Lectures on Logarithmic Sobolev Inequalities

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Contents

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Introduction

During the last decade some important progress was achieved in the domain of the infinite dimensional analysis involving applications of coercive inequalities. In particular the understanding of the ergodicity problem for stochastic dynamics of large dimensional interacting systems has considerably improved. Besides other things this contributed to better understanding the relations between equilibrium description and out of equilibrium systems described by stochastic dynamics which play a crucial role in statistical mechanics.

The intention of these lecture notes is to present a self consistent and relatively complete introduction to this rapidly developing subject.

One can quickly introduce as follows the problem which we shall consider. Given a Markov semi-group $P_t \equiv e^{t\mathcal{L}}$, $t > 0$, defined on a space $\mathcal{C}(\Omega)$ of real-valued continuous functions on a Polish space Ω, one can ask the two following questions ;

(I) Is there an invariant measure for P_t, that is a probability measure μ on Ω so that for every $f \in \mathcal{C}(\Omega)$

$$\mu(P_t f) = \mu(f)$$

where $P_t f$ denotes the image of f by P_t and we denoted in short $\mu(g) = \int g(x) d\mu(x)$.

(II) If yes, does $P_t f$ converge towards μ and in which sense ? More precisely, can we find a norm $|| \cdot ||$ on $\mathcal{C}(\Omega)$ such that

$$||P_t f - \mu f|| \leq \gamma(t) |||f|||$$

with a semi-norm $|||\cdot|||$ defined on a dense subset of $\mathcal{C}(\Omega)$ and a rate $\gamma(t) \to_{t\to\infty} 0$ independent of f. In general, $|| \cdot ||$ is either the uniform norm or the norm of $L^p(\mu)$ for some $p \geq 1$.

Frequently, one can introduce P_t so that a given measure μ (or a class of measures) is P_t - invariant and then the question (II) becomes crucial.

One such situation appears in statistical mechanics. In this case the state space Ω one considers is often an infinite product $\Omega \equiv \mathbf{M}^{\mathbb{Z}^d}$ where $\mathbf{M}$ is a finite set or a smooth compact finite dimensional Riemannian manifold. On such a space, one is given an a priori family of conditional expectations E_X indexed by the finite subsets X of $\mathbb{Z}^d$, where E_X integrates over the variables $(\omega_i, i \in X)$. Under some mild hypotheses, it can be shown that there exists a probability measure μ on Ω, so-called Gibbs measure, characterized by the Dobrushin-Lanford-Ruelle's condition

$$\mu(E_X f) = \mu(f) \tag{0.0.1}$$

for every finite subset X of $\mathbb{Z}^d$ and $f \in \mathcal{C}(\Omega)$. One can use these conditional expectations to introduce a class of infinitesimal jump generators formally given by

$$\mathcal{L}^{(X)} f \equiv \sum_{j \in \mathbb{Z}^d} (E_{X+j} f - f), \tag{0.0.2}$$

where $X+j$ denotes the set $\{j+k, k \in X\}$ (see [25, 47, 62] and the references therein). Naturally the measures μ satisfying (0.0.1) are invariant for the associated Markov semi-groups $P_t^{(X)} = e^{t\mathcal{L}^{(X)}}$. However, in general it is not clear that the solutions of (0.0.1) are the only invariant measures of $P_t^{(X)}$.. In [25], [47] and [62], one finds sufficient conditions so that $P_\cdot^{(X)}$ is uniformly ergodic and hence has a unique invariant measure. In this set up, the conditional expectations $(E_X, X \in \mathbb{Z}^d)$ often depend on some parameters (such as the temperature T or magnetic field h). As these parameters vary, the solutions of equation (0.0.1) change as well as their properties. In [23] and later in [19] constructive conditions were given to insure uniqueness of the solutions of (0.0.1) in terms of the kernels $E_{\{j\}}$ and E_{Y+j} associated with the single point sets $(\{j\}, j \in \mathbb{Z}^d)$ and the cubes with given finite size $(Y+j, j \in \mathbb{Z}^d)$, respectively. These conditions are known as the uniqueness conditions of Dobrushin and Dobrushin-Shlosman, respectively. In general, the second is valid in a broader domain of values of the parameters.

Dobrushin-Shlosman's uniqueness condition was used by Aizenman and Holley in [2] where they showed that a jump dynamics constructed as in (0.0.2) is ergodic in the uniform norm with an exponential rate when the conditional expectations E_{X+j}, $j \in \mathbb{Z}^d$, satisfy this condition. This result in particular implies the ergodicity of the semi-group in $L^2(\mu)$ and the existence of a spectral gap for the spectrum of the self-adjoint operator $\mathcal{L}^{(X)}$ in $L^2(\mu)$. Moreover, as the quadratic forms associated with the generators $\mathcal{L}^{(Y)}$ constructed with other finite sets Y are equivalent to that of $\mathcal{L}^{(X)}$, all the corresponding dynamics are ergodic in $L^2(\mu)$.

Unfortunately, for many interesting models describing systems in a neighbourhood of a "critical point", the size of the cubes for which Dobrushin-Shlosman's condition is satisfied grows to infinity when one approaches the critical point. In particular, Aizenman-Holley strategy fails even though one can show that there exists a unique Gibbs measure satisfying (0.0.1). (In some specific models such as ferromagnetic systems where the conditional expectations preserve monotonicity properties this strategy was extended in a clever way in [70] (Part I).

To overcome these difficulties, a new clever strategy based on the use of hypercontractivity property was introduced in [55] and [56]. The main idea is to deduce the uniform ergodicity from $L^2(\mu)$ ergodicity and hypercontractivity. To this end one first approximates the semi-group with another semi-group in finite dimension. In finite dimension, one can bound uniform norms with L^2 norms by using the ultracontractivity property of the semi-group realized in a unit time. The price to pay for this is a large coefficient growing with the dimension of the space which fortunately can be controlled thanks to the hypercontractivity property.

We recall that the hypercontractivity property allows us to bound the $L^p(\mu)$ norm of $P_t f$ by the $L^q(\mu)$ norm of f for $p = p(t) = 1 + (q-1)e^{\frac{2}{c}t}$ and a constant $c \in (0, \infty)$. It looks rather unusual since p can be, for large times, much bigger than q. This property was introduced first in the constructive quantum field theory (see [87] and [44]). Similarly to the contractivity property, the hypercontractivity allows an equivalent infinitesimal description given by the following logarithmic Sobolev inequality

$$\mu(f^2 \log |f|) \leq c\mu(f(-\mathcal{L}f)) + \mu(f^2) \log \mu(f^2) \tag{0.0.3}$$

with a constant $c \in (0, \infty)$ independent of a function f from the quadratic form domain of $\mathcal{L}$. The logarithmic Sobolev inequality was already formally considered in [36], but the equivalence of the hypercontractivity property and of (0.0.3) was first proven in the seminal work [48] opening the door to further progress. In this paper L. Gross proved also that the logarithmic Sobolev inequality has a product property. He showed that this together with (0.0.3) for a uniform measure on the two point set (proven in the same paper) imply the corresponding inequality for a class of Gaussian measures. The first breakthrough which tremendously increased the class of probability measures satisfying (0.0.3) was due to Bakry and Emery who introduced in [6] a very nice sufficient condition. Roughly speaking, this condition, when formulated in the context of a smooth compact finite dimensional Riemannian manifold $\mathbf{M}$, requires the positivity of the Ricci curvature of $\mathbf{M}$. When considered in the setting of a probability measure $\mu(dx) = Z^{-1}e^{-U(x)}dx$ on $\mathbb{R}^n$, it necessitates that the smallest eigenvalue of the Hessian of U is uniformly bounded below by a positive constant. One of the first interesting applications of the Bakry-Emery condition was published in [10] where it was used to establish the logarithmic Sobolev inequality for a wide class of Gibbs measures on a product space over the unit sphere of dimension N, $N \geq 2$, at high temperatures. Thus [10] provided the first non trivial class of examples of neither product nor Gaussian measures in infinite dimension for which (0.0.3) holds.

Afterwards, a new idea based on the specific structure of Gibbs measures was introduced to deal with Riemannian manifolds with possibly negative Ricci curvature as well as with discrete settings. In particular, in [103] and [104], Sobolev inequality was proved for systems with short range interaction on $(S^1)^{\mathbf{Z}^d}$ at high temperature and $\{-1, +1\}^{\mathbf{Z}}$ for any temperature. An extension of these results to systems with long range interactions (of the same type as those studied in the uniqueness theory of Dobrushin) on Riemannian manifolds at high temperature appeared in [105].

Later these results were extended to a variety of directions described in a rich literature including [70], [64], [60], [49] - [50], [11] - [12], [69], ..., [92] - [95], [99], [100] - [101], [100], [3], [4], ..., [107] -[109] ... Besides the above mentioned control of ergodicity of a hypercontractive Markov semi-group, one of most interesting consequences of the research in this domain is the proof of the equivalence of strong mixing property and the log-Sobolev inequality.

In view of the restricted time and space that we were given to present this area, we will not be able to consider many new and interesting developments of this field, such as the extension to non-compact manifolds $\mathbf{M}$ (see [109] and more recent works of Yoshida and Bodineau, Helffer), the study and the applications of other coercivity inequalities (such as Nash's inequalities (see [7])), the link with isoperimetric's inequalities, the extension to loop spaces (see [32]...), to non-commutative spaces ([73]...)...

The contents of these notes are as follows.

We begin by introducing the objects under study : Markov semi-groups, their infinitesimal construction via infinitesimal generators, invariant and reversible measures.

The second chapter is devoted to L^2 ergodicity and the simplest coercivity inequality – the spectral gap inequality. After some general discussion, we shall show the stability of this property under perturbations, that is by change of the initial measure by bounded densities, and tensorisation (product property).

In chapter 3, we consider classical Sobolev inequality and its consequences : ultracontractivity property and the classical Nash inequality.

In chapter 4, we study the general properties of log-Sobolev inequalities including stability by perturbation, product property, equivalence with hypercontractivity property and its implications concerning spectral gap inequality. We also present the Bakry-Emery criterion and its equivalent forms in terms of semi-groups. We finally give an example of an infinite dimensional system defined by conditional expectations and discuss its log-Sobolev properties.

Our journey in statistical mechanics begins from chapter 5. The reader willing to learn how to prove log-Sobolev inequality in infinite-dimensional settings can go directly to this chapter. We begin by introducing the statistical mechanics framework of spin systems on a lattice. Then, we describe the general strategy introduced in [104, 92, 93] to prove log-Sobolev inequalities for systems with finite range interactions. It is based on a study of an auxiliary Markov chain constructed as the convolution of conditional expectations on cubes of given finite size as described in 5.2. In section 5.3, we restrict ourselves to one-dimensional lattice whereas in sections 5.4 we consider higher dimensional lattices. Under a strong mixing condition hypothesis, this Markov chain will be shown to converge towards the underlying Gibbs measure, the log-Sobolev inequality being then, in a certain sense, derived as a consequence of the product property. In sections 5.3 and 5.4, we prove the key point of this strategy called sweeping out relations. Our method is constructive in the sense that we always consider finite volume expectations. Our formalism will always cover both discrete and continuous settings, i.e. involving product spaces build upon discrete sets as well as smooth compact connnected Riemannian manifolds.

In chapter 6, we apply the ideas of the previous chapter to a slightly different context. We indeed show that the formalism developped in chapter 5 extends naturally to the case where the measure μ is not given a priori as a Gibbs measure but is described as the stationary measure of a cellular automata. In such a setting, one does not know in general even whether μ is a Gibbs measure for some interaction. Moreover the transition matrix of the cellular automata may not be evnen symmetric for μ.

Another strategy to prove logarithmic Sobolev inequalities is to make a martingale expansion of relative entropy by considering, instead of Markov chains constructed as convolutions of conditional expectations, conditional expectations on an increasing sequences of finite subsets of the lattice. This idea was originally introduced in [64] for finite range interaction systems. We illustrate it in chapter 7 where we study systems with long range interaction (though with a strength decreasing in a sufficiently fast way with the distance between the spins to insure that the potential satisfies Dobrushin's uniqueness condition). Again, sweeping out relations are the key of this approach.

In chapter 8, we apply the previous log-Sobolev inequalities to deduce ergodic properties for the associated semi-group. We first present in section 8.1 a simple construction of these semi-groups which yields an exponential approximation property. In section 8.2, we study the ergodic properties of these semi-groups when log-Sobolev property is satisfied. In section 8.3, we describe the equivalence theorem (see [93]) which establishes the correspondence between properties of the Gibbs measure such as strong mixing, uniform analyticity and other conditions introduced by Dobrushin and Shlosman and properties of the dynamics such as hypercontractivity or L^2 ergodicity.

In chapter 9, we explore systems with random interactions and apply the ideas of chapters 5-7 to study the ergodic properties of their dynamics at high temperature.

Finally, we describe a few results concerning L^2 ergodicity of Markov semi-groups at low temperature where strong mixing property fails.

We give a long but still very incomplete bibliography of the subject.

These lecture notes represent several attempts to try to understand and organize the diverse evolutions of the subject. Part of this course was already given by B. Zegarlinski in Bochum in 1992, who would like to thank S. Albeverio for giving him the opportunity to spend a long, happy and fruitful period at Ruhr. We could also continue this work during two semesters at Institut Henri Poincaré of Paris where we had the pleasure to give courses. The second of these courses gave birth to these lecture notes originally in French. We are very grateful to the organizers of these two sessions, as well as to Institut Henri Poincaré team. B. Zegarlinski wishes also to thank D. Stroock for their long and fruitful collaboration. Finally, our collaboration was made possible thanks to the financial help of the European Stochastic Analysis Network and EPSRC.

Chapter 1

Markov Semi-groups

This chapter reviews some classical facts from the theory of Markov semi-groups which can be found for instance in [45] or [15]. We first recall the definition of Markov semi-groups and generators, giving examples in exercises. We then introduce the notions of invariant and reversible measures of Markov semi-groups. At the end of the chapter, we briefly sketch the relation between Markov semi-groups and Markov processes. For more extended treatment of the related theory of Dirichlet forms and their links with Markov semi-groups the reader is encouraged to look at [40], [65] and [77].

1.1 Markov Semi-groups and Generators

Definition 1.1 *A family $(P_t)_{t\geq 0}$ of linear operators on a Banach space $(\mathcal{B}, ||\cdot||)$ is called a semi-group iff it satisfies the following conditions*

(1) *$P_0 = I$, the identity on $\mathcal{B}$.*

(2) *The map $t \to P_t$ is continuous in the sense that for all $f \in \mathcal{B}$, $t \to P_t f$ is a continuous map from $I\!R^+$ into $\mathcal{B}$.*

(3) *For any $f \in \mathcal{B}$ and $(t, s) \in (I\!R^+)^2$,*

$$P_{t+s} f = P_t P_s f.$$

The space $\mathcal{B}$ under consideration in most cases will be the set $\mathcal{C}(\Omega)$ of real-valued bounded continuous functions on a Polish space Ω equipped with the uniform norm. However, in some important cases one needs to consider a Banach space given by a set of more regular functions, such as for example the set of uniformly bounded continuous functions from a Polish space Ω into $I\!R$ furnished with the uniform norm. This is one of the reasons why we prefer to introduce a general Banach space $\mathcal{B}$ setting. In the sequel, we always consider spaces of (nice) real-valued functions. $\mathcal{B}$ will be equipped with a partial order $\geq$.

Definition 1.2 *A semi-group $(P_t)_{t\geq 0}$ is Markov iff*

(4) *For any $t \in I\!R^+$,*

$$P_t \mathbb{1} = \mathbb{1}.$$

(5) *For any* $t \in \mathbb{R}^+$, P_t *preserve the positivity, i.e for any* $f \in \mathcal{B}$ *and* $t \in \mathbb{R}^+$,

$$f \geq 0 \qquad \rightarrow \qquad P_t f \geq 0.$$

Properties (4) and (5) imply that, for any $t \in \mathbb{R}^+$, P_t is contractive, that is

Definition 1.3 P_t *is contractive iff for any* $f \in \mathcal{B}$,

$$||P_t f|| \leq ||f|| \tag{1.1.1}$$

where $||\cdot||$ *denotes the norm on* $\mathcal{B}$.

The notion of Markov semi-groups can be illustrated by the following examples

Exercise 1.4

- a) *Let* A *be a non negative linear bounded operator,* $||A|| \leq 1$, *so that, if* 1 *is the identity in* $\mathcal{B}$, $A1 = 1$. *Let* $\lambda > 0$. *Verify that*

$$P_t = e^{t\lambda(A-I)} = \sum_{n\geq 0} \frac{(\lambda t)^n}{n!}(A-I)^n$$

is a Markov semi-group on $\mathcal{B}$. *Here, we have denoted*
$||A|| = \sup_{f\in\mathcal{B}, f\neq 0} ||f||^{-1}||Af||$ *with* $||\cdot||$ *the norm on* $\mathcal{B}$.

- b) *Let* μ *be a probability measure on a* Ω *equipped with the* σ *- algebra* $\mathcal{F}$ *and let* $m > 0$. *For any* $f \in \mathcal{C}(\Omega)$, *we define*

$$P_t f(\omega) = e^{-mt} f(\omega) + (1 - e^{-mt})\mu(f).$$

Verify that $(P_t)_{t\geq 0}$ *is a Markov semi-group on* $\mathcal{C}(\Omega)$ *equipped with the uniform topology.*

- c) *(d-dimensional Brownian motion) For* $f \in \mathcal{C}(\mathbb{R}^d)$ *and* $\omega \in \mathbb{R}^d$, *we set*

$$P_t f(\omega) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int f(y) e^{-\frac{|\omega-y|^2}{2t}} dy.$$

Show that $(P_t)_{t\geq 0}$ *is a Markov semi-group on the space* $\mathcal{CU}(\Omega)$ *of uniformly continuous functions equipped with the uniform topology.*

- d) *(Brownian motion on the circle) If* $\Omega = S^1$, *one can represent any continuous function on* Ω *by its Fourier expansion*

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}.$$

We then let

$$P_t f(x) = \sum_{n\in\mathbb{Z}} e^{-tn^2} a_n e^{inx}.$$

Verify that $(P_t)_{t\geq 0}$ *is a Markov semi-group on* $\mathcal{C}(\Omega)$ *endowed with the uniform topology.*

- e) *(Poisson Process) Let* $\lambda \in \mathbb{R}^+$, $\Omega = \mathbb{R}^d$. *For any* $f \in \mathcal{C}(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$ *, we define*

$$P_t f(x) = e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^k}{k!} f(x - k\lambda)$$

Show that $(P_t)_{t \geq 0}$ *is a Markov semi-group on the set* $\mathcal{CU}(\Omega)$ *endowed with the uniform topology.*

- f) *(Ornstein-Uhlenbeck's Process) If* $\Omega = \mathbb{R}$, *prove that*

$$P_t f(x) = \int f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

defines a Markov process on $\mathcal{C}(\Omega)$.

Definition 1.5 *The infinitesimal generator* $\mathcal{L}$ *of a semi-group* P_t *is defined by the formula*

$$\mathcal{L}f := \lim_{t \downarrow 0} \frac{1}{t}(P_t - I)f \tag{1.1.2}$$

for any function f *for which the limit makes sense. The domain* $\mathcal{D}(\mathcal{L})$ *of* $\mathcal{L}$ *is the set of functions of* $\mathcal{C}(\Omega)$ *for which the limit* (1.1.2) *exists.*

Exercise 1.6 *In the examples given in exercise 1.4, show that the infinitesimal generators are given, in the same order, by*

- a) $\mathcal{L}f = \lambda(A - I)f$, $\mathcal{D}(\mathcal{L}) = \mathcal{B}$.
- b) $\mathcal{L}f = \mu f - f$, $\mathcal{D}(\mathcal{L}) = \mathcal{C}(\Omega)$.
- c) *In the case* $d = 1$ *to simplify,* $\mathcal{L}f = (1/2)f''$, $\mathcal{D}(\mathcal{L})$ *is the set of twice continuously differentiable functions* f *such that* f' *and* f'' *are uniformly bounded continuous.*

  Hints : Show that

$$g_\lambda(x) := \lambda \int_0^\infty e^{-\lambda t} P_t f(x) dt = \int_{-\infty}^{\infty} \sqrt{\frac{\lambda}{2}} f(y) \exp\{-\sqrt{2\lambda}|x - y|\} dy$$

  by using the identity

$$\int_0^\infty \exp -\{y^2 + \frac{c^2}{y^2}\} dy = \frac{\sqrt{\pi}}{2} e^{-2c}$$

  for all $c > 0$. *Deduce that* $2\lambda(g_\lambda - f) = g''_\lambda$. *On the other hand, prove that* $g_\lambda \in \mathcal{D}(\mathcal{L})$ *and*

$$\mathcal{L}(g_\lambda) = \lambda(g_\lambda - f).$$

  Using the above identity, deduce that $\mathcal{L}g_\lambda = (1/2)g''_\lambda$ *and conclude by letting* λ *go to infinity and using that* $\mathcal{L}$ *is closed (see below).*

- d) $\mathcal{L}f = (1/2)f''$, *$\mathcal{D}(\mathcal{L})$ is the set of twice continuously differentiable functions.*
- e) *For $\lambda \in I\!R^+$, show that $\mathcal{L}f(x) = \lambda(f(x-\lambda) - f(x))$. $\mathcal{D}(\mathcal{L}) = \mathcal{CU}(\Omega)$.*
- f) $\mathcal{L}f(x) = f''(x) - xf'(x)$, *$\mathcal{D}(\mathcal{L}) = \mathcal{C}^3(I\!R)$, the set of 3 times continuously differentiable functions on $I\!R$ with bounded derivatives (see [5]).*

The following theorem characterizes the infinitesimal generators :

Theorem 1.7 *(Hille-Yoshida's theorem for Markov semi-groups) A linear operator $\mathcal{L}$ is the infinitesimal generator of a Markov semi-group $(P_t, t \in I\!R^+)$ on $\mathcal{B}$ iff*

- *$1\!\!I \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}1\!\!I = 0$.*
- *$\mathcal{D}(\mathcal{L})$ is dense in $\mathcal{B}$.*
- *$\mathcal{L}$ is closed.*
- *For any $\lambda > 0$, $(\lambda I - \mathcal{L})$ is invertible. Its inverse $(\lambda I - \mathcal{L})^{-1}$ is bounded with*
$$\sup_{||f||\leq 1} ||(\lambda I - \mathcal{L})^{-1} f|| \leq \frac{1}{\lambda}$$
and preserves positivity (i.e for all $f \geq 0$, $(\lambda I - \mathcal{L})^{-1} f \geq 0$) .

Remark 1.8: Recall that an operator $\mathcal{L}$ is closed iff for any sequence f_n of $\mathcal{D}(\mathcal{L})$ converging (in the sense of the topology inherited from the norm $|| \cdot ||$ on $\mathcal{B}$) towards a function f and such that $\mathcal{L}f_n$ converges, then

$$\lim_{n\to\infty} \mathcal{L}f_n = \mathcal{L}f.$$

Proof of theorem 1.7 *Necessary condition.*

The first point is a direct consequence of (4) of definition (1.2) and of the definition of the generator. To show the second point, we shall see that

$$\mathcal{D}_0 := \left\{ \frac{1}{t}\int_0^t ds P_s f, \quad f \in \mathcal{C}(\Omega), t > 0 \right\} \subset \mathcal{D}(\mathcal{L}). \tag{1.1.3}$$

Since $P_. f$ is continuous for any $f \in \mathcal{B}$, $\mathcal{D}_0$ is dense in $\mathcal{B}$ which completes the proof of this point.

To prove (1.1.3), note that for any $\tau \in I\!R^+$, we have

$$\begin{aligned} \frac{1}{\tau}(P_\tau - I)\int_0^t ds P_s f &= \frac{1}{\tau}\int_\tau^{\tau+t} ds P_s f - \frac{1}{\tau}\int_0^t ds P_s f \\ &= \frac{1}{\tau}\int_t^{\tau+t} ds P_s f \frac{1}{\tau}\int_0^\tau ds P_s f \end{aligned} \tag{1.1.4}$$

by property (3) of definition 1.1. Then (2) of definition 1.1 gives the convergence of the right hand side of (1.1.4) so that we obtain

$$\mathcal{L}\int_0^t dsP_sf = P_tf - f, \tag{1.1.5}$$

resulting with (1.1.3). Remark as well that from (1.1.4), we also deduce by letting τ going to zero that for any $f \in \mathcal{D}(\mathcal{L})$,

$$\mathcal{L}\int_0^t dsP_sf = P_tf - f = \int_0^t dsP_s\mathcal{L}f. \tag{1.1.6}$$

To show that $\mathcal{L}$ is closed, let us take a sequence f_n in $\mathcal{D}(\mathcal{L})$ converging to a function f and such that, for some $g \in \mathcal{B}$,

$$\lim_{n\to\infty} \mathcal{L}f_n = g.$$

By (1.1.5), we obtain that

$$P_tf_n - f_n = \mathcal{L}\int_0^t dsP_sf_n = \int_0^t dsP_s\mathcal{L}f_n. \tag{1.1.7}$$

Hence, letting n go to infinity,we obtain $P_tf - f = \int_0^t dsP_sg$. Dividing by t and taking $t \downarrow 0$, we conclude that $\mathcal{L}f = g$ for every $f \in \mathcal{D}(\mathcal{L})$.

To finish the proof, let us consider the resolvent $R(\lambda, \mathcal{L})$ of the operator $\mathcal{L}$ defined by

$$R(\lambda, \mathcal{L}) := (\lambda I - \mathcal{L})^{-1}$$

and show that

$$R(\lambda, \mathcal{L})f = \int_0^\infty dse^{-\lambda s}P_sf. \tag{1.1.8}$$

In fact, if one considers the semi-group $\widetilde{P}_s = e^{-\lambda s}P_s$ with generator $(\mathcal{L} - \lambda I)$, we obtain, according to (1.1.5) that

$$(\mathcal{L} - \lambda I)\int_0^t dse^{-\lambda s}P_sf = e^{-\lambda t}P_tf - f$$

for any bounded continuous function f. Taking the limit as $t \to \infty$ and using that $\mathcal{L}$ is closed, we conclude that, for any bounded continuous function f, $\int_0^\infty dse^{-\lambda s}P_sf$ belongs to $\mathcal{D}(\mathcal{L})$ and

$$(\mathcal{L} - \lambda I)\int_0^\infty dse^{-\lambda s}P_sf = -f. \tag{1.1.9}$$

Consequently, $(\mathcal{L} - \lambda I)$ is bijective and (1.1.8) is satisfied. In particular, since P_s is contractive,

$$\sup_{||f||=1} ||R(\lambda, \mathcal{L})f|| \leq \int_0^\infty dse^{-\lambda s} = \frac{1}{\lambda}.$$

$R(\lambda, \mathcal{L})$ is thus a bounded operator. Also, it is clear from (1.1.8) that $R(\lambda, \mathcal{L})$ preserves positivity.

Sufficient condition.

For any $\lambda > 0$, we introduce the Yoshida approximation $\mathcal{L}_\lambda$ of $\mathcal{L}$

$$\mathcal{L}_\lambda := \mathcal{L}\lambda I(\lambda I - \mathcal{L})^{-1} = \lambda^2(\lambda I - \mathcal{L})^{-1} - \lambda I. \tag{1.1.10}$$

Since

$$||(\lambda I - \mathcal{L})^{-1}|| \leq \frac{1}{\lambda}, \tag{1.1.11}$$

$\mathcal{L}_\lambda$ is a bounded operator and we can define a family of operators $P^\lambda_.$ by

$$P_t^\lambda := e^{t\mathcal{L}_\lambda} := \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{L}_\lambda^n = e^{-t\lambda I} e^{t\lambda^2(\lambda I - \mathcal{L})^{-1}}. \tag{1.1.12}$$

P^λ is a Markov semi-group for any $\lambda > 0$ (see exercise 1.4.a)). It remains to show that P_t^λ converges towards P_t as λ goes to infinity and then that P possesses all the properties of a Markov semi- group. To begin with, notice that for any f in the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$,

$$\lim_{\lambda\to\infty} \mathcal{L}_\lambda f = \mathcal{L}f. \tag{1.1.13}$$

Indeed, the identity

$$\lambda(\lambda I - \mathcal{L})^{-1} - (\lambda I - \mathcal{L})^{-1}\mathcal{L} = I \tag{1.1.14}$$

holds on $\mathcal{D}(\mathcal{L})$. The second term goes to zero as λ goes to infinity so that we find

$$\lim_{\lambda\to\infty} \lambda(\lambda I - \mathcal{L})^{-1} g = g \tag{1.1.15}$$

for every $g \in \mathcal{D}(\mathcal{L})$, and hence for any $g \in \overline{\mathcal{D}(\mathcal{L})} = \mathcal{B}$. Since $\mathcal{L}$ is closed, we deduce that $\mathcal{L}_\lambda$ converges towards $\mathcal{L}$ as λ goes to infinity. We can now show that P_t^λ converges towards P_t as $\lambda \to \infty$. In fact, for any couple (λ_1, λ_2) of positive real numbers and any $f \in \mathcal{D}(\mathcal{L})$, we have the interpolation formula

$$e^{t\mathcal{L}_{\lambda_1}} f - e^{t\mathcal{L}_{\lambda_2}} f = t\int_0^1 ds e^{t(s\mathcal{L}_{\lambda_1} + (1-s)\mathcal{L}_{\lambda_2})}(\mathcal{L}_{\lambda_1} - \mathcal{L}_{\lambda_2})f \tag{1.1.16}$$

since $\mathcal{L}_{\lambda_1}$ are $\mathcal{L}_{\lambda_2}$ bounded commuting operators. We can see as above that the semi-group $Q_t := e^{t(s\mathcal{L}_{\lambda_1} + (1-s)\mathcal{L}_{\lambda_2})}$ is contractive as $s \in [0, 1]$ and conclude that

$$||e^{t\mathcal{L}_{\lambda_1}} f - e^{t\mathcal{L}_{\lambda_2}} f|| \leq t||(\mathcal{L}_{\lambda_1} - \mathcal{L}_{\lambda_2})f||. \tag{1.1.17}$$

Thus, the convergence of the $\mathcal{L}_\lambda$'s implies that of the P_t^λ's. Let P_t be the linear operator defined by

$$P_t = \lim_{\lambda\to\infty} P_t^\lambda \tag{1.1.18}$$

on $\mathcal{D}(\mathcal{L})$. It is clear that the semi-group properties, contractivity property, positivity properties as well as conservation of unit property of the semi-groups P^λ can be extended to P. Hence P is a Markov semi-group. Also, for any $f \in \mathcal{D}(\mathcal{L})$, we obtain by interpolation

$$P_t f - f = \lim_{\lambda\to\infty} (P_t^\lambda - I)f = \lim_{\lambda\to\infty} \int_0^t ds P_s^\lambda \mathcal{L}_\lambda f = \int_0^t ds P_s \mathcal{L} f. \qquad (1.1.19)$$

so that P is continuous with generator $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$. By condition (4) of the theorem, the resolvent of the semi-group corresponds to that of $\mathcal{L}$ on $\mathcal{B}$ so that $\mathcal{L}$ is the generator of P. ◇

Exercise 1.9 *Show that the generators $\mathcal{L}$ with domains $\mathcal{D}(\mathcal{L})$ defined in exercise 1.6 are infinitesimal generators of Markov semi-groups by using Hille-Yoshida's theorem.*

Exercise 1.10 *Show that for any integer number n, $\mathcal{D}(\mathcal{L}^n)$ is dense in $\mathcal{B}$.*
Hint : Generalize (1.1.5).

1.2 Invariant Measures of a semi-group

We recall that

Definition 1.11 *Let $(P_t)_{t\geq 0}$ be a Markov semi-group. A probability measure μ on (Ω, Σ) is invariant with respect to the semi-group $(P_t)_{t\geq 0}$ iff for any $f \in \mathcal{C}(\Omega)$ and any $t \geq 0$*

$$\mu(P_t f) = \mu(f). \qquad (1.2.20)$$

The set of invariant measures for a semi-group $(P_t)_{t\geq 0}$ will be denoted hereafter $\mathcal{J} = \mathcal{J}(P)$.

Remark 1.12: Note that a semi-group may have no invariant probability measure (consider for instance the semi-group associated with the Brownian motion).
The invariant probability measures are also characterized by

Property 1.13 *μ on (Ω, Σ) is invariant with respect to the semi-group $(P_t)_{t\geq 0}$ iff for any $f \in \mathcal{D}(\mathcal{L})$,*

$$\mu(\mathcal{L} f) = 0. \qquad (1.2.21)$$

Proof : By (1.2.20), we obtain for any t

$$\mu[\frac{1}{t}(P_t - I)f] = 0.$$

Taking the limit $t \to 0$, we deduce (1.2.21). Reciprocally, since we already noticed that

$$(P_t - I)f = \mathcal{L} \int_0^t ds P_s f,$$

by integrating both sides with respect to μ, we get that (1.2.21) implies (1.2.20) for every bounded continuous function since we saw in (1.1.5) that $\int_0^t dsP_s f$ belongs to $\mathcal{D}(\mathcal{L})$. $\diamond$

The semi-group $(P_t)_{t\geq 0}$ has been defined until now on the Banach space $\mathcal{B}$, which a' priori is a subspace of $\mathcal{C}(\Omega)$. In fact, we can extend it as shown in

Property 1.14 *Let $\mu \in \mathcal{J}(P)$ with respect to a Markov semi-group $(P_t)_{t\geq 0}$. $(P_t)_{t\geq 0}$ can be extended to any $L^p(\mu)$ for $p \geq 1$.*

Proof : The operators P_t being linear, positive and with total mass $P_t 1 = 1$, we have by Jensen's inequality for any $p \geq 1$ and any $f \in \mathcal{B}$,

$$|P_t f|^p \leq P_t |f|^p. \tag{1.2.22}$$

Integrating both sides with respect to $\mu \in \mathcal{J}$, we deduce that

$$\mu |P_t f|^p \leq \mu |f|^p. \tag{1.2.23}$$

Thus, Hahn-Banach's theorem (see [83], theorem 5.16) shows that we can extend P_t to $L^p(\mu)$.

$\diamond$

Definition 1.15 *A Markov semi-group $\{P_t, t \geq 0\}$ is $L^q(\mu)$ ergodic for $q \in]1, \infty[$ and $\mu \in \mathcal{J}(P_t)$ iff for any function $f \in L^q(\mu)$,*

$$\lim_{t\to\infty} \int (P_t f - \mu f)^q d\mu = 0.$$

A Markov semi-group $\{P_t, t \geq 0\}$ is uniformly ergodic iff $\mathcal{J}(P_t)$ is reduced to a unique probability measure and

$$\lim_{t\to\infty} ||P_t f - \mu f||_\infty = 0.$$

In the rest of this course our goal will be to develop tools to study the ergodic properties of Markov semi-groups. We shall often consider semi-groups P and measures μ satisfying the following property stronger than invariance.

Definition 1.16 *A probability measure μ on (Ω, Σ) is reversible for a Markov semi-group $\{P_t, t \geq 0\}$ iff for any $(f, g) \in \mathcal{B}$ and time $t \geq 0$,*

$$\mu(gP_t f) = \mu(fP_t g). \tag{1.2.24}$$

Equivalently, one says that $(P_t)_{t\geq 0}$ satisfies the detailed balance condition for the probability measure μ.

The set of reversible measures of a Markov semi-group $\{P_t, t \geq 0\}$ will be denoted $\mathcal{J}_0(P_t)$. Clearly

$$\mathcal{J}_0(P_t) \subset \mathcal{J}(P_t)$$

since we can take $g = 1\!\!I$ and that $P_t 1\!\!I = 1\!\!I$. (1.2.24) shows more precisely that the semi-group $\bar{P}_t$ obtained as the extension of P_t in $L^2(\mu)$ (as indicated in property 1.14) is self adjoint in $L^2(\mu)$. We can associate to $\bar{P}_t$ a closed infinitesimal generator $\bar{\mathcal{L}}$ (see the proof of Hille-Yoshida's theorem) which coincides with the infinitesimal generator of P on its domain. Then, it is not hard to see that condition (1.2.24) is equivalent to

$$\mu(g\bar{\mathcal{L}}f) = \mu(f\bar{\mathcal{L}}g) \tag{1.2.25}$$

for any f, g in the domain of $\bar{\mathcal{L}}$. In other words, since $\bar{\mathcal{L}}$ is closed, $\bar{\mathcal{L}}$ is self-adjoint in $L^2(\mu)$.

Exercise 1.17 *Verify that the probability measure μ is invariant but not reversible for the generator $\mathcal{L}$ in the two following examples*

- *Let (Ω, Σ, μ) be a probability space and $(\mathbf{E}_1, \mathbf{E}_2)$ two different conditional expectations, $\mathbf{E}_1 = \mu(\ |A_1)$ et $\mathbf{E}_2 = \mu(\ |A_2)$, $A_1 \neq A_2$. We define, if I denotes the identity in $L^2(\mu)$, the operator $\mathcal{L}$ in $L^2(\mu)$ by*

$$\mathcal{L} = \mathbf{E}_1\mathbf{E}_2 - I.$$

- *To $U \in \mathcal{C}^1(I\!\!R^d)$, we associate the following Gibbs measure on $I\!\!R$*

$$\mu(dx) = \frac{1}{Z}\exp\{-U(x_1, .., x_d)\}dx_1..dx_d.$$

provided that $0 < Z < \infty$. For a function $\alpha \in \mathcal{C}^1(I\!\!R^d, I\!\!R^d)$, consider the operator on $\mathcal{C}^1(I\!\!R^d)$ defined by

$$\mathcal{L}f(x) = \alpha \cdot \nabla f(x) = \sum_{i=1}^{d} \alpha_i(x)\partial_{x_i} f(x).$$

If

$$div(\alpha) = \alpha \cdot \nabla U,$$

check that μ is invariant for $\mathcal{L}$. However, it is not reversible in general.

Exercise 1.18 *Show that the standard Gaussian law on $I\!\!R$ is reversible for the Ornstein-Uhlenbeck semi-group generated by $\mathcal{L} = \Delta - x\partial_x$. More generally, let Λ be a finite subset of $Z\!\!\!Z^d$ and $\Omega = I\!\!R^\Lambda$. Let U be a continuously differentiable function on Ω diverging to infinity in a sufficiently fast way so that $Z_\Lambda = \int \exp\{-U(x)\}dx_\Lambda$ is finite. Let μ be the probability measure*

$$\mu(dx) = \frac{1}{Z_\Lambda}\exp\{-U(x)\}dx_\Lambda.$$

Consider the generator on $\mathcal{C}_b^2(I\!\!R^{|\Lambda|})$ defined by

$$\mathcal{L} = \sum_{i\in\Lambda}(\Delta_i - g_i\nabla_i)$$

where ∇_i (resp. Δ_i) denotes the derivation (resp. the Laplacian) acting on the i-th coordinate. Here the g_i's are bounded Lipschitz functions. Give sufficient conditions on the g_i's so that μ is reversible for the semi-group associated with $\mathcal{L}$.

Exercise 1.19 *Let Λ be a finite subset of $\mathbb{Z}^d$ and $\Omega = \{-1, +1\}^\Lambda$. Let*

$$\mathcal{L} = \sum_{i \in \Lambda} c_i(\sigma)\partial_i$$

with $\partial_i f = f(\sigma^i) - f(\sigma)$ if

$$\sigma^i_j = \begin{cases} +\sigma_j & \text{if } j \neq i \\ -\sigma_i & \text{if } j = i. \end{cases}$$

Consider the probability measure μ on Ω of the form

$$\mu(f) = \frac{1}{\sum_{\sigma \in \Omega} e^{-U(\sigma)}} \sum_{\sigma \in \Omega} e^{-U(\sigma)} f(\sigma).$$

Give sufficient conditions on the c_i so that μ is reversible for the semi-group associated with $\mathcal{L}$.

Exercise 1.20 *Let Λ be a finite subset of $\mathbb{Z}^d$. Consider $X \subset \mathbb{Z}^d$ a finite subset of $\mathbb{Z}^d$. Set*

$$X + i = \{j + i; j \in X\}.$$

Let μ be a probability measure on a probability space $\Omega = \mathbf{M}^\Lambda$ for a Polish space $\mathbf{M}$. Let Σ^Λ_{X+i} be the sigma-algebra generated by $(x_j)_{j \in (X+i)^c \cap \Lambda}$. We denote

$$\mathbf{E}_{X+i} = \mu(\quad | \Sigma_{X+i})$$

the conditional expectation of μ knowing $(x_j)_{j \in (X+i)^c \cap \Lambda}$. Let $\mathcal{L}$ be the generator given, for $f \in L^2(\mu)$, by

$$\mathcal{L}f(x) = \sum_{i \in \Lambda} (\mathbf{E}_{X+i} - I) f(x).$$

Show that μ is reversible for the semi-group associated with $\mathcal{L}$.

1.3 Markov Processes

The notion of Markov Semi-group is intimately related with that of Markov Processes. In this section we recall some aspects of such processes and in particular their links with Markov Semi-groups.

We define the Markov property as follows.

Definition 1.21 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ be a probability space; an adapted process $X = \{X_t : \Omega \to I\!\!R^n, t \geq 0, \}$, $n \in I\!\!N$, will be said to be Markov with respect to $(\mathcal{F}_t)$ iff for all measurable bounded set Γ and any times s, t,*

$$I\!\!P(X_t \in \Gamma | \mathcal{F}_s) = I\!\!P(X_t \in \Gamma | X_s).$$

Exercise 1.22 *Show that the Brownian motion, or more generally any process with independent increments, is a Markov process.*

In the sequel, we denote for any $x \in \Omega$,

$$I\!\!P^x(X_t \in \Gamma) := I\!\!P(X_t \in \Gamma | X_0 = x)$$

the law of the Markov process with initial condition x.

Later on, homogeneous Markov processes will be of interest to us. They are described as follows

Definition 1.23 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), I\!\!P)$ be a probability space. A Markov process $X = \{X_t, t \geq 0\}$ with respect to $(\mathcal{F}_t)$ will be said to be homogeneous iff for every bounded measurable set G and any times $s, t, u \geq 0$,*

$$I\!\!P(X_t \in \Gamma | X_s) = I\!\!P(X_{t+u} \in \Gamma | X_{s+u}).$$

Exercise 1.24 *Show that Brownian motion $B_.$ is an homogeneous Markov process with respect to its natural filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$.*

If we denote by $I\!\!P^x$ the law of a homogeneous Markov process starting from $x \in \Omega$, we can define a family $(P_t)_{t\geq 0}$ of linear operators on the space of bounded measurable functions by

$$P_t f(x) = \int f(X_t) dI\!\!P^x.$$

$(P_t)_{t\geq 0}$ is then a Markov semi-group.

We can reformulate the homogeneous Markov property as follows

Property 1.25 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), I\!\!P)$ be a probability space. A process $X = \{X_t, t \geq 0\}$ is a homogeneous Markov process iff for any bounded measurable set Γ and any times s, t*

$$I\!\!P^x(X_t \in \Gamma | \mathcal{F}_s) = (P_t 1\!\!I_\Gamma)(X_s) \quad I\!\!P^x - a.s$$

A proof may be found in [57], p. 75.

Notice that the notion of Markov semi-group is in fact equivalent to the definition of Feller-Markov processes defined below. This point is illustrated in the following exercises.

Exercise 1.26 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a probability space and $X = \{X_t, t \geq 0\}$ a Feller-Markov process (i.e a homogeneous Markov process such that for any bounded continuous function f, $P_t f$ is bounded continuous). Show that the operator family $(P_t)_{t\geq 0}$ is a Markov semi-group on $(\mathcal{C}(\Omega), ||\cdot||_\infty)$. Conversely, any Markov semi-group on $(\mathcal{C}(\Omega), ||\cdot||_\infty)$ defines a unique Feller-Markov process.*
Hint : For the second point,

a) Use Riesz 's theorem to see that for any time t and any $x \in \Omega$, there exist a unique probability $\mathbf{E}(t, x, dy)$ so that for any $f \in \mathcal{D}(\mathcal{L})$,

$$P_t(f)(x) = \int f(y)\mathbf{E}(t, x, dy).$$

b) Let $\mathcal{A}$ be the algebra of cylinder functions on $\mathcal{C}(\mathbb{R}^+, \Omega)$, that is the algebra of functions of the form

$$F(\xi) = f(\xi_{t_1}, .., \xi_{t_p})$$

for some finite integer number p and times $(t_1, .., t_p)$. Define , for a probability measure μ, a normalized linear functional $\mathbf{E}^\mu$ on $\mathcal{A}$ such that, for F as above, $\mathbf{E}^\mu(F)$ is equal to

$$\int f(y_1, ., y_p)\mu(dy_0)\mathbf{E}(t_1, y_0, dy_1)\mathbf{E}(t_2 - t_1, y_1, dy_2)..\mathbf{E}(t_p - t_{p-1}, y_{p-1}, dy_p).$$

The Stone-Weierstrass theorem implies that $\mathcal{A}$ is dense in $\mathcal{C}(\mathbb{R}^+, \Omega)$. Conclude by Riesz's theorem that there is a unique probability measure P^μ on this space such that for any $F \in \mathcal{A}$,

$$\int F(\xi)dP^\mu(\xi) = \mathbf{E}^\mu(F).$$

Show that $(\mathbf{E}^{\delta_x}, x \in \Omega)$ is the law of a Markov process.

