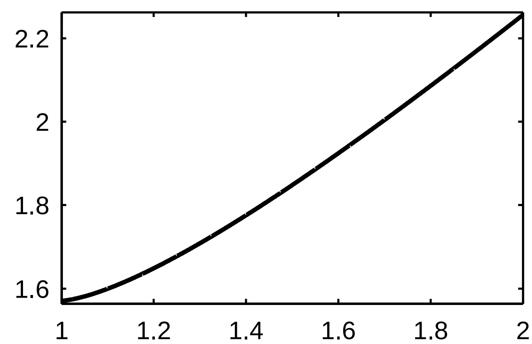

Figure A.2: Entropy functional ϕ_{u} derived from the arcsine distribution with partial constraints $T_{\pm,1}(x) = x \mathbb{1}_{\mathcal{X}_\pm}(x)$.

Appendix A.6. The gamma distribution and (partial) p-order moment(s)

As a very special case, consider here this distribution, expressed as

$$f_X(x) = \frac{\beta^\alpha x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)} \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}_+.$$

Let us concentrate on the case $\alpha > 1$ for which the distribution is non-monotonous, unimodal, where the mode is located at $x = x_{\mathrm{m}}$, and $f_X(\mathbb{R}_+) = \left[0\,;\,\frac{1}{\tau\,e^{\alpha-1}}\right]$ with

$$x_{\mathrm{m}} = \frac{\alpha-1}{\beta} \qquad \text{and} \qquad \tau = \frac{\Gamma(\alpha)}{\beta\,(\alpha-1)^{\alpha-1}}$$

Thus, here again it cannot be viewed as a maximum entropy constraint neither by any p-order moment. Here, we can again interpret it as a maximum entropy constrained by partial moments

$$T_{k,1}(x) = x^p, \quad k \in \{0,-1\} \qquad \text{over} \qquad \mathcal{X}_0 = [0\,;\,x_{\mathrm{m}}) \qquad \text{and} \qquad \mathcal{X}_{-1} = [x_{\mathrm{m}}\,;\,+\infty).$$

or as an extremal entropy constrained by the moment

$$T_1(x) = x^p \quad \text{over} \quad \mathcal{X} = \mathbb{R}_+$$

where $p > 0$. Inverting $y = f_X(x)$ leads to the equation

$$-\frac{x}{x_{\mathrm{m}}}\exp\left(-\frac{x}{x_{\mathrm{m}}}\right) = -(\tau\,y)^{\frac{1}{\alpha-1}}$$

to be solved. As expected, this equation has two solutions. These solutions can be expressed via the multivalued Lambert-W function W defined by $z = \mathrm{W}(z)\exp(\mathrm{W}(z))$, i.e., W is the inverse function of $u \rightsquigarrow u\exp(u)$[97, § 1], leading to the inverse functions

$$f_{X,k}^{-1}(y) = -x_{\mathrm{m}}\ \mathrm{W}_k\left(-(\tau y)^{\frac{1}{\alpha-1}}\right), \qquad y \in \left[0\,;\,\frac{1}{\tau e^{\alpha-1}}\right],$$

where k denotes the branch of the Lambert-W function. $k = 0$ gives the principal branch and here it is related to the entropy part on $\mathcal{X}_0$, while $k = -1$ gives the secondary branch, related to $\mathcal{X}_{-1}$ here.

One has thus to solve the equation

$$\phi_k'(y) = \lambda_0\tau + \lambda_{k,1}\tau\left[-\mathrm{W}_k\left(-(\tau y)^{\frac{1}{\alpha-1}}\right)\right]^p$$

where the positive factor are absorbed in the $\lambda_0, \lambda_{k,1}$ and where to insure the convexity of the ϕ_k,

$$(-1)^k\lambda_{k,1} > 0$$

The same approach allows to design $\widetilde{\phi}_k$, with a unique λ_1 instead of the $\lambda_{k,1}$. Integrating the previous expression is not an easy task. Relation $u\,(1+\mathrm{W}_k(u))\ \mathrm{W}_k'(x) = \mathrm{W}_k(u)$ [97, Eq. 3.2] suggests that a way to make the integration is to search for $\phi_k(y) = \phi_{k,\mathrm{u}}(\tau y)$ where $\phi_{k,\mathrm{u}}(u)$ is searched as the product of $u\left[-\mathrm{W}_k\left(-u^{\frac{1}{\alpha-1}}\right)\right]^p$ and a series of $\left[-\mathrm{W}_k\left(-u^{\frac{1}{\alpha-1}}\right)\right]$ and then to recognize the coefficients of the series. Such an approach leads to the family of entropic functional $\phi_k(y) = \phi_{k,\mathrm{u}}(\tau y)$ with

$$\begin{aligned}\phi_{k,\mathrm{u}}(u) \;&=\; c_k + \lambda_0\,u\\ &+\lambda_{k,1}\,u\left[-\mathrm{W}_k\left(-u^{\frac{1}{\alpha-1}}\right)\right]^p\left[1-\tfrac{p}{p+\alpha-1}\ {}_1F_1\left(1\,;\,p+\alpha\,;\,(1-\alpha)\,\mathrm{W}_k\left(-u^{\frac{1}{\alpha-1}}\right)\right)\right]\mathbb{1}_{(0\,;\,e^{1-\alpha})}(u)\end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric (or Kummer's) function [98, § 13] and c_k are integration constants. One can verify a posteriori that these functions are the ones we search for. The integration constant can be chosen such that ϕ_k coincide in 0 for instance, that gives

$$c_{-1} - c_0 \;=\; \frac{p\,\Gamma(p+\alpha-1)}{(\alpha-1)^{p+\alpha-1}}\,\lambda_{-1,1}$$

using successively [97, Eq. 3.1] and [98, Eq. 13.1.2] for W_0, and successively [98, Eq. 13.1.4] (W_{-1} tending to $-\infty$ in 0^-), $\mathrm{W}_{-1}(u)\exp(\mathrm{W}_{-1}(u)) = u$, and [97, Eq. 4.6 & lines that follow] for W_{-1}. The same algebra leads to the same expression for the $\widetilde{\phi}_k$, except that $\lambda_{k,1}$ are replaced by a unique λ_1.

The multivalued function ϕ_{u} in the concave context is represented figure A.3 for $p = 2, \alpha = 2$ and $\alpha = 5$, and with the choices $c_{-1} = \lambda_0 = 0, \lambda_{0,1} = 1, \lambda_{-1,1} = -0.1$.

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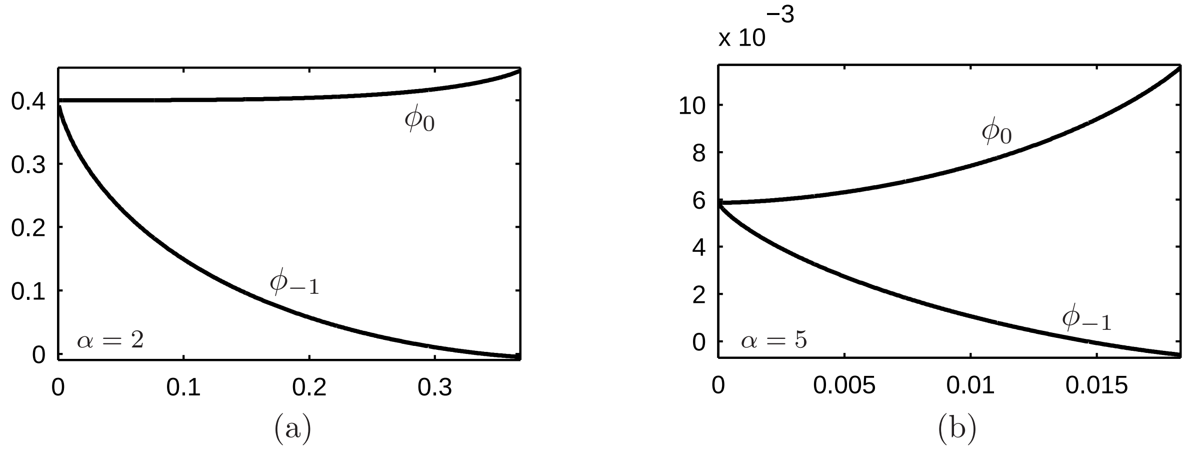

Figure A.3: Multiform entropy functional ϕ_{u} derived from the gamma distribution with the partial moment constraints $T_{k,1}(x) = x^2 \mathbb{1}_{\mathcal{X}_k}(x)$, $k \in \{0, -1\}$. (a): $\alpha = 2$; (b): $\alpha = 5$.

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