Since we saw that the Feller-Markov semi-groups and homogeneous Markov processes are in bijection, the Hille-Yoshida theorem establishes a bijection between Markov processes and infinitesimal generators. This connection may in fact be made more directly by following Stroock and Varadhan [96] who introduced the notion of martingale problems.

To associate a generator to a Markov process, we define

Definition 1.27 *Let $\mathcal{L}$ be an infinitesimal generator. A probability measure $\mathbb{P}$ on $\mathcal{C}(\mathbb{R}^+, \Omega)$ is said to be solution of the martingale problem for $\mathcal{L}$ with initial condition η iff*

1.

$$\mathbb{P}[\xi \in \mathcal{C}(\mathbb{R}^+, \Omega) : \xi_0 = \eta] = 1.$$

2. If we denote by x the canonical process under $\mathbb{P}$, $f(x_t) - \int_0^t \mathcal{L}f(x_s)ds$ is a martingale under $\mathbb{P}$ for the canonical filtration

$$\mathcal{F}_t = \sigma(x_u, u \leq t).$$

We then have the following theorem (see Proposition 4.2 in [57])

Theorem 1.28 *Let $\mathcal{L}$ be an infinitesimal generator. Let $(\mathbf{E}^\eta, \eta \in \Omega)$ be the unique Feller-Markov process associated to $\mathcal{L}$. Then, for every $(\eta \in \Omega)$, $\mathbf{E}^\eta$ is the unique solution of the martingale problem for $\mathcal{L}$ with initial condition η.*

Finally, let us give as an exercise the following example of Markov process

Exercise 1.29 *Let $(h_i)_{1 \le i \le d}$ be continuously differentiable Lipschitz functions on $I\!R^d$. Show that the stochastic differential system*

$$dx_t^i = h_i(x_t)dt + dB_t^i \qquad 1 \le i \le d,$$

with a d-dimensional Brownian motion $(B^i, 1 \le i \le d)$, admits a unique strong solution which is a Markov process. Describe the generator of its associated Markov semi-group.

For further informations on Markov processes the reader may like to consult the books [57] and [76].

Chapter 2

Spectral gap inequalities and L^2 ergodicity

Before considering Sobolev inequalities and entering the heart of the matter, we study the L^2 ergodicity of semi-groups satisfying the detailed balance condition with respect to some probability measure μ. As we already mentioned in the last section, this property allows us to consider their infinitesimal generators as self-adjoint operators in $L^2(\mu)$ so that the study of these operators in that space is rather natural.

Let $\mathcal{L}$ be an infinitesimal generator. Let μ be a probability measure on (Ω, Σ) and $(P_t)_{t\geq 0}$ a Markov semi-group satisfying the detailed balance condition with respect to μ. We define

Definition 2.1 *A probability measure $\mu \in \mathcal{J}_0(\mathcal{L})$ satisfies the spectral gap inequality iff there exists a positive real number m such that*

$$m\mu(f - \mu f)^2 \leq \mu(f(-\mathcal{L})f) \tag{2.0.1}$$

for any function $f \in L^2(\mu) \cap \mathcal{D}(\mathcal{L})$ such that the right hand side of (2.0.1) is finite. The largest positive real number m satisfying (2.0.1) is called the spectral gap of the self-adjoint operator $\mathcal{L}$.

The corresponding Dirichlet form is given on elements of the domain of the generator by

$$\mathcal{E}(f,g) = \mu(f(-\mathcal{L})g)$$

It can as well be defined by the carré du champ

$$\Gamma_1(f,f) := \frac{1}{2}\left(\mathcal{L}f^2 - 2f\mathcal{L}f\right).$$

Indeed, for any $\mu \in \mathcal{J}_0(\mathcal{L})$, we have

$$\mathcal{E}(f,f) = \mu(\Gamma_1(f,f)).$$

The operator carré du champ is non negative since

$$\Gamma_1(f,f) = \lim_{t\downarrow 0} \frac{1}{2t}\left(P_t f^2 - (P_t f)^2\right) \geq 0$$

by Cauchy-Schwarz's inequality. It will play an important role latter.

Example 2.2 *Let Λ be a finite subset of $\mathbb{Z}^d$.*

• *Let $\Omega = (S^{d-1})^\Lambda = \{x = (x_i)_{i\in\Lambda}, x_i \in S^{d-1} \quad \forall i \in \Lambda\}$ with S^{d-1} the unit sphere in $\mathbb{R}^d$. Consider the Langevin dynamics generator*

$$\mathcal{L} = \sum_{i\in\Lambda} (\Delta_i - \nabla_i U \cdot \nabla_i)$$

for some function $U \in \mathcal{C}^1(\Omega)$. Then

$$\Gamma_1(f,f) = \sum_{i\in\Lambda} (\nabla_i f)^2.$$

• *If $\Omega = \{-1,+1\}^\Lambda$ and we consider the Glauber dynamics generator*

$$\mathcal{L} = \sum_{i\in\Lambda} c_i \partial_i$$

for some functions $c_i \geq 0$, then

$$\Gamma_1(f,f) = \sum_{i\in\Lambda} c_i(\partial_i f)^2.$$

Exercise 2.3 *1) Let (Ω,Σ,μ) be a probability space . Let $\mathcal{L}$ be a generator with domain $L^1(\mu)$ given by $\mathcal{L}f = \mu f - f$. Then, show that $\mu \in \mathcal{J}_0(\mathcal{L})$ satisfies a spectral gap inequality with $m = 1$.*

2) If $\Omega = \{-1,+1\}$ and $\mu = p\delta_{-1} + (1-p)\delta_{+1}$ for some $p \in (0,1)$, we define the generator $\mathcal{L}$ on the set of measurable functions by $\mathcal{L}f = c(\sigma)(f(-\sigma) - f(\sigma))$ for some non negative function c. Choose c so that $\mu \in \mathcal{J}_0(\mathcal{L})$ and show that μ satisfies a spectral gap inequality.

The spectral gap property is equivalent with the notion of L^2 ergodicity of the semi-groups

Property 2.4 *$\mu \in \mathcal{J}_0(P_t)$ satisfies a spectral gap inequality with constant m iff for any $t \geq 0$ and any function $f \in L^2(\mu)$*

$$\mu(P_t f - \mu f)^2 \leq e^{-2mt}\mu(f - \mu f)^2.$$

Proof : Note that if f is a centered function, $\int f d\mu = 0$, then $f_t = P_t f$ is also centered since μ is invariant. Hence, (2.0.1), applied to $f_t = P_t f$, gives

$$-\partial_t \mu(P_t f)^2 \geq 2m\mu(P_t f)^2$$

and so, as $P_0 = I$,

$$\mu(P_t f)^2 \le e^{-2mt}\mu(f^2). \tag{2.0.2}$$

Conversely, if $f \in \mathcal{D}(\mathcal{L})$ satisfies (2.0.2), then $t \to e^{mt}\mu(P_t f - \mu f)^2$ is decreasing. Hence,

$$\frac{d}{dt} e^{mt}\mu(P_t f - \mu f)^2 \le 0.$$

In $t = 0$, we deduce the spectral gap inequality.

$\diamond$

The notion of spectral gap inequality can easily be extended in infinite dimension. In fact, we have the following product property

Theorem 2.5 *Let $(\mu_i)_{i=1,2}$ be two probability measures on $(\Omega_i, \Sigma_i)_{i=1,2}$ satisfying the spectral gap inequality with coefficients $(m_i)_{i=1,2}$ for some generators $(L_i)_{i=1,2}$. Let $\Omega = \Omega_1 \times \Omega_2$ be equipped with the product σ-algebra. For any function f on Ω, we note*

$$f_{1,\omega}(x) = f(x,\omega) \qquad f_{2,\omega}(x) = f(\omega, x)$$

and extend $(L_i)_{i=1,2}$ to functions on Ω by

$$\mathcal{L}_1 f(x_1, x_2) = L_1 f_{1,x_2}(x_1) \quad \mathcal{L}_2 f(x_1, x_2) = L_2 f_{2,x_1}(x_2).$$

Then, if $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ and $m = \min(m_1, m_2)$, the product law $\mu_1 \otimes \mu_2$ satisfies the spectral gap inequality with constant m

$$m\mu_1 \otimes \mu_2(f - \mu_1 \otimes \mu_2 f)^2 \quad \le \quad \mu_1 \otimes \mu_2(f(-\mathcal{L})f)) \tag{2.0.3}$$

for any measurable function f for which the right hand side is finite.

Proof : Integrating with respect to one variable, we obtain

$$\mu_1 \otimes \mu_2(f - \mu_1 \otimes \mu_2 f)^2 = \mu_2\left[\mu_1(f - \mu_1 f)^2\right] + \mu_2(\mu_1 f - \mu_1 \otimes \mu_2 f)^2.$$

Applying spectral gap inequality for μ_1 and μ_2, we deduce

$$\mu_1 \otimes \mu_2(f - \mu_1 \otimes \mu_2 f)^2 \le m^{-1}\mu_1 \otimes \mu_2\left(f(-\mathcal{L}_1)f + \mu_1 f(-\mathcal{L}_2)\mu_1 f\right).$$

On the other hand, $f \to \Gamma_1(f,f)$ is convex since

$$\Gamma_1(f,f)(x) = \lim_{t\downarrow 0} \frac{1}{2t} P_t(f^2 - P_t f(x))^2(x).$$

Thus,

$$\mu_2\left(\mu_1 f(-\mathcal{L}_2)\mu_1 f\right) = \mu_2\left(\Gamma_1^{\mathcal{L}_2}(\mu_1 f, \mu_1 f)\right) \le \mu_1 \otimes \mu_2\left(\Gamma_1^{\mathcal{L}_2}(f,f)\right)$$

which finishes the proof.

$\diamond$

Further, the spectral gap property is preserved when we perturb a measure by a bounded density.

Property 2.6 *Let μ be a probability measure on a space (Ω, Σ) satisfying the spectral gap inequality for the generator $\mathcal{L}$ with constant m. Let U be a bounded measurable function and consider the probability measure ν given by*

$$\nu = \frac{1}{Z} e^{-U} d\mu$$

with

$$Z = \int e^{-U} d\mu.$$

Then ν satisfies the spectral gap inequality for the generator $\mathcal{L}$ with constant $me^{-2osc(U)}$.

Proof : This property is clear once one notices that, if $\operatorname{osc}(U) = \sup U - \inf U$, for every measurable set A,

$$e^{-\operatorname{osc}(U)} \mu(A) \leq \nu(A) \leq e^{\operatorname{osc}(U)} \mu(A). \tag{2.0.4}$$

Indeed, one then obtains, with the additional observation that for every constant C, $\nu(f - \nu f)^2 \leq \nu(f - C)^2$,

$$\begin{aligned} \nu(f - \nu f)^2 &\leq \nu(f - \mu f)^2 \\ &\leq e^{\operatorname{osc}(U)} \mu(f - \mu f)^2 \\ &\leq \frac{e^{\operatorname{osc}(U)}}{m} \mu\Gamma_1(f, f) \\ &\leq \frac{e^{2\operatorname{osc}(U)}}{m} \nu\Gamma_1(f, f) \end{aligned}$$

◇

In fact, this property can be improved when one considers finite volume Gibbs measures with short range interaction. We then have (see [86] or [106])

Property 2.7 *Let (Ω, Σ, μ) be a probability space and assume that μ satisfies a spectral gap inequality with constant m for the carré du champ Γ_1. Let $\Lambda = [-L, L] \times [-l, l]^{d-1}$ for some $(l, L) \in (\mathbb{N}^*)^2$, $l \leq L$. Let U be a real valued bounded continuous function on Ω^2. We denote by $\omega \in \Omega^{|\partial\Lambda|}$ the boundary condition . Let*

$$H_\Lambda^\omega(x) = \sum_{i,j \in \Lambda, |i-j|=1} U(x_i, x_j) + \sum_{i \in \Lambda, j \in \Lambda^c, |i-j|=1} U(x_i, \omega_j)$$

and set

$$d\mu_\Lambda^\omega(x) = \frac{1}{Z_\Lambda^\omega} e^{-\beta H_\Lambda^\omega(x)} d\mu^{\otimes|\Lambda|}(x).$$

Then, if $\bar{\Gamma}_1 = \sum_{i \in \Lambda} \Gamma_1^i$, there exists a finite constant c (depending on U) so that

$$\mu_\Lambda^\omega(f - \mu_\Lambda^\omega f)^2 \leq |\Lambda| e^{cl^{d-1}} \mu_\Lambda^\omega(\bar{\Gamma}_1(f, f))$$

for any measurable function f such that the above right hand side is finite.

Remark that the previous property would only yield in this example

$$\mu_\Lambda^\omega(f-\mu_\Lambda^\omega f)^2 \le e^{cl^dL}\mu_\Lambda^\omega(\bar\Gamma_1(f,f)).$$

Hence, property 2.7 is a real improvement when L is large. It already points out that in dimension 1, the spectral gap is bounded below by the inverse of the volume of Λ. In fact, we shall see later that it is of order one, a fact linked with absence of phase transition for systems with finite range interactions on one dimensional lattice. Property 2.7 is optimal (modulo the multiplication by the volume of Λ in the right hand side and the choice of the constant) in such a generality (see the lower bound proved by Thomas [98] at low temperatures).

Proof : Let us choose a lexicographic order in Λ and denote by $\{i_1,..,i_k,...\}$ the points in Λ in this order. We choose it so that $(i_k^1)_{1\le k\le|\Lambda|}$, the first coordinate of $(i_k)_{1\le k\le|\Lambda|}$, increases slowly, that is that we fill in the faces of volume l^{d-1} of Λ with coordinate i^1 one after the other. Set $I_k=\{i_1,..,i_k\}$.

We shall proceed by interpolation : note first that

$$\mu_\Lambda^\omega(f-\mu_\Lambda^\omega f)^2 = \frac{1}{2}\int\int\left(f(x_{i_1},..,x_{i_{|\Lambda|}})-f(\widetilde{x}_{i_1},..,\widetilde{x}_{i_{|\Lambda|}})\right)^2 d\mu_\Lambda^\omega(x)d\mu_\Lambda^\omega(\widetilde{x}).$$

Writing, with $\Delta_pf(x,\widetilde{x})\equiv f(x_{i_1},.,x_{i_p},\widetilde{x}_{i_{p+1}}.,\widetilde{x}_{i_{|\Lambda|}})-f(x_{i_1},.,x_{i_{p+1}},\widetilde{x}_{i_{p+2}}.,\widetilde{x}_{i_{|\Lambda|}})$,

$$f(x_{i_1},..,x_{i_{|\Lambda|}})-f(\widetilde{x}_{i_1},..,\widetilde{x}_{i_{|\Lambda|}}) = \sum_{p=0}^{|\Lambda|-1}\Delta_pf(x,\widetilde{x}),$$

and using Jensen's inequality we obtain

$$\mu_\Lambda^\omega(f-\mu_\Lambda^\omega f)^2 \le |\Lambda|\sum_{p=0}^{|\Lambda|-1}(\mu_\Lambda^\omega)^{\otimes 2}\left(\Delta_pf(x,\widetilde{x})\right)^2. \tag{2.0.5}$$

In each term of the right hand side, only the variable at the i_{p+1}-th site can differ. Notice as well that the points of $J_p^+=\{i_m, m\ge p+c_dl^{d-1}\}$ and of $J_l^-=\{i_m,m\le p-c_dl^{d-1}\}$ are at distance larger than one from i_{p+1} for some well chosen finite constant c_d. Consequently, bounding the density of μ_Λ^ω uniformly on the $\{x_{i_k},|k-p|\le c_dl^{d-1}\}$, we get, if $S_p=\{i_k\in\Lambda;|k-p|\le c_dl^{d-1}\}$, that for any function $F\ge 0$,

$$\mu_\Lambda^\omega F\le e^{4c_dl^{d-1}||U||_\infty}\mu^0_{J_p^-}\otimes\mu_{S_p}\otimes\mu^0_{J_p^+}F.$$

Here$d\mu_{S_p}(x_i,i\in S_p)=d\mu^{\otimes|S_p|}(x_i,i\in S_p)$ and

$$d\mu^0_{J_p^\epsilon}=\frac{1}{Z_{J_p^\epsilon}}e^{-\beta H_{J_p^\epsilon}}d\mu^{\otimes|J_p^\epsilon|}$$

with the convention that for $\epsilon=+$ or $-$,

$$H_{J_p^\epsilon}=\sum_{i,j\in J_p^\epsilon,|i-j|=1}U(x_i,x_j).$$

¿From the above bound, we obtain that

$$(\mu_\Lambda^\omega)^{\otimes 2}\left(\Delta_p f(x,\widetilde{x})\right)^2 \leq e^{4c_d l^{d-1}||U||_\infty}(\mu^0_{J_p^-}\otimes\mu_{S_p}\otimes\mu^0_{J_p^+})^{\otimes 2}\left(\Delta_p f(x,\widetilde{x})\right)^2. \quad (2.0.6)$$

Using the spectral gap inequality for μ, we deduce that

$$(\mu_\Lambda^\omega)^{\otimes 2}\left(\Delta_p f(x,\widetilde{x})\right)^2 \leq \frac{1}{m}e^{4c_d l^{d-1}||U||_\infty}\mu^0_{J_p^-}\otimes\mu_{S_p}\otimes\mu^0_{J_p^+}\left(\Gamma_1^{p+1}(f,f)\right) \quad (2.0.7)$$

with Γ_1^{p+1} acting on the i_{p+1}-th variable.

To find back our initial probability measure, we input by force the potential to find

$$(\mu_\Lambda^\omega)^{\otimes 2}\left(\Delta_p f(x,\widetilde{x})\right)^2 \leq \frac{e^{6c_d l^{d-1}||U||_\infty}}{m}\mu_\Lambda^\omega\left(\Gamma_1^{p+1}(f,f)\right). \quad (2.0.8)$$

Thanks to (2.0.5) and (2.0.8), we can conclude that

$$\mu_\Lambda^\omega(f-\mu_\Lambda^\omega f)^2 \leq \frac{|\Lambda|}{m}e^{6c_d l^{d-1}||U||_\infty}\sum_{l=0}^{|\Lambda|-1}\mu_\Lambda^\omega\left(\Gamma_1^{l+1}(f,f)\right). \quad (2.0.9)$$

$\diamond$

Exercise 2.8 *Generalize the last property to the case where the interaction has finite range, (but not necessarily of nearest neighbours type), and the interaction inhomogeneous, that is that the Hamiltonian is given by*

$$H_\Lambda^\omega(x) = \sum_{X\in\mathcal{X}:X\cap\Lambda\neq\emptyset} U_X(x)$$

for a family $\mathcal{X}$ of finite subsets of $\mathbb{Z}^d$ with uniformly bounded diameters and a family $(U_X)_{X\in\mathcal{X}}$ of local functions so that, for any $X\in\mathcal{X}$, $U_X(x)$ only depends on $(x_i, i\in X)$.

The next exercise gives a link between the spectral gap inequality and the concentration of measure phenomenon

Exercise 2.9 ([52]) *If μ satisfies a spectral gap inequality with constant m on $\mathbb{R}^d$ for a carré du champ Γ_1 satisfying the Leibnitz rule*

$$\Gamma_1(f,gh) = \Gamma_1(f,g)h + \Gamma_1(f,h)g,$$

then for all $f\in\mathcal{D}(\mathcal{L})$, and t sufficiently small so that

$$\frac{t^2}{2m}||\Gamma_1(f,f)||_\infty \leq 1$$

we have

$$\mu e^{tf} \leq const.e^{\frac{t^2}{m}||\Gamma_1(f,f)||_\infty}e^{t\mu f}.$$

Hint : Write

$$\mu(e^{tf}) = \mu(e^{\frac{tf}{2}}, e^{\frac{tf}{2}}) + (\mu(e^{\frac{tf}{2}}))^2$$

and apply the spectral gap inequality to $\mu(e^{\frac{tf}{2}}, e^{\frac{tf}{2}})$*. Observe that when* Γ_1 *satisfies the Leibnitz rule,*

$$\Gamma_1(e^{\frac{tf}{2}}, e^{\frac{tf}{2}}) = \frac{t^2}{4} e^{tf} \Gamma_1(f, f).$$

Proceed inductively.

We shall continue our discussion on the carré du champ and Leibnitz rule when studying Bakry-Emery criterion.

In the next exercise, we suggest proofs of the spectral gap inequality in a few simple cases.

Exercise 2.10

- $\Omega = [0, a]$ *for some positive real number* a*. For* $f \in \mathcal{C}^2(\Omega)$ *such that* $f(a) = f(0) = 0$*, we set*

$$\mathcal{L} = \frac{d^2}{dx^2}.$$

  If μ *is the normalized Lebesgue measure on* Ω*, then* $\mathcal{L}$ *is self-adjoint in* $L^2(\mu)$ *and the spectral gap inequality is satisfied with* $m \geq (1/a^2)$*. Hint : Use integration by parts formula.*

- *Let* $\Omega = \{-1, +1\}$ *and* μ *be the Bernouilli law* $p\delta_{-1} + (1-p)\delta_{+1}$*. Let* $\mathcal{L}f(x) = f(-x) - \nu(f)$*. Show that the spectral gap inequality is satisfied with* $m = 1$*.*

For further general informations about spectral gap inequality, the reader can read [51] and references therein. Even though one most often try to bound the spectral gap constant from below, it can be useful also to obtain upper bounds. We then recommend to read [14]. Finally, we remind the reader that in forthcoming chapter 5 we shall be interested in the logarithmic Sobolev inequalities which imply spectral gap inequalities. Nevertheless, in most cases we shall consider, one can obtain directly (and by similar methods) lower bounds for the spectral gap constant.

To obtain a stronger control on the asymptotic behaviour of $P_t f$ than the L^2 ergodicity studied above, we shall need contractivity inequalities. It is the role played by Sobolev inequalities.

Chapter 3

Classical Sobolev inequalities and Ultracontractivity

In this chapter, we study classical Sobolev inequalities and their links with uniform ergodicity of Markov semi-groups. We shall see that classical Sobolev inequalities is equivalent to Nash's inequalities, which are themselves equivalent to the ultracontractivity of the associated semi-groups. This last property entails the uniform ergodicity of the semi-groups. Unfortunately, we shall see that in general these inequalities cannot be true in infinite dimension (see exercise 3.7). It is one of the main motivations to study log-Sobolev inequalities which satisfy a product property and therefore can hold in infinite dimension.

Definition 3.1 *A probability measure $\mu \in \mathcal{J}_0(P_t)$ satisfies a classical Sobolev inequality iff for some $p \in]2, \infty[$ and two finite constants $(a, b) \in [0, \infty[^2$, we have*

$$||f||_p^2 \leq a\mu\Gamma_1(f, f) + b\mu f^2 \tag{3.0.1}$$

for any measurable function f such that $\mu\Gamma_1(f, f)$ and $||f||_2$ are finite and, in the case where $b = 0$, so that $\int f d\mu = 0$.

In the next exercise, we describe some classes of probability measures for which a Sobolev inequality holds.

Exercise 3.2 • *Let $\mathcal{O} \subset I\!R^d$ be a bounded convex open subset of $I\!R^d$ and μ be the normalized Lebesgue measure on $\mathcal{O}$. Show that μ satisfies a classical Sobolev inequality restricted to the functions which vanish on the boundary of $\mathcal{O}$.*

Hint : For $d > 2$ use Taylor's formula to write

$$|f(x)| = \prod_{i=1}^{d} \left| \int_{x_0^i}^{x_i} \partial_i f(x_{i^c} \circ y_i) dy_i \right|^{\frac{1}{d}}$$

for any function f which vanishes outside $\mathcal{O}$ and any point $(x_0^i, 1 \leq i \leq d)$ with coordinates outside of $\mathcal{O}$. Multiply all Taylor's formulas and rise to the power $p = (1/d - 1)$. Integrate with respect to the d dimensional Lebesgue measure, use inductively Hölder inequality and the geometric - arithmetic mean inequality to arrive at an useful inequality. By substituting to this inequality a power of the absolute value of a sufficiently smooth local function one can obtain a family of useful inequalities including the desired one (with optimal exponent).

- *Show that, if μ satisfies a Sobolev inequality, any probability measure $\nu \ll \mu$ such that there exists $\lambda \in [1, \infty[$ for which*

$$\frac{1}{\lambda} \leq \frac{d\nu}{d\mu} \leq \lambda$$

also satisfies a Sobolev inequality.

Note that when one considers a probability measure μ on the entire real line $\mathbb{R}$, it was shown in [79] that if a probability measure μ has a tail like $e^{-ax^{2+\epsilon}}$ for a and ϵ strictly positive real numbers, then μ satisfies a Sobolev inequality. This result is false when $\epsilon = 0$ since the heat semi-group on $\mathbb{R}$ is not ultracontractive; (the latter as we shall see is equivalent to a Sobolev inequality).

Setting

$$||f||_\infty = \sup\{\mu(fg), ||g||_1 = 1\},$$

we shall see that

Theorem 3.3 *If $\mu \in \mathcal{J}_0(P_t)$ satisfies a Sobolev inequality, there exists a positive real number γ and a finite constant c such that for any $f \in L^1(\mu)$*

$$||P_t f - \mu f||_\infty \leq c(\frac{a}{2et} + b)^\gamma ||f - \mu f||_1 \tag{3.0.2}$$

In particular, if $b = 0$, P_t converges to μ with a polynomial rate.

Moreover, if μ satisfies both the spectral gap inequality and the Sobolev inequality, there exists a finite constant c' so that for any $f \in L^2(\mu)$

$$||P_t f - \mu f||_\infty \leq c' e^{-mt} ||f - \mu f||_2. \tag{3.0.3}$$

Let us point out that the above control is not really uniform on the entire space but with probability one with respect to μ. However, if μ is compactly supported and is absolutely continuous with respect to the uniform measure on the underlying space, the theorem implies uniform ergodicity.

Theorem 3.3 relates Classical Sobolev inequalities with a contractivity property of the associated semi-groups as follows. To prove it, let us recall the standard notation

Definition 3.4 *For $(p, q) \in [1, \infty[^2$,*

$$||P_t||_{q,p} = \sup\{||P_t f||_p : f \in \mathcal{C}(\Omega), ||f||_q = 1, \mu(f) = 0\}$$

where $||f||_q = \left(\int f^q d\mu\right)^{\frac{1}{q}}$. This definition extends to the case $p = \infty$.

With this definition, the first point of the theorem is equivalent to

Property 3.5 *Under the hypotheses of theorem* 3.3, *there exists a positive constant* γ *and a finite constant* c *so that*

$$||P_t||_{1,\infty} \leq c(\frac{a}{2et} + b)^{\gamma}$$

This property is itself equivalent to

Lemma 3.6 *Under the hypotheses of theorem* 3.3, *there exists a positive constant* $\widetilde{\gamma}$ *and a finite constant* $\widetilde{c}$ *so that*

$$||P_t||_{1,2} \leq \widetilde{c}(\frac{a}{4et} + b)^{\widetilde{\gamma}}$$

Indeed, it is clear that property 3.5 implies

$$||P_t||_{1,2} \leq ||P_t||_{1,\infty} \leq c(\frac{a}{2et} + b)^{\gamma}$$

so that the lemma is verified. Conversely, the semi-group property implies that

$$||P_t||_{1,\infty} \leq ||P_{\frac{t}{2}}||_{1,2}||P_{\frac{t}{2}}||_{2,\infty}. \tag{3.0.4}$$

But, by duality,

$$\begin{aligned} ||P_{\frac{t}{2}}||_{2,\infty} &= \sup\{\mu(f_{\frac{t}{2}}g), ||f||_2 = 1, ||g||_1 = 1\} \\ &= \sup\{\mu(g_{\frac{t}{2}}f), ||f||_2 = 1, ||g||_1 = 1\} \\ &= ||P_{\frac{t}{2}}||_{1,2} \end{aligned}$$

which gives with (3.0.4) the desired equivalence.
Proof of Lemma 3.6 [106]

To prove the lemma, we first remark that the Sobolev inequality implies the following classical Nash inequality

$$||f||_2 \leq (a\mu\Gamma_1(f,f) + b\mu f^2)^{\frac{\alpha}{2}}||f||_1^{\beta} \tag{3.0.5}$$

with $\alpha = \frac{1}{2}\frac{p}{p-1}$ and $\beta = 1 - \alpha$. This inequality is in fact directly derived via Hölder 's inequality

$$||f||_2 \leq ||f||_1^{\frac{1}{2u}}||f||_{v+1}^{\frac{v+1}{2v}}$$

for every couple (u, v) of conjugate exponents between one and infinity, by taking $v + 1 = p$. Using (3.0.5) with f_t we get

$$||f_t||_2 \leq (a\mu\Gamma_1(f_t, f_t) + b\mu f^2)^{\frac{\alpha}{2}}||f_t||_1^{\beta} \tag{3.0.6}$$

To estimate $\Gamma_1(f_t, f_t)$, note first that the spectral theorem allows us to write

$$P_t = \int_0^{\infty} e^{-\lambda t} d\mathbf{E}_{\lambda}^{\mu}$$

with $\{\mathbf{E}^\mu_\lambda, \lambda \geq 0\}$ the projections on the eigenspaces of $\mathcal{L}$ in $L^2(\mu)$ (see Bourbaki, théorie spectrale or [78]). With these notations

$$\begin{aligned}\mu\Gamma_1(f_t, f_t) &= -\mu(P_t f \mathcal{L} P_t f)\\ &= \int \lambda e^{-2\lambda t} d(\mathbf{E}^\mu_\lambda f, f)_{L^2(\mu)}.\end{aligned}$$

Noticing that for any real number u

$$ue^{-2u} \leq \frac{1}{2e}$$

we deduce

$$\mu\Gamma_1(f_t, f_t) \leq \frac{1}{2et}\int d(\mathbf{E}^\mu_\lambda f, f)_{L^2(\mu)} = \frac{1}{2et}\mu(f^2). \tag{3.0.7}$$

Plugging this result into (3.0.6) and noticing that $\mu|P_t f|^q \leq \mu|f|^q$ for $q \geq 1$, we deduce that for any $t \geq 0$,

$$||f_t||_2 \leq (\frac{a}{2et} + b)^{\frac{\alpha}{2}} ||f||_2^\alpha ||f||_1^\beta. \tag{3.0.8}$$

Choosing $t_0 = t/2$, we obtain by induction

$$||f_t||_2 \leq \prod_{n=1}^\infty \Big(\frac{a}{2e\frac{t}{2^n}} + b\Big)^{\frac{\alpha^n}{2}} ||f||_1^{\beta \sum_{n\geq 0}\alpha^n}. \tag{3.0.9}$$

Since

$$\Big(\frac{a}{2e\frac{t}{2^n}} + b\Big) \leq 2^n(\frac{a}{2et} + b)$$

we deduce that

$$||f_t||_2^2 \leq 2^{\sum_{n\geq 1} n\alpha^n} \Big(\frac{a}{2et} + b\Big)^{\frac{\alpha}{1-\alpha}} ||f||_1^{\frac{2\beta}{1-\alpha}}. \tag{3.0.10}$$

The lemma is established.

$\diamond$

To complete the proof of theorem 3.3, it is enough to notice that, when the spectral gap inequality holds,

$$||P_t f - \mu f||_\infty \leq ||P_1||_{2,\infty} ||P_{t-1} f - \mu f||_2 \leq ce^{-mt} ||f - \mu f||_2.$$

$\diamond$

Unfortunately, classical Sobolev inequality is not satisfied in infinite dimension in the sense that it does not satisfies a product property as the spectral gap inequality.

Exercise 3.7 *Let $P_t = e^{t\mathcal{L}}$ be a Markov semi-group on $\mathcal{C}(\Omega)$. Let $\mu \in \mathcal{J}_0(P_t)$ satisfying the Sobolev inequality*

$$\mu(f^p)^{\frac{2}{p}} \le a\mu\Gamma_1(f,f) + b\mu f^2.$$

For any integer number n, we define a semi-group $P_t^n = e^{t\mathcal{L}^n}$ with

$$\mathcal{L}^n = \sum_{i=1}^n \mathcal{L}_i$$

if $\mathcal{L}_i$ acts on the i-th variable. Show that for sufficiently large integer number n, one cannot have

$$\mu^{\otimes n}(f^p)^{\frac{2}{p}} \le a\mu^{\otimes n}\Gamma_1^n(f,f) + b\mu^{\otimes n} f^2$$

for all functions f for which the right hand side is finite.

Hint : Assume the above equality true and obtain a contradiction by taking for some $g \in \mathcal{D}(\mathcal{L})$,

$$f(\omega) = \prod_{i=1}^n g(\omega_i).$$

We saw that if $\mu \in \mathcal{J}_0(P_t)$ satisfies a Sobolev inequality, P_t is ultracontractive, that is that for sufficiently large times t

$$||P_t||_{2,\infty} < \infty.$$

This property was the key point to obtain uniform ergodicity. In fact, such a property can be obtained directly when the state space is a compact Riemaniann manifold as one can see by doing the following exercise.

Exercise 3.8 *Let $\mathcal{S}^1$ be the unit circle in $\mathbb{R}^2$. Let $\Lambda \subset\subset \mathbb{Z}^d$ and U_Λ be a twice continuously bounded differentiable function on $\mathcal{S}^1 \times \mathcal{S}^1$. Set*

$$H_\Lambda(x) = \sum_{i,j\in\Lambda, |i-j|=1} U_\Lambda(x_i, x_j).$$

We consider the semi-group associated with

$$\mathcal{L} = \sum_{i\in\Lambda} (\Delta_i - \nabla_i H_\Lambda(x).\nabla_i)$$

where $\Delta_i = (1/2)\frac{d^2}{dx_i^2}$ et $\nabla_i = \frac{d}{dx_i}$ on $\mathcal{S}^1$ for $i \in \Lambda$.

Show by use of Girsanov formula (see theorem 5.1 in [57]) that there exists a finite constant c which only depends on U_Λ so that for any function $f \ge 0$,

$$P_t f \le e^{ct|\Lambda|} \int f(y) \prod_{i\in\Lambda} p_t(x_i, dy_i).$$

Here, p_t is the semi-group associated with the Laplacian on the circle. Show that there exists, for any $t \geq 0$, a finite constant c_t such that

$$\sup_{x_i} p_t(x_i, dy_i) \leq c_t \lambda(dy_i)$$

with λ the uniform measure on the circle. Conclude that there exists a finite constant c' so that

$$||P_t||_{1,\infty} \leq (c_t)^{|\Lambda|} e^{c' t|\Lambda|}.$$

Finally, let us point out that Nash's inequality is equivalent to the control we obtained for $||P_t||_{1,2}$. Indeed, we have

Exercise 3.9 *Let $\mu \in \mathcal{J}_0(P_t)$ and assume that there exists $\gamma, a, b > 0$ such that*

$$||P_t||_{1,2} \leq (\frac{a}{2et} + b)^{\gamma}.$$

Then, there exists $A, B > 0$ such that

$$||f||_2 \leq (A\mu\Gamma_1(f,f) + B\mu f^2)^{\frac{1}{2}\alpha} ||f||_1^{\beta}$$

with $\alpha = \gamma(\gamma+1)$ and $\beta = 1 - \alpha$.
Hint : Note that for all $t \geq 0$,

$$||f_t||_2^2 \geq ||f||_2^2 - 2t\mu\Gamma_1(f,f)$$

and deduce from the hypothesis that

$$||f||_2^2 \leq 2t\mu\Gamma_1(f,f) + (\frac{a}{2et} + b)^{2\gamma} ||f||_1^2$$

Conclude by optimization over the time parameter t.

In the set up of finite Markov chains, Diaconis and Saloff-Coste [21] gave simple proofs of Nash's inequalities. In [7], generalized log-Nash and Nash inequalities were introduced.

Chapter 4

Logarithmic Sobolev inequalities and Hypercontractivity

We begin this chapter by describing the equivalence theorem of Gross contained in his seminal work [48] which inspired intensive interest and development in logarithmic Sobolev Inequalities. Then, we study the properties of logarithmic Sobolev Inequalities, such as its stability by product (see property 4.4) and its stability by perturbation (see property 4.6 and exercise 4.7). We then compare logarithmic Sobolev inequality with spectral gap inequality, showing in particular that logarithmic Sobolev inequality implies spectral gap inequality (see theorem 4.9). Finally, we discuss Bakry-Emery criterion, giving the examples of probability measures satisfying logarithmic Sobolev inequality.

Let $P_t = e^{t\mathcal{L}}$ be a Markov semi-group and $\mu \in \mathcal{J}_0(P_t)$. Given $p \in]1, \infty[$ and a constant $c \in]0, \infty[$, we define

$$q(t) := q(t, p, c) = 1 + (p-1)e^{\frac{2t}{c}}, \quad t \in I\!R^+.$$

Theorem 4.1 *[Gross' Integration Lemma] Let $c \in]0, \infty[$ and $d \in]0, \infty[$. The following statements are equivalent :*
(i)

$$\mu\Big(f^2 \log \frac{f}{||f||_2}\Big) \leq c\mu\Gamma_1(f,f) + d||f||_2^2 \tag{4.0.1}$$

for all non negative functions for which the right hand side is finite.
(ii) For any $q \in [2, \infty[$,

$$\mu\Big(f^q \log \frac{f}{||f||_q}\Big) \leq \frac{2c}{q}\mu\Gamma_1(f^{\frac{q}{2}}, f^{\frac{q}{2}}) + \frac{2d}{q}||f||_q^q \tag{4.0.2}$$

for all non negative functions for which the right hand side is finite.

(iii) Inequality (4.0.2) for some $q \geq 1$.
(iv) For all $p \in]1, \infty[$ *and* $p \leq q \leq q(t, p, c)$*, we have*

$$||P_t||_{p,q} \leq \exp\Big(2d\Big(\frac{1}{p} - \frac{1}{q}\Big)\Big) \tag{4.0.3}$$

(v) For all $q \in [2, \infty[$, $t \in I\!R^+$,

$$||P_t||_{2,q} \leq \exp\Big(2d\Big(\frac{1}{2} - \frac{1}{q}\Big)\Big) \tag{4.0.4}$$

Properties (i)-(v) imply
(vi) For all $t \in I\!R^+$ *and any non negative function* f *such that* $||f||_1 = 1$ *we have*

$$S_f(t) = \mu f_t \log f_t \leq e^{\frac{-2t}{c}} S_f(0) + 2d(1 - e^{\frac{-2t}{c}}) \tag{4.0.5}$$

for all $t \in I\!R^+$.

Inequality (4.0.1) is called the logarithmic Sobolev (or in short log-Sobolev) inequality.

Constants (c, d), $c \in (0, \infty)$ and $d \in [0, \infty)$, such that (i) is satisfied will be called log-Sobolev coefficients.

Remark 4.2: If additionally

$$\mu\left((\mathcal{L}f)\log f\right) = -4\mu\Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}}) \tag{4.0.6}$$

then (vi) is equivalent to (i)-(v). (4.0.6) is in particular satisfied when Γ_1 satisfies the Leibnitz rule (see 4.13).

Exercise 4.3

- *Show that Sobolev inequality implies logarithmic Sobolev inequality.*

  Hint : write

  $$\mu\Big[f^2 \log \frac{f^2}{||f||_2^2}\Big] = ||f||_2^2 \mu\Big[\frac{f^2}{||f||_2^2} \log \frac{f^2}{||f||_2^2}\Big],$$

  and use Jensen's inequality together with Sobolev inequality .

- *Prove the logarithmic Sobolev inequality for a Bernoulli law* $\mu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$*, with* $(a, b) \in I\!R^2$*, by a direct computation (see [48]). The generalization to* $\mu = p\delta_a + (1-p)\delta_b$ *for* $p \in (0, 1)$ *can be found for instance in [85].*

Note that the theorem shows that the logarithmic Sobolev inequality with $d = 0$ implies that the associated semi-group is hypercontractive. In this situation one shows that the spectral gap inequality is satisfied, see theorem 4.9 below.

Proof : The equivalence between (i), (ii) and (iii) is easily obtained by substitution (For (i)→(ii), we take $f = g^{\frac{q}{2}}$, (ii) can be clearly reduced to (iii) and (iii)→(i) by taking $f = g^{\frac{2}{q}}$).

Let us show that (ii) implies (iv). To do so, we remark that

$$
\begin{aligned}
\partial_t \log ||f_t||_{q(t)} &= -\frac{\partial_t q(t)}{q(t)^2} \log \mu f_t^{q(t)} + \frac{1}{q(t)\mu f_t^{q(t)}} \mu(f_t^{q(t)} \partial_t q(t) \log f_t) \\
&= -\frac{\partial_t q(t)}{q(t)^2} \log \mu f_t^{q(t)} + \frac{1}{q(t)\mu f_t^{q(t)}} \mu((\partial_t q(t)) f_t^{q(t)} \log f_t \\
&\quad + q(t) f_t^{q(t)-1} \mathcal{L} f_t) \qquad (4.0.7) \\
&= \frac{\partial_t q(t)}{q(t)\mu f_t^{q(t)}} \left(\mu(f_t^{q(t)} \log \frac{f_t}{||f_t||_{q(t)}}) + \frac{c q(t)}{2(q(t)-1)} \mu(f_t^{q(t)-1} \mathcal{L} f_t) \right)
\end{aligned}
$$

where in the last line we used that

$$
\partial_t q(t) = \frac{2}{c}(q(t) - 1).
$$

Consider now more closely the last term in the bracket on the right hand side of (4.0.7). By definition, for any $q \geq 1$ and any non negative function g,

$$
\begin{aligned}
\mu(g^{q-1} \mathcal{L} g) &= \lim_{\tau \to 0} \frac{1}{\tau} (g^{q-1}(P_\tau - 1) g) \\
&= -\lim_{\tau \to 0} \frac{1}{2\tau} \int \mu(d\omega) P_\tau(\omega, d\widetilde{\omega})(g^{q-1}(\omega) - g^{q-1}(\widetilde{\omega}))(g(\omega) - g(\widetilde{\omega})) \\
&\leq -\frac{4(q-1)}{q^2} \lim_{\tau \to 0} \frac{1}{2\tau} \int \mu(d\omega) P_\tau(\omega, d\widetilde{\omega})(g^{\frac{q}{2}}(\omega) - g^{\frac{q}{2}}(\widetilde{\omega}))^2 \quad (4.0.8) \\
&= -\frac{4(q-1)}{q^2} \mu \Gamma_1(g^{\frac{q}{2}}, g^{\frac{q}{2}}).
\end{aligned}
$$

In (4.0.8), we used the following inequality satisfied for any non negative (x, y)

$$
(x^{\frac{q}{2}} - y^{\frac{q}{2}})^2 \leq \frac{q^2}{4(q-1)} (x^q - y^q)(x - y).
$$

(4.0.7) and (4.0.8) yield

$$
\begin{aligned}
&\partial_t \log ||f_t||_{q(t)} \\
\leq &\ \frac{\partial_t q(t)}{q(t)\mu f_t^{q(t)}} \left(\mu(f_t^{q(t)} \log \frac{f_t}{||f_t||_{q(t)}}) - 2\frac{c}{q(t)} \mu \Gamma_1(f_t^{\frac{q(t)}{2}}, f_t^{\frac{q(t)}{2}}) \right) \quad (4.0.9)
\end{aligned}
$$

Under hypothesis (ii), we deduce that

$$
\partial_t \log ||f_t||_{q(t)} \leq 2d \frac{\partial_t q(t)}{q^2(t)}
$$

and hence, since $q(0) = p$,

$$
||f_t||_{q(t)} \leq ||f_t||_p \exp\{2d(\frac{1}{p} - \frac{1}{q(t)})\}.
$$

Finally, for any given $q \leq q(t)$, we can find a $t_0 \leq t$ such that $q = q(t_0)$. Using the definition of the semi-group and its contractivity property in any L^p, $p \geq 1$, we obtain

$$\begin{aligned}||f_t||_q = ||f_t||_{q(t_0)} = ||P_{t_0} f_{t-t_0}||_{q(t_0)} &\leq ||f_{t-t_0}||_p \exp\{2d\Big(\frac{1}{p} - \frac{1}{q}\Big)\} \\ &\leq ||f||_p \exp\{2d\Big(\frac{1}{p} - \frac{1}{q}\Big)\}\end{aligned}$$

which extends the property to any $q \leq q(t)$.

To show that (iv) implies (i), we note first that (iv) induces that for any non negative function

$$\phi_f(t) : t \to \exp\{-2d(\frac{1}{p} - \frac{1}{q(t)})\}||f_t||_{q(t)}$$

is decreasing. Indeed, it implies that for all $s, t \geq 0$,

$$\begin{aligned}\phi_f(t+s) &\leq \exp\{-2d\Big(\frac{1}{p} - \frac{1}{q(t+s)}\Big)\}||P_t||_{q(s),q(t+s)}||f||_{q(s)} \\ &\leq \exp\{-2d\Big(\frac{1}{p} - \frac{1}{q(s)}\Big)\}||f||_{q(s)}.\end{aligned}$$

Consequently, $\log \phi_f(t)$ is differentiable and also decreasing. We thus obtain

$$\partial_t \log ||f_t||_{q(t)} \leq \partial_t\{2d\Big(\frac{1}{p} - \frac{1}{q(t)}\Big)\} = 2d\frac{\partial_t q}{q^2}(t).$$

Going back to (4.0.7), we deduce

$$\mu\Big(f_t^q \log \frac{f_t}{||f_t||_q}\Big) + \frac{cq}{2(q-1)}\mu\Big(f_t^{q-1}\mathcal{L}f_t\Big) \leq 2d\frac{\mu f_t^q}{q}(t).$$

Taking $t = 0$ and $p = 2$, we have

$$\mu(f^2 \log \frac{f}{||f_t||_2}) \leq c\mu\Gamma_1(f, f) + \frac{2d}{q}||f||_2^2.$$

Finally, to show that (iv) implies (vi), we differentiate with respect to the time variable the following function

$$\psi_f(t) = e^{\frac{2t}{c}} S_f(t)$$

defined with a non negative f from the domain of the generator. Assuming that $||f_t||_1 = ||f||_1 = 1$, we find

$$\partial_t e^{\frac{2t}{c}} S_f(t) = \frac{2}{c} e^{\frac{2t}{c}} S_f(t) + e^{\frac{2t}{c}} \mu(\mathcal{L}f_t \log f_t). \tag{4.0.10}$$

To estimate the last term, we note that for any function $F \in \mathcal{D}(\mathcal{L})$,

$$\mu[(\mathcal{L}F)\log F] = \lim_{\tau \to 0} \frac{-1}{2\tau} \int \mu(d\omega) P_\tau(\omega, d\widetilde{\omega})(F(\omega) - F(\widetilde{\omega}))(\log F(\omega) - \log F(\widetilde{\omega})).$$

Since for any non negative x, y,

$$(\log x - \log y)(x - y) \geq (x^{\frac{1}{2}} - y^{\frac{1}{2}})^2,$$

gives

$$\mu[(\mathcal{L}F)\log F] \quad \leq \quad -4\mu\Gamma_1(F^{\frac{1}{2}}, F^{\frac{1}{2}}). \tag{4.0.11}$$

We deduce from (4.0.10) and (4.0.11) that

$$\partial_t e^{\frac{2t}{c}} S_f(t) \quad \leq \quad \frac{2}{c} e^{\frac{2t}{c}} \left(S_f(t) - 2c\mu(\Gamma_1(f_t^{\frac{1}{2}}, f_t^{\frac{1}{2}}) \right) \tag{4.0.12}$$

and, if logarithmic Sobolev inequality is satisfied,

$$\partial_t e^{\frac{2t}{c}} S_f(t) \quad \leq \quad \frac{4d}{c} e^{\frac{2t}{c}}. \tag{4.0.13}$$

(vi) follows from the integration with respect to the time variable.

$\diamond$

4.1 Properties of logarithmic Sobolev inequality

We begin by showing that, similarly to the spectral gap inequality, the logarithmic Sobolev inequality has the product property.

Theorem 4.4 *Let $(\mu_i)_{i=1,2}$ be two probability measures on probability spaces satisfying the logarithmic Sobolev inequality with coefficients $(c_i, d_i)_{i=1,2}$ for the infinitesimal generators $(\mathcal{L}_i, i = 1, 2)$. Let $\mathcal{L}$ be the generator on the product space $\Omega_1 \times \Omega_2$ defined as in property 2.5 and Γ_1 its carré du champ. Then, the product probability measure $\mu_1 \otimes \mu_2$ satisfies*

$$\mu_1 \otimes \mu_2 (f^2 \log \frac{f}{||f||_2}) \quad \leq \quad c\mu_1 \otimes \mu_2(\Gamma_1(f, f) + d\mu_1 \otimes \mu_2 f^2 \tag{4.1.14}$$

$$\textit{with} \qquad c \quad = \quad \max(c_1, c_2) \tag{4.1.15}$$

$$d \quad = \quad d_1 + d_2 \tag{4.1.16}$$

for any non negative function f for which the right hand side of (4.1.14) is finite. Consequently, if μ is a probability measure satisfying a logarithmic Sobolev inequality with a coefficient $c < \infty$ and $d = 0$, then the product probability measure $\mu^{\otimes n}$ satisfies the logarithmic Sobolev inequality with the same coefficient c for any integer number n.

Exercise 4.5 (Gross) *Deduce from exercise 4.3 and property 4.4 that the Gaussian law satisfies a logarithmic Sobolev inequality (see [48]).*
Hint : Use the central limit theorem.

Proof : Let f be a bounded measurable function on $\Omega_1 \times \Omega_2$ in the domain of $\mathcal{L}$. Then, using the logarithmic Sobolev inequality for the probability measure μ_1, we obtain

$$\begin{aligned}\mu_1 \quad \otimes \quad & \mu_2(f^2 \log \frac{f^2}{||f||_2^2}) = \mu_2 \left(\mu_1(f^2 \log \frac{f^2}{\mu_1(f^2)}) + \mu_1(f^2) \log \frac{\mu_1(f^2)}{\mu_1 \otimes \mu_2(f^2)} \right) \\ \leq \quad & \mu_2 \left(2c_1 \mu_1 \Gamma_{1,1}(f,f) + 2d_1 \mu_1(f^2) + \mu_1(f^2) \log \frac{\mu_1(f^2)}{\mu_1 \otimes \mu_2(f^2)} \right)\end{aligned}$$

Applying the logarithmic Sobolev inequality to $g = \sqrt{\mu_1(f^2)}$ and Cauchy-Schwarz's inequality, we deduce

$$\begin{aligned}\mu_1 \otimes \mu_2(f^2 \log \frac{f^2}{||f||_2^2}) \quad \leq \quad & 2c_1 \mu_2 \otimes \mu_1 \Gamma_{1,1}(f,f) + 2c_2 \mu_2 \Gamma_{1,2}(g,g) \\ & + 2(d_1 + d_2) \mu_2 \otimes \mu_1(f^2).\end{aligned}$$

We can estimate the last term of the above inequality by noticing that

$$\begin{aligned}\mu_2 \Gamma_{1,2}(g,g) \quad &= \quad \mu_2 \Gamma_{1,2}(\sqrt{\mu_1(f^2)}, \sqrt{\mu_1(f^2)}) \\ &= \quad \lim_{\tau \downarrow 0} \frac{1}{2\tau} \int \mu_2(d\omega) P_{\tau,2}(\omega, d\widetilde{\omega}) \left(\sqrt{\mu_1(f^2)}(\omega) - \sqrt{\mu_1(f^2)}(\widetilde{\omega}) \right)^2 \\ &\leq \quad \lim_{\tau \downarrow 0} \frac{1}{2\tau} \int \mu_2(d\omega) P_{\tau,2}(\omega, d\widetilde{\omega}) d\mu_1(\eta) \left(f(\omega, \eta) - f(\widetilde{\omega}, \eta) \right)^2 \\ &= \quad \mu_1 \otimes \mu_2 \left(\Gamma_{1,2}(f,f) \right)\end{aligned}$$

where the last inequality is due to Cauchy-Schwarz's inequality. We hence have shown that

$$\begin{aligned}& \mu_1 \otimes \mu_2(f^2 \log \frac{f^2}{||f||_2^2}) \\ \leq \quad & 2c_1 \mu_2 \otimes \mu_1 \Gamma_{1,1}(f,f) + 2c_2 \mu_2 \otimes \mu_1(\Gamma_{1,2}(f,f)) + 2(d_1 + d_2) \mu_2 \otimes \mu_1(f^2) \\ \leq \quad & 2 \max(c_1, c_2) \mu_2 \otimes \mu_1 \Gamma_1(f,f) + 2(d_1 + d_2) \mu_2 \otimes \mu_1(f^2)\end{aligned}$$

which was the announced statement.

$\diamond$

Logarithmic Sobolev inequality is also stable under perturbation.

Property 4.6 *Let U be a bounded measurable function on a probability space (Ω, Σ, μ). Assume that μ satisfies the logarithmic Sobolev inequality with coefficients (c, d) for an infinitesimal generator $\mathcal{L}$. Then*

$$d\mu_U := \frac{1}{Z_U} e^{-U} d\mu \qquad with \quad Z_U = \int e^{-U} d\mu$$

satisfies the logarithmic Sobolev inequality for the same generator $\mathcal{L}$ with coefficients bounded above by $(ce^{2osc(U)}, de^{2osc(U)})$ with $osc(U) = \sup U - \inf U$.

Proof : The proof is based on the formula

$$\mu(f^2 \log \frac{f^2}{\mu f^2}) = \inf_{t \geq 0} \mu \left(f^2 \log(\frac{f^2}{t}) - f^2 + t \right)$$

which is easily checked. Moreover,

$$\phi(t, x) = f^2(x) \log(\frac{f^2(x)}{t}) - f^2(x) + t = t \left(z(t, x) \log z(t, x) - z(t, x) + 1 \right),$$

with $z(t, x) = (f^2(x)/t)$, is clearly non negative. Hence, we can use (2.0.4) to deduce

$$\mu_U(f^2 \log \frac{f^2}{\mu_U f^2}) \leq e^{\mathrm{osc}(U)} \mu(f^2 \log \frac{f^2}{\mu f^2}) \leq e^{\mathrm{osc}(U)} \left(c\mu(\Gamma_1(f, f) + d\mu(f^2) \right). \tag{4.1.17}$$

By further use of (2.0.4), we finish the proof of the lemma.

$\diamond$

We recall that for a finite volume Gibbs measure with finite range interaction we could improve the estimate of property 2.6 for the spectral gap inequality . In the next exercise, we propose to generalize this result to logarithmic Sobolev inequality in the discrete case.

Exercise 4.7 *Let ν be the uniform measure on $\{-1, +1\}$ and $\Lambda = [-L, L] \times [-l, l]^{d-1}$ for $(l, L) \in (I\!N^*)^2$, $l \leq L$. Let $U(x, y)$ be a function on $\{-1, +1\}^2$ and set*

$$H_\Lambda^\omega(x) = \sum_{i,j \in \Lambda, |i-j|=1} U(x_i, x_j) + \sum_{i \in \Lambda, j \in \Lambda^c, |i-j|=1} U(x_i, \omega_j).$$

Let

$$d\mu_\Lambda^\omega(x) = \frac{1}{Z_\Lambda^\omega} e^{-\beta H_\Lambda^\omega(x)} d\mu^{\otimes |\Lambda|}(x).$$

Then, with $\bar{\Gamma}_1(f, f) = \sum_{i \in \Lambda} (\partial_i f)^2$ if $\partial_i f(\sigma) = f(\sigma_1, ..\sigma_{i-1}, -\sigma_i, \sigma_{i+1}, ..) - f(\sigma)$, prove that there exist two constants $(c, C) \in (0, \infty)$ depending only on U so that *for every $\omega \in \Omega$, μ_Λ^ω satisfies logarithmic Sobolev inequality for the carré du champ $\bar{\Gamma}_1$ with coefficient bounded by $C|\Lambda|^2 e^{cl^{d-1}}$.*

Hint : Choose a lexicographic order $(i_j, j \in \{1, .., |\Lambda|\})$ of Λ as in property 2.7. Denoting $\Lambda_k = \{i_j, j \leq k\}$ and, for a non negative bounded function f, $f_k(\omega) = \mu_{\Lambda_k}^\omega f$, $f_0 = f$, write

$$\mu_\Lambda^\omega(f \log \frac{f}{\mu_\Lambda^\omega(f)}) = \sum_{k=0}^{|\Lambda|-1} \mu_\Lambda^\omega \mu_{\Lambda_{k+1}} \left(f_k \log \frac{f_k}{f_{k+1}} \right).$$

Using the previous proof and the product property 4.4, show that there exists a finite constant C such that

$$\mu_{\Lambda_{k+1}}^\omega \left(f_k \log \frac{f_k}{f_{k+1}} \right) \leq e^{Cl^{d-1}} \mu_{\Lambda_{k+1}}^\omega (\partial_{i_{k+1}} f_k)^2$$

Remark that there exists a finite constant D such that

$$(\partial_{i_{k+1}} f_k)^2(\omega) \leq 2\mu_{\Lambda_k}^\omega |\partial_{i_{k+1}} f|^2 + D\mu_{\Lambda_k}^\omega (f - \mathbf{E}_{\Lambda_k} f)^2$$

and conclude by property 2.7.

The next exercise links the logarithmic Sobolev inequality with a property of concentration of measure

Exercise 4.8 (see e.g. [61], [1]) *Let (Ω, Σ, μ) be a probability space with a probability measure μ satisfying the logarithmic Sobolev inequality with coefficients $(c, 0)$ for a carré du champ Γ_1 satisfying Leibnitz rule. Show that for any $z \geq 0$, any measurable function f satisfying $||\Gamma_1(f, f)||_\infty < \infty$,*

$$\mu(e^{z(f - \mu(f))}) \leq e^{\frac{c||\Gamma_1(f,f)||_\infty}{4} z^2}.$$

Hint : Show that

$$\partial_z \frac{1}{z} \log \int e^{zf} d\mu \leq \frac{c}{4} ||\Gamma_1(f, f)||_\infty$$

by applying the logarithmic Sobolev inequality to $g = e^{zf}$.

For other connections between logarithmic Sobolev inequalities and concentration of measure phenomenon, we refer the reader to [8] and [61].

4.2 Logarithmic Sobolev and Spectral Gap inequalities

We have the following relationship between logarithmic Sobolev inequality and spectral gap inequality.

Theorem 4.9 *If μ is a probability measure satisfying the logarithmic Sobolev inequality with coefficients $(c, 0)$, then μ satisfies the spectral gap inequality with a coefficient*

$$m \geq \frac{1}{c}.$$

Conversely, if μ satisfies logarithmic Sobolev inequality with coefficients (c, d) as well as a spectral gap inequality with coefficient m, then μ satisfies logarithmic Sobolev inequality with coefficients $(c', 0)$ with

$$c' \leq c + \frac{d+1}{m}.$$

This result can be found in [81], (see also [20] or [106]).

Proof : The first result is obtained as follows. Let $f \in \mathcal{D}(\mathcal{L})$ be a bounded measurable function centered with respect to μ. For ϵ sufficiently small, $1 + \epsilon f$ is a positive bounded function. Hence, applying logarithmic Sobolev inequality, we obtain

$$\mu(1+\epsilon f)^2 \log(1+\epsilon f)^2 \leq 2c\epsilon^2 \mu(\Gamma_1(f,f)) + \mu(1+\epsilon f)^2 \log \mu(1+\epsilon f)^2. \quad (4.2.18)$$

By Taylor's expansion and identifying the first order terms (in ϵ^2), we get

$$\mu(1+\epsilon f)^2 \log(1+\epsilon f)^2 = 3\epsilon^2 \mu f^2 + O(\epsilon^3)$$

and

$$\mu(1+\epsilon f)^2 \log \mu(1+\epsilon f)^2 \approx \epsilon^2 \mu f^2 + O(\epsilon^3)$$

so that

$$\mu f^2 \leq c\mu(\Gamma_1(f,f)).$$

To prove the second point, it is enough to notice that for any measurable function f,

$$\mu(f^2 \log \frac{f^2}{||f||_2^2}) \leq \mu((f-\mu f)^2 \log \frac{|f-\mu f|^2}{||f-\mu f||_2^2}) + 2||f-\mu f||_2^2. \quad (4.2.19)$$

The statement is then obtained by using both the logarithmic Sobolev inequality and the spectral gap inequality in the right hand side. To show (4.2.19), note first that for $\mu f \neq 0$ it is equivalent to the following inequality

$$\mu((1+t\phi)^2 \log \frac{(1+t\phi)^2}{1+t^2}) \leq t^2 \mu\phi^2 \log \phi^2 + 2t^2 \quad (4.2.20)$$

with

$$t = \frac{||f-\mu f||_2}{\mu f} \qquad \phi = \frac{f-\mu f}{||f-\mu f||_2}.$$

We shall use the following observations concerning ϕ ;

$$\mu\phi = 0, \mu(\phi^2) = 1. \quad (4.2.21)$$

Up to replacement ϕ by $-\phi$, we can and do assume that $t \geq 0$. We shall bound the second derivative of

$$F(t) = \mu((1+t\phi)^2 \log \frac{(1+t\phi)^2}{1+t^2}).$$

Using (4.2.21), we find

$$F'(t) = \mu(2\phi(1+t\phi)\log \frac{(1+t\phi)^2}{1+t^2})$$

and then

$$F''(t) = \mu(2\phi^2 \log \frac{(1+t\phi)^2}{1+t^2}) + 4 - \frac{4t}{1+t^2}.$$

Using Jensen's inequality applied to the probability measure $\mu(\phi^2.)$, we remark that

$$\mu\left(\phi^2 \log \frac{(1+t\phi)^2}{\phi^2(1+t^2)}\right) \leq \log \mu\left(\phi^2 \frac{(1+t\phi)^2}{\phi^2(1+t^2)}\right) = 0,$$

and since we assumed $t \geq 0$, we get

$$F''(t) \leq 2\mu\phi^2 \log \phi^2 + 4.$$

Integrating twice with the initial conditions $F'(0) = F(0) = 0$ gives the result.

Note here that F may not be differentiable. Applying the same arguments with

$$F_\delta(t) = \mu([\delta^2 + (1+t\phi)^2] \log \frac{[\delta^2 + (1+t\phi)^2]}{1+t^2+\delta^2})$$

for some non negative real number δ, one can show that

$$F''_\delta(t) \leq 2\mu\phi^2 \log \phi^2 + 4(1+\delta^2)$$

and conclude as above with further use of dominated convergence theorem.

◇

Exercise 4.10 *In the set up of property 4.6 and with additional assumption that the carré du champ $\Gamma_1(f,f)$ is equal to $|\nabla f|^2$, show that if μ_U satisfies a spectral gap inequality with coefficient m, μ_U satisfies a logarithmic Sobolev inequality with coefficients (c',0) with*

$$c' \leq c(1 + \frac{1}{4}||\nabla U||_\infty) + \frac{1}{m}\left((c||\nabla U||_\infty^2 + 1)^2 + d + ||U||_\infty\right).$$

Compare this result with the one obtained by property 4.6.
Hint : Denoting $\rho_U = \frac{e^{-U}}{Z_U}$, and being given a function $f \in \mathcal{D}(\mathcal{L})$ centered with respect to μ, apply logarithmic Sobolev inequality under μ to the function $g = \rho^{\frac{1}{2}} f$ to find

$$\mu_U\left(f^2 \log \frac{\rho_U f^2}{\mu_U(f^2)}\right) \leq \left(c + \frac{1}{4}||\log \rho_U||_\infty\right)\mu_U(\Gamma_1(f,f)) +$$
$$\left(c||\frac{|\nabla \rho_U^{\frac{1}{2}}|^2}{\rho_U}||_\infty + \frac{c}{2}||\nabla \log \rho_U^{\frac{1}{2}}||_\infty + d\right)\mu_U\left(f^2\right).$$

4.3 Bakry-Emery Criterion

We shall assume in this section that the carré du champ Γ_1 of the generator under study satisfies the Leibnitz rule

$$\Gamma_1(f, gh) = \Gamma_1(f, g)h + \Gamma_1(f, h)g.$$

This rule was already encountered previously but we shall now describe it more precisely. First, let us consider the following examples.

Example 4.11 *Consider the Brownian motion on the real line $I\!R$ or on the unit circle $\mathcal{S}^1$, $\mathcal{L} = \Delta$, and show that*

$$\Gamma_1(f, f) = |\nabla f|^2$$

satisfies Leibnitz rule. More generally, consider the operator

$$\mathcal{L} = \sum_{1 \le i,j \le d} a_{ij} \partial_i \partial_j - \sum_{1 \le j \le d} \beta_j \partial_j$$

with a positive definite matrix $A = (a_{ij})_{ij}$ with real valued entries and some vector $(\beta_j)_j$ and show that

$$\Gamma_1(f, f) = \sum_{i,j} a_{ij} (\partial_i f)(\partial_j f)$$

satisfies Leibnitz rule.

On the other hand, Leibnitz rule may not be satisfied if the state space Ω is discrete ; take $\Omega = \{-1, +1\}$ and $\mathcal{L} = \partial$ to be the discrete derivative ; $\partial f(\sigma) = f(-\sigma) - f(\sigma)$. Then, show that

$$\Gamma_1(f, f) = |f(-\sigma) - f(\sigma)|^2$$

does not satisfy Leibnitz rule.

More generally, considering a measurable space (Ω, Σ), a family of probability measures $(\nu^\omega)_{\omega \in \Omega}$ on (Ω, Σ), and the generators $\mathcal{L}f(\omega) = \nu^\omega f - f(\omega)$, show that Leibnitz rule fails.

Necessary and sufficient conditions for a semi-group to have a generator satisfying Leibnitz rule can be found in [58] and [80].

One of the consequence of Leibnitz rule is the following fact.

Lemma 4.12 *Assume that Γ_1 satisfies the Leibnitz rule. Then, for any entire function v on $I\!R$, we have*

$$\Gamma_1(g, v \circ f) = \Gamma_1(g, f) v' \circ f$$

for any functions $(f, g) \in \mathcal{D}(\mathcal{L})$.

Proof : Since for any monomial function x^n, $n \in I\!N$, one easily check

$$\Gamma_1(g, f^n) = n\Gamma_1(g,f)f^{n-1}$$

the result is straightforward for entire functions. $\diamond$
We notice also the following interesting fact.

Property 4.13 *Assume that Γ_1 satisfies the Leibnitz rule. Then, the last condition (vi) of Gross's theorem [48] is equivalent to the others.*

Proof : It is sufficient to show that, for any non negative function f, we have, thanks to the previous lemma,

$$\mu[(\mathcal{L}f)\log f] = -\mu(\Gamma_1(f, \log f)) = -4\mu\Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}}).$$

Plugging this equality in the proof of Gross's theorem [48] yields the result. $\diamond$

Hereafter, we shall consider so called the carré du champ itéré , (or simply Gamma two), given, for f in the domain of $\mathcal{L}$, by

$$\begin{aligned}\Gamma_2(f,f) &= \frac{1}{2}\frac{d}{dt}\left(P_t(\Gamma_1(f,f) - \Gamma_1(P_tf, P_tf)\right)|_{t=0} \\ &= \frac{1}{2}\{\mathcal{L}\Gamma_1(f,f) - 2\Gamma_1(f, \mathcal{L}f)\}.\end{aligned}$$

We shall define the Bakry-Emery condition by

Definition 4.14 *We say that Bakry-Emery's condition (denoted* (BE) *) is satisfied if there exists a positive constant $c > 0$ such that*

$$\Gamma_2(f,f) \geq \frac{1}{c}\Gamma_1(f,f) \tag{4.3.22}$$

for any function f for which $\Gamma_1(f,f)$ and $\Gamma_2(f,f)$ are well defined.

We have the following characterization of Bakry-Emery's criterion (see [106])

Theorem 4.15 *Condition* (BE) *is satisfied iff for any $t \geq 0$,*

$$\Gamma_1(P_tf, P_tf) \leq e^{-\frac{2}{c}t}P_t\Gamma_1(f,f) \tag{4.3.23}$$

for any function f so that $\Gamma_1(f,f)$ and $\Gamma_2(f,f)$ are well defined.

Proof : Let us assume that (4.3.23) is satisfied. Then, for any $t > 0$, we have

$$\begin{aligned}0 &\leq \frac{1}{t}\left(e^{-\frac{2}{c}t}P_t\Gamma_1(f,f) - \Gamma_1(P_tf,P_tf)\right) \\ &= \frac{e^{-\frac{2}{c}t}-1}{t}P_t\Gamma_1(f,f) + \frac{1}{t}\left(P_t\Gamma_1(f,f) - \Gamma_1(P_tf,P_tf)\right) \qquad (4.3.24)\end{aligned}$$

Taking the limit $t \downarrow 0$, we deduce, according to the definition of Γ_2 that

$$0 \le -\frac{2}{c}\Gamma_1(f,f) + 2\Gamma_2(f,f)$$

that is condition (BE).

Conversely, if condition (BE) is satisfied, for any $t \ge 0$ and $s \in [0,t]$, the function

$$F : s \to e^{\frac{2}{c}s} P_{t-s}\Gamma_1(P_s f, P_s f)$$

is decreasing. In fact,

$$\begin{aligned}\frac{d}{ds}F(s) &= e^{\frac{2}{c}s}P_{t-s}\left(\frac{2}{c}\Gamma_1(P_sf,P_sf) - \mathcal{L}\Gamma_1(P_sf,P_sf) + 2\Gamma_1(\mathcal{L}P_sf,P_sf)\right)\\ &= 2e^{\frac{2}{c}s}P_{t-s}\left(\frac{1}{c}\Gamma_1(P_sf,P_sf) - \Gamma_2(P_sf,P_sf)\right) \le 0.\end{aligned}$$

In particular, $F(t) \le F(0)$, that is

$$e^{\frac{2}{c}t}\Gamma_1(P_tf,P_tf) \le P_t\Gamma_1(f,f)$$

that is (4.3.23).

◇

The main application of Bakry-Emery's criterion is the following

Theorem 4.16 *Let $\mathcal{L}$ be the generator of a Markov semi-group $P_t, t \in I\!R^+$ with carré du champ Γ_1 satisfying Leibnitz rule. Let $\mu \in \mathcal{J}_0(P_t)$ so that P_t is weakly ergodic, i.e*

$$\lim_{t\to\infty} P_tf(\omega) = \mu f \qquad \mu - a.s. \tag{4.3.25}$$

for any bounded continuous function f. Then, Bakry-Emery's criterion implies that μ satisfies logarithmic Sobolev inequality.

Proof : Let f be a positive bounded continuous function so that $\mu f = 1$. We set $f_t = P_tf$ and let

$$S_f(t) = \mu(f_t \log f_t).$$

Under the weak ergodic hypothesis, we have

$$\lim_{t\to\infty} S_f(t) = 0.$$

Hence,

$$S_f(0) = -\int_0^\infty dt\frac{d}{dt}S_f(t) = \int_0^\infty dt\mu\Gamma_1(f_t,\log f_t). \tag{4.3.26}$$

Next using the fact that P_t is symmetric together with the Schwartz inequality, we get

$$\mu\Gamma_1(f_t,\log f_t) = \mu\Gamma_1\left(f, P_t(\log f_t)\right) \le \tag{4.3.27}$$

$$\leq \left(\mu\frac{\Gamma_1(f,f)}{f}\right)^{\frac{1}{2}} (\mu f \Gamma_1(P_t \log f_t, P_t \log f_t))^{\frac{1}{2}}$$

Applying to the last term our condition equivalent to (BE) with the function $\log f_t$, we obtain

$$(\mu f \Gamma_1(P_t \log f_t, P_t \log f_t))^{\frac{1}{2}} \leq \left(\mu f e^{-\frac{2}{c}t} P_t \Gamma_1(\log f_t, \log f_t)\right)^{\frac{1}{2}} \tag{4.3.28}$$

$$= e^{-\frac{1}{c}t} \left(\mu f_t \Gamma_1(\log f_t, \log f_t)\right)^{\frac{1}{2}} = e^{-\frac{1}{c}t} \left(\mu \Gamma_1(f_t, \log f_t)\right)^{\frac{1}{2}}$$

where in the last stage we have used symmetry of the semigroup and the Leibnitz rule for Γ_1. The inequalities (4.3.27) and (4.3.28) imply the following bound

$$\mu\Gamma_1(f_t, \log f_t) \leq e^{-\frac{2}{c}t} \mu \frac{\Gamma_1(f,f)}{f} = 4e^{-\frac{2}{c}t} \mu \Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}}) \tag{4.3.29}$$

Using this one arrives at

$$S_f(0) \leq \int_0^\infty 4e^{-\frac{2t}{c}} dt \mu \Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}})) = 2c\mu\Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}})$$

which completes the proof. $\diamond$

One can also show that

Proposition 4.17 *If* (BE) *is satisfied,*

$$P_t(f \log f) \leq 2c(1 - e^{-\frac{2}{c}t}) P_t \Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}}) + P_t f \log P_t f$$

for any function f for which the right hand side is well defined.

The proof of this proposition is very similar to the previous one. It is given by Bakry and Emery, [6], Proposition 5.

Let us give a few examples where (BE) is fulfilled.

Exercise 4.18 • *Let U be a twice continuously differentiable function so that $Z = \int e^{-U(x)} dx$ is well define and finite. Set*

$$\mu(dx) = Z^{-1} e^{-U(x)} dx.$$

Let $\mathcal{L} = \Delta - U'(x).\partial_x$. Show that, if U'' is uniformly bounded below by a positive constant, Bakry-Emery criterion holds.

• *Let μ_G be the Gaussian measure on $\mathbb{R}^n$ with covariance $G = \{G_{ij}\}_{1 \leq i,j \leq n}$. Given $U \in \mathcal{C}^2(\mathbb{R}^n)$ such that*

$$0 < \mu_G(e^{-U}) < \infty,$$

we define

$$\mu_U(f) = \frac{\mu_G(f e^{-U})}{\mu_G(e^{-U})}.$$

Let $\mathcal{L}$ be the Markov generator given by

$$\mathcal{L}f = \Delta f - \sum_i \beta_i \partial_{x_i} f$$

and, with $M = G^{-1}$,

$$\beta_i = \partial_{x_i} U + \sum_j M_{ij} x_j .$$

Then, μ_U is reversible for $\mathcal{L}$. Show that

$$\Gamma_1(f,f) = |\nabla f|^2$$

and

$$\Gamma_2(f,f) = \sum_{i,j} \left(|\partial_{x_i}\partial_{x_j} f|^2 + (\partial_{x_j}\beta_i)\partial_{x_i} f \cdot \partial_{x_j} f \right).$$

Prove that, if we denote $N(x)$ the matrix with entries $\partial_{x_i}\partial_{x_j} U$, and if for all x, the smallest eigenvalue of the symmetric matrix $M + N(x)$ is bounded below by $(1/c)$, then Γ_1 satisfies (BE) with coefficient c. In particular, μ satisfies logarithmic Sobolev inequality with coefficient c for the carré du champ Γ_1.

The interest of Bakry-Emery criterion is also to obtain log-Sobolev inequalities in smooth compact Riemaniann manifolds. We send the reader to the original article [6], as well as to the more recent paper [13], for such applications.

Exercise 4.19 $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. *For a finite subset Λ of $\mathbb{Z}^d$, a constant $m \neq 0$, we denote $\mu_G^{\partial\Lambda}$ the Gaussian measure on $\mathbb{R}^\Lambda$ with covariance*

$$G = (-\Delta_\delta^{\partial\Lambda} + m^2)^{-1}$$

where $\Delta_\delta^{\partial\Lambda}$ is the discrete Laplacian with Dirichlet boundary conditions. In other words, if we denote $x_\Lambda = (x_i)_{i\in\Lambda}$ and $i \approx j$ for $|i-j| = 1$, $\omega \in \Omega$,

$$\mu_G^{\partial\Lambda,\omega}(dx_\Lambda) = \frac{1}{Z_G^{\partial\Lambda}} \exp\{-\frac{1}{2} \sum_{i\in\Lambda, i\approx j} (x_i - x_j)^2 - \frac{m^2}{2} \sum_{i\in\Lambda} x_i^2\} dx_\Lambda$$

with $x|_{\Lambda^c} = \omega|_{\Lambda^c}$. Let V be a bounded below $\mathcal{C}^2(\mathbb{R}^\Lambda)$ function and set

$$U_\Lambda(x_\Lambda) = \sum_{i\in\Lambda} V(x_i).$$

For instance, for $\lambda > 0$, $V(x) = \lambda x^4 + a x^2$. We can then define

$$\mu_U^{\partial\Lambda,\omega}(dx_\Lambda) = \frac{1}{Z_\Lambda^{\partial\Lambda,\omega}} e^{-U_\Lambda} \mu_G^{\partial\Lambda,\omega}(dx_\Lambda).$$

Following the previous example, see that if, for some $c > 0$,

$$-\Delta_\delta^{\partial\Lambda} + m^2 + D(x) \geq \frac{1}{c} I$$

with D the diagonal matrix $D_{ij}(x) = \delta_{ij} V''(x_i)$ and the inequality is understood in the sense of quadratic forms, then $\mu_U^{\partial\Lambda,\omega}$ satisfies the logarithmic Sobolev inequality with coefficient bounded by c. Moreover, if the following limit exists

$$\mu = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_\Lambda^{\partial\Lambda,\omega} \quad ,$$

μ satisfies also logarithmic Sobolev inequality with coefficient bounded by c.

Let us remark that such a measure may not be unique and that then the logarithmic Sobolev inequality is satisfied by any such limit. The proof of the existence of a limit is in general not easy. We note that for any $\Lambda \subset \mathbb{Z}^d$ and $\Lambda_0 \subset \Lambda$, the conditional expectation $\mu_\Lambda^{\partial\Lambda,\omega}$ knowing $\Sigma_{\Lambda_0^c}$, the σ-algebra generated by $\{x_i, i \in \Lambda_0^c\}$, is independent of Λ and is defined by $\delta_\omega(\mu_{\Lambda_0}^{\partial\Lambda_0,\omega}(\cdot))$ (exercise).

One can use this idea to define the infinite volume measure as follows.

Let a family of conditional expectations $(\mathbf{E}_\Lambda^\omega, \omega \in \Omega, \Lambda \subset \mathbb{Z}^d)$ be given so that

(1) $\mathbf{E}_\Lambda^\omega 1 = 1 \qquad \forall \omega \in \Omega, \qquad \forall \Lambda \subset \mathbb{Z}^d$

(2) $\omega \to \mathbf{E}_\Lambda^\omega(f)$ is Σ_{Λ^c} measurable for any bounded measurable function f.

(3)If $\Lambda_1 \subset \Lambda_2$, then $\Sigma_{\Lambda_2^c} \subset \Sigma_{\Lambda_1^c}$ and

$$\mathbf{E}_{\Lambda_2}^\omega \mathbf{E}_{\Lambda_1}^{\cdot} f = \mathbf{E}_{\Lambda_2}^\omega f.$$

The Gibbs measures in infinite volume μ associated with the specification $(\mathbf{E}_\Lambda^\omega, \omega \in \Omega, \Lambda \subset \mathbb{Z}^d)$ are described as the solutions of the equation (DLR)

$$\mu \mathbf{E}_\Lambda^{\cdot} = \mu.$$

This equation may have several solutions.

To be more precise, let us consider the case where $U = 0$. Denote $\mathbf{E}_{\mu_G}^\omega(f|\Sigma_{\Lambda^c})$ the conditional expectation of μ_G with respect to Σ_{Λ^c}. The family of these conditional expectation is called local specification. We can consider the Gibbs measure associated with this local specification. It is not hard to check that

$$\mathbf{E}_{\mu_G}^\omega(f|\Sigma_{\Lambda^c}) = \mu_G^{\partial\Lambda}\left(f(. + \phi_\omega^{\partial\Lambda})\right) \tag{4.3.30}$$

where

$$\begin{cases} G^{-1}\phi_\omega^{\partial\Lambda}(i) = 0 & \forall i \in \Lambda \\ \phi_\omega^{\partial\Lambda}(j) = \omega_j & \forall j \in \Lambda^c \end{cases}$$

Consequently, if ϕ is a global solution of

$$G^{-1}\phi = 0,$$

any probability measure

$$\mu_{G,\phi}(f) = \mu_G(f(. + \phi))$$

have the same conditional expectations. Hence, they define Gibbs measures in infinite volume with the same local specification. All these measures satisfy the logarithmic Sobolev inequality with the same coefficient.

Chapter 5

Logarithmic Sobolev inequalities for spins systems on a lattice

This chapter will be concerned with logarithmic Sobolev inequalities for Gibbs measures. We begin by describing the statistical mechanics context in which we shall work. In particular, we define local specifications and the associated Gibbs measures. We introduce as well some Markov semi-groups, (via their generators), satisfying the detailed balance condition with respect to a Gibbs measure. Then, we present a general strategy to prove logarithmic Sobolev inequalities for a given Gibbs measure with Dirichlet forms of relevant generators. The application of this strategy requires to check four general conditions. In section 5.3 we show that they are satisfied for all one dimensional systems with bounded finite range potential. For "geometric" reasons, this case is slightly easier to handle than the higher dimensional case. In section 5.4, assuming some mixing condition, we show that the requirements of the general stratedy listed in section 5.2 hold for systems with bounded finite range potential in dimension $d \geq 2$. The proof is similar to that of section 5.3, except that one needs an extra argument taking into account the mixing condition (based on the so-called sweeping out relations defined in paragraph 5.4.1).

5.1 Notation and definitions, statistical mechanics

In this chapter, we shall consider random variables, (in applications representing spins or particles and therefore frequently called by these names), with values in a Polish space $\mathbf{M}$. We shall assume that either $\mathbf{M}$ is a finite set, (frequently being simply given as a two point set $\mathbf{M} = \{+1, -1\}$), or a finite dimensional smooth (compact) connected Riemaniann manifold. These two cases will be called discrete and continuous settings respectively. In most of the proofs, we will consider in more detail the discrete case which is usually more complicated to deal with, (mainly because the discrete derivative used there does not satisfy the Leibnitz rule). The spins will be "located" on the lattice $\mathbb{Z}^d$, for some positive integer number $d \in \mathbb{N}^*$. We equipp $\mathbb{Z}^d$ with a distance $d(x,y) = \max_{1\le i\le d} |x_i - y_i|$ for (x,y) in $\mathbb{Z}^d$. The notation $\Lambda \subset\subset \mathbb{Z}^d$ will be used to say that Λ is a finite subset of $\mathbb{Z}^d$. Let $\Omega = \mathbf{M}^{\mathbb{Z}^d}$ be the state space. For $\Lambda \subset\subset \mathbb{Z}^d$, we shall denote by

$$\sigma_\Lambda : \Omega \to \mathbf{M}^\Lambda$$

the projection defined by

$$\begin{array}{rcl} \sigma_\Lambda : \Omega & \to & \mathbf{M}^\Lambda \\ \omega = (\omega_j)_{j\in\mathbb{Z}^d} & \to & \sigma_\Lambda(\omega) = (\omega_j)_{j\in\Lambda} \end{array}$$

In particular, a projection σ_i corresponding to single point set $\{i\}$ is called a spin at site i. For $\Lambda \subset \mathbb{Z}^d$, we shall denote by Σ_Λ the smallest σ-algebra for which all the spins $\{\sigma_i, i \in \Lambda\}$ are Σ_Λ-measurable. We shall say that a function f on Ω is localized in Λ if it is Σ_Λ-measurable. We denote by Λ_f (or alternatively $\Lambda(f)$) the smallest subset of $\mathbb{Z}^d$ for which f is localized in Λ_f. Later on, a function f for which the cardinality of Λ_f is finite will be called a local function.

Let $\phi = (\phi_X)_{X\subset\mathbb{Z}^d}$ be a potential with finite range R, that is a family $(\phi_X)_{X\subset\mathbb{Z}^d}$ of continuous functions on Ω so that for all $X \subset \mathbb{Z}^d$, ϕ_X is localized in X and $\phi_X \equiv 0$ if the diameter of X is greater than R. We shall assume hereafter that

$$||\phi|| = \sup_{i\in\mathbb{Z}} \sum_{X:i\in X} ||\phi_X||_\infty < \infty. \tag{5.1.1}$$

The energy U_Λ in a finite volume $\Lambda \subset\subset \mathbb{Z}^d$ is then well defined by

$$U_\Lambda = \sum_{X\cap\Lambda\neq\emptyset} \phi_X. \tag{5.1.2}$$

Let ν be the uniform measure on $\mathbf{M}$ and let

$$\nu_\Lambda(d\sigma_\Lambda) = \otimes_{i\in\Lambda}\nu(d\sigma_i).$$

We can define a local Gibbs measure in a finite volume Λ and with boundary conditions $\omega \in \Omega$ by

$$\mathbf{E}^\omega_\Lambda f = \frac{\delta_{\sigma_{\Lambda^c}=\omega} \otimes \nu_\Lambda\left(e^{-U_\Lambda} f\right)}{\delta_{\sigma_{\Lambda^c}=\omega} \otimes \nu_\Lambda\left(e^{-U_\Lambda}\right)}. \tag{5.1.3}$$

Frequently it will be convenient to use the following notation

$$\mathbf{E}_\Lambda f(\omega) = \mathbf{E}^\omega_\Lambda f.$$

We shall call local specification the family of local Gibbs measures $\{\mathbf{E}^\omega_\Lambda\}_{\Lambda \subset\subset \mathbb{Z}^d, \omega\in\Omega}$.

By μ, we shall denote a Gibbs measure in infinite volume associated to the local specification $\{\mathbf{E}^\omega_\Lambda\}_{\Lambda \subset\subset \mathbb{Z}^d, \omega\in\Omega}$, that is a solution of the (DLR) equation, (DLR being a shorthand for Dobrushin, Landford and Ruelle), which is given by

$$\mu \mathbf{E}^{\cdot}_\Lambda f = \mu f \tag{5.1.4}$$

for all local bounded measurable function f and all $\Lambda \subset\subset \mathbb{Z}^d$.

For an introduction to the theory of Gibbs measures the reader may like to look at the classical reference [75]. Concerning the uniqueness versus non uniqueness problem (a phase transition phenomenon) the literature is very wide; the reader may consult [90] and [41] for more detailed discussion and further guide to the literature. We note that, except in the last chapter of these notes, we will only be concerned with Gibbs measures for which log-Sobolev inequalities hold for all local Gibbs measures with uniformly bounded coefficients. This entails the uniqueness of the infinite volume Gibbs measure.

The Markov generators under study will be defined as follows. In the continuous setting where we consider a smooth connected Riemaniann manifold $\mathbf{M}$ equipped with the Laplace-Baltrami operator Δ and a gradient ∇, for any $i \in \mathbb{Z}^d$, we set Δ_i and ∇_i to be the corresponding operators acting on the i^{th} variable ω_i. We shall then consider the operator defined on the set of local twice continuously differentiable functions by

$$\mathcal{L}f = \sum_{i\in\mathbb{Z}^d} (\Delta_i - \nabla_i U_\Lambda . \nabla_i) f$$

with Λ any finite subset of $\mathbb{Z}^d$ so that $d(\Lambda^c, \Lambda_f) > R$.

In the discrete setting, for any finite subset $X \subset \mathbb{Z}^d$, we introduce a generator by means of the local Gibbs measures as follows

$$\mathcal{L}^{(X)} = \sum_{j\in\mathbb{Z}^d} \mathcal{L}_{X+j} f$$

with

$$\mathcal{L}_{X+j} f(\omega) = \mathbf{E}^\omega_{X+j} f - f(\omega)$$

for any $j \in \mathbb{Z}^d$ and f any integrable local function.

It is not hard to see that μ is reversible for such operators. Moreover, $\mathcal{L}$ (resp. $\mathcal{L}^{(X)}$) is non positive and with dense domain in $L^2(\mu)$ for any Gibbs measure μ satisfying (5.1.4) for the specification $(\mathbf{E}^{\cdot}_\Lambda, \Lambda \subset\subset \mathbb{Z}^d)$. Consequently, the semi-group $P_t = e^{t\mathcal{L}}$ (or $P_t = e^{t\mathcal{L}^{(X)}}$) is symmetric and leaves μ invariant. We shall call in the sequel the standard Dirichlet form the quantity

$$\mathcal{E}(f,f) = \mu|\nabla f|^2$$

with

$$|\nabla f|^2 = \sum_{j \in \mathbb{Z}^d} |\nabla_j f|^2.$$

where in the discrete setting, we denote $\nabla f = (\partial_{\sigma_j} f)_{j \in \mathbb{Z}^d}$ with

$$\partial_{\sigma_j} f = \int f(\sigma_k, k \neq j, \bar{\sigma}) d\nu(\bar{\sigma}) - f(\sigma).$$

We shall denote, whenever it makes sense,

$$|||f||| = \left(\sum_{i \in \mathbb{Z}^d} ||\nabla_i f||_\infty^2 \right)^{\frac{1}{2}}$$

and say that a function f is of class $\mathcal{C}^1$ iff $|||f|||$ is well defined and finite. $\mathcal{E}(f,f)$ is clearly well defined for any function f of class $\mathcal{C}^1$.

Exercise 5.1 *Show that if Φ is a bounded potential with finite range, for any $X \subset\subset \mathbb{Z}^d$, the operator $\mathcal{L}^{(X)}$ has a quadratic form $\mu\Gamma_1^{(X)}(f,f)$ equivalent to the standard Dirichlet form, that is that there exists two constants c_X and C_X, $0 < c_X \leq C_X < \infty$, so that*

$$c_X \mathcal{E}(f,f) \leq \mu\Gamma_1^{(X)}(f,f) \leq C_X \mathcal{E}(f,f).$$

5.2 Strategy to prove the logarithmic Sobolev inequality

To prove logarithmic Sobolev inequality for the Gibbs measure μ, the idea is to use the local Gibbs measures to define an auxiliary Markov chain on (Ω, Σ) with transition matrix $\mathbf{E}$ satisfying the following conditions

(Ci)

$$\mu \mathbf{E} f = \mu f$$

for any bounded measurable function f.

(Cii) There exists a positive finite constant $\bar{c}$ so that

$$\mu \left(\mathbf{E}(f \log f) - (\mathbf{E} f) \log(\mathbf{E} f) \right) \leq 2\bar{c}\mu |\nabla f^{\frac{1}{2}}|^2.$$

(Ciii) There exists $\lambda \in (0,1)$ so that

$$\mu |\nabla (\mathbf{E} f)^{\frac{1}{2}}|^2 \leq \lambda \mu |\nabla f^{\frac{1}{2}}|^2.$$

(Civ) Denoting for any bounded measurable function f, $(f_n)_{n \in \mathbb{N}}$ the sequence of measurable functions given by $f_0 = f$ and $f_n = \mathbf{E}(f_{n-1})$,

$$\lim_{n\to\infty} f_n = \mu(f) \qquad \mu \; - a.s.$$

Before proving that a construction of such transition matrix $\mathbf{E}$ is possible and giving a sufficient condition for this, we first show that conditions (C) imply that the Gibbs measure μ satisfies a logarithmic Sobolev inequality .

To this end for a fixed non negative function f we consider the sequence (f_n) given by

$$f_0 \;=\; f, \qquad f_n = \mathbf{E} f_{n-1}.$$

We notice that by using (Ci), we have

$$\mu(f\log\frac{f}{\mu f}) = \mu\left(\mathbf{E}(f\log f) - (\mathbf{E}f)\log(\mathbf{E}f)\right) + \mu\left((\mathbf{E}f)\log\frac{(\mathbf{E}f)}{\mu f}\right). \quad (5.2.5)$$

and hence by induction

$$\mu(f\log\frac{f}{\mu f}) = \sum_{n=0}^{N-1}\mu\left(\mathbf{E}\left(f_n\log f_n\right) - (\mathbf{E}f_n)\log(\mathbf{E}f_n)\right) + \mu(f_N\log\frac{f_N}{\mu f}) \quad (5.2.6)$$

On the other hand, by (Cii),

$$\mu\left(\mathbf{E}\left(f_n\log f_n\right) - (\mathbf{E}f_n)\log(\mathbf{E}f_n)\right) \le 2\bar{c}\mu|\nabla f_n^{\frac{1}{2}}|^2. \quad (5.2.7)$$

Applying (Ciii), we obtain by induction that

$$\mu|\nabla f_n^{\frac{1}{2}}|^2 \le \lambda\mu|\nabla f_{n-1}^{\frac{1}{2}}|^2 \le \lambda^n\mu|\nabla f^{\frac{1}{2}}|^2. \quad (5.2.8)$$

Combining (5.2.6), (5.2.7) and (5.2.8), we get for any $N \in I\!N$,

$$\mu(f\log\frac{f}{\mu f}) \le \frac{2\bar{c}}{1-\lambda}\mu|\nabla f^{\frac{1}{2}}|^2 + \mu(f_N\log\frac{f_N}{\mu f}). \quad (5.2.9)$$

Finally, hypothesis (Civ) implies that the last term on the right hand side of the inequality (5.2.9) goes to zero as N goes to infinity. Indeed, the convexity of $x \to x\log x$ together with Jensen's inequality imply that $\mu(f_N\log\frac{f_N}{\mu f}) \ge 0$. Since we also have that for any $\epsilon > 0$, any $f \ge 0$ which is not identically null, $f_N\log\frac{f_N+\epsilon}{\mu f}$ is uniformly bounded, the monotone convergence theorem implies that

$$\limsup_{N\to\infty}\mu(f_N\log\frac{f_N}{\mu f}) \le \limsup_{N\to\infty}\mu(f_N\log\frac{f_N+\epsilon}{\mu f}) = \mu(f)\log(\frac{\mu f+\epsilon}{\mu f})$$

which allows us to conclude by letting ϵ going to zero.

Hence, we proved that μ satisfies a logarithmic Sobolev inequality with coefficient bounded by $c = \frac{\bar{c}}{1-\lambda}$.

5.3 Logarithmic Sobolev inequality in dimension 1 ; an example

Log-Sobolev inequalities on the one dimensional lattice were first studied for discrete spins in [56]. In that paper a bound on logarithmic Sobolev inequality coefficients (logarithmically growing in the dimension of the configuration space bounds) were obtained and used to prove a form of ergodicity of the associated semi-group (although in general not an exponential decay in the uniform norm). A first proof of logarithmic Sobolev inequality on infinite the one dimensional lattice was obtained in [104].

In this section we give a new proof of that result (with improved estimates on logarithmic Sobolev inequality coefficients).

Theorem 5.2 *Assume $d = 1$ and consider the local Gibbs measures on $\Omega = \{-1, 1\}^{\mathbb{Z}}$ constructed with a potential Φ with finite range R as in (5.1.3). Then, the unique Gibbs measure μ on Ω solving the corresponding (5.1.4) satisfies the logarithmic Sobolev inequality .*

Consequently, logarithmic Sobolev inequality is satisfied (with possibly different coefficients) for any quadratic form associated with the generator $\mathcal{L}^{(X)}$ for any finite subset $X \subset\subset \mathbb{Z}$ (see exercise 5.1). The semi-groups $P_t^{(X)} = e^{t\mathcal{L}^{(X)}}$ are thus hypercontractive (and therefore, as we shall see later, uniformly ergodic).

In dimension greater or equal to 2, such a result is obtained in general only under some additional mixing conditions that will be considered in the next section.

5.3.1 Construction of the auxiliary Markov chain

Let

$$\Lambda_0 = [0, 2(L + R)]$$

with R the range of the interaction. L is an integer number the value of which will be properly chosen later. For $k \in \mathbb{Z}$, set

$$\Lambda_k = \Lambda_0 + 2k(L + 2R).$$

With such a choice, we have

$$\mathrm{dist}(\Lambda_k, \Lambda_{k+1}) = 2R. \tag{5.3.10}$$

For $l = 0$ or 1, we shall denote

$$\Gamma_l = \cup_{k\in\mathbb{Z}} \{\Lambda_k + l(L + 2R)\}.$$

In this manner, we have constructed two sets (each composed with disjoint sets at a distance greater or equal to $2R$), the union of which covers the whole lattice.

$$\mathbb{Z} = \Gamma_0 \cup \Gamma_1. \tag{5.3.11}$$

The goal of this construction is to compare the coefficient of the log-Sobolev inequality satisfied by μ with the maximum of those satisfied by $(\mathbf{E}^{\omega}_{\Gamma_l}, l = 0, 1)$

$$\mathbf{E}^{\omega}_{l} = \mathbf{E}^{\omega}_{\Gamma_l} = \otimes_k \mathbf{E}^{\omega}_{\Lambda_k + l(L+2R)}, \qquad l = 0, 1.$$

with properly chosen L (sufficiently large).

The coefficient in the inequality corresponding to $\mathbf{E}^{\omega}_{l}$ can easily be estimated thanks to the product property and the estimates of the log-Sobolev coefficients for local Gibbs measures.

We define a Markov chain on (Ω, Σ) by the transition matrix

$$\mathbf{E}^{\omega} = \mathbf{E}^{\omega}_{1} \mathbf{E}^{\cdot}_{0}$$

By definition $\mathbf{E}$ preserves the unit and positivity. Moreover, $\mathbf{E}$ preserves the set of cylindrical functions.

5.3.2 Checking of conditions (C)

Condition (Ci) is clearly satisfied by $\mathbf{E}$ since, by property of local Gibbs measures,

$$\mu \mathbf{E}^{\cdot}_{1} \mathbf{E}^{\cdot}_{0} f = \mu \mathbf{E}^{\cdot}_{0} f = \mu f.$$

Conditions (Cii), (Ciii) and (Civ) result from conditions (a) and (b) of the following auxiliary lemma.

Lemma 5.3

(a) *For any finite subset Λ of $\mathbb{Z}$, there exist non negative constants $B_1(\Lambda)$ and $B_2(\Lambda)$ so that for any $i \in \mathbb{Z}$, we have*

$$\mu |\nabla_i (\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2 \leq B_1(\Lambda) \mu |\nabla_i f^{\frac{1}{2}}|^2 + B_2(\Lambda) \mu |\nabla_\Lambda f^{\frac{1}{2}}|^2$$

for any non negative function f for which the right hand side is finite. Moreover, for any $l \in \mathbb{N}$,

$$B_1(l) = \sup_{\Lambda : |\Lambda| \leq l} B_1(\Lambda) < \infty, \qquad B_2(l) = \sup_{\Lambda : |\Lambda| \leq l} B_2(\Lambda) < \infty.$$

(b) *There exists $L_0 \in \mathbb{N}$ and a constant $\bar{\lambda} \in (0, 1)$, satisfying*

$$\bar{\lambda} \max\{R B_1(l), R^2 B_2(l)\} < 1$$

for $l \equiv 2(L + R)$, with $L \geq L_0$, such that for any finite $\Lambda \subset\subset \mathbb{Z}$ of size l and any $\widetilde{\Lambda} \subset \Lambda$ for which dist$(\widetilde{\Lambda}, \Lambda^c) \geq L - R$, for any $i \in \mathbb{Z}$, we have

$$\mu |\nabla_i (\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2 \leq \bar{\lambda} \mu |\nabla_\Lambda f^{\frac{1}{2}}|^2$$

for any differentiable $\Sigma_{\Lambda^c \cup \widetilde{\Lambda}}$-measurable function $f \geq 0$

We shall first show that

Proof of (Cii) assuming lemma 5.3 (a)

Remark that

$$\mu[\mathbf{E}(f\log f)-(\mathbf{E}f)\log(\mathbf{E}f)] \quad = \quad \mu[\mathbf{E}_0(f\log\frac{f}{\mathbf{E}_0 f})]+\mu[\mathbf{E}_0(f)\log\frac{\mathbf{E}_0(f)}{\mathbf{E}_1(\mathbf{E}_0 f)}] \tag{5.3.12}$$

Since, for any $\omega\in\Omega$, $\mathbf{E}_0^\omega$ (resp. $\mathbf{E}_1^\omega$) is a product measure, the product property of logarithmic Sobolev inequality , (theorem 4.4), shows that $\mathbf{E}_0^\omega$ (resp. $\mathbf{E}_1^\omega$) satisfies a logarithmic Sobolev inequality with coefficient bounded above by

$$c_0=\sup_{\omega\in\Omega}\sup_{\Lambda_k\in\Gamma_0\cup\Gamma_1} c(\mathbf{E}^\omega_{\Lambda_k}).$$

In particular,

$$\mathbf{E}_0(f\log\frac{f}{\mathbf{E}_0 f}) \quad \leq \quad 2c_0\mathbf{E}_0|\nabla_{\Gamma_0} f^{\frac{1}{2}}|^2 \tag{5.3.13}$$

and

$$\mathbf{E}_1\left[\mathbf{E}_0(f)\log\frac{\mathbf{E}_0(f)}{\mathbf{E}_1(\mathbf{E}_0 f)}\right] \quad \leq \quad 2c_0\mathbf{E}_1|\nabla_{\Gamma_1}(\mathbf{E}_0 f)^{\frac{1}{2}}|^2. \tag{5.3.14}$$

Moreover, by definition of $\mathbf{E}_0$, we have

$$|\nabla_{\Gamma_1}(\mathbf{E}_0 f)^{\frac{1}{2}}|^2 \quad = \quad \sum_{i\in\Gamma_1}|\nabla_i(\mathbf{E}_0 f)^{\frac{1}{2}}|^2=\sum_{i\in\Gamma_1\backslash\Gamma_0}|\nabla_i(\mathbf{E}_0 f)^{\frac{1}{2}}|^2 \tag{5.3.15}$$

Since the range of the interaction is finite, it is not hard to see that for any $i\in\Gamma_1\backslash\Gamma_0$, there exists a subset $\Lambda^{(i)}$ of Γ_0 with length larger or equal to $l=2(L+R)$ so that $|\nabla_i\mathbf{E}_0 f|=|\nabla_i\mathbf{E}_0\mathbf{E}_{\Lambda^{(i)}}f|\leq\mathbf{E}_0|\nabla_i\mathbf{E}_{\Lambda^{(i)}}f|$. Then, we have

$$\mu|\nabla_i(\mathbf{E}_0 f)^{\frac{1}{2}}|^2\leq\mu|\nabla_i(\mathbf{E}_{\Lambda^{(i)}}f)^{\frac{1}{2}}|^2. \tag{5.3.16}$$

Assuming (a) of the lemma and setting for short $B_1=B_1(l)$ and $B_2=B_2(l)$, we deduce from (5.3.14) the following bound

$$\begin{aligned}
&\mu\left[\mathbf{E}_0(f)\log\frac{\mathbf{E}_0(f)}{\mathbf{E}_1(\mathbf{E}_0 f)})\right]\\
\leq\quad & 2c_0\sum_{i\in\Gamma_1\backslash\Gamma_0}\left(B_1\mu|\nabla_i(f)^{\frac{1}{2}}|^2+B_2\mu|\nabla_{\Lambda^{(i)}}(f)^{\frac{1}{2}}|^2\right)\\
=\quad & 2c_0B_1\sum_{i\in\Gamma_1\backslash\Gamma_0}\mu|\nabla_i(f)^{\frac{1}{2}}|^2+2c_0RB_2\sum_{i\in\Gamma_0}\mu|\nabla_i(f)^{\frac{1}{2}}|^2. \qquad (5.3.17)
\end{aligned}$$

Plugging (5.3.13) and (5.3.17) in (5.3.12), we conclude that

$$\begin{aligned}\mu[\mathbf{E}(f\log f)-(\mathbf{E}f)\log(\mathbf{E}f)] \quad &\leq \quad 2c_0B_1\sum_{i\in\Gamma_1\backslash\Gamma_0}\mu|\nabla_i(f)^{\frac{1}{2}}|^2\\ &\quad +2c_0(1+RB_2)\sum_{i\in\Gamma_0}\mu|\nabla_i(f)^{\frac{1}{2}}|^2\end{aligned}\tag{5.3.18}$$

which gives (Cii) with $\bar{c}=c_0\max\{B_1,1+RB_2\}$.

Proof of (Ciii) assuming lemma 5.3 (a) and (b)

Let us notice first that $\mathbf{E}_0f$ is $\Sigma_{\Gamma_1\backslash\Gamma_0}$ measurable and that $\mathbf{E}f=\mathbf{E}_1(\mathbf{E}_0f)$ is $\Sigma_{\Gamma_0\backslash\Gamma_1}$ measurable. By our choice of Γ_0 and Γ_1,

$$\text{dist}(\Gamma_1\backslash\Gamma_0,\Gamma_0\backslash\Gamma_1)=L$$

and

$$\mu|\nabla_{\Gamma_0}(\mathbf{E}f)^{\frac{1}{2}}|^2=\sum_{i\in\Gamma_0\backslash\Gamma_1}\mu|\nabla_i(\mathbf{E}_1\mathbf{E}_0f)^{\frac{1}{2}}|^2.\tag{5.3.19}$$

For any $i\in\Gamma_0\backslash\Gamma_1$, following the arguments of (5.3.16) and denoting by $\Lambda^{(i)}$ the corresponding subset of Γ_1 with diameter $l=2(L+R)$, we deduce that

$$\mu|\nabla_i(\mathbf{E}_1\mathbf{E}_0f)^{\frac{1}{2}}|^2\leq\mu|\nabla_i(\mathbf{E}_{\bar{\Lambda}^{(i)}}\mathbf{E}_0f)^{\frac{1}{2}}|^2.\tag{5.3.20}$$

Since $\Lambda^{(i)}$ is at a distance smaller or equal to R of $i\in\Gamma_0\backslash\Gamma_1$, the function $\mathbf{E}_{\Gamma_0}f$ is localized in a subset of $\Gamma_1\backslash\Gamma_0$ at distance greater or equal to L of i. Thus, if $L\geq L_0$ for suitable $L_0\geq 0$, we can apply lemma (b) to obtain

$$\mu|\nabla_i(\mathbf{E}_1\mathbf{E}_0f)^{\frac{1}{2}}|^2\leq\bar{\lambda}\mu|\nabla_{\Lambda^{(i)}}(\mathbf{E}_0f)^{\frac{1}{2}}|^2.\tag{5.3.21}$$

Introducing for any $j\in\Lambda^{(i)}$ the sets $\Lambda^{(j)}\subset\Gamma_0$, (in a similar way as $\Lambda^{(i)}$), we have

$$\mu|\nabla_j(\mathbf{E}_0f)^{\frac{1}{2}}|^2\leq\mu|\nabla_j(\mathbf{E}_{\Lambda^{(j)}}f)^{\frac{1}{2}}|^2.$$

We deduce from (5.3.21) by lemma (a) that

$$\mu|\nabla_i(\mathbf{E}_1\mathbf{E}_0f)^{\frac{1}{2}}|^2\leq\bar{\lambda}B_1\sum_{j\in\Lambda^{(i)}}\mu|\nabla_j(f)^{\frac{1}{2}}|^2+\bar{\lambda}RB_2\sum_{j\in\Lambda^{(i)}}\mu|\nabla_{\Lambda^{(j)}}(f)^{\frac{1}{2}}|^2.\tag{5.3.22}$$

By (5.3.20) and (5.3.22), and since the diameters of $\Lambda^{(j)}$ and $\Lambda^{(i)}$ are equal to $l=2(L+R)$, we conclude that

$$\mu|\nabla_0(\mathbf{E}f)^{\frac{1}{2}}|^2\leq\bar{\lambda}RB_1\sum_{i\in\Gamma_1\backslash\Gamma_0}\mu|\nabla_i(f)^{\frac{1}{2}}|^2+\bar{\lambda}R^2B_2\sum_{i\in\Gamma_0}\mu|\nabla_i(f)^{\frac{1}{2}}|^2.\tag{5.3.23}$$

Hence, (Ciii) holds with

$$\lambda\leq\bar{\lambda}R\max\{B_1,RB_2\}.$$

Proof of (Civ) assuming lemma 5.3 (a) and (b)

According to theorem 4.9, $\mathbf{E}_0$ and $\mathbf{E}_1$ satisfy a spectral gap inequality with coefficient bounded below by $(1/c_0)$ so that

$$\begin{aligned} \mu(f-\mathbf{E}f)^2 &\le 2\mu\mathbf{E}_{\Gamma_1}(\mathbf{E}_{\Gamma_0}f-\mathbf{E}_{\Gamma_1}\mathbf{E}_{\Gamma_0}f)^2+2\mu\mathbf{E}_{\Gamma_0}(f-\mathbf{E}_{\Gamma_0}f)^2 \\ &\le 2c_0\left(\mu(|\nabla_{\Gamma_1}\mathbf{E}_{\Gamma_0}f|^2)+|\nabla_{\Gamma_0}f|^2\right)\le K\mu|\nabla f|^2 \end{aligned} \tag{5.3.24}$$

with the constant K obtained from lemma 5.3 (a). Now, if $f_0=f$ and $f_{n+1}=\mathbf{E}f_n$, we deduce from (Ciii) that

$$\mu|\nabla f_n^{\frac12}|^2\le\lambda\mu|\nabla f_{n-1}^{\frac12}|^2\le\lambda^n\mu|\nabla f^{\frac12}|^2$$

which converges towards zero as n goes to infinity. Hence, (5.3.24) implies that the sequences $(f_n-\mu f_n)_{n\in\mathbb{N}}$ and $(|\nabla f_n^{\frac12}|)_{n\in\mathbb{N}}$ converges μ-almost surely by the Borel-Cantelli lemma. The limit of $f_n-\mu(f_n)=f_n-\mu(f)$ is therefore constant, and hence identically zero.

Proof of Lemma 5.3(a)

Let us recall first that, if ν is the uniform Bernoulli law on $\{-1,1\}$,

$$\nabla_jF = \nu_jF-F=\frac12(T_jF-F)$$

with $T_jF:=F(\omega^{(j)})$ where

$$\omega^{(j)}:=\begin{cases}-\omega_k & \text{for } k=j,\\ +\omega_k & \text{otherwise.}\end{cases}$$

We have the following discrete analogue of Leibnitz rule

$$\nabla_j(FG)=F\nabla_jG+(\nabla_jF)T_jG. \tag{5.3.25}$$

In particular,

$$\nabla_j(F^2)=2A_j(F)\nabla_jF \tag{5.3.26}$$

with

$$A_j(F):=\frac12(F+T_jF).$$

Consequently, to estimate $\nabla_j(\mathbf{E}_\Lambda f)^{\frac12}$, we have to find a bound of the form

$$|\nabla_j(\mathbf{E}_\Lambda f)|\le 2A_j\left((\mathbf{E}_\Lambda f)^{\frac12}\right)\times \text{ desired terms.}$$

Let us recall that

$$\mathbf{E}_\Lambda f=\nu_\Lambda(\rho_\Lambda f) \text{ with } \rho_\Lambda=\frac{e^{-U_\Lambda}}{\nu_\Lambda(e^{-U_\Lambda})}.$$

Hence, we have that for any $j \in \Lambda^c$,

$$\begin{aligned}
\nabla_j \mathbf{E}_\Lambda f &= \nu_\Lambda\left[\nabla_j \rho_\Lambda f\right] \\
&= \nu_\Lambda\left[(\nabla_j f) T_j \rho_\Lambda + f \nabla_j \rho_\Lambda\right] \\
&= (T_j \mathbf{E}_\Lambda)[\nabla_j f] + \mathbf{E}_\Lambda[f(\rho_\Lambda^{-1} \nabla_j \rho_\Lambda)]
\end{aligned} \tag{5.3.27}$$

In the first term of (5.3.27), one of the boundary conditions is inverted. To estimate this term, we use the discrete Leibnitz rule $\nabla_j(f) = 2A_j(f^{\frac{1}{2}})\nabla_j f^{\frac{1}{2}}$ to obtain by Cauchy-Schwarz 's inequality

$$|(T_j \mathbf{E}_\Lambda)(\nabla_j f)| \leq 2\left((T_j \mathbf{E}_\Lambda)(A_j(f^{\frac{1}{2}}))^2\right)^{\frac{1}{2}}\left((T_j \mathbf{E}_\Lambda)(|\nabla_j f^{\frac{1}{2}}|^2)\right)^{\frac{1}{2}}. \tag{5.3.28}$$

We note that

$$\rho_\Lambda^{-1} T_j \rho_\Lambda = \frac{e^{-2\nabla_j U_\Lambda}}{\mathbf{E}_\Lambda[e^{-2\nabla_j U_\Lambda}]} \leq e^{4||\nabla_j U_\Lambda||_\infty}, \tag{5.3.29}$$

for any non negative measurable function F

$$(T_j \mathbf{E}_\Lambda)(F) \leq e^{4||\nabla_j U_\Lambda||_\infty} \mathbf{E}_\Lambda F.$$

Thus, we can bound the first term on the right hand side of (5.3.28) by

$$\begin{aligned}
2\left((T_j \mathbf{E}_\Lambda)(A_j(f^{\frac{1}{2}})^2)\right)^{\frac{1}{2}} &\leq ((T_j \mathbf{E}_\Lambda)(f))^{\frac{1}{2}} + ((T_j \mathbf{E}_\Lambda)(T_j f))^{\frac{1}{2}} \\
&\leq 2e^{2||\nabla_j U_\Lambda||_\infty} A_j[(\mathbf{E}_\Lambda f)^{\frac{1}{2}}]
\end{aligned} \tag{5.3.30}$$

This last inequality together with (5.3.28) yields

$$|(T_j \mathbf{E}_\Lambda)(\nabla_j f)| \leq 2e^{4||\nabla_j U_\Lambda||_\infty} A_j[(\mathbf{E}_\Lambda f)^{\frac{1}{2}}]\left(\mathbf{E}_\Lambda |\nabla_j f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}. \tag{5.3.31}$$

To estimate the second term in the right hand side of (5.3.27), we first notice that $\mathbf{E}_\Lambda(\rho_\Lambda^{-1}\nabla_j \rho_\Lambda) = 0$ and henceforth

$$\mathbf{E}_\Lambda\left(f\rho_\Lambda^{-1}\nabla_j\rho_\Lambda\right) = \mathbf{E}_\Lambda\left[(f - \mathbf{E}_\Lambda f)\rho_\Lambda^{-1}\nabla_j\rho_\Lambda\right] \tag{5.3.32}$$
$$= \frac{1}{2}\int\int (f(\sigma) - f(\widetilde{\sigma}))(\rho_\Lambda^{-1}\nabla_j\rho_\Lambda(\sigma) - \rho_\Lambda^{-1}\nabla_j\rho_\Lambda(\widetilde{\sigma}))d\mathbf{E}_\Lambda(\sigma)d\mathbf{E}_\Lambda(\widetilde{\sigma}).$$

We immediately obtain the following bound

$$|\mathbf{E}_\Lambda(f\rho_\Lambda^{-1}\nabla_j\rho_\Lambda)| \leq \tag{5.3.33}$$
$$\leq \sup_{\sigma,\widetilde{\sigma}} |\rho_\Lambda^{-1}\nabla_j\rho_\Lambda(\sigma) - \rho_\Lambda^{-1}\nabla_j\rho_\Lambda(\widetilde{\sigma})|\frac{1}{2}\int\int |f(\sigma) - f(\widetilde{\sigma})|d\mathbf{E}_\Lambda(\sigma)d\mathbf{E}_\Lambda(\widetilde{\sigma}).$$

Moreover,

$$\begin{aligned}\sup_{\sigma,\widetilde{\sigma}} |\rho_\Lambda^{-1}\nabla_j\rho_\Lambda(\sigma) - \rho_\Lambda^{-1}\nabla_j\rho_\Lambda(\widetilde{\sigma})| &= \frac{1}{2}\sup_{\sigma,\widetilde{\sigma}}\left|\frac{e^{-2\nabla_j U_\Lambda(\sigma)}}{\mathbf{E}_\Lambda[e^{-2\nabla_j U_\Lambda(\sigma)}]} - \frac{e^{-2\nabla_j U_\Lambda(\widetilde{\sigma})}}{\mathbf{E}_\Lambda[e^{-2\nabla_j U_\Lambda(\widetilde{\sigma})}]}\right| \\ &\leq e^{4||\nabla_j U_\Lambda||_\infty}. \qquad (5.3.34)\end{aligned}$$

Furthermore Cauchy-Schwarz's inequality shows that for any $f \geq 0$

$$\begin{aligned}&\left(\int\int |f(\sigma) - f(\widetilde{\sigma})| d\mathbf{E}_\Lambda(\sigma) d\mathbf{E}_\Lambda(\widetilde{\sigma})\right)^2 \\ \leq\ & \mathbf{E}_\Lambda^{\otimes 2}[(\sqrt{f(\sigma)} + \sqrt{f(\widetilde{\sigma})})^2]\mathbf{E}_\Lambda^{\otimes 2}[(\sqrt{f(\sigma)} - \sqrt{f(\widetilde{\sigma})})^2] \\ \leq\ & 4\mathbf{E}_\Lambda[f]\mathbf{E}_\Lambda[(\sqrt{f} - \mathbf{E}_\Lambda\sqrt{f})^2] \\ \leq\ & 16(A_j\sqrt{\mathbf{E}_\Lambda[f]})^2\mathbf{E}_\Lambda[(\sqrt{f} - \mathbf{E}_\Lambda\sqrt{f})^2] \qquad (5.3.35)\end{aligned}$$

where the last inequality is trivial. Finally, we saw in property 2.7 that there exists a finite constant C so that for any $\Lambda \subset\subset \mathbb{Z}$,

$$\mathbf{E}_\Lambda[(\sqrt{f(\sigma)} - \mathbf{E}_\Lambda\sqrt{f})^2] \leq C|\Lambda|\mathbf{E}_\Lambda[|\nabla_\Lambda f^{\frac{1}{2}}|^2]. \qquad (5.3.36)$$

¿From (5.3.33)-(5.3.36), we deduce

$$\begin{aligned}&|\mathbf{E}_\Lambda(f\rho_\Lambda^{-1}\nabla_j\rho_\Lambda)| \leq \\ &\leq 2\sqrt{C|\Lambda|}e^{4||\nabla_j U_\Lambda||}(A_j\mathbf{E}_\Lambda[f]^{\frac{1}{2}})\mathbf{E}_\Lambda[|\nabla_\Lambda f^{\frac{1}{2}}|^2]^{\frac{1}{2}} \qquad (5.3.37)\end{aligned}$$

Plugging this estimate into (5.3.27) and using (5.3.31), we obtain

$$\begin{aligned}&|\nabla_j(\mathbf{E}_\Lambda f)^{\frac{1}{2}}| = \frac{\nabla_j\mathbf{E}_\Lambda f}{2A_j(\mathbf{E}_\Lambda f)^{\frac{1}{2}}} \\ \leq\ & e^{4||\nabla_j U_\Lambda||}\left(\mathbf{E}_\Lambda|\nabla_j f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} + \sqrt{C|\Lambda|}e^{4||\nabla_j U_\Lambda||}\left(\mathbf{E}_\Lambda|\nabla_\Lambda f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} (5.3.38)\end{aligned}$$

This ends the proof of lemma 5.3 (a) with the constants

$$B_1(\Lambda) = 2\sup_j e^{8||\nabla_j\phi||} \qquad B_2(\Lambda) = 2C|\Lambda|\sup_j e^{8||\nabla_j\phi||}.$$

$\diamond$

Proof of Lemma 5.3 (b)

The proof of this second property relies on several facts already developed in the course of the preceding proof. In fact, going back to equation (5.3.27) of the previous proof but with f being $\Sigma_{\widetilde{\Lambda}}$-measurable with a set $\widetilde{\Lambda} \subset \Lambda$ such that

$$\mathrm{dist}(\widetilde{\Lambda}, \Lambda^c) \geq L > R,$$

we find out that the first term on the right hand side of (5.3.27) is equal to zero. Hence, we need only to improve the estimate of the second term.

To this end, let us remark that, if R is the range of the interaction, for j at positive distance smaller than R from Λ, $\rho_\Lambda^{-1}\nabla_j\rho_\Lambda$ is Σ_{Λ_1}-measurable for

$$\Lambda_1 = \{i; dist(i, \partial\Lambda) \leq R\}.$$

Thus, we can replace the estimates (5.3.33)-(5.3.35) by the following bound

$$\begin{aligned} |\mathbf{E}_\Lambda[f\rho_\Lambda^{-1}\nabla_j\rho_\Lambda] \quad | &= |\mathbf{E}_\Lambda[(f - \mathbf{E}_\Lambda f)\mathbf{E}_{\Lambda\setminus\widetilde{\Lambda}}\rho_\Lambda^{-1}\nabla_j\rho_\Lambda]| \\ &\leq 4 \quad \left(A_j(\mathbf{E}_\Lambda f)^{\frac{1}{2}}\right) \mathrm{Var}_\Lambda \left(\mathbf{E}_{\Lambda\setminus\widetilde{\Lambda}}\rho_\Lambda^{-1}\nabla_j\rho_\Lambda\right) \left(\mathbf{E}_\Lambda(\sqrt{f} - \mathbf{E}_\Lambda\sqrt{f})^2\right)^{\frac{1}{2}} \end{aligned}$$

with the notation $\mathrm{Var}_X G = \sup_{\omega_{X^c}=\widetilde{\omega}_{X^c}} |G(\omega) - G(\widetilde{\omega})|$.

Moreover, if f is localized in $\widetilde{\Lambda}$, denoting by $\mathbf{E}_{\Lambda,\widetilde{\Lambda}}$ the restriction of $\mathbf{E}_\Lambda$ to $\Sigma_{\widetilde{\Lambda}}$, we have

$$\mathbf{E}_\Lambda(f^{\frac{1}{2}} - \mathbf{E}_\Lambda f^{\frac{1}{2}})^2 = \mathbf{E}_{\Lambda,\widetilde{\Lambda}}(f^{\frac{1}{2}} - \mathbf{E}_{\Lambda,\widetilde{\Lambda}} f^{\frac{1}{2}})^2 \leq m(\widetilde{\Lambda})^{-1}\mathbf{E}_\Lambda|\nabla_{\widetilde{\Lambda}} f^{\frac{1}{2}}|^2 \tag{5.3.39}$$

with $m(\widetilde{\Lambda})^{-1} \leq C|\widetilde{\Lambda}|$, according to property 2.7. Hence, we obtain that

$$|\nabla_j(\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2 \leq \bar{\lambda}\mathbf{E}_\Lambda|\nabla_\Lambda f^{\frac{1}{2}}|^2 \tag{5.3.40}$$

where

$$\bar{\lambda} = \frac{1}{8}\left(\mathrm{Var}_{\widetilde{\Lambda}}(\mathbf{E}_{\Lambda\setminus\widetilde{\Lambda}}(\rho_\Lambda^{-1}\nabla_j\rho_\Lambda))\right)^2 \cdot m(\widetilde{\Lambda})^{-1}.$$

In dimension $d = 1$, we have the following estimate

$$\mathrm{Var}_{\widetilde{\Lambda}}(\mathbf{E}_{\Lambda\setminus\widetilde{\Lambda}}(\rho_\Lambda^{-1}\nabla_j\rho_\Lambda)) \leq e^{-M(d(\widetilde{\Lambda},\partial\Lambda)-R)}C(\Phi), \tag{5.3.41}$$

for a finite constant $C(\Phi)$ which only depends on the potential Φ. This result is standard (see [41] and [84]) and is given as an exercise below.

As a consequence, since $B_1(l)$ and $B_2(l)$ grow at most polynomially with $l = 2(L+R)$ according to the previous estimates, we can choose L large enough so that

$$\frac{1}{8}\max\{RB_1(l), R^2B_2(l)\}\sup\left\{\left(\mathrm{Var}_{\widetilde{\Lambda}}(\mathbf{E}_{\Lambda\setminus\widetilde{\Lambda}}(\rho_\Lambda^{-1}\nabla_j\rho_\Lambda))\right)^2 \cdot m(\widetilde{\Lambda})^{-1}\right\} < 1$$

where the supremum runs over all the sets $(\Lambda, \widetilde{\Lambda})$ such that $\widetilde{\Lambda} \subset \Lambda$, $|\Lambda| \leq 2(L+R)$ and $d(\widetilde{\Lambda}, \Lambda^c) \geq L - R$. The proof of the second point of lemma 5.3 is thus complete. ◇

Exercise 5.4 *Proof of estimate (5.3.41). Hints : Assume that f is localized in $\Lambda_0 \equiv [a_0, b_0]$ and let $\Lambda \equiv [a, b]$ be so that $\Lambda_0 \subset \Lambda$. Note that, by anti-symmetry properties, if $\Lambda \cap \Lambda(f) = \emptyset$,*

$$\mathbf{E}_\Lambda^\omega f - \mathbf{E}_\Lambda^{\widetilde{\omega}} f = \int \mathbf{E}_\Lambda^\omega(dx) \otimes \mathbf{E}_\Lambda^{\widetilde{\omega}}(dy)(f(x) - f(y))$$

$$= \int \mathbf{E}_\Lambda^\omega(dx) \otimes \mathbf{E}_\Lambda^{\widetilde{\omega}}(dy) \left(G_\Lambda^{\omega,\widetilde{\omega}}(x,y)(f(x) - f(y)) \right)$$

with

$$G_\Lambda^{\omega,\widetilde{\omega}}(x,y) \equiv \frac{e^{-W_\Lambda(x_\Lambda \bullet \omega_{\Lambda^c}) - W_\Lambda(y_\Lambda \bullet \widetilde{\omega}_{\Lambda^c})} - e^{-W_\Lambda(y_\Lambda \bullet \omega_{\Lambda^c}) - W_\Lambda(x_\Lambda \bullet \widetilde{\omega}_{\Lambda^c})}}{e^{-W_\Lambda(x_\Lambda \bullet \omega_{\Lambda^c}) - W_\Lambda(y_\Lambda \bullet \widetilde{\omega}_{\Lambda^c})} + e^{-W_\Lambda(y_\Lambda \bullet \omega_{\Lambda^c}) - W_\Lambda(x_\Lambda \bullet \widetilde{\omega}_{\Lambda^c})}}$$

where $W_\Lambda = \sum_{X \cap \Lambda \neq \emptyset, X \cap \Lambda^c \neq \emptyset} \Phi_X$*,*

Consider an increasing sequence of subsets $\Lambda_l \equiv [a, b_l]$*,* $l = 1, \cdots, k$ *so that* $b_{l+1} = b_l + R + 1 < b_0$*. Denote* $f_l \equiv \mathbf{E}_{\Lambda_l} f$*,* $G_l = G_{\Lambda_l}$ *and show by induction that*

$$\mathbf{E}_{\Lambda_k}^\omega f - \mathbf{E}_{\Lambda_k}^{\widetilde{\omega}} f = \mathbf{E}_{\Lambda_k}^\omega \otimes \mathbf{E}_{\Lambda_k}^{\widetilde{\omega}} \left(\left(\prod_{l=1}^k G_l \right) \cdot (f - \widetilde{f}) \right)$$

to deduce

$$||\mathbf{E}_{\Lambda_k}^\omega(f) - \mathbf{E}_{\Lambda_k}^{\widetilde{\omega}}(f)||_\infty \leq \prod_{l=1}^k ||G_l||_\infty \mathrm{Var}(f).$$

Show that when the state space is finite and the potential Φ *uniformly bounded,* $\sup_l ||G_l||_\infty \leq e^{-MR}$ *for a constant* $M > 0$*.*

5.4 Logarithmic Sobolev inequalities in dimension $d \geq 2$

In dimension $d \geq 2$, log-Sobolev inequalities for local Gibbs measures with constant bounded independently of the volume can be obtained when some mixing conditions are satisfied. This last property gives a spatial decorrelation which roughly allows to approximate the system by a system of particles which are interacting only in cubes of finite size for which one can prove logarithmic Sobolev inequality .

This rough idea was developed in different ways in the literature ; the reader may look at [64], [69], [92] and [104]. Here we present a formalism quite close to the one developed in [92] and which is based on the strategy described in the beginning of this chapter. It will clearly rely on a key intermediate property characterized by what we shall call sweeping out relations. (One should realize that our strategy does not rely on an a priori spectral gap property of the Gibbs measure as described in [94].)

In order to illustrate the general considerations above, we will begin by describing what are the so-called sweeping out relations and show how they lead to log-Sobolev inequalities. Later we shall show how these relations can be deduced from strong mixing hypotheses.

We shall again restrict ourselves to the setting introduced at the beginning of this chapter where we are given a local specification $(E_\Lambda^{\cdot}, \Lambda \subset\subset \mathbb{Z}^d)$ described by a finite range potential Φ. Later we also consider the case where the range of the interaction is infinite.

5.4.1 Sweeping out relations

We shall say in the following that sweeping out relations are satisfied for a finite subset X_0 of $\mathbb{Z}^d$ if
For any set $\Lambda = \{j + X_0\}$ for some $j \in \mathbb{Z}^d$, there exist constants $\alpha_{ij}^{(\Lambda)} \in (0, \infty)$, such that for any $i \in \mathbb{Z}^d \backslash \Lambda$ so that $dist(i, \Lambda) \leq R$,

$$\left(q_i |\nabla_i (\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \leq \alpha_{ii}^{(\Lambda)} \left(\mathbf{E}_\Lambda q_i |\nabla_i f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} + \sum_{j \in \Lambda \cup \{i\}} \alpha_{ij}^{(\Lambda)} \left(\mathbf{E}_i \mathbf{E}_\Lambda q_j |\nabla_j f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \tag{5.4.42}$$

with

$$\alpha_{ij}^{(\Lambda)} \leq D e^{-\varepsilon |i-j|} \tag{5.4.43}$$

where $D, \varepsilon \in (0, \infty)$ are finite constants which are independent of i, j and Λ. Here, in the discrete setting $q_i = \nu_i(x_i)$ where ν_i is an isomorphic copy of the uniform measure on the finite set $\mathbf{M}$ (acting on the particle x_i at site i), whereas in the continuous setting q_i equal to the identity.

Let us note that in the following, we shall always restrict ourselves to cubes X_0 and shall subdivide large subsets $\Lambda \subset \mathbb{Z}^d$ (or even $\mathbb{Z}^d$ itself) in small cells of the type $\{j + X_0, j \in \mathbb{Z}^d\}$. In this way, we shall obtain bounds on log-Sobolev coefficients for E_Λ (or μ) in terms of the log-Sobolev coefficients of the local Gibbs measures $\{E_{X_0+j}, j \in \mathbb{Z}^d\}$ when the sweeping out relations are satisfied for X_0. The strategy is then to optimize on the choice of X_0. The fact that we choose homogeneous partitions is of course irrelevant, except that it helps to write the formulae more easily.

5.4.2 Proof of logarithmic Sobolev inequalities assuming sweeping out relations

We assume in this part that the sweeping out relations are satisfied. $\mathbf{M}$ shall be either a connected compact and smooth Riemannian manifold of finite dimension or a finite set.

To obtain log-Sobolev inequalities, we shall follow the strategy of section 5.2 and first define a suitable auxiliary Markov chain $\mathbf{\Pi}$, analogous to the chain $\mathbf{E}$ studied in the last section.

To this end, for an integer number $L \in \mathbb{N}$, for $k \in \mathbb{Z}^d$, we denote by $X_k \equiv k + ([0, 2(L+R)])^d$, the translation by the vector k of the cube $X_0 = [0, 2(L+R)]^d$ containing the origin. Later on it will be convenient to represent $s \in \mathbb{N}$ as follows $s = \sum_{l=1,\dots,d} v_s^{(l)} \, 2^{l-1}$ with a family $\{\mathbf{v}_s\}_{s=0,\dots,2^d-1} \subset \{0,1\}^d$. We denote $\mathcal{T}_s = \{k \in \mathbb{Z}^d; k \in (L+2R)\mathbf{v}_s + (2(L+2R)\mathbf{Z})^d\}$ and $\mathcal{T} = \cup_{s=0}^{2^d-1} \mathcal{T}_s$. Set

$$\Gamma_s \equiv \bigcup \{X_k \,:\, k \in \mathcal{T}_s\}.$$

Γ_s is the union of disjoint cubes of shape X_0 at distance greater to $2R$ of each other. Let us also remark that $\cup_{1 \leq s \leq 2^d-1} \Gamma_s$ covers $\mathbb{Z}^d$. It is important to note that, by construction, for any $i \in \mathbb{Z}^d$, there exists an $s \in \{0, \ldots, 2^d - 1\}$ and a cube in $X_k \subset \Gamma_s$ such that $i \in X_k$ and $d(i, X_k^c) \geq (L/2)$.

Setting $\mathbb{E}_s \equiv \otimes_{X_k \subset \Gamma_s} \mathbf{E}_{X_k}$, we shall consider in the following

$$\mathbf{\Pi}(f)(\omega) \equiv \mathbb{E}^\omega_{2^d-1}(\cdots \mathbb{E}_1(\mathbb{E}_0 f)). \tag{5.4.44}$$

We shall assume hereafter that the local Gibbs measures $\mathbf{E}^\omega_{X_k}$ satisfy a log-Sobolev inequality with coefficient $\widetilde{c} \in (0, \infty)$ independent of $k \in \mathbb{Z}^d$,

$$\mathbf{E}^\omega_{X_k} f \log f - \mathbf{E}^\omega_{X_k} f \log \mathbf{E}^\omega_{X_k} f \leq 2\widetilde{c}\, \mathbf{E}^\omega_{X_k} |\nabla f^{\frac{1}{2}}|^2 \tag{5.4.45}$$

for any non negative function f of class $\mathcal{C}^1$. This assumption is naturally satisfied when the local specification is defined by a finite range potential Φ according to property 4.6; the constant $\widetilde{c}$ is then bounded by $c_0 e^{4||\Phi||(2(L+R))^d}$ if c_0 is the log-Sobolev constant for the uniform measure ν (see also exercise after 4.6).

Our goal is to prove that when sweeping out relations hold, $\mathbf{\Pi}$ satisfies the conditions (Ci)-(Civ) of section 5.2. By definition of the local Gibbs measures $\mathbf{E}_{X_k}$, it is clear that (Ci) is true. Hence, we need to consider conditions (Cii) and (Ciii);

Proof of properties (Cii) and (Ciii) assuming the sweeping out relations

In this part, we shall prove that

Lemma 5.5 *If X_0 satisfies the sweeping out relations (5.4.42), there exist finite constants $\lambda \in (0, \infty)$ and $\bar{c} \in (0, \infty)$ such that*

$$|\nabla (\mathbf{\Pi} f)^{\frac{1}{2}}|^2 \leq \lambda\, \mathbf{\Pi}\, |\nabla f^{\frac{1}{2}}|^2 \tag{5.4.46}$$

and

$$\mathbf{\Pi} f \log f - (\mathbf{\Pi} f) \log(\mathbf{\Pi} f) \leq 2\bar{c}\, \mathbf{\Pi}\, |\nabla f^{\frac{1}{2}}|^2. \tag{5.4.47}$$

Moreover, if there exists $L_0 \geq 0$ such that for any $L \geq L_0$, any $v \in \mathcal{T}$,

$$\alpha^{(X_v)}_{ij} \leq D e^{-\varepsilon|i-j|} \tag{5.4.48}$$

with constants $D, \varepsilon \in (0, \infty)$ independent of L, $v \in \mathcal{T}$, $i \in Y^c$ so that $d(i, Y) \leq R$ and of $j \in X \cup \{i\}$, then we can choose L sufficiently large so that (5.4.46) is satisfied with $\lambda \in (0, 1)$. Consequently, if for $n \in \mathbb{N}$, we define inductively $\mathbf{\Pi}^{n,\omega} f \equiv \mathbf{\Pi}^n f(\omega) = \mathbf{\Pi}^{n-1} \mathbf{\Pi} f(\omega)$, $\mathbf{\Pi}^0 f = f$, we have for any $\omega \in \Omega$,

$$\mathbf{\Pi}^{n,\omega} f \log f - (\mathbf{\Pi}^{n,\omega} f) \log(\mathbf{\Pi}^{n,\omega} f) \leq 2 \frac{\bar{c}}{1-\lambda} \mathbf{\Pi}^{n,\omega} |\nabla f^{\frac{1}{2}}|^2 \tag{5.4.49}$$

<u>*Remark*</u> 5.6: The constants $(\lambda, \bar{c})$ can be chosen as follows. Setting

$$\alpha \equiv \sup_i \left(\alpha^{(X_{k(i)})}_{ii} + \sum_{j \in X_{k(i)} \cup \{i\}} \alpha^{(X_{k(i)})}_{ij} \right), \qquad \eta^{(s)} = \sup_j \sum_{i \in \mathbf{Z}^d} \eta^{(s)}_{ij}$$

were for $s \in \{0, \ldots, 2^d - 1\}$

$$\eta_{ij}^{(s)} \equiv \sum_{j_1 \in X_{k(i)} \cup i \setminus \Gamma_{s-2}} \cdots \sum_{j_{s-2} \in X_{k(j_{s-3})} \cup j_{s-3} \setminus \Gamma_1} \alpha_{ij_1}^{(X_{k(i)})} \alpha_{j_1 j_2}^{(X_{k(j_1)})} \cdots \alpha_{j_{s-2}\ j}^{(X_{k(j_{s-2})})}$$

with $X_{k(j_l)}$ being the cube of Γ_{s-l} at distance less than or equal to R from j_l for $l = 0, \ldots, s-1$. Then we can choose

$$\bar{c} \equiv 2^d \widetilde{c} \sup_{s \in \{0, \ldots, 2^d - 1\}} \lambda^{(s)}$$

with

$$\lambda^{(s)} \equiv b^2 e^{2^{d+1} ||\Phi||} \alpha^s \eta^{(s)},$$

with appropriately chosen constant b dependent only on $\mathbf{M}$, and choose $\lambda \equiv \lambda^{(2^d - 1)}$

In a few cases encountered for instance in random media, (considered in Chapter 9), the decay (5.4.48) of the coefficients $\alpha_{ij}^{(\Lambda)}$ appearing in the sweeping out relations for the local Gibbs measure $\mathbf{E}_\Lambda$ is satisfied only for points (i, j) at a distance greater or equal to some typical length $l(\Lambda)$ depending on Λ. For the study of such situations in Chapter 9, it will be useful to make the following exercise.

Exercise 5.7 *Let us assume that for any cube $\Lambda_L = [-L, L]^d$, the decay (5.4.48) of the coefficients $\alpha_{ij}^{(\Lambda)}$ appearing in the sweeping out relations for the local Gibbs measure is satisfied only for cubes $Y_0 \subset \Lambda_L$ of side greater or equal to a function $d(L)$ of L. Show that $\mathbf{E}_{\Lambda_L}$ satisfies a logarithmic Sobolev inequality with coefficient bounded by $Cd(L)^{2d} e^{cd(L)^{d-1}}$ for two finite constants $(c, C) \in (0, \infty)$. Show in particular that if $d(L) = A \log L$ for some finite constant A, the log-Sobolev constant increases at most polynomially with the length of the side of the cube Λ in dimension 2.*
Hints : Use cubes of sides of length of order $d(L)$ to construct $\mathbf{\Pi}$, the bound on log-Sobolev constants obtained in exercise 4.7 for the local Gibbs measures of these cubes and the above controls.

Proof of lemma 5.5.

Let f be a differentiable function of $L^1(\mu)$. Denoting $f_{-1} = f$ and $f_k = \mathbf{E}_{\Gamma_k} \mathbf{E}_{\Gamma_{k-1}} \cdots \mathbf{E}_{\Gamma_0} f$ for $k = 0, \ldots, 2^d - 1$, we first notice that

$$\mathbf{\Pi}^\omega f \log f - (\mathbf{\Pi}^\omega f) \log(\mathbf{\Pi}^\omega f) = \sum_{s=0,\ldots,2^d-1} \mathbb{E}_{2^d-1}^\omega \cdots \left[\mathbb{E}_s f_{s-1} \log \frac{f_{s-1}}{\mathbb{E}_s f_{s-1}} \right]. \tag{5.4.50}$$

According to the hypothesis (5.4.45) and the product property of theorem 4.4, E_{Γ_k} satisfies, for $k = \{0, \ldots, 2^d - 1\}$, a logarithmic Sobolev inequality with coefficient bounded above by $\widetilde{c}$. Consequently, we deduce from (5.4.50) that

$$\mathbf{\Pi}^\omega f \log f - \mathbf{\Pi}^\omega f \log \mathbf{\Pi}^\omega f \leq 2\widetilde{c} \sum_{s=0,\ldots,2^d-1} \left(\mathbb{E}_{2^d-1} \cdots \left(\mathbb{E}_s |\nabla_{\Gamma_s} f_{s-1}^{\frac{1}{2}}|^2 \right) \right) \tag{5.4.51}$$

The sweeping out relations are now going to be useful to control the gradient terms in the right hand side of (5.4.51). To this end, let us observe that for any $i \in \Gamma_s$, we have the following two possibilities ; either $i \in \Gamma_{s-1}$ and then $\nabla_i f_{s-1}$ is null, or there exists a unique $X_k \equiv X_{k(i)} \subset \Gamma_{s-1}$ so that $i \in X^c_{k(i)}$ and $d(i, X_{k(i)}) \le R$. In this second case, we use lemma 5.10 (see the next section) which tells us that

$$\begin{aligned} |\nabla_i f_{s-1}^{\frac{1}{2}}|^2 &= |\nabla_i \left(I\!\!E_{\Gamma_{s-1}\setminus X_{k(i)}} I\!\!E_{X_{k(i)}} f_{s-2}\right)^{\frac{1}{2}}|^2 \\ &\le b^2\, I\!\!E_{\Gamma_{s-1}\setminus X_{k(i)}} q_i |\nabla_i \left(I\!\!E_{X_{k(i)}} f_{s-2}\right)^{\frac{1}{2}}|^2 \end{aligned} \tag{5.4.52}$$

with a finite constant b, equal to one in the continuous setting or when $|\mathbf{M}| = 2$, and otherwise bounded by $\sqrt{|\mathbf{M}|}$. We can now deduce from the sweeping out relations and from Hölder's inequality that

$$\begin{aligned} q_i|\nabla_i(\mathbf{E}_{X_{k(i)}} f)^{\frac{1}{2}}|^2 \le\ & \alpha \cdot \Bigg(\alpha_{ii}^{(X_{k(i)})} \mathbf{E}_{X_{k(i)}} q_i |\nabla_i f^{\frac{1}{2}}|^2 \\ &+ \sum_{j\in X_{k(i)}\cup\{i\}} \alpha_{ij}^{(X_{k(i)})} \mathbf{E}_i \mathbf{E}_{X_{k(i)}} q_j |\nabla_j f^{\frac{1}{2}}|^2 \Bigg) \end{aligned} \tag{5.4.53}$$

where

$$\alpha \equiv \sup_i \left(\alpha_{ii}^{(X_{k(i)})} + \sum_{j\in X\cup\{i\}} \alpha_{ij}^{(X_{k(i)})} \right).$$

With (5.4.52), we thus obtain that

$$\begin{aligned} |\nabla_i f_{s-1}^{\frac{1}{2}}|^2 \le b^2 \alpha\ I\!\!E_{\Gamma_{s-1}\setminus X_{k(i)}} & \Bigg(\alpha_{ii}^{(X_{k(i)})} \mathbf{E}_{X_{k(i)}} q_i |\nabla_i f_{s-2}^{\frac{1}{2}}|^2 + \\ &+ \sum_{j\in X_{k(i)}\cup\{i\}} \alpha_{ij}^{(X_{k(i)})} \mathbf{E}_i \mathbf{E}_{X_{k(i)}} q_j |\nabla_j f_{s-2}^{\frac{1}{2}}|^2 \Bigg) \end{aligned} \tag{5.4.54}$$

for any $s \ge 1$. We can reinterpret this last result in the following form

$$|\nabla_i f_{s-1}^{\frac{1}{2}}|^2 \le b^2 \alpha \left(\sum_{j\in X_{k(i)}\cup i} \left((1-\delta_{ij})\mathbf{E}_j + \delta_{ij}\right) \alpha_{ij}^{(X_{k(i)})} I\!\!E_{s-1} q_j |\nabla_j f_{s-2}^{\frac{1}{2}}|^2 \right). \tag{5.4.55}$$

Repeating inductively the above arguments, we find for $s \ge 2$

$$|\nabla_i f_{s-1}^{\frac{1}{2}}|^2 \le b^2\alpha^2 \sum_{j_1\in X_{k(i)}\cup i\setminus\Gamma_{s-2}} \sum_{j_2\in X_{k(j_1)}\cup j_1\setminus\Gamma_{s-3}} \alpha_{ij_1}^{(X_{k(i)})} \alpha_{j_1 j_2}^{(X_{k(j_1)})} \tag{5.4.56}$$

$$((1-\delta_{ij_1})\mathbf{E}_{j_1} + \delta_{ij_1})\, \mathbb{E}_{s-1}\mathbb{E}_{s-2} q_{j_2} |\nabla_{j_2} f_{s-3}^{\frac{1}{2}}|^2$$

where, since $j_1 \in X_{k(i)} \cup \{i\}$, we did not need to write the term $(1-\delta_{j_1 j_2})\mathbf{E}_{j_1} + \delta_{j_1 j_2}$. By induction, we arrive at the bound

$$\mathbb{E}_{2^d-1} \cdots \mathbb{E}_s |\nabla_i f_{s-1}^{\frac{1}{2}}|^2 \leq b^2 \alpha^{s-1} \sum_{j \in \mathbf{Z}^d} \eta_{ij}^{(s)} \mathbf{\Pi} q_j |\nabla_j f^{\frac{1}{2}}|^2 \tag{5.4.57}$$

with

$$\eta_{ij}^{(s)} \equiv \sum_{j_1 \in X_{k(i)} \cup i \setminus \Gamma_{s-2}} \cdots \sum_{j_{s-2} \in X_{k(j_{s-3})} \cup j_{s-3} \setminus \Gamma_1} \alpha_{ij_1}^{(X_{k(i)})} \alpha_{j_1 j_2}^{(X_{k(j_1)})} \cdots \alpha_{j_{s-2}\, j}^{(X_{k(j_{s-2})})} \tag{5.4.58}$$

and where $X_{k(j_l)} \subset \Gamma_{s-l}$ for $l = 0, \ldots, s-1$.

Noticing that

$$\mathbf{\Pi} q_j |\nabla_j f^{\frac{1}{2}}|^2 \leq e^{2^d ||\Phi||} \mathbf{\Pi} |\nabla_j f^{\frac{1}{2}}|^2 \tag{5.4.59}$$

and summing over all the $i \in \Gamma_s$ in (5.4.57), we conclude that for $1 \leq s \leq 2^d - 1$,

$$\mathbb{E}_{2^d-1} \cdots \mathbb{E}_s |\nabla_{\Gamma_s} f_{s-1}^{\frac{1}{2}}|^2 \leq b^2 \alpha^{s-1} \eta^{(s)} e^{2^d ||\Phi||}\ \mathbf{\Pi} |\nabla f^{\frac{1}{2}}|^2 \tag{5.4.60}$$

with

$$\eta^{(s)} \equiv \sup_j \sum_{i \in \mathbf{Z}^d} \eta_{ij}^{(s)}.$$

With (5.4.50) and (5.4.51), we obtain (5.4.47).

To prove (5.4.46), let us come back to (5.4.55) and note that for $s = 2^d - 1$, we inductively obtain

$$|\nabla_i f_{s-1}^{\frac{1}{2}}|^2 \leq b^2 \alpha^{2^d} \sum_{j_1 \in X_{k(i)} \cup i \setminus \Gamma_{s-2}} \sum_{j_2 \in X_{k(j_1)} \cup j_1 \setminus \Gamma_{s-3}} \cdots \sum_{j_{s-2} \in X_{k(j_{s-3})} \cup j_{s-3} \setminus \Gamma_1} \tag{5.4.61}$$
$$\alpha_{ij_1}^{(X_{k(i)})} \alpha_{j_1 j_2}^{(X_{k(j_1)})} \cdots \alpha_{j_{s-2} j}^{(X_{k(j_{s-2})})} ((1-\delta_{ij_1})\mathbf{E}_{j_1} + \delta_{ij_1})\, \mathbf{\Pi} q_j |\nabla_j f^{\frac{1}{2}}|^2.$$

Since $\cup_{s=0}^{2^d} \Gamma_s = \mathbb{Z}^d$, for any j we can find an $s \in \{0, \ldots, 2^d - 1\}$ so that $j \in \Gamma_s$

$$\mathbf{E}_i \mathbf{\Pi} q_j |\nabla_j f^{\frac{1}{2}}|^2 \leq e^{2^d ||\Phi||} \mathbf{\Pi} q_j |\nabla_j f^{\frac{1}{2}}|^2 \leq e^{2^{d+1} ||\Phi||} \mathbf{\Pi} |\nabla_j f^{\frac{1}{2}}|^2 \tag{5.4.62}$$

where we used (5.4.59) to obtain the last bound. Summing up over all the $i \in \mathbb{Z}^d$, we deduce (5.4.46) from (5.4.61).

Finally, to prove the second part of the lemma, we remark that under the additional assumption (5.4.43), η can be chosen as small as one wishes as long as L is chosen large enough. Indeed, by construction, for any $j \in \mathbb{Z}^d$, there exists $r \in \{0, \ldots, 2^d - 1\}$ and a cube $X_k \subset \Gamma_r$ so that for any $j \in X_k$, $d(j, X_k^c) \geq L/2$. Henceforth, in any path $\mathcal{W}_{ij} \equiv \{i \equiv j_0, j_1 \in X_{k(i)} \cup i \setminus \Gamma_s, \ldots, j_l \in X_{k(j_{l-1})} \cup j_{l-1} \setminus \Gamma_{s-l}, \cdots, j_s \in X_{k(j_{s-1})} \cup j_{s-1} \setminus \Gamma_0, j \equiv j_s\}$, there is at least one couple of points (j_{l-1}, j_l) at distance greater or equal to $L/2$. This in particular implies that in the sum defining of $\eta_{ij}^{2^d-1}$, there is at least one term $\alpha_{j_{l-1},j_l}^{(X_{k(j_{l-1})})} \leq De^{-\varepsilon \cdot L/2}$. We hence obtain the bound

$$\sum_i \eta_{ij}^{(2^d-1)} \leq \alpha^{2^d-2} De^{-\varepsilon \cdot L/2} \tag{5.4.63}$$

Since, under hypothesis (5.4.48), α is clearly bounded independently of L, we can choose L sufficiently large so that $\lambda < 1$.

Finally, the last part of the lemma can be proved following (5.2.6)-(5.2.9) but without bothering about the limiting Gibbs measure μ.

$\diamond$

Proof of condition (Civ) when the sweeping out relations are satisfied

To show that $\mathbf{\Pi}^n$ converges, as n goes to infinity, towards a measure μ, we are going to prove that $(\mathbf{\Pi}^{n,\omega} f, n \in \mathbb{N})$ is a Cauchy sequence for any continuously differentiable function f localized in a finite subset of $\mathbb{Z}^d$, uniformly in $\omega \in \Omega$. This in turn implies the weak convergence of $\mathbf{\Pi}^n$ towards a unique measure μ. Indeed, given a finite subset Λ of $\mathbb{Z}^d$ and considering the set $\mathcal{F}_\Lambda$ of continuous functions f localized in Λ, we note that $\mathcal{F}_\Lambda$ is separable as $\mathbf{M}^{|\Lambda|}$ is compact since $\mathbf{M}$ is. Thus, we can consider a countable subset $(f_i, i \in \mathbb{N})$ of $\mathcal{F}_\Lambda$ dense in $\mathcal{F}_\Lambda$. By the standard diagonalization procedure, we see that if $(\mathbf{\Pi}^{n,\omega} f_i, n \in \mathbb{N})$ is Cauchy for any $i \in \mathbb{N}$, uniformly in $\omega \in \Omega$, $(\mathbf{\Pi}^{n,\omega} f_i, i \in \mathbb{N})$ converges simultaneously along some subsequence. The limit then defines a probability measure. By the property of local Gibbs measures, μ is independent of the choice of the finite set Λ so that we can define μ on the set Ω of all configurations. Finally, uniqueness of μ can be deduced from the uniformity of the convergence with respect to boundary conditions. Hence, we need to prove that, in view of (5.4.46), $(\mathbf{\Pi}^{n,\omega} f, n \in \mathbb{N})$ is a Cauchy sequence for any $f \in \mathcal{F}_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^d$. We can of course restrict ourselves to non negative functions f. Let $n, m \in \mathbb{N}$, $n < m$ and a non negative function $f \in \mathcal{F}_\Lambda$ be given. Then we have

$$\begin{aligned} |\mathbf{\Pi}^n f(\omega) - \mathbf{\Pi}^m f(\omega)| &\leq \int \mathbf{\Pi}^{m-n,\omega}(d\widetilde{\omega})|\mathbf{\Pi}^n f(\omega) - \mathbf{\Pi}^n f(\widetilde{\omega})| \\ &\leq \sup_{\omega,\widetilde{\omega}} |\mathbf{\Pi}^n f(\omega) - \mathbf{\Pi}^n f(\widetilde{\omega})| \leq |||\mathbf{\Pi}^n f|||. \end{aligned} \tag{5.4.64}$$

Moreover, by construction of $\mathbf{\Pi}$, $\mathbf{\Pi}^n f$ is localized in a set Λ_n so that $|\Lambda_n| \leq [diam(\Lambda) + n2(L+R)]^d$. Thus, we deduce that

$$\begin{aligned} &|||\mathbf{\Pi}^n f||| \leq |\Lambda_n| \cdot ||f^{\frac{1}{2}}||_\infty \cdot \sup_{i \in \Lambda n} ||\nabla_i (\mathbf{\Pi}^n f)^{\frac{1}{2}}||_\infty \\ &\leq |\Lambda_n| \cdot ||f^{\frac{1}{2}}||_\infty \cdot || \sum_{i \in \Lambda_n} |\nabla_i (\mathbf{\Pi}^n f)^{\frac{1}{2}}|^2||_\infty^{\frac{1}{2}} \leq \lambda^{n/2} |\Lambda_n| \cdot ||f^{\frac{1}{2}}||_\infty \cdot |||\nabla f^{\frac{1}{2}}|^2||_\infty^{\frac{1}{2}} \end{aligned} \tag{5.4.65}$$

where in the last line we made an inductive use of (5.4.46). Since the volume of Λ_n increases at most polynomially in n, we conclude that, when $\lambda < 1$, $|||\mathbf{\Pi}^n f|||$

goes to zero when n goes to infinity, which gives with the help of (5.4.64) the desired result.

$\diamond$

5.4.3 Proof of sweeping out relations

In this section, we relate the sweeping out relations, (used in the previous part to prove logarithmic Sobolev inequality), to strong mixing conditions satisfied by local Gibbs measures. We first describe these mixing conditions and show that they are equivalent to decay of correlations properties. Then, we prove that they imply that sweeping out relations hold. At the end of this section, we prove that mixing conditions are necessary to obtain logarithmic Sobolev inequality for local Gibbs measures with uniformly bounded constants. Thus, this last result holds iff mixing conditions are satisfied. We shall finally see that mixing conditions are satisfied in some situations such as for example a high temperature regime.

Strong mixing conditions

For a subset $Y \subset \mathbf{Z}^d$, we denote by $|||f|||_Y$ a semi-norm of f given by

$$|||f|||_Y \equiv \sum_{i \in Y} ||\nabla_i f||_\infty.$$

Here, when $\mathbf{M}$ is a connected smooth Riemannian manifold, $\nabla_i f$ denotes as usual the gradient operator acting on the variable at i, and, in case $\mathbf{M}$ is a finite set, $\nabla_i f \equiv \nu_i f - f$. If $Y = \mathbf{Z}^d$, we use a short notation $|||f|||_{\mathbf{Z}^d} \equiv |||f|||$. In the following, we shall measure the variations of a function f with respect to the coordinates $\omega_Y \equiv (\omega_i : i \in Y)$ by

$$\mathrm{Var}_Y(f) \equiv \sup_{\omega_{Y^c} = \widetilde{\omega}_{Y^c}} |f(\omega_Y) - f(\widetilde{\omega}_Y)|.$$

If $Y = \{j\}$, we simplify the notation as $\mathrm{Var}_Y(f) \equiv \mathrm{Var}_j(f)$. We remark that $\mathrm{Var}_Y(f) \leq 2||f||_\infty$ and that the variation of f is related to the triple semi-norm by

$$\mathrm{Var}_Y(f) \leq a|||f|||_Y \tag{5.4.66}$$

with, in the discrete setting, $a \leq 2$ and in the continuous setting, $a \leq s_{\mathbf{M}} \equiv \sup_{x,y \in \mathbf{M}} \inf l_{x,y}$, with $l_{x,y}$ the length of a geodesic containing x and y and the infimum being taken on the set of these geodesics.

Exercise 5.8 *Prove (5.4.66).*
Hints : Proceed by interpolation between ω and $\widetilde{\omega}$ by a sequence $\omega^{(j_k)}$ (for $j_k \in Y$, $k = 1, \ldots, |Y|$ a lexicographical order in Y) such that each term of this sequence only differs from the previous by one coordinate to deduce that

$$\mathrm{Var}_Y(f) \leq \sum_{k=1,\ldots,|Y|} \mathrm{Var}_{j_k}(f).$$

Prove the result for $Y = \{j\}$ and conclude.

With the above notations, we consider the following mixing conditions :

Strong Mixing Condition (SMC):

There exists a constant $\varepsilon \in (0, \infty)$ such that for any $X \subset \mathbb{Z}^d$, and any cube $Y \subset X^c$,

$$\mathrm{Var}_Y(E_X f) \leq RC(Y) e^{-\varepsilon d(Y, \Lambda_f \cap X)} |||f||| \tag{5.4.67}$$

with a finite constant $C(Y)$ depending only on the size of Y.

Let us point out that we shall only use in the sequel (**SMC**)for sets X defined as the union of a finite number of sets obtained by translation of a given cube Y_0. As we will show below this condition is equivalent to the following strong decay of correlations property

Strong Decay of Correlations (SDC):

There exists a constant $\varepsilon \in (0, \infty)$ such that for any functions f, g with finite triple norm on $\mathbb{Z}^d$ and, for any boundary conditions $\omega \in \Omega$, we have

$$|E_X^\omega(g, f)| \leq A e^{-\varepsilon d(\Lambda_g, \Lambda_f)} |||g||| \cdot |||f||| \tag{5.4.68}$$

for a constant $A \in (0, \infty)$ depending only on $\min(|X \cap \Lambda_f|, |X \cap \Lambda_g|)$.
We have the following property.

Theorem 5.9 *The strong mixing condition* (**SMC**) *is equivalent to the strong decay of correlations* (**SDC**).

Proof : Let us first prove that (**SDC**)$\Longrightarrow$ (**SMC**). Fix $Y \subset \mathbb{Z}^d$, a set $X \subset Y^c$, two elements $(\omega, \widetilde{\omega}) \in \Omega$ so that $\omega_{Y^c} = \widetilde{\omega}_{Y^c}$ and a function f localized in $\Lambda_f \subset X$. Since, by definition, the local Gibbs measures $(E_X^\omega, \omega \in \Omega)$ are all equivalent to ν_X, we can write

$$E_X^\omega(f) - E_X^{\widetilde{\omega}}(f) = E_X^{\widetilde{\omega}}(\xi_{\omega,\widetilde{\omega}}; f) \tag{5.4.69}$$

with

$$\xi_{\omega,\widetilde{\omega}} = \frac{dE_X^\omega}{dE_X^{\widetilde{\omega}}}$$

and $E_X^{\widetilde{\omega}}(g; f) = E_X^{\widetilde{\omega}}(gf) - E_X^{\widetilde{\omega}}(f) E_X^{\widetilde{\omega}}(g)$. In the case where the range of the interaction is finite (say R), one easily sees that $d(Y, \Lambda(\xi_{\omega,\widetilde{\omega}})^c) \leq R$ so that the triangle inequality implies

$$d(\Lambda_f, \Lambda(\xi_{\omega,\widetilde{\omega}})) \geq d(\Lambda_f, Y) - R.$$

Applying (**SDC**) to (5.4.69) and using the last remark, we deduce that

$$|E_X^\omega(f) - E_X^{\widetilde{\omega}}(f)| \leq C e^{-\varepsilon d(Y, \Lambda_f)} |||f||| \tag{5.4.70}$$

with
$$C = Ae^{\varepsilon R}|||\xi_{\omega,\widetilde{\omega}}|||.$$

(**SMC**)$\Longrightarrow$ (**SDC**) Conversely, for any functions (f,g) on Ω such that $\Lambda_f \cap X$ and $\Lambda_g \cap X$ are disjoint, we have

$$E_X^\omega(g,f) = E_X^\omega(E^{\cdot}_{X\setminus\Lambda_f}(g),f) = \frac{1}{2}E_X^\omega \otimes \widetilde{E}_X^\omega\left((E^{\cdot}_{X\setminus\Lambda_f}(g) - \widetilde{E}^{\cdot}_{X\setminus\Lambda_f}(g))(f-\widetilde{f})\right).$$

Consequently, if (**SMC**) is satisfied, we obtain

$$\begin{aligned} |E_X^\omega(g,f)| &\leq \mathrm{Var}_{\Lambda_f\cap X}(E^{\cdot}_{X\setminus\Lambda_f}(g)) \cdot \mathrm{Var}_{\Lambda_f\cap X}(f) \quad (5.4.71)\\ &\leq Ae^{-\varepsilon d(\Lambda_g\cap X,\,\Lambda_f\cap X)}|||g|||\cdot|||f||| \end{aligned}$$

with $A \leq 2C$. $\diamond$

Study of sweeping out relations

We shall in this paragraph show that (**SMC**) results with sweeping out relations. In fact, we show in lemma 5.11 that inequality (5.4.42) holds with the coefficients $(\alpha_{ij}^{(\Lambda)})$ described in (5.4.80) and (5.4.81) respectively. We estimate these coefficients at the end of the paragraph and show that mixing conditions result with the decay (5.4.43) of these coefficients, second condition for sweeping out relations to hold. To prove lemma 5.11, a key ingredient is the following estimate.

Lemma 5.10 *Let (Θ,Λ) be two subsets of $\mathbb{Z}^d$ such that $dist(\Theta,\Lambda) > R$. Set $\mathbf{E}_{\Theta\cup\Lambda} \equiv \mathbf{E}_\Theta \otimes \mathbf{E}_\Lambda$. Then, for any $i \in \mathbb{Z}^d$ such that $dist(i,\Lambda) \leq R$, we have*

$$|\nabla_i(\mathbf{E}_{\Theta\cup\Lambda}f)^{\frac{1}{2}}| \leq b\left(\mathbf{E}_\Theta q_i|\nabla_i(\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \quad (5.4.72)$$

with $b \leq |\mathbf{M}|^{\frac{1}{2}}$ if $|\mathbf{M}| < \infty$ and $b = 1$ in the case $|\mathbf{M}| = 2$ or if $\mathbf{M}$ is a smooth connected Riemannian manifold.

Proof : Let f be a measurable function of $L^1(\mathbf{E}_\Lambda)$ and $F \equiv \mathbf{E}_\Lambda f$. Let us first consider the discrete setting. We then have

$$|\nabla_i(\mathbf{E}_{\Theta\cup\Lambda}f)^{\frac{1}{2}}| = |\nabla_i(\mathbf{E}_\Theta F)^{\frac{1}{2}}| = |\nu_i(\mathbf{E}_\Theta F)^{\frac{1}{2}} - (\mathbf{E}_\Theta F)^{\frac{1}{2}}| \quad (5.4.73)$$

and so, as ν_i is the uniform measure on $\mathbf{M}$,

$$|\nabla_i(\mathbf{E}_{\Theta\cup\Lambda}f)^{\frac{1}{2}}| \leq |\mathbf{M}|^{\frac{1}{2}}\left(\nu_i|\nu_i(\mathbf{E}_\Theta F)^{\frac{1}{2}} - (\mathbf{E}_\Theta F)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}. \quad (5.4.74)$$

Moreover, Minkowski's and Cauchy-Schwarz's inequalities imply

$$\begin{aligned}\nu_i|\nu_i(\mathbf{E}_\Theta F)^{\frac{1}{2}} - (\mathbf{E}_\Theta F)^{\frac{1}{2}}|^2 &= \frac{1}{2}\nu_i \otimes \widetilde{\nu}_i \left((\mathbf{E}_\Theta F)^{\frac{1}{2}} - (\mathbf{E}_\Theta \widetilde{F})^{\frac{1}{2}}\right)^2 \\ &\leq \frac{1}{2}\nu_i \otimes \widetilde{\nu}_i \left(\mathbf{E}_\Theta \left(F^{\frac{1}{2}} - \widetilde{F}^{\frac{1}{2}}\right)^2\right).\end{aligned} \tag{5.4.75}$$

Since $dist(i,\Theta) > R$, $\mathbf{E}_\Theta$ is independent of the coordinate ω_i, and we have

$$\frac{1}{2}\nu_i \otimes \widetilde{\nu}_i \left(\mathbf{E}_\Theta \left(F^{\frac{1}{2}} - \widetilde{F}^{\frac{1}{2}}\right)^2\right) = \mathbf{E}_\Theta \nu_i (F^{\frac{1}{2}} - \nu_i F^{\frac{1}{2}})^2 = \mathbf{E}_\Theta \nu_i |\nabla_i F^{\frac{1}{2}}|^2.$$

Putting (5.4.73)-(5.4.75) together, we obtain the desired estimate.
In the case where $|\mathbf{M}| = 2$, we can improve this bound in the following way ; we observe that

$$\begin{aligned}|\nabla_i(\mathbf{E}_{\Theta\cup\Lambda} f)^{\frac{1}{2}}| = |\nabla_i(\mathbf{E}_\Theta F)^{\frac{1}{2}}| &= \frac{1}{2}|(\mathbf{E}_\Theta F_{|\omega_i=+1})^{\frac{1}{2}} - (\mathbf{E}_\Theta F_{|\omega_i=-1})^{\frac{1}{2}}| \\ &\leq \frac{1}{2}\left(\mathbf{E}_\Theta (F^{\frac{1}{2}}_{|\omega_i=+1} - F^{\frac{1}{2}}_{|\omega_i=-1})^2\right)^{\frac{1}{2}} = \left(\mathbf{E}_\Theta |\nabla_i F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}\end{aligned} \tag{5.4.76}$$

with $|\nabla_i F^{\frac{1}{2}}|^2 = \nu_i |\nabla_i F^{\frac{1}{2}}|^2$. In the continuous case, the same estimate is true since

$$|\nabla_i(\mathbf{E}_{\Theta\cup\Lambda} f)^{\frac{1}{2}}| = \frac{1}{2}\frac{|\mathbf{E}_\Theta \nabla_i F|}{(\mathbf{E}_\Theta F)^{\frac{1}{2}}} = \frac{|\mathbf{E}_\Theta F^{\frac{1}{2}} \nabla_i F^{\frac{1}{2}}|}{(\mathbf{E}_\Theta F)^{\frac{1}{2}}} \leq \left(\mathbf{E}_\Theta |\nabla_i F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}$$

thanks to Cauchy-Schwarz's inequality.

◇

By $m_{X,Y}$ we shall denote the best constant in the spectral gap inequality satisfied by the measure $E_X|_{\Sigma_{(X\setminus Y)^c}}$. Let $(\mathbf{E}^{0_i}_X, X \subset\subset \mathbb{Z}^d)$ be the Gibbs measures constructed with the potential $\{\Phi_X \delta_{\{i \notin X\}}, X \subset\subset \mathbb{Z}^d\}$, that is the Gibbs measures where the interaction with the spin at i has been removed. We shall denote

$$\xi_{i,\Lambda} \equiv \frac{d\mathbf{E}^{0_i}_\Lambda|_{\Sigma_\Lambda}}{d\mathbf{E}_\Lambda|_{\Sigma_\Lambda}}.$$

Finally, for any $X \subset \Lambda$ and $i \in \Lambda^c$, we introduce a constant $\eta^{(\Lambda)}_{iX}$, given in the discrete setting by

$$\eta^{(\Lambda)}_{iX} \equiv \mathrm{Var}_{\Lambda\cap X}\left(\mathbf{E}_{\Lambda\setminus X}(\xi_{i,\Lambda})\right) \tag{5.4.77}$$

and in the continuous setting, if the potential U_Λ is $\nu_{\mathbb{Z}^d}$-almost surely differentiable, by

$$\eta^{(\Lambda)}_{iX} \equiv \mathrm{Var}_{\Lambda\cap X}\left(\mathbf{E}_{\Lambda\setminus X}(|\nabla_i U_\Lambda|)\right). \tag{5.4.78}$$

By definition, $\eta^{(\Lambda)}_{iX}$ will be zero for any other couple (i,X).

Lemma 5.11 *Let $\Lambda \subset\subset \mathbb{Z}^d$ and F be a measurable function. Let $i \in \Lambda^c$.*
• *In the continuous setting, assuming that F is almost surely differentiable, we have*

$$|\nabla_i(\mathbf{E}_\Lambda F)^{\frac{1}{2}}| \leq \left(\mathbf{E}_\Lambda|\nabla_i F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} + \sum_{j\in\Lambda_F\cap\Lambda} \epsilon_{ij}^{(\Lambda)} \left(\mathbf{E}_i\mathbf{E}_\Lambda|\nabla_j F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \tag{5.4.79}$$

with

$$\epsilon_{ij}^{(\Lambda)} = \left(\frac{2}{m_{\Lambda,j}}\right)^{\frac{1}{2}} \eta_{ij}^{(\Lambda)}. \tag{5.4.80}$$

• *In the discrete setting, the following estimate is satisfied*

$$\left(\nu_i|\nabla_i(\mathbf{E}_\Lambda F)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \leq e^{\frac{1}{2}\eta^{(\Lambda)}_{i,\Lambda_F\cap\Lambda}} \left(\mathbf{E}_\Lambda\nu_i|\nabla_i F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \tag{5.4.81}$$

$$+2\left(\frac{|M|}{m_{\Lambda,\Lambda_F}}\right)^{\frac{1}{2}} e^{||\Phi||}\eta^{(\Lambda)}_{i,\Lambda_F\cap\Lambda} \sum_{j\in\Lambda\cap\Lambda_F} \left(\mathbf{E}_i\mathbf{E}_\Lambda\nu_j|\nabla_j F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}.$$

Proof : Here we shall only focus on the discrete setting, the proof in the continuous being slightly easier and following almost the same lines.
Writing

$$(\mathbf{E}_\Lambda F)^{\frac{1}{2}} = (\mathbf{E}_\Lambda^{0_i} F)^{\frac{1}{2}} + \left((\mathbf{E}_\Lambda F)^{\frac{1}{2}} - (\mathbf{E}_\Lambda^{0_i} F)^{\frac{1}{2}}\right) \tag{5.4.82}$$

and applying the same arguments as in (5.4.75), we obtain

$$\begin{aligned}\left(\nu_i|\nabla_i(\mathbf{E}_\Lambda F)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \quad &\leq \quad \left(\frac{1}{2}\nu_i\otimes\widetilde{\nu}_i\left((\mathbf{E}_\Lambda^{0_i}F)^{\frac{1}{2}} - (\mathbf{E}_\Lambda^{0_i}\widetilde{F})^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ &\quad + \left(\frac{1}{2}\nu_i\otimes\widetilde{\nu}_i\left((\mathbf{E}_\Lambda F)^{\frac{1}{2}} - (\mathbf{E}_\Lambda^{0_i}F)^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ &\quad + \left(\frac{1}{2}\nu_i\otimes\widetilde{\nu}_i\left((\widetilde{\mathbf{E}}_\Lambda \widetilde{F})^{\frac{1}{2}} - (\mathbf{E}_\Lambda^{0_i}\widetilde{F})^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.4.83}$$

We can estimate the first term in the right hand side of (5.4.83) by using Minkowski' s inequality so that

$$\begin{aligned}\left(\frac{1}{2}\nu_i\otimes\widetilde{\nu}_i\left((\mathbf{E}_\Lambda^{0_i}F)^{\frac{1}{2}} - (\mathbf{E}_\Lambda^{0_i}\widetilde{F})^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \quad &\leq \quad \left(\frac{1}{2}\nu_i\otimes\widetilde{\nu}_i\,\mathbf{E}_\Lambda^{0_i}\left(F^{\frac{1}{2}} - \widetilde{F}^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ &= \quad \left(\mathbf{E}_\Lambda^{0_i}\nu_i|\nabla_i F^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.4.84}$$

Moreover, by definition of $\xi_{i,\Lambda}$, we notice that for any non negative function G, we have

$$\begin{aligned}
\mathbf{E}_\Lambda^{0_i} G &= \mathbf{E}_\Lambda G + \mathbf{E}_\Lambda \left(\xi_{i,\Lambda} - 1\right) G \\
&= \mathbf{E}_\Lambda G + \mathbf{E}_\Lambda \left(\mathbf{E}_{\Lambda\setminus\Lambda_G}\xi_{i,\Lambda} - 1\right) G \qquad (5.4.85)\\
&\leq \left(1 + \mathrm{Var}_{\Lambda\cap\Lambda_G}\left(\mathbf{E}_{\Lambda\setminus\Lambda_G}(\xi_{i,\Lambda})\right)\right)\mathbf{E}_\Lambda G = (1+\eta^{(\Lambda)}_{i,\Lambda_G})E_\Lambda G.
\end{aligned}$$

Using this remark with $G = \nu_i|\nabla_i F^{\frac12}|^2$, we deduce from (5.4.84) and (5.4.85) that

$$\left(\frac12 \nu_i \otimes \widetilde{\nu}_i \left((\mathbf{E}_\Lambda^{0_i} F)^{\frac12} - (\mathbf{E}_\Lambda^{0_i}\widetilde{F})^{\frac12}\right)^2\right)^{\frac12} \leq \left[1+\eta^{(\Lambda)}_{i,\Lambda_F}\right]^{\frac12}\left(\mathbf{E}_\Lambda \nu_i |\nabla_i F^{\frac12}|^2\right)^{\frac12} \tag{5.4.86}$$

which provides the desired estimate of the first term in the right hand side of (5.4.83). The other terms only depend on the difference $(\mathbf{E}_\Lambda F)^{\frac12} - (\mathbf{E}_\Lambda^{0_i} F)^{\frac12}$ that we are going to study. To this end, let us note that

$$|(\mathbf{E}_\Lambda^{0_i} F)^{\frac12} - (\mathbf{E}_\Lambda F)^{\frac12}| = \frac{|\mathbf{E}_\Lambda(\xi_{i,\Lambda}-1)F|}{(\mathbf{E}_\Lambda^{0_i} F)^{\frac12} + (\mathbf{E}_\Lambda F)^{\frac12}}. \tag{5.4.87}$$

The above numerator can be bounded above by noticing that

$$\begin{aligned}
|\mathbf{E}_\Lambda(\xi_{i,\Lambda}-1)F| &= |\frac12 \mathbf{E}_\Lambda \otimes \hat{\mathbf{E}}_\Lambda\left((\mathbf{E}_{\Lambda\setminus\Lambda_F}\xi_{i,\Lambda} - \hat{\mathbf{E}}_{\Lambda\setminus\Lambda_F}\hat{\xi}_{i,\Lambda})(F-\hat{F})\right)| \qquad (5.4.88)\\
&\leq \frac12 \eta^{(\Lambda)}_{i,\Lambda_F}\mathbf{E}_\Lambda \otimes \hat{\mathbf{E}}_\Lambda |F-\hat{F}|,
\end{aligned}$$

with $\hat{\mathbf{E}}_\Lambda$ is a copy of $\mathbf{E}_\Lambda$. Moreover, further computations give

$$\begin{aligned}
\frac12 \mathbf{E}_\Lambda \otimes \hat{\mathbf{E}}_\Lambda |F-\hat{F}| &\leq (\mathbf{E}_\Lambda F)^{\frac12}\left(\frac12 \mathbf{E}_\Lambda \otimes \hat{\mathbf{E}}_\Lambda (F^{\frac12} - \hat{F}^{\frac12})^2\right)^{\frac12} \qquad (5.4.89)\\
&\leq \left(\frac{1}{m_{\Lambda,\Lambda_F}}\right)^{\frac12}(\mathbf{E}_\Lambda F)^{\frac12}\left(\mathbf{E}_\Lambda |\nabla_{\Lambda\cap\Lambda_F} F^{\frac12}|^2\right)^{\frac12}.
\end{aligned}$$

Inequalities (5.4.87) - (5.4.89) give the following control

$$|(\mathbf{E}_\Lambda^{0_i} F)^{\frac12} - (\mathbf{E}_\Lambda F)^{\frac12}| \leq \left(\frac{1}{m_{\Lambda,\Lambda_F}}\right)^{\frac12}\eta^{(\Lambda)}_{i,\Lambda_F}\left(\mathbf{E}_\Lambda |\nabla_{\Lambda\cap\Lambda_F} F^{\frac12}|^2\right)^{\frac12}. \tag{5.4.90}$$

Thanks to (5.4.90), we obtain the following bound for the second term in the right hand side of (5.4.83)

$$\left(\frac12 \nu_i \otimes \widetilde{\nu}_i \left((\mathbf{E}_\Lambda F)^{\frac12} - (\mathbf{E}_\Lambda^{0_i} F)^{\frac12}\right)^2\right)^{\frac12} \leq \frac{1}{m^{\frac12}_{\Lambda,\Lambda_F}}\eta^{(\Lambda)}_{i,\Lambda_F}\left(\nu_i \mathbf{E}_\Lambda |\nabla_{\Lambda\cap\Lambda_F} F^{\frac12}|^2\right)^{\frac12}$$

$$\leq \frac{e^{||\Phi||}}{m_{\Lambda,\Lambda_F}^{\frac{1}{2}}} \eta_{i,\Lambda_F}^{(\Lambda)} \left(\mathbf{E}_i \mathbf{E}_\Lambda |\nabla_{\Lambda \cap \Lambda_F} F^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \tag{5.4.91}$$

$$\leq \frac{e^{||\Phi||} |M|^{\frac{1}{2}}}{m_{\Lambda,\Lambda_F}^{\frac{1}{2}}} \eta_{i,\Lambda_F}^{(\Lambda)} \sum_{j \in \Lambda \cap \Lambda_F} \left(\mathbf{E}_i \mathbf{E}_\Lambda \nu_j |\nabla_j F^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}}.$$

Similarly, we estimate the third term in the right hand side of (5.4.83) and, with (5.4.83), (5.4.86) and (5.4.91), conclude that

$$\left(\nu_i |\nabla_i (\mathbf{E}_\Lambda F)^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \leq \left[1 + \eta_{i,\Lambda_F}^{(\Lambda)} \right]^{\frac{1}{2}} \left(\mathbf{E}_\Lambda \nu_i |\nabla_i F^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \tag{5.4.92}$$

$$+2 \left(\frac{|M|}{m_{\Lambda,\Lambda_F}} \right)^{\frac{1}{2}} e^{||\Phi||} \eta_{i,\Lambda_F}^{(\Lambda)} \sum_{j \in \Lambda \cap \Lambda_F} \left(\mathbf{E}_i \mathbf{E}_\Lambda \nu_j |\nabla_j F^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}}$$

that is the desired result in the discrete setting.

$\diamond$

For further use, we present here the following lemma

Lemma 5.12 *For any* $\Lambda \subseteq \mathbf{Z}^d$

$$m_{\Lambda,Y} \geq m_0 \, e^{-4||\Phi|| \cdot |Y|} \tag{5.4.93}$$

with m_0 *the spectral gap for the product measure* $\nu_{\mathbb{Z}^d}$.

The proof is a direct consequence of property 2.6. In fact, in finite range interaction models, this result can be improved as seen in property 2.7. The main interest of lemma 5.12 is that the estimate of $m_{\Lambda,Y}$ there is independent of Λ.

We can now establish sweeping out relations.

Theorem 5.13 *(***Sweeping out relations***)*
For any finite subset Λ *of* $\mathbb{Z}^d$ *which can be represented as the union of a finite numbers of cubes obtained by translation of the same cube* Y_0*, there exist constants* $\alpha_{ij}^{(\Lambda)} \in (0,\infty)$*, so that for any* $i \in \Lambda^c$*, dist*$(i,\Lambda) \leq R$*, we have*

$$\left(\nu_i |\nabla_i (\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \leq \alpha_{ii}^{(\Lambda)} \left(\mathbf{E}_\Lambda \nu_i |\nabla_i f^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} + \sum_{j \in \Lambda \cup \{i\}} \alpha_{ij}^{(\Lambda)} \left(\mathbf{E}_i \mathbf{E}_\Lambda \nu_j |\nabla_j f^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \tag{5.4.94}$$

*If we additionnaly assume that the strong mixing condition (***SMC***)is satisfied, then*

$$\alpha_{ij}^{(\Lambda)} \leq D e^{-\varepsilon|i-j|} \tag{5.4.95}$$

for constants $D, \varepsilon \in (0,\infty)$ *independent of* $i \in \Lambda^c$, $d(i,\Lambda) \leq R$ *and* $j \in \Lambda \cup \{i\}$*, but eventually depending on the size of* Y_0.

We notice that the dependence of D and ε on the size of Y_0 is not essential since the sets Λ can be as large as one wishes, for a given size of Y_0.
Proof : The idea of the proof is to apply inductively lemma 5.11 and the uniform bound of lemma 5.12.

Let us assume that Λ is defined as the union of a family of disjoint cubes $\{Y_k \equiv Y_0 + L_0 y_k\}$, $y_k \in \mathbf{Z}^d$ with $k = 1, \ldots, N$, for an integer number N, and a cube Y_0 with diameter L_0. Let us remark that for any cube $\{Y_k, k \in \{1, \ldots, N\}\}$, lemma 5.11 implies

$$\begin{aligned} \left(\nu_i|\nabla_i(\mathbf{E}_\Lambda f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} &= \left(\nu_i|\nabla_i(\mathbf{E}_\Lambda(\mathbf{E}_{\Lambda\setminus Y_k} f))^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \\ &\leq e^{\frac{1}{2}\eta^{(\Lambda)}_{i,Y_k}} \left(\mathbf{E}_\Lambda \nu_i|\nabla_i(\mathbf{E}_{\Lambda\setminus Y_k} f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \\ &\quad + C\eta^{(\Lambda)}_{i,Y_k} \sum_{j\in Y_k} \left(\mathbf{E}_i \mathbf{E}_\Lambda \nu_j|\nabla_j(\mathbf{E}_{\Lambda\setminus Y_k} f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \end{aligned} \tag{5.4.96}$$

with $C \equiv 2(|M|/m_{\Lambda,Y_k})^{\frac{1}{2}} e^{||\Phi||}$. Using lemma 5.12, we know that

$$C \leq A \equiv 2(|M|/m_0)^{\frac{1}{2}} e^{||\Phi||(1+2L_0^d)}.$$

In order to bound the first term on the right hand side of (5.4.96), we apply inductively this relation. It will soon appear that the optimal sequence of cubes $\{Y_k, k \in \{1, \ldots, N\}\}$ one can choose is to order them in decreasing order of their distance to i. We hence let $(k^i_j, 1 \leq j \leq N)$ be a lexicographical order on $\{1, \ldots, N\}$ such that $j \to d(i, Y_{k^i_j})$ is decreasing and let $\Lambda^l_i = \bigcup_{m=l}^N Y_{k^i_m}$. To simplify the notations, we set $\eta^{(\Lambda^l_i)}_{i,Y_{k^i_l}} = \eta_{i,l}$ and $Y_{k^i_l} = Y_{i,l}$, $i, l \in \{1, \ldots, N\}$. Using repeatedly (5.4.96), we obtain

$$\begin{aligned} \left(\nu_i|\nabla_i(\mathbf{E}_X f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} &\leq e^{\frac{1}{2}\sum_{l=1}^N \eta_{i,l}} \left(\mathbf{E}_X \nu_i|\nabla_i f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \\ &\quad + \sum_{l=1}^N A \cdot \eta_{i,l} \sum_{j\in Y_{i,l}} \left(\mathbf{E}_i \mathbf{E}_\Lambda \nu_j|\nabla_j(\mathbf{E}_{\Lambda^l_i} f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{5.4.97}$$

To bound the second term in the above inequality,

- we apply, for $j \in Y_{i,l}$ so that $d(j, \Lambda^l_i) > R$, lemma 5.10 (with $\Theta = \Lambda^l_i$ and $\Lambda = \emptyset$).
- We notice that for the $j \in Y_{i,l}$ so that $d(j, \Lambda^l_i) \leq R$, we can write a bound for Λ^l_i similar to that obtained in (5.4.97),

$$\left(\nu_j|\nabla_j(\mathbf{E}_{\Lambda^l_i} f)^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \leq e^{\frac{1}{2}\sum_{l_2=1}^{N-l} \eta^{(2)}_{j,l_2}} \left(\mathbf{E}_X \nu_j|\nabla_j f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \tag{5.4.98}$$

$$+\sum_{l_2=1}^{N-l} A \cdot \eta^{(2)}_{j,l_2} \sum_{j_2 \in Y_{j,l_2}} \left(\mathbf{E}_i \mathbf{E}_\Lambda \nu_{j_2} |\nabla_j (\mathbf{E}_{\Lambda_j^{l,2}} f)^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}}$$

where this time $(k^j_{l_2}, 1 \leq l_2 \leq N-l)$ is a lexicographical order on $(k^i_m, m \in \{l,\ldots,N\})$ so that $l_2 \to d(j, Y_{k^j_{l_2}})$ is decreasing, $\Lambda_j^{m,2} = \bigcup_{l_2=1}^{N-l} Y_{k^j_{l_2}}$ and we set $Y_{j,l_2} = Y_{k^j_{l_2}}$, $\eta^{(\Lambda_j^{l_2,2})}_{j,Y_{j,l_2}} = \eta^{(2)}_{j,l_2}$.

Applying these two arguments inductively, we finally arrive at

$$\left(\nu_i |\nabla_i (\mathbf{E}_X f)^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \leq \alpha^{(\Lambda)}_{ii} \left(\mathbf{E}_X \nu_i |\nabla_i f^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} + \sum_{j \in \Lambda} \alpha^{(\Lambda)}_{ij} \left(\mathbf{E}_i \mathbf{E}_X \nu_j |\nabla_j f^{\frac{1}{2}}|^2 \right)^{\frac{1}{2}} \tag{5.4.99}$$

with

$$\alpha^{(\Lambda)}_{ii} \leq e^{\frac{1}{2}\sum_{l=1}^N \eta^i_l} \tag{5.4.100}$$

and

$$\alpha^{(\Lambda)}_{ij} \leq \sum_{1 \leq n \leq N} \sum_{Y^i_1, Y^i_2, \ldots, Y^i_n} A^n \eta^{(\Lambda)}_{i,Y^i_1} \eta^{(\Lambda_2)}_{Y^i_1,Y^i_2} \cdots \eta^{(\Lambda_n)}_{Y^i_{n-1},Y^i_n} \max\{b, \max_{i \in \Lambda} \alpha^{(\Lambda)}_{ii}\} \tag{5.4.101}$$

where we have denoted $\eta^{(\Lambda)}_{X,Y} = \max_{i \in X} \eta^{(\Lambda)}_{i,Y}$ and where the sum holds over all the random walks $(Y^i_1, Y^i_2, \ldots, Y^i_n)$ on the cubes in Λ such that $d(i, Y^i_1) = \max_{1 \leq k \leq N} d(i, Y_k)$ and

$$d(Y^i_k, Y^i_{k-1}) = \max_{Y \in \Lambda \setminus \cup_{p=1}^{k-1} Y^i_p} d(Y_{k-1}, Y) \tag{5.4.102}$$

and $j \in Y^i_n$. Further, $\Lambda_k = \Lambda \setminus \cup_{p=1}^{k-1} Y^i_p$.

Now, assuming that condition (**SMC**) is satisfied, we see that by definition of the coefficients $\eta^{(\Lambda)}_{i,X}$ there exist two constants $C, \varepsilon \in (0,\infty)$ such that for any $\Lambda \subset \mathbb{Z}^d$, any $X \subset \mathbb{Z}^d$ and $i \in \mathbb{Z}^d$,

$$\eta^{(\Lambda)}_{i,X} \leq C e^{-\varepsilon d(i,X)}$$

and hence

$$\eta^{(\Lambda)}_{X,Y} \leq C e^{-\varepsilon d(Y,X)}.$$

In particular, if $d(Y^i_k, Y^i_{k-1}) \geq D_0 \equiv (2/\epsilon) \left(||\Phi||(1+2L_0^d) + \frac{1}{2}\log 2(|M|/m_0) \right)$,

$$A \eta^{(\Lambda_k)}_{Y^i_{k-1}, Y^i_k} \leq C e^{-(\varepsilon/2) d(Y^i_k, Y^i_{k-1})}. \tag{5.4.103}$$

We then conclude that

$$\alpha^{(\Lambda)}_{ij} \leq C_0 e^{-C_1|i-j|} \tag{5.4.104}$$

for constants $C_0, C_1 \in (0,\infty)$ depending only on L_0. The proof is complete.

◇

5.4.4 Comments on the strong mixing assumptions

To complete this section, we comment here on the mixing conditions we assumed to prove the logarithmic Sobolev inequality . We shall see that they are optimal with respect to our strategy in the sense that when $\mathbf{\Pi}$ satisfies condition (Ciii), some strong mixing property will be satisfied. We shall also show how to check that (**SDC**)(and hence (**SMC**)) is satisfied in high temperature models.

Optimality of strong mixing assumption

Let us consider the symmetric transfer matrix

$$\mathbf{T} \equiv \mathbf{\Pi}^* \mathbf{\Pi}$$

with $\mathbf{\Pi}$ as in section 5.4.2. Its adjoint $\mathbf{\Pi}^*$ is given by a similar formula but with the conditional expectations coming in reverse order. It is clear that by using the same argument as in the last paragraph 5.4.3, and assuming (**SMC**), one finds the following analogous of property (5.4.46) for $\mathbf{T}$

$$|\nabla\left(\mathbf{T}f\right)^{\frac{1}{2}}|^2 \leq \lambda^2 |\nabla f^{\frac{1}{2}}|^2 \tag{5.4.105}$$

with a constant $\lambda \in (0,1)$. We shall see that this bound implies itself the strong mixing assumption. For further use, we recall we have the following estimate of the variance of $\mathbf{T}^N f$, for $N \in I\!N$,

$$\begin{aligned} \mu(\mathbf{T}^N f; \mathbf{T}^N f) &= \frac{1}{2} \mu \otimes \widetilde{\mu} \left(\mathbf{T}^N f - \widetilde{\mathbf{T}}^N f \right)^2 \\ &\leq 2 || \left(\mathbf{T}^N f\right)^{\frac{1}{2}} ||_\infty \cdot \sum_i || \nabla_i \left(\mathbf{T}^N f\right)^{\frac{1}{2}} ||_\infty . \end{aligned} \tag{5.4.106}$$

If f is localized in $\Lambda(f)$, then by construction of $\mathbf{T}$, the function $\mathbf{T}^N f$ is localized in the set $\Lambda_N \equiv \{ i \ : \ d(i, \Lambda(f)) \leq 4d(L+R)N \}$ satisfying $|\Lambda_N| \leq CN^d$ with a constant $C \leq \left(4d(L+R) + D(f)\right)^d$ if $D(f)$ is the diameter of $\Lambda(f)$. We obtain in particular that

$$\sum_i || \nabla_i \left(\mathbf{T}^N f\right)^{\frac{1}{2}} ||_\infty \leq C N^d \left(|| \, |\nabla \left(\mathbf{T}^N f\right)^{\frac{1}{2}} |^2 \, ||_\infty \right)^{\frac{1}{2}} . \tag{5.4.107}$$

Thus, applying (5.4.105) with (5.4.106)-(5.4.107), we arrive at the following bound

$$\mu(\mathbf{T}^N f; \mathbf{T}^N f) \leq 2 || f^{\frac{1}{2}} ||_\infty \cdot C N^d \lambda^N \left(|| \, |\nabla f^{\frac{1}{2}}|^2 \, ||_\infty \right)^{\frac{1}{2}} . \tag{5.4.108}$$

Since $\mathbf{T}$ is self-adjoint with non negative eigenvalues, its spectral radius (see Bourbaki, [9], p. 15) is given by

$$\sup_{f\in\mathcal{C}_1}\lim_{N\to\infty}\left(\mu(\mathbf{T}^N f;\mathbf{T}^N f)\right)^{\frac{1}{2N}} \leq \lambda^{\frac{1}{2}} \tag{5.4.109}$$

and so

$$\mu(\mathbf{T}^N f;\mathbf{T}^N f) \leq \lambda^N \mu(f;f). \tag{5.4.110}$$

Let us now assume that we are given two functions f and g localized in two disjoint sets $\Lambda(f)$ and $\Lambda(g)$. Denoting $N \equiv [d(\Lambda(f),\Lambda(g))/C]$, and assuming N even (up to replace it by $N+1$) we see that

$$\Lambda(\mathbf{T}^{\frac{N}{2}} f)\cap\Lambda(\mathbf{T}^{\frac{N}{2}} g) = \emptyset$$

so that $\mathbf{T}^{\frac{N}{2}}(fg) = \mathbf{T}^{\frac{N}{2}}(f)\mathbf{T}^{\frac{N}{2}}(g)$. In particular, as μ is invariant for $\mathbf{T}$,

$$|\mu(f;g)| = |\mu(\mathbf{T}^{\frac{N}{2}} f;\mathbf{T}^{\frac{N}{2}} g)| \leq \left(\mu(\mathbf{T}^{\frac{N}{2}} f;\mathbf{T}^{\frac{N}{2}} f)\right)^{\frac{1}{2}}\left(\mu(\mathbf{T}^{\frac{N}{2}} g;\mathbf{T}^{\frac{N}{2}} g)\right)^{\frac{1}{2}} \tag{5.4.111}$$

$$\leq \lambda^{N/2}\left(\mu(f;f)\right)^{\frac{1}{2}}\left(\mu(g;g)\right)^{\frac{1}{2}}$$

Since $(\mu(f;f))^{\frac{1}{2}} \leq C|||f|||$ for a finite constant C, we conclude that we also have

$$|\mu(f;g)| \leq C^2 e^{-M[dist(\Lambda(f),\Lambda(g))-1]}|||f|||\cdot|||g||| \tag{5.4.112}$$

with

$$M \geq -\frac{1}{2}\log\lambda$$

Similar arguments can be developed in any finite volume (by modifying $\mathbf{T}$ accordingly) so that the strong mixing condition is indeed equivalent with condition (5.4.105).

Proof of (SMC)in the high temperature models

Here, we consider local Gibbs measures $(\mathbf{E}^\omega_\Lambda, \Lambda \subset\subset \mathbb{Z}^d, \omega\in\Omega)$ defined by a finite range potential $\mathbf{\Phi}$. We shall assume that $||\Phi||$ is sufficiently small (corresponding to the high temperature situations) and shall show that the strong decay of correlation property (or equivalently the strong mixing property) is satisfied. We follow here the papers [59] and [31].

On $\Lambda \subset\subset \mathbb{Z}^d$, let us consider a lexicographical order $(j_i, i = 1,\ldots,|\Lambda|)$ and set

$$V^i_\Lambda = \sum_{*}\Phi_X$$

where the sum is taken over all the finite subsets X with non empty intersection with Λ, containing j_i but not $(j_k, k<i)$. With such a definition,

$$U_\Lambda(x) = \sum_{i=1}^{|\Lambda|} V^i_\Lambda(x).$$

Up to replace U_Λ by $U_\Lambda - \int U_\Lambda d\nu_\Lambda$, we may always assume that the energy U_Λ is centered with respect to the product measure ν_Λ. Then, Jensen's inequality implies that the partition function Z^ω_Λ is bounded below by 1.

Let f and g be two bounded measurable functions localized respectively in $\Lambda(f)$ and $\Lambda(g)$. We are going to prove that, if $||\Phi||$ is sufficiently small, there exists a constant $\epsilon > 0$ so that

$$E^\omega_\Lambda(f;g) \leq e^{-\epsilon d(\Lambda(f),\Lambda(g))} |||f|||.|||g|||. \tag{5.4.113}$$

To this end, let us first notice that

$$E^\omega_\Lambda(f;g) = \frac{1}{2(Z^\omega_\Lambda)^2} \int (f - \widetilde{f})(g - \widetilde{g}) e^{-U_\Lambda - \widetilde{U}_\Lambda} d\nu_\Lambda(x) d\nu_\Lambda(\widetilde{x}). \tag{5.4.114}$$

Denoting $z^i_\Lambda = e^{-V^i_\Lambda - \widetilde{V}^i_\Lambda} - 1$, we can write

$$e^{-U_\Lambda - \widetilde{U}_\Lambda} = \prod_{i=1}^{|\Lambda|} (z^i_\Lambda + 1) = \sum_{\mathbf{i}} \prod_{k \in \mathbf{i}} z^k_\Lambda$$

where the sum $\mathbf{i}$ goes over all the subsets of $\{1, \ldots, |\Lambda|\}$. Observe that the $(z^i_\Lambda, i \in \{1, \ldots, N\})$ are localized into sets of radius bounded by the range of the interaction R. Hence, introducing this decomposition into the right hand side of (5.4.114), we see that only the $\mathbf{i}$' such that the $(j_k, k \in \mathbf{i})$ make a path of points in Λ at distance less or equal to R joining $\Lambda(f)$ and $\Lambda(g)$ will contribute. Denote $\mathcal{W}_{\Lambda(f),\Lambda(g)}$ the set of these paths. We then have, since $Z^\omega_\Lambda \geq 1$,

$$|E^\omega_\Lambda(f;g)| \leq \frac{1}{2} \sum_{\mathbf{i} \in \mathcal{W}_{\Lambda(f),\Lambda(g)}} \int |f - \widetilde{f}||g - \widetilde{g}| \prod_{k \in \mathbf{i}} |z^k_\Lambda| d\nu_\Lambda(x) d\nu_\Lambda(\widetilde{x}) \tag{5.4.115}$$

According to (5.4.66), we have $|f - \widetilde{f}| \leq a|||f|||$. Moreover, the following uniform bound

$$|z^k_\Lambda| \leq \delta \equiv 2||\Phi|| e^{2||\Phi||}$$

holds. Further, we observe that, in any dimension d, we can find a finite constant $C_{d,R}$ depending on the number of neighbours of a point in $\mathbb{Z}^d$ and of the range R so that

$$|\mathcal{W}_{\Lambda(f),\Lambda(g)}| \leq C_{d,R}^{d(\Lambda(f),\Lambda(g))}.$$

Hence, we conclude according to (5.4.115) that

$$|E^\omega_\Lambda(f;g)| \leq a^2 C_{d,R}^{d(\Lambda(f),\Lambda(g))} \delta^{\frac{1}{R} d(\Lambda(f),\Lambda(g))} |||f|||.|||g|||$$

which gives the announced statement if $\delta^{\frac{1}{R}} C_{d,R} < 1$, that is if $||\Phi||$ is sufficiently small.

Chapter 6

Logarithmic Sobolev inequalities and cellular automata

In this part, we introduce and study cellular automata. This approach to obtain log-Sobolev inequalities for measures possibly non related to a given potential was introduced by one of the authors (as an extension of the result for finite convolutions contained in [103]). It was later studied by G. Gielis ([42]) who used an idea based on disagreement of percolation to cover an extended high temperature domain. Here we shall use cellular automata to establish logarithmic Sobolev inequality for dynamical systems with possibly infinite range of interaction. The transition probability of a parallel cellular automata is described by a product probability ; if $\mathcal{C}$ is a countable set, we consider the transition probability on $\Omega = \mathbf{M}^{\mathcal{C}}$ given by

$$\mathcal{P}f(\omega) = \int f(\widetilde{\omega}) \otimes_{i\in\mathcal{C}} p_i^{\omega}(d\widetilde{\omega}_i) \tag{6.0.1}$$

where $(p_i^{\omega}, i \in \mathcal{C})$ are probability measures on $\mathbf{M}$ which are absolutely continuous with respect to the uniform measure

$$p_i^{\omega}(d\widetilde{\omega}_i) = \rho_i^{\omega}(\widetilde{\omega}_i)\nu_i(d\widetilde{\omega}_i).$$

As before, we shall concentrate on the case where $\mathbf{M}$ is a finite set, the continuous setting being easier to analyze. The operator $\mathcal{P}$ given by (6.0.1) can be considered as the transition matrix of a Markov chain.

Let $(\mathcal{P}^{n,\omega}, n \in \mathbb{N}^*)$ be the family of transition probability measures defined by induction as follows $\mathcal{P}^{n+1,\omega}f = \mathcal{P}(\mathcal{P}^{n,\cdot}f)(\omega)$, $\mathcal{P}^{0,\omega}f = f(\omega)$. We shall show that, under some conditions specified later, the following logarithmic Sobolev inequality is satified with a constant $c \in (0,\infty)$ independent of $n \in \mathbb{N}^*$ and $\omega \in \Omega$

$$\mathcal{P}^{n,\omega}\left(f\log f\right)-\mathcal{P}^{n,\omega}f\log\mathcal{P}^{n,\omega}f\leq c\,\mathcal{P}^{n,\omega}|\nabla f^{\frac{1}{2}}|^{2}. \tag{6.0.2}$$

We shall see that under proper assumptions $\mathcal{P}^{n,\omega}$ converges as n goes to infinity towards a probability measure μ on Ω, the limit being independent of the configuration $\omega\in\Omega$. This limit law μ will also satisfy a logarithmic Sobolev inequality . The limiting probability measure has a priori no link with the Gibbs measures introduced in the previous chapter. Also, we point out that the transition matrix $\mathcal{P}$ is in general not symmetric in $L^2(\mu)$.

We begin with the following central proposition

Proposition 6.1 *Suppose that*

$$A\equiv\sup_i\sup_{\omega,\widetilde{\omega}}\prod_k\sup_{\omega_{\{i\}^c}=\widetilde{\omega}_{\{i\}^c}}\left\|\frac{\rho_k^{\omega}}{\rho_k^{\widetilde{\omega}}}\right\|_\infty<\infty \tag{6.0.3}$$

and

$$\sup_i\max\left[\sum_{j\in\mathcal{C}}\xi_{ij}\,,\,\sum_{j\in\mathcal{C}}\xi_{ji}\right]<\infty \tag{6.0.4}$$

with the notation

$$\xi_{ij}\equiv\sup_{\omega_{\{i\}^c}=\widetilde{\omega}_{\{i\}^c}}\left\|\frac{\rho_j^{\omega}}{\rho_j^{\widetilde{\omega}}}-1\right\|_\infty. \tag{6.0.5}$$

Then, there exists a constant $\lambda\in(0,\infty)$ such that for any non negative function f, we have

$$|\nabla(\mathcal{P}f)^{\frac{1}{2}}|^2\leq\lambda\mathcal{P}|\nabla f^{\frac{1}{2}}|^2. \tag{6.0.6}$$

Remark 6.2: The constant λ can be chosen for example as follows

$$\lambda\equiv b^2A^4c_0\sup_i\max\left[\sum_{j\in\mathcal{C}}\xi_{ij}\,,\,\sum_{j\in\mathcal{C}}\xi_{ji}\right]^2 \tag{6.0.7}$$

with $b\leq|\mathbf{M}|^{\frac{1}{2}}$ if $|\mathbf{M}|\geq 3$ and $b=1$ if $\mathbf{M}$ has cardinality 2 or is a Riemannian manifold. The constant c_0 can be chosen equal to $\sup_{j,\omega}\|\rho_j^{\omega}\|_\infty\cdot\|\left(\rho_j^{\omega}\right)^{-1}\|_\infty$.

Proof of Proposition 6.1. For any $i\in\mathcal{C}$ and any non negative function f, we can show as in (5.4.72) that

$$|\nabla_i(\mathcal{P}f)^{\frac{1}{2}}|^2\leq b^2\,\nu_i|\nabla_i(\mathcal{P}f)^{\frac{1}{2}}|^2=b^2\,\nu_i\left((\mathcal{P}f)^{\frac{1}{2}};(\mathcal{P}f)^{\frac{1}{2}}\right) \tag{6.0.8}$$

The right hand side of this last inequality can be rewritten as

$$\nu_i\left((\mathcal{P}f)^{\frac{1}{2}};(\mathcal{P}f)^{\frac{1}{2}}\right)=\frac{1}{2}\int\nu_i(d\omega_i)\otimes\nu_i(d\widetilde{\omega}_i)\left(\frac{\mathcal{P}f(\omega)-\mathcal{P}f(\widetilde{\omega})}{(\mathcal{P}f)^{\frac{1}{2}}(\omega)+(\mathcal{P}f)^{\frac{1}{2}}(\widetilde{\omega})}\right)^2 \tag{6.0.9}$$

with the notation $\widetilde{\omega} \equiv \omega_{\{i\}^c} \circ \widetilde{\omega}_i$. Using a lexicographic order $\{j_l \in \mathcal{C}\}_{l\in\mathbf{N}}$, we obtain

$$\mathcal{P}f(\omega) - \mathcal{P}f(\widetilde{\omega}) = \sum_{l\in\mathbf{N}} \mathcal{P}_l \left(p^{\omega}_{j_l} f - p^{\widetilde{\omega}}_{j_l} f \right) \tag{6.0.10}$$

where $\mathcal{P}_l \equiv \mathcal{P}_{l,i,\omega,\widetilde{\omega}} \equiv \otimes_{s<l}\, p^{\omega}_{j_s}\ \otimes_{s>l}\ p^{\widetilde{\omega}}_{j_s}$. With the notation (6.0.5), we have

$$\begin{aligned} |p^{\omega}_{j_l} f - p^{\widetilde{\omega}}_{j_l} f| &= |p^{\omega}_{j_l}(\frac{\rho^{\widetilde{\omega}}_{j_l}}{\rho^{\omega}_{j_l}}; f)| \le 2^{\frac12} \|\xi_{ij_l}\|_\infty \left(p^{\omega}_{j_l}(f)\right)^{\frac12} \left(p^{\omega}_{j_l}(f^{\frac12}; f^{\frac12})\right)^{\frac12} \\ &\le 2^{\frac12} c_0^{\frac12}\, \|\xi_{ij_l}\|_\infty \left(p^{\omega}_{j_l}(f)\right)^{\frac12} \left(p^{\omega}_{j_l} |\nabla_{j_l} f^{\frac12}|^2\right)^{\frac12} \end{aligned} \tag{6.0.11}$$

with a constant $c_0 \le \sup_{j,\omega} \|\rho^{\omega}_j\|_\infty \cdot \| \left(\rho^{\omega}_j\right)^{-1} \|_\infty$. We now deduce from (6.0.10) and (6.0.11) that

$$|\mathcal{P}f(\omega) - \mathcal{P}f(\widetilde{\omega})| \le \sum_{l\in\mathbf{N}} 2^{\frac12} c_0^{\frac12} \|\xi_{ij_l}\|_\infty \left(\mathcal{P}_l \otimes p^{\omega}_{j_l}(f)\right)^{\frac12} \left(\mathcal{P}_l \otimes p^{\omega}_{j_l} |\nabla_{j_l} f^{\frac12}|^2\right)^{\frac12}. \tag{6.0.12}$$

Now, if

$$A \equiv \sup_i \sup_{\omega,\widetilde{\omega}\in\Omega} \prod_{k\in\mathcal{C}} \sup_{\omega_{\{i\}^c} = \widetilde{\omega}_{\{i\}^c}} \left\| \frac{\rho^{\omega}_k}{\rho^{\widetilde{\omega}}_k} \right\|_\infty ,$$

let us observe that for any $l \in \mathbf{N}$, and all $\omega, \widetilde{\omega} \in \Omega$, we have

$$A^{-1} \cdot \mathcal{P}F(\omega) \le \mathcal{P}_l \otimes p^{\omega}_{j_l} F(\omega, \widetilde{\omega}) \le A \cdot \mathcal{P}F(\omega) \tag{6.0.13}$$

so that we can estimate the right hand side of (6.0.9) by

$$\begin{aligned} \left(\frac{\mathcal{P}f(\omega) - \mathcal{P}f(\widetilde{\omega})}{(\mathcal{P}f)^{\frac12}(\omega) + (\mathcal{P}f)^{\frac12}(\widetilde{\omega})} \right)^2 &\le \left(\sum_{j\in\mathcal{C}} \left[2^{\frac12} A c_0^{\frac12} \xi_{ij} \right] \left(\mathcal{P}|\nabla_j f^{\frac12}|^2 \right)^{\frac12} \right)^2 (\omega) \\ &\le \left[2A^2 c_0 \sum_{j\in\mathcal{C}} \xi_{ij} \right] \sum_{j\in\mathcal{C}} \xi_{ij} \mathcal{P} |\nabla_j f^{\frac12}|^2(\omega) \end{aligned} \tag{6.0.14}$$

where - we recall - that ω and $\widetilde{\omega}$ only differ at the site i. Inserting this result in (6.0.9) and using (6.0.8), we obtain

$$\begin{aligned} |\nabla_i (\mathcal{P}f)^{\frac12}|^2 &\le b^2 \left[A^2 c_0 \sum_{j\in\mathcal{C}} \xi_{ij} \right] \sum_{j\in\mathcal{C}} \xi_{ij} \int \nu_i(d\omega_i) \mathcal{P} |\nabla_j f^{\frac12}|^2 \\ &\le b^2 \left[A^4 c_0 \sum_{j\in\mathcal{C}} \xi_{ij} \right] \sum_{j\in\mathcal{C}} \xi_{ij} \mathcal{P} |\nabla_j f^{\frac12}|^2 \end{aligned} \tag{6.0.15}$$

which gives, after summation over the $i \in \mathcal{C}$,

$$|\nabla(\mathcal{P}f)^{\frac{1}{2}}|^2 \le \left(b^2 A^4 c_0 \sup_i \max \left[\sum_{j\in\mathcal{C}} \xi_{ij} \, , \, \sum_{j\in\mathcal{C}} \xi_{ji} \right]^2 \right) \mathcal{P}|\nabla f^{\frac{1}{2}}|^2 \qquad (6.0.16)$$

which finishes the proof of the proposition. $\diamond$

Proposition 6.1 will be the key ingredient in the proof of the following theorem.

Theorem 6.3 *Assume λ of (6.0.6) is strictly smaller than one. Then there exists $c \in (0, \infty)$ so that*

$$\mathcal{P}^{n,\omega}(f \log f) - \mathcal{P}^{n,\omega} f \log \mathcal{P}^{n,\omega} f \le c\, \mathcal{P}^{n,\omega} |\nabla f^{\frac{1}{2}}|^2 \qquad (6.0.17)$$

for any $n \in I\!N$ and $\omega \in \Omega$, and every non negative differentiable function f. Consequently, the probability measure $\mu \equiv \lim_{n\to\infty} \mathcal{P}^{n,\omega}$ satisfies the logarithmic Sobolev inequality with the same constant.

Proof. For a non negative differentiable function f, we set $f_n \equiv \mathcal{P} f_{n-1}$ with the convention $f_0 \equiv f$.

$$\mathcal{P}^{\omega} f_{n-1} \log f_{n-1} - \mathcal{P}^{\omega} f_{n-1} \log \mathcal{P}^{\omega} f_{n-1} \le \widetilde{c}_0\, \mathcal{P}^{\omega} |\nabla f_{n-1}^{\frac{1}{2}}|^2 \qquad (6.0.18)$$

By proposition 6.1, we find that

$$\mathcal{P}^{\omega} f_{n-1} \log f_{n-1} - \mathcal{P}^{\omega} f_{n-1} \log \mathcal{P}^{\omega} f_{n-1} \le 2\widetilde{c}_0 \lambda^{n-1}\, \mathcal{P}^{\omega} |\nabla f^{\frac{1}{2}}|^2. \qquad (6.0.19)$$

Thus, we deduce, as in section 5.2 (see (5.2.5) and (5.2.6)), (6.0.17) from (6.0.19) with $c = (\widetilde{c}_0/1 - \lambda)$ when $\lambda < 1$. To prove the second part of the theorem, let us remark that using (6.0.10), for any $(\omega, \widetilde{\omega}) \in \Omega^2$ so that $\omega_{\{i\}^c} = \widetilde{\omega}_{\{i\}^c}$, we have

$$|\mathcal{P}^{\omega} f - \mathcal{P}^{\widetilde{\omega}} f| = \left| \sum_l \mathcal{P}_l \left(p_{j_l} \left((\frac{\rho_j^{\omega}}{\rho_j^{\widetilde{\omega}}} - 1) \nabla_{j_l} f \right) \right) \right| \qquad (6.0.20)$$

and therefore

$$|\mathcal{P}^{\omega} f - \mathcal{P}^{\widetilde{\omega}} f| \le A \sum_{j\in\mathcal{C}} \xi_{ij} \mathcal{P} |\nabla_j f|. \qquad (6.0.21)$$

Now, for any $n, m \in I\!N$, $n > m$, we see that

$$|\mathcal{P}^{n,\omega} f - \mathcal{P}^{m,\omega} f| \le \sup_{\omega,\bar{\omega}} |\mathcal{P}^m f(\omega) - \mathcal{P}^m f(\bar{\omega})|. \qquad (6.0.22)$$

It is not hard to see that

$$|\mathcal{P}^m f(\omega) - \mathcal{P}^m f(\bar{\omega})| \le \sum_{i\in\mathcal{C}} ||\nabla_i \mathcal{P}^m f||_{\infty} \equiv |||\mathcal{P}^m f|||. \qquad (6.0.23)$$

Since (6.0.21) implies that for any i, $||\nabla_i \mathcal{P} f||_\infty \le A \sum_j \xi_{ij} ||\nabla_j f||_\infty$, we deduce that

$$|||\mathcal{P}^m f||| \le \left(A \sup_i \sum_j \xi_{ij} \right)^m |||f||| \tag{6.0.24}$$

which, under our hypotheses, converges towards zero as $m \to \infty$. Hence, we have shown that $(\mathcal{P}^n f)_{n \in \mathbb{N}}$ is a Cauchy sequence, uniformly with respect to the boundary conditions $\omega \in \Omega$. This is sufficient to guarantee, as we saw in part 5.4, that $(\mathcal{P}^n)_{n \in \mathbb{N}}$ converges towards a unique probability measure μ.

$\diamond$

In the last part of this chapter, we discuss the case where the cellular automata is described by a potential. In other words, if for a potential Φ of possibly infinite range of interaction and a subset $\mathcal{C}$ of $\mathbb{Z}^d$, we set $U_j \equiv \sum_{X \subset\subset \mathcal{C} : X \ni j} \Phi_X$, then we define the transition matrix $\mathcal{P}_\Phi$ by (6.0.1) with $\rho_j = e^{-U_j} / \nu_j e^{-U_j}$.

To study the ergodic properties of $\mathcal{P}_\Phi$, we shall try to find natural conditions under which the assumptions of the last proposition are satisfied. With the same notation as above, let us first note that for any $\omega_{\{i\}^c} = \widetilde{\omega}_{\{i\}^c}$, we have

$$\left\| \frac{\rho_k^\omega}{\rho_k^{\widetilde{\omega}}} \right\|_\infty \le \exp\left(2 \sum_{X \subset\subset \mathcal{C} : X \ni i,k} Var_X(\Phi_X) \right) \tag{6.0.25}$$

In this case, we see that

$$A \le \exp\left(2 \sup_i \sum_{X \subset\subset \mathcal{C} : X \ni i} |X| \cdot Var_X(\Phi_X) \right) \le e^{4||\Phi||_{\mathbf{B}_2}} \tag{6.0.26}$$

and that

$$\xi_{ij} \le 2 \sum_{X \subset\subset \mathcal{C} : X \ni i,k} Var_X(\Phi_X) \exp\left(2 \sum_{X \subset\subset \mathcal{C} : X \ni i,k} Var_X(\Phi_X) \right). \tag{6.0.27}$$

Here, we used the notation

$$||\Phi||_{\mathbf{B}_2} \equiv \sup_{i \in \mathbf{Z}^d} \sum_{X \subset\subset \mathbf{Z}^d :\, X \ni i} |X| \cdot ||\Phi_X||_\infty. \tag{6.0.28}$$

We shall assume in the sequel that $\Phi \equiv \{\Phi_X\}_{X \subset\subset \mathbf{Z}^d}$ belongs to the Banach space $\mathbf{B}_2$ of potential with finite $|| \cdot ||_{\mathbf{B}_2}$ norm. Then,

$$\sup_i \max \left[\sum_{j \in \mathcal{C}} \xi_{ij} \,,\, \sum_{j \in \mathcal{C}} \xi_{ji} \right] < 2 e^{2||\Phi||_{\mathbf{B}_2}} ||\Phi||_{\mathbf{B}_2} \tag{6.0.29}$$

Using the previous proposition, we can hence write down the following result.

Theorem 6.4 *Assume that the cellular automata $\mathcal{P}_\Phi$ is described by a potential $\Phi \in \mathbf{B}_2$. If $||\Phi||_{\mathbf{B}_2} < \beta_0 \in (0, \infty)$, for a sufficiently small $\beta_0 \in (0, \infty)$, then there exists a unique invariant measure μ for the semi-group $\mathcal{P} \equiv \mathcal{P}_\Phi$ satisfying for some constant $\varepsilon > 0$ and for any differentiable function f,*

$$||\mathcal{P}^n f - \mu f||_\infty \leq e^{-n\varepsilon}|||f|||. \tag{6.0.30}$$

Moreover, there exists a finite constant $c \in (0, \infty)$ such that

$$\mu f \log f/\mu f \leq 2c\mu|\nabla f^{\frac{1}{2}}|^2 \tag{6.0.31}$$

for any non negative function for which the above right hand side is finite.

Remark 6.5: The reader interested in the ergodicity questions of the cellular automata may like to consult for example [67], [42] and [43] and the references therein. It is interesting to note that the question whether the limiting probability measure of a cellular automata is a Gibbs measure or not has not been addressed in the general setting of a potential $\Phi \in \mathbf{B}_2$.

Chapter 7

Logarithmic Sobolev inequalities for spin systems with long range interaction, Martingale expansion.

We now come back to a spin system described by a potential $\Phi \equiv \{\Phi_X\}_{X \subset\subset \mathbf{Z}^d}$ as introduced in chapter 5. However, we remove the assumption of finite range of the interaction to extend the previous results to $\Phi \in \mathbf{B}_2$, that is satisfying

$$||\Phi||_{\mathbf{B}_2} \equiv \sup_{i \in \mathbf{Z}^d} \sum_{X \subset\subset \mathbf{Z}^d :\, X \ni i} |X| \cdot ||\Phi_X||_\infty < \infty.$$

We shall prove that when the uniqueness condition of Dobrushin is fulfilled, the unique Gibbs measure in infinite volume satisfies a logarithmic Sobolev inequality. To this end, we shall first recall a few facts from the uniqueness theory of Dobrushin ([24], [38], [88], [41]). Then, to prove the logarithmic Sobolev inequality, we follow an approach based on martingale expansion introduced in [64]. To simplify the notations, we restrict ourselves to the case $\mathbf{M} = \{-1, +1\}$.

7.0.5 Long range interaction systems

Given a family of probability kernels $\{\mathbf{E}_k \equiv \mathbf{E}_{k,\Phi}\}_{k \in \mathbf{Z}^d}$ on a probability space Ω equipped with a σ-algebra Σ, the interaction matrix C_{kl}, $k, l \in \mathbf{Z}^d$ of Dobrushin is defined by

$$C_{kl} \equiv \sup_{\omega, \widetilde{\omega} \in \Omega : \omega_{\{l\}^c} = \widetilde{\omega}_{\{l\}^c}} \sup_A |\mathbf{E}_k^\omega(A) - \mathbf{E}_k^{\widetilde{\omega}}(A)| \tag{7.0.1}$$

where the supremum is taken over all Σ-measurable sets A.

Hereafter, we shall assume that the following condition is satisfied

Dobrushin's uniqueness condition

$$\sup_k \sum_{l\in\mathbf{Z}^d} C_{kl} < 1 \tag{7.0.2}$$

Let us recall the very nice result due to H. Föllmer [38]

Theorem 7.1 *Assume Dobrushin's uniqueness condition holds. Then, for any* $\Lambda \subset \mathbf{Z}^d$ *and* $\omega \in \Omega$,*we have*

$$|\mathbf{E}^\omega_\Lambda(f\,;\,g)| \leq \sum_{k,l\in\Lambda} Var_k(f) D_{kl}\, Var_l(g) \tag{7.0.3}$$

with

$$D_{kl} \equiv \sum_{n=0}^{\infty} C^n_{kl}. \tag{7.0.4}$$

Remark 7.2: One can see as in section 5.4.4 that Dobrushin uniqueness condition is satisfied when for a sufficiently small positive real number β_0

$$||\Phi||_{\mathbf{B}_2} < \beta_0$$

but that this condition is not necessary (see [88], [41] for examples of possibly large potentials and for some other types of potentials see [74] , [35]). Theorem 7.1 shows that the local Gibbs measures satisfy a property of decay of correlations, the crucial step to obtain a logarithmic Sobolev inequality in the previous parts. In order to use it to prove a logarithmic Sobolev inequality, we shall first prove an auxiliary lemma which shall be essential to get sweeping out relations. To this end, let us denote as in section 5.4.3,

$$\xi_{i,\Lambda}(\omega) \equiv \frac{d\mathbf{E}^{\omega^{(i)}}_\Lambda}{d\mathbf{E}^\omega_\Lambda} = \frac{e^{-\{U_\Lambda(\omega^{(i)})-U_\Lambda(\omega)\}}}{\left[Z^{\omega^{(i)}}_\Lambda / Z^\omega_\Lambda\right]} \tag{7.0.5}$$

with $\omega^{(i)}_{\mathbf{Z}^d\setminus i} = \omega_{\mathbf{Z}^d\setminus i}$ and $\omega^{(i)}_i = -\omega_i$, then

Lemma 7.3 *If Dobrushin uniqueness condition holds, for any* $\Lambda \subset\subset \mathbf{Z}^d$ *and all* $i \in \Lambda^c$, $j \in \Lambda$, *the quantity*

$$\eta_{ij} \equiv \sup_{\Lambda\subset\subset\mathbf{Z}^d} Var_j(\mathbf{E}_{\Lambda\setminus j}\xi_{i,\Lambda}) \tag{7.0.6}$$

satisfies

$$\sup_i \max\left(\sum_{j\in\mathbf{Z}^d} \eta_{ij}\,,\, \sum_{j\in\mathbf{Z}^d} \eta_{ji}\right) < \infty. \tag{7.0.7}$$

Proof : To estimate the quantities $\mathrm{Var}_j(\mathbf{E}_{\Lambda\setminus j}\xi_{i,\Lambda})$, let us remark that

$$|\mathbf{E}^{\omega}_{\Lambda\setminus j}\xi_{i,\Lambda} - \mathbf{E}^{\omega^{(j)}}_{\Lambda\setminus j}\xi_{i,\Lambda}| \leq |\mathbf{E}^{\omega}_{\Lambda\setminus j}\left(\xi_{i,\Lambda};\xi_{j,\Lambda\setminus j}\right)| + \mathrm{Var}_j(\xi_{i,\Lambda}). \tag{7.0.8}$$

Using theorem 7.1, we obtain

$$|\mathbf{E}^{\omega}_{\Lambda\setminus j}\left(\xi_{i,\Lambda};\xi_{j,\Lambda\setminus j}\right)| \leq \sum_{k,l\in\Lambda\setminus j} \mathrm{Var}_k(\xi_{i,\Lambda}) D_{kl} \mathrm{Var}_l(\xi_{j,\Lambda\setminus j}). \tag{7.0.9}$$

Consequently, to bound (7.0.9) as well as the second term in the right hand side of (7.0.8), we have to bound $\mathrm{Var}_k(\xi_{i,\Lambda})$. To this end, let us go back to the explicit definition (7.0.5), to obtain

$$\mathrm{Var}_k(\xi_{i,\Lambda}) \leq \left\| Z^{\omega}_{\Lambda}/Z^{\omega^{(i)}}_{\Lambda}\right\|_{\infty} \mathrm{Var}_k(e^{-\{U_\Lambda(\omega^{(i)})-U_\Lambda(\omega)\}}) \leq \widetilde{C}_{ik} \tag{7.0.10}$$

where we have set

$$\widetilde{C}_{ik} \equiv 2e^{2||\Phi||} \sum_{X\ni i,k} \mathrm{Var}_X(\Phi_X).$$

To use this bound, we note first that $\widetilde{C}_{ik}$ is summable in i or k. In fact, changing the order of summation according to Fubini's theorem for non negative variables and since $\mathrm{Var}_X(\Phi_X) \leq 2||\Phi_X||_\infty$, we get

$$\frac{1}{2}e^{-2||\Phi||}\sum_{k\in\mathbf{Z}^d}\widetilde{C}_{ik} = \sum_{k\in\mathbf{Z}^d}\sum_{X\ni i,k}\mathrm{Var}_X(\Phi_X) \leq 2\sum_{X\ni i}|X|\cdot||\Phi_X||_\infty \leq 2||\Phi||_{\mathbf{B}_2} \tag{7.0.11}$$

with the right hand side finite with our hypothesis. We can thus bound (7.0.9) as follows

$$|\mathbf{E}^{\omega}_{\Lambda\setminus j}\left(\xi_{i,\Lambda};\xi_{j,\Lambda\setminus j}\right)| \leq \sum_{k,l\in\Lambda\setminus j}\widetilde{C}_{ik}D_{kl}\widetilde{C}_{jl}. \tag{7.0.12}$$

Since under the Dobrushin uniqueness condition, the family D_{kl} is summable with respect to k,l according to theorem 7.1, we deduce from (7.0.10) and (7.0.11) that the right hand side of (7.0.12) is also summable over i and j. Combining this remark with the bounds (7.0.8) and (7.0.9) completes the proof. $\diamond$

Lemma 7.3 is the key to the following theorem.

Theorem 7.4 *Assume that the Dobrushin uniqueness condition is satisfied for a system with potential $\Phi \in \mathbf{B}_2$. Then, there exists a constant $c \in (0,\infty)$ such that for any $\Lambda \subset \mathbf{Z}^d$ and $\omega \in \Omega$, we have*

$$\mathbf{E}^{\omega}_{\Lambda} f\,log\,(\frac{f}{\mathbf{E}^{\omega}_{\Lambda}f}) \leq c\,\mathbf{E}^{\omega}_{\Lambda}|\nabla_\Lambda f^{\frac{1}{2}}|^2 \tag{7.0.13}$$

for any non negative function f such that the right hand side is well defined.

E. Laroche [60] proved a similar result for exponentially fast decreasing interactions. (Also, note that the method used in [70] proves similar results for exponentially decaying interactions.) Here we present a proof based on lemma 7.3 and a martingale expansion of relative entropy, (introduced in [64]).

Proof : Let us consider a lexicographical order $\{i_k, k=1,..,N=|\Lambda|\}$ of Λ and set $\Lambda_1=\{i_1\}$, $\Lambda_{n+1}=\Lambda_n\cup\{i_{n+1}\}$. Let $\mathbf{E}_n=\mathbf{E}_{\Lambda_n}$, and, for a non negative measurable function f, set $f_n=\mathbf{E}_n f$, $f_0=f$. We then deduce from lemma 7.5 applied with cubes reduced to single point sets that

$$\mathbf{E}_N f\log\frac{f}{\mathbf{E}_N f}\le c_0\sum_{n=1}^{N}\mathbf{E}_N|\nabla_{i_n}f_{n-1}^{\frac{1}{2}}|^2 \tag{7.0.14}$$

with c_0 the smallest constant in the log-Sobolev inequalities satisfied by all the local Gibbs measures restricted to a single spin σ algebra related to points $\{i_k, 1\le k\le N\}$, uniformly with respect to boundary conditions. Applying lemma 7.3 and proceeding as in section 5.4.3, it is not difficult to see that there exist non negative coefficients $\alpha_{ij}^{(\Lambda)}$ so that for any $n\in\{1,..,N\}$,

$$|\nabla_{i_{n+1}}(\mathbf{E}_n f)^{\frac{1}{2}}|\le\sum_{j\in\Lambda_n\cup\{i_{n+1}\}}\alpha_{i_{n+1}j}^{(\Lambda_n)}\left(\mathbf{E}|\nabla_j f^{\frac{1}{2}}|^2\right)^{\frac{1}{2}} \tag{7.0.15}$$

with

$$\gamma_{ij}\equiv\sup_{\Lambda\subset\subset\mathbf{Z}^d}\alpha_{ij}^{(\Lambda)} \tag{7.0.16}$$

satisfying

$$\gamma\equiv\sup_i\max\left(\sum_{j\in\mathbf{Z}^d}\gamma_{ij},\sum_{j\in\mathbf{Z}^d}\gamma_{ji}\right)<\infty. \tag{7.0.17}$$

It is then easy to deduce from (7.0.14) that

$$\mathbf{E}_N f\log\frac{f}{\mathbf{E}_N f}\le c_0\cdot\gamma^2\mathbf{E}_N|\nabla_{\Lambda_N}f^{\frac{1}{2}}|^2. \tag{7.0.18}$$

Indeed, inequality (7.0.15) gives

$$\begin{aligned}|\nabla_i(\mathbf{E}_n f)^{\frac{1}{2}}|^2 &\le \left(\sum_{j\in\Lambda_n\cup\{i\}}\alpha_{ij}^{(\Lambda_n)}\right)\sum_{j\in\Lambda_n\cup\{i\}}\alpha_{ij}^{(\Lambda_n)}\mathbf{E}|\nabla_j f^{\frac{1}{2}}|^2 \qquad (7.0.19)\\ &\le \gamma\cdot\sum_{j\in\Lambda_n}\gamma_{i_nj}\mathbf{E}_n|\nabla_j f^{\frac{1}{2}}|^2\end{aligned}$$

so that

$$\begin{aligned}\sum_{n=1}^{N}\mathbf{E}_N|\nabla_{Y_n}f_{n-1}^{\frac{1}{2}}|^2 &\le \gamma\sum_{n=1}^{N}\cdot\sum_{j\in\Lambda_n\cup\{i\}}\gamma_{ij}\mathbf{E}_N|\nabla_j f^{\frac{1}{2}}|^2 \qquad (7.0.20)\\ &\le \gamma^2\sum_{j\in\Lambda_N}\mathbf{E}_N|\nabla_j f^{\frac{1}{2}}|^2=\gamma^2\mathbf{E}_N|\nabla_{\Lambda_N}f^{\frac{1}{2}}|^2.\end{aligned}$$

(7.0.15) thus implies the desired estimate. $\diamond$

7.0.6 Martingale expansions

In this section we present a different useful way of organizing the proofs using a martingale expansion of relative entropy(introduced first in [64]; see also [70]). We consider spin variables with values in a finite set or a smooth connected Riemaniann manifold $\mathbf{M}$. Let $L_0 \in 2\mathbb{N}$ and $Y_k \equiv Y_0 + L_0 \cdot y_k$ be the translation of the cube $Y_0 = [-L_0/2, L_0/2]^d$ centered at the origin by the vector $L_0 y_k$, $y_k \in \mathbb{Z}^d$. The vectors $(y_k, k \in \mathbb{N})$ are ordered according to a lexicographic order compatible with the distance $d(\cdot,\cdot)$ on $\mathbb{Z}^d$. We define a sequence of finite subsets of $\mathbb{Z}^d$ by $\Lambda_1 \equiv Y_1$ and $\Lambda_{n+1} \equiv \Lambda_n \cup Y_{n+1}$. We denote in short $\mathbf{E}_n \equiv \mathbf{E}_{\Lambda_n}$. For a continuously differentiable function f, we set $f_0 \equiv f$ and $f_{n+1} \equiv \mathbf{E}_{n+1} f_n = \mathbf{E}_{n+1} f$. We then have the following

Lemma 7.5 *There exists a constant $c_0 \in (0,\infty)$ depending only on the size of Y_0 such that for any $N \in \mathbb{N}$ and any continuously differentiable non negative function f, we have*

$$\mathbf{E}_N f \log \frac{f}{\mathbf{E}_N f} \leq c_0 \sum_1^N \mathbf{E}_N |\nabla_{Y_n} f_{n-1}^{\frac{1}{2}}|^2. \tag{7.0.21}$$

Proof : Let us first notice that

$$\mathbf{E}_n \left(f_{n-1} \log \frac{f_{n-1}}{\mathbf{E}_n f_{n-1}} \right) = \mathbf{E}_{n,Y_n} \left(f_{n-1} \log \frac{f_{n-1}}{\mathbf{E}_{n,Y_n} f_{n-1}} \right) \tag{7.0.22}$$

with $\mathbf{E}_{n,Y_n}$ the restriction of $\mathbf{E}_n$ to the σ-algebra Σ_{Y_n}. According to lemma 7.6 below, at any point $\omega \in \Omega$, the measure $\mathbf{E}^{\omega}_{n,Y_n}$ satisfies a logarithmic Sobolev inequality with a constant $c_0 \in (0,\infty)$ independent of $\omega \in \Omega$ and $n \in \mathbb{N}$. Consequently, we get

$$\mathbf{E}_n \left(f_{n-1} \log \frac{f_{n-1}}{\mathbf{E}_n f_{n-1}} \right) \leq c_0 \, \mathbf{E}_n |\nabla_{Y_n} f_{n-1}^{\frac{1}{2}}|^2 \tag{7.0.23}$$

Using this inequality with (7.0.22), we obtain lemma 7.5. It thus remains to prove the

Lemma 7.6 *Let $c_{Y,X}$ be the constant in the logarithmic Sobolev inequality for the restriction $\mathbf{E}_X|\Sigma_{(X\setminus Y)^c}$ of $\mathbf{E}_X$ to the σ-algebra $\Sigma_{(X\setminus Y)^c}$. Then, for any continuously differentiable non negative function f localized in a set $Y \subset X$, we have*

$$\mathbf{E}_X f \log \frac{f}{\mathbf{E}_X f} \leq c_{Y,X} \, \mathbf{E}_X |\nabla_Y f^{\frac{1}{2}}|^2 \tag{7.0.24}$$

with

$$0 < c_{Y,X} < \widetilde{c}_0 \cdot e^{4||\Phi||\cdot|Y|} \tag{7.0.25}$$

if c_0 is the constant in the logarithmic Sobolev inequality for the probability measure ν.

This lemma is a direct consequence of property 4.6. Indeed, the probability measure $\mathbf{E}_X^\omega$ is absolutely continuous with respect to the probability measure $\widetilde{\mathbf{E}}_{X,Y}^\omega \equiv \mathbf{E}_X^\omega(e^{U_Y}\cdot)/\mathbf{E}_X^\omega(e^{U_Y}) = \delta_{\omega_{X^c}} \otimes \nu_Y \otimes \widetilde{\mu}_{X\setminus Y}^\omega$ if ν_Y is the product probability measure on $(x_i, i \in Y)$. The corresponding density $\rho_{Y,X}^\omega(\widetilde{\omega}_Y) \equiv \frac{d\mathbf{E}_X^\omega}{d\widetilde{\mathbf{E}}_{X,Y}^\omega}$ satisfies

$$e^{-2||\Phi||\cdot|Y|} \leq \rho_{Y,X}^\omega(\widetilde{\omega}_Y) \leq e^{+2||\Phi||\cdot|Y|} \tag{7.0.26}$$

so that property 4.6 provides the desired estimate.

$\diamond$

Chapter 8

Markov semi-group in infinite volume, ergodic properties

In this chapter, we study Markov semi-groups acting on functions of infinitely many variables of $\mathbf{M}^{\mathbb{Z}^d}$. We first construct them as limits of semi-groups described in section 5.1 with localized potentials. This construction is important to insure that such semi-groups are Feller continuous, but also to be able to approximate them by Markov semi-group in finite volume (see the exponential approximation property of theorem 8.2) which are easier to study. Such a construction can be found in the literature in [60] (for the continuous setting) and [62], [94] (for the discrete setting); see also the references given there. We then study the uniform ergodicity of these infinite volume semi-groups when the corresponding Gibbs measure satisfies a logarithmic Sobolev inequality. We show that they converge uniformly towards this Gibbs measure with an exponential rate. This result is actually part of the so-called equivalence theorem which states equivalence between such a uniform convergence of the semi-groups with a logarithmic Sobolev property of the Gibbs measure, but also with other properties of the Gibbs measure such as spectral gap inequality. We discuss this theorem in the last section of this chapter.

8.1 Construction of Markov semi-groups in infinite volume

In this section, we present a construction of Markov semi-groups in infinite volume when $\mathbf{M}$ is a finite set (the continuous setting being left as an exercise to the reader).

Let $\Omega = \mathbf{M}^{\mathbb{Z}^d}$ be the configuration set and $\{\mathbf{E}_X^{\omega}\}_{X \subset\subset \mathbb{Z}^d, \omega \in \Omega}$ be a local specification constructed with the product measure $\mu_0 = \nu^{\otimes \mathbb{Z}^d}$ and a interaction potential Φ with finite range. We shall use the notations of the chapter 5 (see section 5.1).

We can easily define, for $X \subset\subset \mathbb{Z}^d$, finite volume semi-groups by their generators

$$\mathcal{L}_\Lambda = \sum_{j:X+j\subset\Lambda} \mathcal{L}_{X+j}$$

with

$$\mathcal{L}_{X+j} f = \mathbf{E}_{X+j} f - f.$$

We denote by $P_t^{(\Lambda)}$ the corresponding semi-group.

Exercise 8.1 *Show that for any subset Λ the operator $\mathcal{L}_\Lambda$ formally given above is well defined on the space of functions f for which the following semi-norm is finite*

$$|||f||| \equiv \sum_{j\in\mathbb{Z}^d} ||\nabla_j f||_u$$

were $\nabla_j f \equiv f - \nu_j f$.

The goal of this section is to prove the following result

Theorem 8.2 *For any local function f, the following limit in the uniform norm exists*

$$P_t f := \lim_{\Lambda\uparrow\mathbb{Z}^d} P_t^{(\Lambda)} f$$

and defines a Feller-continuous Markov semi-group on $\mathcal{C}(\Omega)$.

Moreover, we have the following exponential approximation property : for any $A \in \mathbb{R}^+$, there exists a constant $B \in [1,\infty)$ depending only on A such that if $\Lambda \subset \mathbb{Z}^d$ is sufficiently large, contains $\Lambda(f)$ and dist$(\Lambda(f), \Lambda^c) \geq Bt$, then

$$||P_t f - P_t^{(\Lambda)} f||_\infty \leq e^{-At} |||f|||. \tag{8.1.1}$$

Proof :

We will show that for any couple $(\Lambda_1, \Lambda_2) \in (\mathbb{Z}^d)^2$, $\Lambda_1 \subset \Lambda_2$ so that $\Lambda_2\backslash\Lambda_1$ is a cube of a given size, the semi-groups $P^1 := P^{(\Lambda_1)}$ and $P^2 := P^{(\Lambda_2)}$ satisfy

$$||P_t^1 f - P_t^2 f||_\infty \leq e^{-At-AN} |||f||| \tag{8.1.2}$$

if

$$\text{dist}(\Lambda(f), \Lambda_2\backslash\Lambda_1) \geq Bt$$

for a finite constant B depending only on A for

$$N = \left[\frac{\text{dist}(\Lambda(f), \Lambda_2\backslash\Lambda_1)}{R + \text{diam}(X)}\right]$$

with the square bracket denoting the entire part.

To prove this estimate, note that

$$P_t^2 f - P_t^1 f \quad = \quad \int_0^t ds \frac{d}{ds} P_{t-s}^1 P_s^2 f = \int_0^t ds P_{t-s}^1 (\mathcal{L}_2 - \mathcal{L}_1) P_s^2 f \quad (8.1.3)$$

where we have denoted in short $\mathcal{L}_i := \mathcal{L}_{\Lambda_i}$ for $i = 1, 2$.

¿From (8.1.3), we immediately deduce that

$$\begin{aligned} ||P_t^2 f - P_t^1 f||_\infty \quad &\leq \quad \int_0^t ds ||(\mathcal{L}_2 - \mathcal{L}_1) P_s^2 f||_\infty \\ &\leq \quad C_0 \int_0^t ds \sum_{j \in \Lambda_2 \setminus \Lambda_1} ||\nabla_j P_s^2 f||_\infty \end{aligned} \quad (8.1.4)$$

with a finite constant C_0 depending only on the size of $\Lambda_2 \setminus \Lambda_1$ and the interaction Φ.

We are hence naturally interested in the quantities $\nabla_j \widetilde{P}_s f$ for a finite volume semi-group $\widetilde{P}$. We present below a simple study of these quantities.

Let us first remark that

$$\begin{aligned} \nabla_j \widetilde{P}_t f - \widetilde{P}_t \nabla_j f \quad &= \quad \int_0^t ds \frac{d}{ds} \widetilde{P}_{t-s} \nabla_j \widetilde{P}_s f \\ &= \quad \int_0^t ds \widetilde{P}_{t-s} [\nabla_j, \widetilde{\mathcal{L}}] \widetilde{P}_s f \end{aligned} \quad (8.1.5)$$

with $[\nabla_j, \widetilde{\mathcal{L}}] = \nabla_j \widetilde{\mathcal{L}} - \widetilde{\mathcal{L}} \nabla_j$. We also point out that, due to the local structure of the generator, we have

$$[\nabla_j, \widetilde{\mathcal{L}}] = \sum_{k : d(k,j) \leq R + \mathrm{diam} X} [\nabla_j, \mathcal{L}_{X+k}]. \quad (8.1.6)$$

Moreover, one can easily see that for any smooth function F, we have

$$||[\nabla_j, \mathcal{L}_{X+k}] F||_\infty \leq \sum_{l \in \{j\} \cup X + k} \alpha_{jl} ||\nabla_l F||_\infty \quad (8.1.7)$$

for uniformly bounded constants α_{jl}. We deduce from (8.1.5), (8.1.6) and (8.1.7) that

$$||\nabla_j \widetilde{P}_t f||_\infty \leq ||\nabla_j f||_\infty + \int_0^t ds \sum_l D_{jl} ||\nabla_l \widetilde{P}_s f||_\infty \quad (8.1.8)$$

with a matrix D with uniformly bounded coefficients such that

$$D_{jl} = 0 \qquad \text{if} \qquad d(j,l) > R + \mathrm{diam}(X).$$

Since $\nabla_j f = 0$ if j does not belong to $\Lambda(f)$, we can use (8.1.8) inductively to conclude that

$$||\nabla_j \widetilde{P}_t f||_\infty \leq \sum_{n=N_j}^{\infty} \frac{t^n}{n!} \sum_l D_{jl}^{(n)} ||\nabla_l f||_\infty \tag{8.1.9}$$

where

$$N_j = \left[\frac{\mathrm{dist}(j, \Lambda(f))}{R + \mathrm{diam}(X)}\right]$$

and $D_{jl}^{(n)}$ is the jl^{th} entry of the matrix D^n the n-th power of the matrix D. We can estimate these coefficients by noting that if C is a finite constant satisfying

$$D_{jl} \leq \frac{C}{[2(R + \mathrm{diam}(X)) + 1]^d}$$

for any (jl), we have

$$D_{jl}^{(n)} \leq C^n.$$

As a consequence,

$$||\nabla_j \widetilde{P}_t f||_\infty \leq \sum_{n=N_j}^{\infty} \frac{t^n}{n!} C^n |||f||| \leq \frac{(tC)^{N_j}}{N_j!} e^{tC} |||f|||. \tag{8.1.10}$$

Observing that for any $n \in I\!N^*$,

$$n! > e^{n \log n - 2n}$$

and choosing $B \in (0, \infty)$ such that

$$2 - \log B + \log C + \frac{C}{B} \leq -2A$$

we conclude that for

$$N = \inf_j N_j = \left[\frac{\mathrm{dist}(\Lambda_2 \backslash \Lambda_1, \Lambda(f))}{R + \mathrm{diam}(X)}\right] \geq Bt$$

we have

$$||\nabla_j \widetilde{P}_t f||_\infty \leq e^{-At - AN} |||f|||. \tag{8.1.11}$$

Using (8.1.11) with $\widetilde{P} = P^2$ in (8.1.4), we deduce that for any sequence $\{P^{(\Lambda_i)}, i \in I\!N\}$ of Markov semi-groups such that the Λ_i's are a Van Hove sequence (i.e are constructed by addition of the translation of a given cube), $\{P^{(\Lambda_i)} f, i \in I\!N\}$ is Cauchy for any local function f. Hence, the limit

$$P_t f = \lim_{\Lambda \uparrow \mathbb{Z}^d} P_t^{(\Lambda)} f$$

exists for any local function f. Furthermore, if $\Lambda = \Lambda_N$, we have

$$\begin{aligned} \|P_t f - P_t^{(\Lambda)} f\|_\infty &\leq \sum_{n=N}^{\infty} \|P_t^{(\Lambda_{n+1})} f - P_t^{(\Lambda_n)} f\|_\infty \\ &\leq tC_0 \sum_{n=N}^{\infty} \sum_{j \in \Lambda_n \setminus \Lambda_{n-1}} \|\nabla_j P_t^{(\Lambda_n)} f\|_\infty \\ &\leq tC_0 e^{-At} \sum_{n=N}^{\infty} \sum_{j \in \Lambda_n \setminus \Lambda_{n-1}} e^{-AN_n} |||f||| \\ &\leq te^{-At} \left(C_0 |\Lambda_0| \sum_{L=N}^{\infty} L^{d-1} e^{-AL} \right) |||f||| \quad (8.1.12) \end{aligned}$$

with the notation

$$N_n = \left[\frac{\mathrm{dist}(\Lambda_n \setminus \Lambda_{n-1}, \Lambda(f))}{R + \mathrm{diam}(X)} \right].$$

This last estimate completes the proof of the exponential approximation. As a consequence, since in finite volume $P^{(\Lambda)}$ is strongly continuous, the same is true for P. Since the local functions are dense in $\mathcal{C}(\Omega)$, P extends naturally to a Feller-continuous Markov semi-group with generator

$$\mathcal{L} = \sum_{j \in \mathbb{Z}^d} \mathcal{L}_{X+j}.$$

Exercise 8.3 *Generalize theorem 8.2 to the continuous case, i.e. to the case where* $\mathbf{M}$ *is a smooth connected Riemannian manifold and* $\mathcal{L}_\Lambda$ *has the form*

$$\mathcal{L}_\Lambda = \sum_{i \in \mathbb{Z}^d} (\Delta_i + \nabla_i H_\Lambda . \nabla_i).$$

where $H_\Lambda \equiv \sum_{X \subset \Lambda} \Phi_X$. *Hint : Follow the arguments of the proof of theorem 8.2.*

Exercise 8.4 *In the case where the potential* Φ *is not of finite range, but satisfies*

$$\|\Phi\|_{\mathbf{B}_2} \equiv \sup_{i \in \mathbb{Z}^d} \sum_{X \ni i} |X| \cdot \|\Phi_X\|_u < \infty$$

show that the following exponential approximation property holds ; for any $A > 0$ *and any* $\delta > 0$, *there exists a finite constant* C *so that*

$$\|P_t f - P_t^{(\Lambda)} f\|_\infty \leq e^{-At} |||f|||$$

provided that

$$d(\Lambda, \Lambda(f)^c) \geq Ce^{\delta t}.$$

8.2 Uniform ergodicity of Markov semi-groups in infinite volume

In this section, we summarize the main links between the log-Sobolev inequality and the uniform ergodicity of Markov semi-groups in infinite volume, (see [55],[56] and [92]-[95]).

Again, we are given a local specification $\{\mathbf{E}_X^{\omega}\}_{X\subset\subset \mathbb{Z}^d,\omega\in\Omega}$ defined by a uniform product measure $\mu_0=\nu^{\otimes \mathbb{Z}^d}$ and a finite range interaction potential Φ.

Then, the following relations are satisfied

Theorem 8.5 *Assume that we can define a Gibbs measure μ in infinite volume and that it satisfies the logarithmic Sobolev inequality with a coefficient c.*

In finite volume, assume that there exists a finite constant C_1 such that for any non negative function f, any $\Lambda\subset\subset \mathbb{Z}^d$,

$$||P_1^{(\Lambda)}f||_\infty \le e^{C_1|\Lambda|}\mathbf{E}_\Lambda[f]. \tag{8.2.13}$$

Moreover, assume that the finite volume exponential approximation is satisfied, that is that for any local function f, any $A\in \mathbb{R}^+$

$$||P_tf-P_t^{(\Lambda)}f||_\infty \le e^{-At}|||f||| \tag{8.2.14}$$

provided that for sufficiently large Λ,

$$dist(\Lambda(f),\Lambda^c)\ge Bt$$

for a finite constant B depending only on A.

Then, for any $\theta\in(0,1)$, there exist a finite constant $C(\theta,\Lambda(f))$ and a positive real number $m\in[1/c,\infty[$ such that for any $t\ge 0$,

$$||P_tf-\mu f||_\infty \le C(\theta,\Lambda(f))e^{-\theta m t}|||f|||.$$

<u>Remark</u> 8.6: If $\{E_X, X\subset\subset \mathbb{Z}^d\}$ satisfy a classical Nash inequality with constants growing at most exponentially with the volume, (8.2.13) is satisfied according to theorem 3.3.

Proof : Let us first note that the following decomposition holds

$$|P_tf-\mu f|\le |P_t^{(\Lambda)}f-\mu f|+|P_t^{(\Lambda)}f-P_tf|. \tag{8.2.15}$$

The second term will be estimated by the exponential approximation property. For the first term on the right hand side of (8.2.15), we note that, by Hölder's inequality, for any $q\ge 1$,

$$\begin{aligned} |P_t^{(\Lambda)}f-\mu f| &= |P_1^{(\Lambda)}(P_{t-1}^{(\Lambda)}f-\mu f)| \\ &\le |P_1^{(\Lambda)}|P_{t-1}^{(\Lambda)}f-\mu f|^q|^{\frac{1}{q}} \\ &\le e^{\frac{C_1|\Lambda|}{q}}\left(\mathbf{E}_\Lambda|P_{t-1}^{(\Lambda)}f-\mu f|^q\right)^{\frac{1}{q}} \end{aligned} \tag{8.2.16}$$

where we used hypothesis (8.2.13). Moreover, by definition of $P_t^{(\Lambda)}$, for any time t, $P_t^{(\Lambda)}f - \mu f$ is Σ_{Λ^R} measurable if f is when $\Lambda^R = \{i, d(i,\Lambda) \leq R\}$. Hence, (using the assumption that the potential Φ is bounded and of finite range), we can find a finite constant C_2 so that

$$\mathbf{E}_\Lambda |P_{t-1}^{(\Lambda)} f - \mu f|^q \leq e^{C_2|\partial\Lambda|} \mu(P_{t-1}^{(\Lambda)} f - \mu f)^q.$$

Noting as well that

$$\left(\mu|P_{t-1}^{(\Lambda)} f - \mu f|^q\right)^{\frac{1}{q}} \leq ||P_{t-1}^{(\Lambda)} f - P_{t-1} f||_\infty + (\mu|P_{t-1}f - \mu f|^q)^{\frac{1}{q}} \qquad (8.2.17)$$

we deduce from (8.2.16) that

$$\begin{aligned} ||P_t^{(\Lambda)} f - \mu f||_\infty \quad &\leq \quad e^{\frac{C_1|\Lambda| + C_2|\partial\Lambda|}{q}} \left\{ (\mu|P_{t-1}f - \mu f|^q)^{\frac{1}{q}} + ||P_{t-1}^{(\Lambda)} f - P_{t-1} f||_\infty \right\} \\ &\quad + ||P_t^{(\Lambda)} f - P_t f||_\infty. \qquad (8.2.18) \end{aligned}$$

By the exponential approximation property, we conclude that if f is a local function and Λ is chosen sufficiently large, so that $\Lambda(f) \subset \Lambda$ and its diameter is of order $t \geq 1$,

$$||P_t^{(\Lambda)} f - \mu f||_\infty \quad \leq \quad e^{\frac{C_1'(t+1)t^{d-1}}{q}} (\mu(P_{t-1}f - \mu f)^q)^{\frac{1}{q}} + e^{\frac{C_1'(t+1)t^{d-1}}{q}} e^{-At} |||f|||$$

for a finite constant C_1'. If moreover μ satisfies a logarithmic Sobolev inequality with a coefficient $c < \infty$, let us recall that theorem 4.1 implies that for any $\theta \in (0,1)$ such that $q \leq q(\frac{t-1}{\theta t}, 2, c) = 1 + e^{\frac{2(1-\theta)t-2}{c}}$,

$$(\mu|P_{t-1}f - \mu f|^q)^{\frac{1}{q}} \leq \left(\mu(P_{\theta t} f - \mu f)^2\right)^{\frac{1}{2}} \leq e^{-m\theta t} ||f - \mu f||_2$$

where m is the spectral gap of $\mathcal{L}$ which is bounded below by c^{-1} (see theorem 4.9). With such a choice of q, the factor $e^{\frac{C_1'(t+1)t^{d-1}}{q}}$ is uniformly bounded in t and we obtain the desired estimate.

$\diamond$

Exercise 8.7 *Extend the theorem to the case where $\Phi \in \mathbf{B}_2$.*

8.3 Equivalence Theorem

We consider a local specification $\{\mathbf{E}_\Lambda^\omega\}_{\Lambda \subset\subset \mathbb{Z}^d, \omega \in \Omega}$ defined by a product measure $\mu_0 = \nu^{\otimes \mathbb{Z}^d}$ and an interaction potential $\Phi = (\phi_X)_{X \subset\subset \mathbb{Z}^d}$ with finite range R. When needed, we shall indicate more explicitly the potential defining the local specification by the following notation $\{\mathbf{E}_{\Lambda,\Phi}^\omega\}_{\Lambda \subset\subset \mathbb{Z}^d, \omega \in \Omega}$.

For $\Lambda \subset\subset \mathbb{Z}^d$, we shall denote $P_t^{(\Lambda,\omega)} = e^{t\mathcal{L}_{\Lambda,\omega}}$ the Markov semi-group generated by

$$\mathcal{L}_{\Lambda,\omega} = \delta_{\omega_{\Lambda^c}}(\sum_{i\in\Lambda} \mathcal{L}_i)$$

that is the semi-group acting on the variables $\{\sigma_i, i \in \Lambda\}$ and with fixed boundary conditions. There, we used the notation

$$(\delta_{\omega_{\Lambda^c}} \mathcal{L}_i) f(\widetilde{\omega}) = \mathbf{E}_{\{i\}} f(\widetilde{\omega}_{\Lambda\cap\{i\}^c} \circ \omega_{\Lambda^c\cap\{i\}^c}) - f(\widetilde{\omega}_\Lambda \circ \omega_{\Lambda^c})$$

in the discrete case and

$$(\delta_{\omega_{\Lambda^c}} \mathcal{L}_i) f(\widetilde{\omega}) = (\Delta_i - \nabla_i U_i(\widetilde{\omega}_\Lambda \circ \omega_{\Lambda^c}).\nabla_i) f(\widetilde{\omega}_\Lambda \circ \omega_{\Lambda^c})$$

in the continuous setting.

Now, we present the result from [92] relating mixing conditions and ergodicity.

Theorem 8.8 *The following conditions are equivalent*

(i) *Strong mixing conditions*

There exists $M \in (0,\infty)$ such that for any $\Lambda \subset\subset \mathbb{Z}^d$, any $\omega \in \Omega$, and any local functions f and g localized in Λ,

$$|\mathbf{E}^\omega_\Lambda(f,g)| \leq e^{-Md(\Lambda(f),\Lambda(g))} |||f|||.|||g|||.$$

(ii) *Complete analyticity condition :*

For any $n \in \mathbb{N}$ and any potentials $\{\psi_k, k = 1,..,n\}$ with finite range, the map

$$(z_1,..,z_n) \to \mathbf{E}^\omega_{\Lambda,\Phi+\sum_{k=1}^n z_k\psi_k}(f)$$

is analytic in a neighborhood $\cap\{|z_k| \leq \epsilon\}$ of the origin for some ϵ independent of $\Lambda \subset\subset \mathbb{Z}^d$, $\omega \in \Omega$ and of the local function f.

(iii) *Uniform spectral gap inequality :*

There exists a constant $m \in (0,\infty)$ such that for any $\Lambda \subset\subset \mathbb{Z}^d$, and all $\omega \in \Omega$,

$$m\ \mathbf{E}^\omega_\Lambda(f - \mathbf{E}^\omega_\Lambda f)^2 \leq \mathbf{E}^\omega_\Lambda(|\nabla f|^2)$$

for any function f for which the right hand side of the above inequality is well defined and finite.

(iv) *Uniform logarithmic Sobolev inequality :*

There exists a constant $c \in (0,\infty)$ such that for any $\Lambda \subset\subset \mathbb{Z}^d$, and all $\omega \in \Omega$,

$$\mathbf{E}^\omega_\Lambda(f \log \frac{f}{\mathbf{E}^\omega_\Lambda f}) \leq c\mathbf{E}^\omega_\Lambda(|\nabla f|^2)$$

for any non negative function f for which the right hand side of the above inequality is well defined and finite.

Remarks :

1) The equivalence between (i) and (ii) is due to Dobrushin and Shlosman [19] who proved the equivalence between some 14 conditions encountered in statistical mechanics. The proof of the equivalence with (iii) and (iv) was given first in [92].

2) Similar assumptions remain equivalent if one replaces the sequence of finite sets by a Van Hove sequence. In this setting, the equivalence between (i) and (ii) is due to E. Olivieri and P. Picco [72], the others to S.Lu and H.T. Yau [64] as well as F.Martinelli and E. Olivieri [70]. These results allow to extend the domain of validity of logarithmic Sobolev inequalities to a larger class of potentials.

3) The implication (iv) gives (iii) was already seen and is a direct consequence of the definition of logarithmic Sobolev inequality . The converse implication is in general rather unusual. It is intimately related to the assumption that the Spectral Gap inequalities are assumed to be obtained uniformly with respect to the finite sets Λ and the boundary conditions.

4) The implication (i) gives (iv) can be proved as discussed in details in the previous chapters.

5) The fact that (iii) implies (i) is due to the exponential approximation property. Indeed, we have

$$\begin{aligned}
\mathbf{E}^{\omega}_{\Lambda}(f,g) &= \mathbf{E}^{\omega}_{\Lambda}(fg) - \mathbf{E}^{\omega}_{\Lambda}(f)\mathbf{E}^{\omega}_{\Lambda}(g) \\
&= \mathbf{E}^{\omega}_{\Lambda}(P_t^{(\Lambda,\omega)}(fg)) - \mathbf{E}^{\omega}_{\Lambda}(f)\mathbf{E}^{\omega}_{\Lambda}(g)
\end{aligned}$$

Using the exponential approximation property, we get

$$\left| P_t^{(\Lambda,\omega)} fg - P_t^{(\Lambda,\omega)} f P_t^{(\Lambda,\omega)} g \right| \leq e^{-At} |||f|||.|||g|||$$

as long as

$$d(\Lambda(f), \Lambda(g)) \approx Bt$$

for some $B = B(A) \in (0, \infty)$. As a consequence,

$$\begin{aligned}
|\mathbf{E}^{\omega}_{\Lambda}(f,g)| &\leq |\mathbf{E}^{\omega}_{\Lambda}(P_t^{(\Lambda,\omega)} f, P_t^{(\Lambda,\omega)} g)| + e^{-At}|||f|||.|||g||| \\
&\leq \left(\mathbf{E}^{\omega}_{\Lambda}(P_t^{(\Lambda,\omega)} f - \mathbf{E}^{\omega}_{\Lambda} f)^2 \mathbf{E}^{\omega}_{\Lambda}(P_t^{(\Lambda,\omega)} g - \mathbf{E}^{\omega}_{\Lambda} g)^2 \right)^{\frac{1}{2}} + e^{-At}|||f|||.|||g|||
\end{aligned}$$

Using the spectral gap inequality, we obtain (i).

Chapter 9

Disordered systems ; uniform ergodicity in the high temperature regime

The simplest example of disordered systems we consider is described by the following formal Hamiltonian

$$H(\sigma) = \sum_{|i-j|=1} J_{ij}\sigma_i \cdot \sigma_j$$

with the spins σ_i in $\{-1,+1\}$ for the Ising type models, and for continuous models such as the rotator, the spins take on values in a smooth manifold such as an N-dimensional sphere. (For example if $N = 2$, one has the representation

$$\sigma_i.\sigma_j = \cos(\phi_i - \phi_j)$$

with $\phi_i \in [0, 2\pi]$.)

In this chapter, the J_{ij} will be taken at random. It is assumed that the J_{ij} are independent and identically distributed.

One can easily imagine that in such systems on the infinite lattice $\mathbb{Z}^d$, large regions will have strong couplings. Thus the interaction in such regions will be of low temperature type. However, if the inverse of the temperature β, (used to scale the Hamiltonian in the formal expresion for the Gibbs measure), is sufficiently small, "most" of the spins in the system will effectively interact weakly. The system is therefore mostly of the type studied previously, but exhibits with probability one (with respect to the realization of the couplings) large regions of strong interaction. As a consequence of this general picture, the Glauber dynamics should still converge at high temperature towards the unique Gibbs measure but its convergence will be slowed down due to these regions with strong couplings.

To illustrate these ideas, we shall first describe the proof given in [104] of the absence of spectral gap at any temperature for models in which the J_{ij} can be as large as one wishes with positive probability.

Secondly, we shall bound almost surely the growth of constant in logarithmic Sobolev inequality for local Gibbs measures in dimension 2 as a function of the volume and deduce a stretched exponential decay of the dynamics towards equilibrium (that is a decay with rate going to zero as the exponential of $-t^{\theta}$ for some $\theta \in (0,1)$).

9.1 Absence of spectral gap for disordered ferromagnetic Ising model

Here, we restrict ourselves to the Ising model

$$\sigma_i \in \{-1,+1\}, \qquad \Omega = \{-1,+1\}^{\mathbb{Z}^d}.$$

We shall also assume $d \geq 2$, We consider the Hamiltonian given, for any $\Lambda \subset\subset \mathbb{Z}^d$, by

$$H_\Lambda^\omega(\sigma) \equiv H_\Lambda^\omega(\mathbf{J},\sigma) \equiv -\sum_{|i-j|=1, i,j\in\Lambda} J_{ij}\sigma_i\sigma_j + \sum_{|i-j|=1, i\in\Lambda, j\in\Lambda^c} J_{ij}\sigma_i\omega_j.$$

where the couplings $(\mathbf{J}) = (J_{ij}, (i,j) \in \mathbb{Z}^d)$ are independent identically distributed real valued random variables on a probability space $(\mathbb{J}, \mathcal{B}_{\mathbf{J}}, \mathbb{P})$. We denote by $\mathbb{E}$ the expectation under $\mathbb{P}$. We shall assume that $\mathbb{E}[|J_{ij}|]$ is finite. We can associate to H_Λ^ω the local Gibbs measure $\mathbf{E}_{\Lambda,\mathbf{J}}^\omega$ on $\{-1,+1\}^\Lambda$ given by

$$\mathbf{E}_{\Lambda,\mathbf{J}}^\omega(d\sigma_\Lambda) = \frac{1}{Z_{\Lambda,\mathbf{J}}^\omega} e^{-\beta H_\Lambda^\omega(\sigma)} d\mu_0^{\otimes|\Lambda|}(\sigma_\Lambda)$$

with

$$d\mu_0(\sigma) = \frac{1}{2}(\delta_{\sigma=-1} + \delta_{\sigma=+1}).$$

In case when the couplings $(\mathbf{J})$ can take with positive probability values as small as one wishes, that is ($\mathbb{P}(|J_{ij}| \leq \epsilon) > 0$ for any $\epsilon > 0$), and $\mathbb{E}J_{ij} = 0$, for any $\beta > 0$ one can define with $\mathbb{P}$-probability one a unique Gibbs measure $\mu_{\mathbf{J}}$ in infinite volume as a limit of the local Gibbs measures $\mathbf{E}_{\Lambda,\mathbf{J}}^\omega$ (see [39], [26], [31]). Indeed, with $\mathbb{P}$-probability one, the couplings J_{ij}) will be as small as needed in a set $C_L^l = \{i \in \mathbb{Z}^d, |i_1| + |i_2| + .. + |i_d| \in [L, L+l]\}$ with l as large as one wishes provided L is large enough (as a consequence of the Borel-Cantelli lemma). Thus, the expectation under $\mathbf{E}_{\Lambda,\mathbf{J}}^\omega$, $C_L^l \subset \Lambda$, of functions localized in the sphere $S_L = \{i \in Z^d, |i_1| + |i_2| + .. + |i_d| \leq L\}$ will weakly depend on the boundary conditions ω (with a correction going to zero exponentially with l growing to infinity). Consequently, with $\mathbb{P}$-probability one, one can construct a Gibbs measure in infinite volume and it is unique. [This phenomenon is even more sharp when one deals with couplings $(\mathbf{J})$ which can be null with positive probability; as soon as one can draw a closed loop surrounding the origin of null couplings, the expectation inside this loop is independent of the spins outside it and therefore does not depend on the choice of the local Gibbs measure $\mathbf{E}_\Lambda^\omega$ as soon as Λ contains this loop.]

Despite this decay of correlations and uniqueness of Gibbs measures, we will show that the generators of the corresponding Glauber dynamics have no spectral gap. To this end, let us consider the Glauber dynamics generated by

$$\mathcal{L}_{\mathbf{J}} f = \sum_{i \in \mathbb{Z}^d} (\mathbf{E}_{\{i\}} f - f).$$

The construction of the associated infinite volume semi-group $P^{\mathbf{J}}$ can be done in a similar way as in nonrandom case considered before, (see [49]). We shall see that the following result is true.

Theorem 9.1 *For any $\beta > 0$, if*

$$\mathbb{P}(J_{ij} \geq J) \quad > 0 \qquad et \quad \mathbb{P}(|J_{ij}| \leq a) \quad > 0 \tag{9.1.1}$$

for a sufficiently large $J > 0$ and a constant $a > 0$ sufficiently small (depending on β), with $\mathbb{P}$-probability one,

$$\inf_{f \perp 1} \frac{\mu_{\mathbf{J}}\left(-\mathcal{L}_{\mathbf{J}}(f) f\right)}{\mu_{\mathbf{J}}\left(f - \mu_{\mathbf{J}} f\right)^2} = 0 \tag{9.1.2}$$

e.g. the generator $\mathcal{L}_{\mathbf{J}}$ has no spectral gap $\mathbb{P}$-almost surely.

Proof : Let us consider the configurations of the couplings $(\mathbf{J})$ such that, if we denote $\Lambda_L = [-L, L]^d$, $(\mathbf{J})$ is in the set

$$\mathcal{P}_{k+\Lambda_L}^{\mathbf{J},a} = \{\forall (i,j) \subset k + \Lambda_L \ J_{ij} \geq J \quad \text{and} \quad \forall i \in \Lambda_L, j \in \partial\Lambda_L, \, |J_{ij}| \leq a\}$$

with $k \in \mathbb{Z}^d$. Here,

$$k + \Lambda_L = \{(i,j) = k + (i', j'); (i', j') \in \Lambda_L\}.$$

We claim that, for any $L \in \mathbb{N}$, with $\mathbb{P}$-probability one, $(\mathbf{J})$ belongs to $\mathcal{P}_{k+\Lambda_L}^{\mathbf{J},a}$ for some point k of the lattice. Indeed, $\mathbb{P}(\mathcal{P}_{k+\Lambda_L}^{\mathbf{J},a}) = \mathbb{P}(\mathcal{P}_{\Lambda_L}^{\mathbf{J},a})$ is strictly positive according to our hypotheses for any $L \in \mathbb{N}$ and $k \in \mathbb{Z}^d$. Consequently, Borel-Cantelli's lemma implies that for any $L \in \mathbb{N}$,

$$\mathbb{P}\Bigg(\bigcup_{k \in \mathbb{Z}^d} \mathcal{P}_{k+\Lambda_L}^{\mathbf{J},a} \Bigg) = 1.$$

We can assume without loss of generality that k is the origin and thus assume that $(\mathbf{J}) \in \mathcal{P}_{\Lambda_L}^{\mathbf{J},a}$.

For such a configuration of the couplings, we shall compare the restriction of the Gibbs measure $\mu_{\mathbf{J}}$ to Σ_{Λ_L} with the measure $\mathbf{E}_{\Lambda_L,\mathbf{J}}^0$ with Dirichlet boundary conditions (e.g. $\omega = 0^{\partial\Lambda_L}$ at the boundary of Λ_L).

In fact, for any function f localized in Λ_L, we have

$$\frac{\mu_{\mathbf{J}}\left(\sum_i(\partial_i f)^2\right)}{\mu_{\mathbf{J}}\left(f-\mu_{\mathbf{J}}f\right)^2} \leq e^{6a\beta|\partial\Lambda|}\frac{\mathbf{E}^0_{\Lambda_L,\mathbf{J}}\left(\sum_i(\partial_i f)^2\right)}{\mathbf{E}^0_{\Lambda_L,\mathbf{J}}\left(f-\mu^0_{\Lambda,\mathbf{J}}f\right)^2}. \tag{9.1.3}$$

Moreover, in Λ_L, if βJ is large enough, we can use the results of Thomas [98]. We then find a constant $\alpha > 0$ such that

$$\inf_{f\perp 1}\frac{\mathbf{E}^0_{\Lambda_L,\mathbf{J}}\left(\sum_i(\partial_i f)^2\right)}{\mathbf{E}^0_{\Lambda_L,\mathbf{J}}\left(f-E^0_{\Lambda_L,\mathbf{J}}f\right)^2} \leq C_1(\mathbf{J})e^{-J\beta\alpha|\partial\Lambda_L|}. \tag{9.1.4}$$

(9.1.3) and (9.1.4) imply that

$$\inf_{f\perp 1}\frac{\mu_{\mathbf{J}}\left(-\mathcal{L}_{\mathbf{J}}(f)f\right)}{\mu_{\mathbf{J}}\left(f-\mu_{\mathbf{J}}f\right)^2} \leq e^{(6a-J\alpha)\beta|\partial\Lambda_L|}. \tag{9.1.5}$$

Since L can be taken as large as desired, we conclude that when $6a < J\alpha$,

$$\inf_{f\perp 1}\frac{\mu_{\mathbf{J}}\left(-\mathcal{L}_{\mathbf{J}}(f)f\right)}{\mu_{\mathbf{J}}\left(f-\mu_{\mathbf{J}}f\right)^2} = 0 \qquad \mathbb{P}a.s. \tag{9.1.6}$$

$\diamond$

Absence of spectral gap was also proved for the rotator model in [49] where the spins take on values in a unit circle and the interaction is described by the potential

$$\Phi_X = \begin{cases} J_{ij}\cos(\varphi_i-\varphi_j) & \text{if } X=\{ij\} \\ 0 & \text{otherwise} \end{cases} \tag{9.1.7}$$

with the (J_{ij}) taking again arbitrarily large and small values with positive probability. Thomas's estimates need then to be replaced, in order to estimate the spectral gap of the generator in the low temperature regions, by the use of Ginibre's inequalities [33].

9.2 Upper bound for the constant of logarithmic Sobolev inequality in finite volume and uniform ergodicity, d=2

In this section, we prove an upper bound for the constant in logarithmic Sobolev inequality in finite volume in dimension 2 and show that it implies the uniform ergodicity for the dynamics of the corresponding infinite system. We describe the strategy developed in [49]. It relies on the controls obtained in [93],[94] and [95] and described in chapter 5. Under some additional but physically quite general assumptions, sharper results were obtained later by Cesi, Maes and Martinelli [11] giving optimal controls in any dimension. These results show that if the tail of the random variables $(J_{ij},(i,j)\in \mathbb{Z}^d)$ decreases faster than exponentially, the log-Sobolev constant $c(\Lambda)$ for the local Gibbs measure localized in Λ decreases very slowly and more precisely, when $\Lambda=[-L,L]^d$,

$$c(\Lambda)\le \exp c\beta(\log\log L)^{d-1}(\log L)^{\frac{d-1}{d}} \qquad I\!P.a.s.$$

However, when the tail of the random variables $(J_{ij},(i,j)\in \mathbb{Z}^d)$ is exponential, $c(\Lambda)$ decreases only polynomially with the volume. In this setting, the estimate obtained in [49] is on the right scale.

We restrict ourselves here to the discrete setting, the generalization to the continuous setting being straightforward and given in sufficient detail in [49]. We shall denote by $\Omega=\mathbf{M}^{\mathbb{Z}^2}$ the configuration space with a finite set $\mathbf{M}$. $(J\!\!J,\mathcal{B}_{\mathbf{J}},I\!P)$ will denote the probability space on which the external random coupling live. The Hamiltonian of the system is given by a potential $\Phi\equiv(\Phi_X)_{X\in\mathbb{Z}^d}$ of real-valued measurable functions on $J\!\!J\times\Omega$ such that

- For any $X\subset\mathbb{Z}^d$, the function $\Phi_X(\mathbf{J},\cdot)$ is continuous and Σ_X measurable.
- For any $i\in\mathbb{Z}^d$, we have

$$\sum_{\substack{X\in\mathbf{F}\\ X\ni i}}||\Phi_X(\mathbf{J},\cdot)||_\infty<\infty$$

-The family $\{\Phi_X(\cdot,\omega),X\in\mathbb{Z}^d\}$ of random variables is mutually independent. Moreover, the random variables $\Phi_{X+j}(\cdot,T_j\omega)$ et $\Phi_X(\cdot,\omega)$ are identically distributed.

Furthermore, we shall assume that the interaction is of finite range, that is there exists a finite positive real number R such that $\Phi_X\equiv 0$ when $\mathrm{diam}(X)>R$, $I\!P$-almost surely.

The Hamiltonian of the system in finite volume Λ is then given by

$$H^\omega_\Lambda(\sigma)=\sum_{X\cap\Lambda\neq\emptyset}\Phi_X(\mathbf{J},\sigma_\Lambda\circ\omega_{\Lambda^c})$$

where $\sigma_\Lambda\circ\omega_{\Lambda^c}$ is the configuration described by σ inside the cube Λ and by ω outside Λ.

Remark 9.2: In the previous section, we considered the particular case (but most commonly studied) where

$$\Phi_X(\mathbf{J},\sigma)=J_{ij}\sigma_i\sigma_j$$

if $X=\{i,j\}$ when $|i-j|=1$ and $\Phi_X\equiv 0$ otherwise.

We associate to Φ the local specification $(\mathbf{E}^\omega_{\Lambda,\mathbf{J}},\Lambda\subset\subset\mathbb{Z}^d)$ as before. We consider the dynamics generated by

$$\mathcal{L}f=\sum_i\{\mathbf{E}_{i+X}f-f\}$$

for a finite set X of $\mathbb{Z}^d$. The construction, for $I\!P$ almost all $\mathbf{J}$, of the semi-group $P^{\mathbf{J}}$ in infinite volume can be achieved as in chapter 8.1. We then obtain the following exponential approximation property.

Theorem 9.3 *Assume that* $\Phi \equiv \{\Phi_X\}_{X\in\mathbf{F}}$ *is a finite range potential such that*

$$\sup_X \sup_{i,j\in \mathbb{Z}^d} \mathbb{E}[||\nabla_j \phi_X||_\infty^K] < \infty$$

for a finite real number $K > d$. *Then, the limit*

$$P_t^{\mathbf{J}} f \equiv \lim_{\Lambda\uparrow\mathbb{Z}^d} P_t^{\mathbf{J},\Lambda} f$$

exists $\mathbb{P}$*-almost surely. Moreover, for any cube* Λ, *any* $A \in \mathbb{R}^+$ *and any local function* f, *we have*

$$||P_t^{\mathbf{J}} f - P_t^{\mathbf{J},\Lambda} f||_\infty \leq e^{-At} D(\mathbf{J}) |||f||| \tag{9.2.8}$$

provided that $d(f,\Lambda^c) \geq \bar{L}$ *for a constant* $\bar{L} \equiv \bar{L}(\Lambda(f), R, \mathbf{J})$ *almost surely finite and* $d(f,\Lambda^c)^{1-\delta} \geq Ct$*for a finite constant* $C \in \mathbb{R}^+$ *depending on* $\mathbf{J}$, $\Lambda(f)$, A *and for some* $\delta \in (0,1)$.

9.2.1 Bound on the log-Sobolev constant

The basic idea to control the log-Sobolev constant is to show that a decay of correlation property is satisfied by the local Gibbs measures $\mathbf{E}^\omega_{\Lambda,\mathbf{J}}$ but that it will depend on the size $|\Lambda|$ of the finite volumes under consideration. More precisely, we prove the following

Property 9.4 *Let* $\Lambda_L = [-L,L]^d$.
Assume that
(H1) For any $\xi \in \mathbb{R}$, *we have*

$$\mathbb{E}\exp\{\xi||\widetilde{\Phi}^{(j)}(\mathbf{J},\cdot)||_1\} < \infty \tag{9.2.9}$$

with

$$||\widetilde{\Phi}^{(j)}(\mathbf{J},\cdot)||_1 \equiv \sum_{\substack{X\in\mathbf{F}\\ X\ni j}} ||\Phi_X(\mathbf{J},\cdot)||_\infty .$$

(H2) There exists $J_0 > 0$ *sufficiently small so that*

$$p_1 \equiv \sup_X \mathbb{P}\{\ ||\Phi_X||_\infty > J_0\} \tag{9.2.10}$$

belongs to $(0, p_c^b(2,R))$ *with* $p_c^b(2,R)$ *a universal critical percolation exponent.*

If (H1) and (H2) are satisfied, there exist a finite constant $r_0 > 0$ *and a constant* $M \in (0,\infty)$ *such that for almost all* $\mathbf{J}$, L *sufficiently large, for any* $\Lambda \subset \Lambda_L$, *and any* $\omega \in \mathbf{M}^{|\Lambda_L|}$,

$$\mathbf{E}^\omega_{\Lambda,\mathbf{J}}(f,g) \leq e^{-Md(\Lambda(f),\Lambda(g))} |||f|||.|||g|||$$

for any functions (f,g) *localized in* Λ_L *such that*

$$d(\Lambda(f),\Lambda(g)) \geq r_0 \log L.$$

Moreover, r_0 *goes to zero when* p_1 *goes to zero.*

Using the result of exercise 5.7, one deduces from this property the following almost-sure bound on the log-Sobolev constants.

Theorem 9.5 *Let $\Lambda_L = [-L, L]^d$ for $L \in I\!N$. If (H1) and (H2) are satisfied, there exists a finite constant $c_0 > 0$ such that $I\!P$ almost surely, for L sufficiently large ($L \geq L(\mathbf{J})$ with $L(\mathbf{J})$ being $I\!P$-a.s. finite)*

$$c(\mathbf{E}^{\omega}_{\Lambda_L,\mathbf{J}}) \leq \exp\{c_0(\log L)^{d-1}\}.$$

Moreover, c_0 goes to zero when p_1 does.

To complete our proof, we shall prove property 9.4. The proof is in fact a slight generalization of the computation presented in section 5.4.4. To simplify the notation, we assume $R = 1$, the generalization being given in details in [49]. Let us notice, following [31], that

$$\mathbf{E}^{\omega}_{\Lambda,\mathbf{J}}(f,g) \quad \leq \quad e^{4\sup_{j\in\Lambda}\sum_{X:j\in X}||\phi_X||_\infty} \sum_{\gamma\in W(f,g)} \prod_{\{i,j\}\in\gamma} z_{i,j} \qquad (9.2.11)$$

with $W(f,g)$ the set of paths in Λ connecting $\Lambda(f)$ and $\Lambda(g)$ and

$$z_{i,j} = \begin{cases} e^{4J_0} - 1 \text{ if } |\phi_{ij}| \leq J_0 \\ 1 \text{ otherwise.} \end{cases}$$

Under (H1), we know by Chebychev's inequality that for any $\epsilon > 0$, there exists a finite constant C_ϵ such that

$$P(\sup_{j\in\Lambda_n} \sum_{X:j\in X} ||\phi_X||_\infty \geq \frac{\epsilon}{4}\log n) \leq C_\epsilon n^{-2}.$$

Hence, for any $\epsilon > 0$, any sufficiently large n

$$\sup_{j\in\Lambda_n} \sum_{X:j\in X} ||\phi_X||_\infty \leq \frac{\epsilon}{4}\log n \qquad (9.2.12)$$

almost surely, by the Borel-Cantelli lemma.

Moreover, according to Kesten [59] , if

$$\lambda_j(r) = \inf_{\gamma\in W(f,g), d(\Lambda(f),\Lambda(g))\geq r} \frac{\mathrm{card}(\{i,j\}\in\gamma : ||\phi_{ij}||_\infty \leq J_0)}{\mathrm{card}(\{i,j\}\in\gamma)}$$

and if

$$p_1 = I\!E[||\phi_{ij}||_\infty \geq J_0] < p_c^b(2,1),$$

there exists $r_0(p_1) > 0$, $r_0(p_1) \to 0$ when $p_1 \to 0$, such that for any $\eta > 0$,

$$I\!P[\bigcap_{n\in I\!N} \bigcup_{n\geq k} \inf_{j\in\Lambda_n} \{\lambda_j(r_0\log n) \geq \eta\}] = 1.$$

With (9.2.11) and (9.2.12), we conclude that almost surely, for any n and all $\Lambda \subset \Lambda_n$, we have

$$\mathbf{E}^{\omega}_{\Lambda,\mathbf{J}}(f,g) \leq \mathrm{const.}(e^{4J_0} - 1)^{\eta d(\Lambda(f),\Lambda(g))} |||f|||.|||g|||$$

provided that $d(\Lambda(f),\Lambda(g)) \geq r_0 \log n$.

$\diamond$

9.2.2 Ergodicity in infinite volume

We shall deduce from theorem 9.5 a stretched exponential decay towards equilibrium of the Markov semi-group in infinite volume. We follow the steps of chapter 8.2. The theorem states as follows.

Theorem 9.6 *Assume that conditions (H 1) and (H 2) are satisfied with* $p_1 > 0$ *sufficiently small. Then, there exists* $\theta \in (0,1)$ *such that for every local function* f*, with* $\mathbb{P}$*-probability one, we have*

$$||P_t^{\mathbf{J}} f - \mu_{\mathbf{J}}(f)||_\infty \leq C(\Lambda(f), \mathbf{J}) e^{-t^\theta} |||f||| \tag{9.2.13}$$

for any $t \geq T(\Lambda(f), \mathbf{J})$ *and with almost surely finite positive random variables* $C(\Lambda(f), \mathbf{J})$ *and* $T(\Lambda(f), \mathbf{J})$.

Proof : Let a local function f be given. Assume n_0 large enough so that $\Lambda(f) \subset \Lambda_{n_0} \equiv [-n_0, +n_0]^2$. Let $\mathbb{J}_0 \subset \mathbb{J}$ be a measurable set with $\mathbb{P}$-probability one such that the conclusions of property 9.4 and theorem 9.3 are satisfied on $\mathbb{J}_0$. Choosing L sufficiently large, we know that for such configurations of the disorder, the conclusions of theorem 9.5 are satisfied. We thus have that

$$\begin{aligned}
& |P_t^{\mathbf{J}} f(\omega) - \mu_{\mathbf{J}} f| \\
\leq \; & |P_t^{\mathbf{J},\Lambda_n} f(\omega) - \mathbf{E}^0_{\mathbf{J},\Lambda_n} f| + ||P_t^{\mathbf{J}} f - P_t^{\mathbf{J},\Lambda_n} f||_\infty + |\mu_{\mathbf{J}} f - \mathbf{E}^0_{\mathbf{J},\Lambda_n} f| \\
\leq \; & |P_t^{\mathbf{J},\Lambda_n} f(\omega) - \mathbf{E}^0_{\mathbf{J},\Lambda_n} f| + e^{-At} |||f||| + e^{-M(n-n_0)} |||f|||
\end{aligned} \tag{9.2.14}$$

if $d(\Lambda(f), \Lambda_n^c)^{1-\eta} \geq Bt$ and $\mathbf{E}^0_{\mathbf{J},\Lambda_n}$ is the Gibbs measure in finite volume with free boundary conditions. We saw above that, with the result of theorem 9.5, we have for any $\delta \in (0,1)$ and $q = 1 + e^{\frac{2\delta t}{c_n}}$,

$$\begin{aligned}
|P_t^{\mathbf{J},\Lambda_n} f(\omega) - \mathbf{E}^0_{\mathbf{J},\Lambda_n} f| & \leq e^{2||H_{\mathbf{J},\Lambda_n}||_\infty / q} \left(\mathbf{E}^0_{\mathbf{J},\Lambda_n} |P_t^{\mathbf{J},\Lambda_n} f(\cdot) - \mathbf{E}^0_{\mathbf{J},\Lambda_n} f|^q \right)^{\frac{1}{q}} \\
& \leq e^{2||H_{\mathbf{J},\Lambda_n}||_\infty / q} e^{\frac{(\delta-1)t}{c_n}}.
\end{aligned} \tag{9.2.15}$$

We thus obtain that for n sufficiently large such that $d(\Lambda(f), \Lambda_n^c)^{1-\eta} \geq Bt$, or in other words for n of order $t^{\frac{1}{1-\eta}}$,

$$|P_t^{\mathbf{J}} f(\omega) - \mu_{\mathbf{J}} f| \leq e^{2||H_{\mathbf{J},\Lambda_n}||_\infty / q} e^{\frac{(\delta-1)t}{c_n}} + e^{-At} |||f||| + e^{-M(n-n_0)} |||f|||. \tag{9.2.16}$$

Clearly, this bound can only help us if $\frac{t}{c_n}$ (that is $\frac{n^{1-\eta}}{c_n}$) goes to infinity with n. In particular, if this ratio goes to infinity faster than logarithmically with n, we almost surely have

$$\lim_{n \to \infty} ||H_{\mathbf{J},\Lambda_n}||_\infty / q = 0.$$

When $d = 2$, we have
$$c_n \approx n^{\xi}$$
with ξ going to zero as p_1 goes to zero. Consequently, when β is sufficiently small, choosing $n^{1-\eta} \approx Bt$, we obtain
$$|P_t^{\mathbf{J}} f(\omega) - \mu_{\mathbf{J}} f| \quad \leq \quad e^{-c_0 t^{\alpha}} |||f||| \tag{9.2.17}$$
with a constant $c_0 > 0$ and
$$\alpha = \frac{1 - \eta - \xi}{1 - \eta}.$$
Since we took $t = n^{1-\eta}/B'$ with $n \geq N(\mathbf{J})$ and an almost surely finite random variable $N(\mathbf{J})$, this result is obtained for $t \geq T(\mathbf{J})$, with an almost surely finite random time $T(\mathbf{J})$. ◇

Chapter 10

Low temperature regime : L^2 ergodicity in infinite volume

At low temperature, it frequently happens that there are several coexisting phases. (For classical references to this phenomenon, we refer to [90], [41] and [84].) For example, let us consider the ferromagnetic Ising model given by the interaction Hamiltonian

$$H_\Lambda^\omega(\sigma) = -\sum_{|i-j|=1, i,j\in\Lambda} \sigma_i\sigma_j - \sum_{|i-j|=1, j\in\Lambda^c} \sigma_i\omega_j$$

and

$$\mathbf{E}_\Lambda^\omega(f) = \frac{1}{Z_\Lambda^\omega}\int f(\sigma)e^{-\beta H_\Lambda^\omega(\sigma)}d\nu_\Lambda$$

with ν_Λ the product measure on the spins $(\sigma_i, i\in\Lambda)$ with marginal law $\nu = (1/2)(\delta_{+1}+\delta_{-1})$. One knows that for sufficiently large β's the following two limits exist and are different,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\mathbf{E}_\Lambda^+ = \mu^+, \qquad \lim_{\Lambda\uparrow\mathbb{Z}^d}\mathbf{E}_\Lambda^- = \mu^-$$

where $+$ (resp. $-$) means that the spins at the boundary of the set Λ take all the value $+1$ (resp. -1). In such a setting, one cannot hope anymore the infinite volume semi-group to be uniformly ergodic. However, one can look for L^2 ergodicity. For instance, it is expected that for ferromagnetic Ising model in dimension 2, there exists a positive constant c such that for any local function f

$$\mu^\epsilon(P_tf-\mu^\epsilon f)^2 \le e^{-c\sqrt{t}}|||f|||$$

with $\epsilon = +$ or $-$ (see Fisher and Huse [37]).

F. Martinelli [68] has shown that for any local function f and any $M \in \mathbb{R}^+$, there exists a finite constant $c(\Lambda(f), M)$ such that for all times $t \geq 1$,

$$\mu^+(P_t f - \mu^+ f)^2 \leq c(\Lambda(f), M) t^{-M} |||f|||.$$

His proof relies on the spectral gap estimate

$$m^+(\Lambda_L) = \inf_f \frac{\mathbf{E}^+_{\Lambda_L}(f(-\mathcal{L}_\Lambda)f)}{\mathbf{E}^+_{\Lambda_L}(f - \mathbf{E}^+_{\Lambda_L} f)^2} \geq e^{-o(L)L} \tag{10.0.1}$$

if $\Lambda_L = [-L, L]^2$ and $\lim_{L\to\infty} o(L) = 0$. If $m^0(\Lambda_L)$ denotes the spectral gap constant with free boundary conditions, he also proved that

$$\lim_{L\uparrow\infty} \frac{1}{2\beta L} \log m^0(\Lambda_L) = -\tau_\beta \tag{10.0.2}$$

where $\tau_\beta > 0$ is the surface tension.

For related results on this subject, one can read [98] and [53].

We shall discuss hereafter the proof of (10.0.1) and (10.0.2), as well as that of the $L^2(\mu^+)$- ergodicity of the semi-group in infinite volume.

10.1 Spectral gap estimate

10.1.1 Strategy

The strategy we propose below slightly differs from that originally proposed by F. Martinelli relying on a clever use of auxiliary bloc dynamics. Here, we may as well consider other auxiliary dynamics. However, both approaches are more or less equivalent and require the same kind of basic estimates.

We shall denote by μ a probability measure and will present a general scheme to establish a spectral gap inequality for μ with respect to a Dirichlet form $\mathcal{E}$.

The basic idea is to decouple the difficulties by introducing auxiliary operators. In fact, we see that if π is an operator in $L^2(\mu)$, for any function $f \in L^2(\mu)$, the triangle inequality yields

$$||f - \mu f||_2 \leq ||f - \pi f||_2 + ||\pi f - \mu f||_2. \tag{10.1.3}$$

Let us now assume that π was chosen so that there exists $\delta > 0$ such that for every bounded measurable function f,

$$||\pi f - \mu f||_\infty \leq (1 - \delta)||f - \mu f||_\infty. \tag{10.1.4}$$

Then, if π is self-adjoint in $L^2(\mu)$, we have

$$||\pi f - \mu f||_2 \leq (1 - \delta)||f - \mu f||_2 \tag{10.1.5}$$

since (see [71] or [9])

$$\sup_{f\perp 1}\frac{||\pi f-\mu f||_2}{||f-\mu f||_2}=\lim_{n\to\infty}\left(\sup_{f\perp 1}\frac{||\pi^n f-\mu f||_\infty}{||f-\mu f||_\infty}\right)^{\frac{1}{n}}.$$

Hence, we deduce from (10.1.3) and (10.1.5) that

$$||f-\mu f||_2\le\frac{1}{\delta}||f-\pi f||_2. \tag{10.1.6}$$

To obtain the desired result for the Dirichlet form under consideration, we need the following additional property

$$\mu(f-\pi f)^2\le C\mathcal{E}(f,f) \tag{10.1.7}$$

for a finite constant C. Then, we conclude that

$$\mu(f-\mu f)^2\le\frac{C}{\delta}\mathcal{E}(f,f). \tag{10.1.8}$$

We recall that F. Martinelli [69], (following an idea of Holley), applied the above strategy with the specific choice of

$$\mu=\mathbf{E}^\omega_\Lambda,\qquad \pi=\frac{1}{n}\sum_{i=1}^n\mathbf{E}^\omega_{\Lambda_i}$$

for subsets Λ_i of Λ with possibly non empty intersection. Such the π generates the so-called bloc spin flip dynamics.

The next exercise shows how it can be used to control the spectral gap in the high temperature regime.

Exercise 10.1 [See [69](Thm 4.5)]

Let $\Lambda_L=[-L,L]^2\subset\mathbb{Z}^d$. We assume in this exercise that the interaction is of finite range and that the local Gibbs measure $\mathbf{E}^\omega_\Lambda$ on Σ_Λ satisfies the mixing condition

$$Var_Y\mathbf{E}^\omega_\Lambda f\le Ce^{-Md(Y,X\cap\Lambda_f)}||f||_\infty$$

for finite constants $(C,M)>0$ (see section 5.4.3 for the notations). As a consequence, there exists a unique Gibbs measure μ in infinite volume for the local specification $(\mathbf{E}^\omega_\Lambda,\Lambda\subset\mathbb{Z}^d)$.

Show that if for any $L\in\mathbb{N}$, we set

$$m_L=\inf_{\max_{1\le i\le d}(k'_i-k_i)\le L}\ \inf_{i\in\mathbb{Z}^d}\ \inf_f\frac{\mathbf{E}^\omega_{i+\Lambda_K,\mathbf{J}}(\sum_i(\partial_i f)^2)}{\mathbf{E}^\omega_{i+\Lambda_K,\mathbf{J}}(f,f)}$$

with $\Lambda_K=[k_1,k'_1]\times\cdots\times[k_d,k'_d]$, there exist two finite constants c,c' such that

$$m(2L)\ge(1-\frac{c}{\sqrt{L}})m(L+c'\sqrt{L}\log L). \tag{10.1.9}$$

Show that for $\rho \in (1,2)$, we can find $L_0 \in \mathbb{N}$ so that $m(L_0) \geq m > 0$ and $\rho L \geq 2^{-1}(L + c'\sqrt{L}\log L)$ for $L \geq L_0$, to conclude that there exists a positive constant m' such that for all $n \in \mathbb{N}$, $m(L_0\rho^n) \geq m' > 0$. Conclude.

Hint: Proceed by induction; Consider a set $\mathcal{R} = k + \prod_{1\leq i\leq d}[0,l_i]$ with $l_i \leq L$ for $1 \leq i \leq d$, with, to simplify the notations $k = 0$ and $l_1 \geq l_2 \geq .. \geq l_d$, $l_1 \geq \sqrt{L}$. Let, for $k,n \in \mathbb{N}$, $kn \leq l_1/2$, $\Lambda_1 = [0, l_1/2 + nk] \times [0,l_2] \times \cdots \times [0,l_d]$ and $\Lambda_2 = [l_1/2 + (n-1)k, l_1] \times [0,l_2] \times \cdots \times [0,l_d]$. Notice that, under our hypotheses and since $d(\Lambda(\mathbf{E}_{\Lambda_2}f), \partial\Lambda_1) = d(\Lambda(\mathbf{E}_{\Lambda_1}f), \partial\Lambda_2) = k$, for $i = 0,1$,

$$||\mathbf{E}_{\Lambda_{i+1}}\mathbf{E}_{\Lambda_{2-i}}f - \mathbf{E}_{\mathcal{R}}f||_\infty \leq e^{-Mk}||f - E_{\mathcal{R}}f||_\infty.$$

Let $\pi = \frac{1}{2}(\mathbf{E}_{\Lambda_1} + \mathbf{E}_{\Lambda_2})$. Show that for any integer number $p \geq 2$

$$||\pi^p f - \mathbf{E}_{\mathcal{R}}f||_\infty \leq \frac{1}{2^p}(1 + e^{-Mk})^p||f - \mathbf{E}_{\mathcal{R}}f||_\infty$$

Deduce that since

$$\left(\mathbf{E}_{\mathcal{R}}(\pi f - f)^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\mathbf{E}_{\mathcal{R}}\left(\mathbf{E}_{\Lambda_1}(f - \mathbf{E}_{\Lambda_1}f)^2\right)^{\frac{1}{2}} + \frac{1}{2}\mathbf{E}_{\mathcal{R}}\left(\mathbf{E}_{\Lambda_2}(f - \mathbf{E}_{\Lambda_2}f)^2\right)^{\frac{1}{2}},$$

we have

$$\left(\mathbf{E}_{\mathcal{R}}(f - \mathbf{E}_{\mathcal{R}}f)^2\right)^{\frac{1}{2}} \leq (1 - e^{-Mk})^{-1}\max\{m(\Lambda_1)^{-1}, m(\Lambda_2)^{-1}\}$$
$$\times\left(\mathbf{E}_{\mathcal{R}}\Big(\sum_{i\in\mathcal{R}}(\partial_i f)^2 + \sum_{i\in\Lambda_1\cap\Lambda_2}(\partial_i f)^2\Big)\right)^{\frac{1}{2}}.$$

Summing over $n \in \{1,\ldots,n_0\}$, bound $\left(\mathbf{E}_{\mathcal{R}}(f - \mathbf{E}_{\mathcal{R}}f)^2\right)^{\frac{1}{2}}$ in terms of $m(\Lambda_i^n)^{-1}, i = 1,2$ and proceed by induction to bound similarly $\max\{m(\Lambda_1)^{-1}, m(\Lambda_2)^{-1}\}$ to arrive at

$$m(L) \geq (1 - e^{-Mk})^d(1 + n_0^{-1})^{-d}m(2^{-1}L + n_0 k).$$

Choose wisely M and n_0 to conclude.

10.1.2 Spectral gap estimate

In the high temperature regime and in dimension 2, we choose, (compare [69]),

$$\mu = \mathbf{E}_\Lambda^+, \qquad \pi = \mathbf{E}_{\Lambda_1}\cdots\mathbf{E}_{\Lambda_{n-1}}\mathbf{E}_{\Lambda_n}\mathbf{E}_{\Lambda_{n-1}}..\mathbf{E}_{\Lambda_1}$$

with, for $i \in \{1,\ldots,n = 2[2L/[\epsilon L]]\}$ and $\epsilon > 0$

$$\Lambda_i = [[\epsilon L](i-1)/2 - L; [\epsilon L](i-1)/2 + [\epsilon L] - L] \times [0,L].$$

Following the strategy stated below, we need to bound $||f - \pi f||_2$ in terms of the Dirichlet form and $\pi f - \mu f$ uniformly. For the first term, the triangular inequality and local Gibbs measures property imply that

$$\begin{aligned} ||f-\pi f||_2 &\leq 2\sum_{i=1}^{n}\left(\mathbf{E}_\Lambda^+\left(\mathbf{E}_{\Lambda_1}\cdots\mathbf{E}_{\Lambda_{i-[i/n]n}}-\mathbf{E}_{\Lambda_1}\cdots\mathbf{E}_{\Lambda_{i+1-[i/n]n}}\right)f\right)^2\Big)^{\frac{1}{2}} \\ &\leq 2\sum_{i=1}^{n}\left(\mathbf{E}_\Lambda^+\left(f-\mathbf{E}_{\Lambda_i}f\right)^2\right)^{\frac{1}{2}}. \end{aligned} \quad (10.1.10)$$

Moreover, property 2.7 shows that there exists a constant $c(\beta)<\infty$ such that

$$\mathbf{E}_{\Lambda_i}\left(f-\mathbf{E}_{\Lambda_i}f\right)^2 \leq e^{c(\beta)\epsilon L}\mathbf{E}_{\Lambda_i}\left(\sum_{j\in\Lambda_i}(\partial_j f)^2\right).$$

Consequently, we obtain

$$\begin{aligned} ||f-\pi f||_2 &\leq 2e^{\frac{1}{2}c(\beta)\epsilon L}\sum_{i=1}^{n}\mathbf{E}_\Lambda^+[|\nabla_{\Lambda_i}f|^2]^{\frac{1}{2}} \\ &\leq 4[\epsilon L]ne^{c(\beta)\epsilon L}\left(\mathbf{E}_\Lambda^+[|\nabla_\Lambda f|^2]\right)^{\frac{1}{2}}. \end{aligned} \quad (10.1.11)$$

Moreover, F. Martinelli proved that for the ferromagnetic Ising model (see [69], p. 68) the following bound is true

Property 10.2 *For any $\epsilon>0$, L sufficiently large,*

$$||\pi f-\mathbf{E}_\Lambda^+ f||_\infty \leq e^{-\epsilon L}||f-\mathbf{E}_\Lambda^+ f||_\infty.$$

Consequently, for this model, we deduce that for any $\epsilon>0$

$$\mathbf{E}_\Lambda^+(f-\mathbf{E}_\Lambda^+ f)^2 \leq \frac{8L}{\epsilon(1-e^{-m\epsilon L})}e^{c(\beta)\epsilon L}\mathbf{E}_\Lambda^+\left(\sum_{j\in\Lambda}(\partial_j f)^2\right) \quad (10.1.12)$$

which gives the desired estimate on the spectral gap.

Property 10.2 relies on the precise knowledge of the fluctuations of the interface between the phases of $+1$ spins and -1 spins in dimension 2 (see [27]) which allows to see that the restriction of $E^{(+,+,+,\omega)}_{[0,L]\times[0,l]}$ to the σ-algebra generated by $(\sigma_i, i\in[0,L]\times[0,l-M\sqrt{L}])$ depends very weakly on the boundary conditions ω. Here, $(+,+,+,\omega)$ denotes the boundary conditions where all the spins on the sides at north, south and west are equal to $+1$ whereas they take the configuration ω on the east side. This observation allows roughly speaking to see that $E_{\Lambda_1}E_{\Lambda_2}f\simeq E_{\Lambda_1\cup\Lambda_2}$ for overlapping sets $\Lambda_1=[k_1,k_1+L]\times[k_1',k_1'+L]$, $\Lambda_2=[k_2,k_2+L]\times[k_2',k_2'+L]$ with intersection of the form $\Lambda_1\cap\Lambda_2=[k,k+M\sqrt{L}]\cap[l,l+L]$, and by induction that $\pi f\simeq\mu f$.

In case of free boundary conditions, F. Martinelli [69] proved that

Property 10.3 *For any $\epsilon > 0$,*

$$||\pi f - \mathbf{E}^0_\Lambda f||_\infty \leq (1 - e^{-\beta\tau_\beta(1+14\epsilon)(2L+1)})||f - \mathbf{E}^+_\Lambda f||_\infty$$

which in turn implies that

$$\mathbf{E}^0_\Lambda(f - \mathbf{E}^0_\Lambda f)^2 \leq \frac{8L}{\epsilon} e^{c(\beta)\epsilon L + \beta\tau_\beta(1+14\epsilon)(2L+1)} \mathbf{E}^+_\Lambda \left(\sum_{j\in\Lambda} (\partial_j f)^2 \right) \tag{10.1.13}$$

and gives a lower bound on the spectral gap. F. Martinelli has also shown that the bound (10.1.13) is on the right scale by bounding above the spectral gap by choosing, in its definition as an infimum on test functions, the test function

$$f_\Lambda(\sigma) = 1\!\!I_{\sum_{i\in\Lambda}\sigma_i > 0} - 1\!\!I_{\sum_{i\in\Lambda}\sigma_i < 0}.$$

10.2 L^2 ergodicity in infinite volume

We obtained in the previous section spectral gap lower bounds in finite volume decreasing with the volume faster than $|\Lambda|^{-\frac{1}{d}}$. Following the remarks we did when we studied the ergodicity for disordered systems, the method employed there is in this case useless. In fact, one does not know in general how to deduce ergodicity in infinite volume from such estimates. This is due to the lack of balance between the finite volume approximation of the semi-group, which says that we can approximate $P_t f$ by $P_t^{\Lambda_n} f$ for n of order t, and the spectral gap estimate which provides an exponential decay to equilibrium of $P_t^{\Lambda_n} f$ with the speed $m_n t$, which is going to zero when n is of order t.

Fortunately, in ferromagnetic models, the finite volume approximation of the semi-group can be avoided. We briefly describe the arguments used in this case restricting ourselves to the discrete spins and considering a generator

$$\mathcal{L}f = \sum_i (\mathbf{E}_{\{i\}} f - f) = \sum_i c_i(\sigma)\partial_i f$$

with

$$c_i(\sigma) = \frac{1}{1 + e^{2\beta\sigma_i \sum_{|i-j|=1}\sigma_j}}$$

and

$$\partial_i f = f(\sigma^{(i)}) - f(\sigma).$$

Then, it is not hard to see that for any configuration (σ, η) such that $\sigma_i \leq \eta_i$ for any i and any $j \in \mathbb{Z}^d$,

$$c_j(\sigma) \leq c_j(\eta) \qquad \text{if } \sigma_j = \eta_j = -1$$

and

$$c_j(\sigma) \geq c_j(\eta) \qquad \text{if } \sigma_j = \eta_j = +1.$$

Thus, the more spins with state +1 will be contained in a given configuration, the higher the probability that the Glauber dynamics will turn the spins into the state +1. This heuristic can be characterized by the construction of a coupling (see [86]) $\rho_t^{\eta,\eta'}(d\sigma,\sigma')$ of $P_t(\eta)$ and $P_t(\eta')$(that is a probability measure on the product space Ω^2 with marginals $P_t(\eta)$ and $P_t(\eta')$) such that, with $\rho_t^{\eta,\eta'}$ probability one, if for any $j \in \mathbb{Z}^d$, $\eta_j \leq \eta'_j$, then

$$\sigma_t(j) \leq \sigma'_t(j) \qquad \forall j \in \mathbb{Z}^d, \rho_t^{\eta,\eta'} - a.s..$$

Also, one can construct a coupling $r_t^{\eta,+,\Lambda}$ of $P_t(\eta)$ with the semi-group$P_t^{\Lambda,+}(+1)\otimes \prod_{j\in\Lambda^c}\delta_{\sigma_j=+1}$ such that

$$\sigma_t(j) \leq \sigma'_t(j) \qquad \forall j \in \mathbb{Z}^d, \rho_t^{\eta,+,\Lambda} - a.s.$$

Similarly, one can couple $P_t(\eta)$, $P_t(\eta')$ and $P_t^{\Lambda,+}(+1)\otimes \prod_{j\in\Lambda^c}\delta_{\sigma_j=+1}$, the process representing the last marginal then dominating the two others. In particular, for any $t \geq 0$,

$$P_t(\sigma_0 = +1)(\eta) \leq P_t^{\Lambda,+}(\sigma_0 = +1)(+1). \tag{10.2.14}$$

This property is the key point to overcome the finite volume approximation of Markov semi-groups.

Note that this kind of property also exists for continuous models when, roughly speaking, the potential is convex (see [50]).

We shall use it to show that

Property 10.4 *For any function f, any $M \in \mathbb{R}^+$, and all $t \geq 1$,*

$$\mu^+(P_t f - \mu^+ f)^2 \leq t^{-M} |||f|||.$$

Proof : Let us first notice that, on the discrete set $\{-1,+1\}^{\mathbb{Z}^d}$, the local functions can be decomposed into sums of monomial functions of the type

$$f(\sigma) = \prod_{i\in X}\sigma_i$$

for finite sets $X \subset \mathbb{Z}^d$.

For such a function, we have

$$\begin{aligned}
\mu^+(P_t f - \mu^+ f)^2 &= \frac{1}{2}\mu^+\otimes\mu^+(P_t f(\eta) - P_t f(\eta'))^2 \\
&= \frac{1}{2}\mu^+\otimes\mu^+(\rho_t^{\eta,\eta'}(\prod_{i\in X}\sigma_i(t) - \prod_{i\in X}\sigma'_i(t))^2 \\
&\leq |X|\sum_{i\in X}\mu^+\otimes\mu^+(\rho_t^{\eta,\eta'}|\sigma_i(t) - \sigma'_i(t)|)^2 \\
&= 4|X|\sum_{i\in X}\mu^+\otimes\mu^+(\rho_t^{\eta,\eta'}[\sigma_i(t)\neq\sigma'_i(t)]) \\
&\leq 4|X|^2\mu^+\otimes\mu^+(\rho_t^{\eta,\eta'}[\sigma_0(t)\neq\sigma'_0(t)])
\end{aligned} \tag{10.2.15}$$

with $\rho_t^{\eta,\eta'}$ a coupling between $P_t(\eta)$ and $P_t(\eta')$. Here we used the invariance by translation of μ^+. Therefore we only need to prove the estimate for $f(\sigma) = \sigma_0$. According to the last remark, we have for any finite cube Λ,

$$\begin{aligned}
& \frac{1}{2}\mu^+ \otimes \mu^+ (\rho_t^{\eta,\eta'}[\sigma_0(t) \neq \sigma_0'(t)]) \\
\leq\ & \mu^+ \otimes \mu^+ (r_t^{\eta,+,\Lambda}[\sigma_0(t) \neq \sigma_0'(t)]) \\
=\ & \mu^+ \otimes \mu^+ (P_t^{\Lambda,+}(\sigma_0 = +1)(+) - P_t(\sigma_0 = +1)(\eta)) \\
=\ & P_t^{\Lambda,+}\sigma_0(\eta) - \mu^+ P_t \sigma_0 \\
=\ & P_t^{\Lambda,+}\sigma_0(+) - \mu^+ \sigma_0 \\
\leq\ & e^{2\beta|\Lambda|}\left(\mathbf{E}_\Lambda^+(P_t^{\Lambda,+}(\sigma_0) - \mathbf{E}_\Lambda^+\sigma_0)^2\right)^{\frac{1}{2}} + \mathbf{E}_\Lambda^+\sigma_0 - \mu^+\sigma_0 \\
\leq\ & e^{2\beta|\Lambda| - m(\Lambda)t} + \mathbf{E}_\Lambda^+\sigma_0 - \mu^+\sigma_0 \qquad (10.2.16)
\end{aligned}$$

where we used the definition of the spectral gap constant and introduced by "force" the Gibbs measure $\mathbf{E}_\Lambda^+$. Finally, let us recall that (see [27]), if $\Lambda = [-L, L]^2$,

$$|\mathbf{E}_\Lambda^+\sigma_0 - \mu^+\sigma_0| \leq e^{-L}.$$

Thus, by the estimate on the spectral gap inequality obtained in the last section,

$$\mu^+(P_t f - \mu^+ f)^2 \quad \leq \quad 4|X|^2\left(e^{2\beta L^2 - e^{-o(L)L}t} + e^{-L}\right). \qquad (10.2.17)$$

Optimizing on L, we take $L = [M \log t]$ for $M \in \mathbb{R}^+$ showing that we can find a finite constant $C(M)$ so that

$$\mu^+(P_t f - \mu^+ f)^2 \quad \leq \quad C(M)|X|^2 t^{-M}. \qquad (10.2.18)$$

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