The Gagliardo-Nirenberg inequality on metric measure spaces

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Abstract

In this paper, we prove that if a metric measure space satisfies the volume doubling condition and the Gagliardo-Nirenberg inequality with the same exponent n $(n \geq 2)$, then it has exactly the n-dimensional volume growth. Besides, two interesting applications have also been given. The one is that we show that if a complete n-dimensional Finsler manifold of nonnegative n-Ricci curvature satisfies the Gagliardo-Nirenberg inequality with the sharp constant, then its flag curvature is identically zero. The other one is that we give an alternative proof to Mao's main result in [23] for smooth metric measure spaces with nonnegative weighted Ricci curvature.

1 Introduction

Let M be an n-dimensional complete non-compact Riemannian manifold, and denote by ∇ the gradient operator on M. Given positive numbers p and q, denote by $\mathscr{D}^{p,q}(M)$ the completion of the space of smooth compactly supported functions on M under the norm $\|\cdot\|_{p,q}$ defined by $\|u\|_{p,q} = \|\nabla u\|_p + \|u\|_q$. Let $1 < p < n$, $p < q \leq \frac{p(n-1)}{n-p}$, $\delta = np - (n-p)q$, $r = p\frac{q-1}{p-1}$, $\theta = \frac{(q-p)n}{(q-1)(np-(n-p)q)}$. For all $u \in \mathscr{D}^{p,q}(\mathbb{R}^n)$, Del Pino-Dolbeault [9, 10] proved that

$$\left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{1}{r}} \leq \Phi \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^n} |u|^q dx\right)^{\frac{1-\theta}{q}}, \tag{1.1}$$

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where Φ is the best constant for the inequality (1.1) and takes the explicit form

$$\Phi = \left(\frac{q-p}{p\sqrt{\pi}}\right)^{\theta}\left(\frac{pq}{n(q-p)}\right)^{\frac{\theta}{p}}\left(\frac{\theta}{pq}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\Gamma\left(\frac{n}{2}+1\right)\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)\Gamma\left(n\frac{p-1}{p}+1\right)}\right)^{\frac{\theta}{n}}.$$

Equality holds in (1.1) if and only if for some $\alpha \in \mathbb{R}$, $\beta > 0$, $\overline{x} \in \mathbb{R}^n$,

$$u(x) = \alpha\left(1+\beta|x-\overline{x}|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}, \qquad \forall x \in \mathbb{R}^n.$$

The inequality (1.1) is usually called the *Gagliardo-Nirenberg inequality*. Moreover, when $q = p\frac{n-1}{n-p}$, then $\theta = 1$, $r = \frac{np}{n-p}$, and (1.1) becomes the optimal Sobolev inequality, which is separately found by Aubin [1] and Telenti [29], having many important applications (see, for instance, [2, 3, 13, 14, 19]). Complete manifolds with nonnegative Ricci curvature on which some Sobolev or Caffarelli-Kohn-Nirenberg type inequality is satisfied were studied in [8, 18, 32].

Let M be a Riemannian manifold. Let dv be the Riemannian volume element on M, and $C_0^\infty(M)$ be the space of smooth functions on M with compact support. Let $B(x,r)$ be the geodesic ball with center $x \in M$ and radius r, and $\mathrm{Vol}[B(x,r)]$ be the volume of $B(x,r)$, which is given by

$$\mathrm{Vol}[B(x,r)] = \int_{B(x,r)} dv.$$

In 2005, Xia [31] studied complete non-compact Riemannian manifolds with nonnegative Ricci curvature on which some Gagliardo-Nirenberg type inequality is satisfied, and proved the following result.

Theorem 1.1. *Let $1 < p < n, p < q \le \frac{p(n-1)}{n-p}, r = p\frac{q-1}{p-1}, \theta = \frac{(q-p)n}{(q-1)(np-(n-p)q)}$, and let $C \ge \Phi$ be a constant. Assume that M is an n-dimensional $(n \ge 2)$ complete non-compact Riemannian manifold with non-negative Ricci curvature and assume that for any $u \in C_0^\infty(M)$, we have*

$$\left(\int_M |u|^r dv\right)^{\frac{1}{r}} \le C\left(\int_M |\nabla u|^p dv\right)^{\frac{\theta}{p}}\left(\int_M |u|^q dv\right)^{\frac{1-\theta}{q}}. \tag{1.2}$$

Then for any $x \in M$, we have

$$\mathrm{Vol}[B(x,r)] \ge \left(C^{-1}\Phi\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}} V_0(r), \qquad \forall r > 0, \tag{1.3}$$

where $V_0(r)$ is the volume of an r-ball in $\mathbb{R}^n$.

Let (X,d) be a metric measure space, and μ be a Borel measure on X such that $0 < \mu(U) < \infty$ for any nonempty bounded open set $U \subset X$. Let $\mathrm{Lip}_0(X)$ be the space of Lipschitz functions with compact support on X, and define $|Du|(x)$ as follows

$$|Du|(x) := \limsup_{y\to x} \frac{|u(y)-u(x)|}{d(x,y)},$$

which is the local Lipschitz constant of u at $x \in X$. The function $x \to |Du|(x)$ is Borel measurable for $u \in \mathrm{Lip}_0(X)$. In 2013, Kristály-Ohta [17] studied metric measure spaces satisfying the volume doubling condition mentioned therein and the Caffarelli-Kohn-Nirenberg inequality with the same exponent $n \geq 3$, and then they proved that those spaces have exactly the n-dimensional volume growth. Inspired by Xia's and Kristály-Ohta's works mentioned above, here we investigate a metric measure space satisfying a volume doubling condition and the Gagliardo-Nirenberg inequality, and successfully prove the following result.

Theorem 1.2. *Let p, q, r, θ, n be as in Theorem 1.1, $x_0 \in X$, $C \geq \Phi$, and $C_0 \geq 1$. Assume that for any $u \in \mathrm{Lip}_0(X)$, the Gagliardo-Nirenberg inequality*

$$\left(\int_X |u(x)|^r d\mu(x)\right)^{\frac{1}{r}} \leq C\left(\int_X |\nabla u|^p(x) d\mu(x)\right)^{\frac{\theta}{p}}\left(\int_X |u(x)|^q d\mu(x)\right)^{\frac{1-\theta}{q}} \tag{1.4}$$

and the volume conditions

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C_0\left(\frac{R}{r}\right)^n, \quad \text{for all } x \in X, \text{ and } 0<r<R, \tag{1.5}$$

$$\liminf_{r\to 0}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))} = 1 \tag{1.6}$$

hold on a proper metric measure space (X,d,μ) of dimension n, where $B(x,r) := \{y \in X : d(x,y) < r\}, \mathbb{B}_n(r) := \{x \in \mathbb{R}^n : |x| < r\}$, and μ_E is the n-dimensional Lebesgue measure. Then, for any $x \in X$ and $\rho > 0$, we have

$$\mu(B(x,\rho)) \geq C_0^{-1}\left(C^{-1}\Phi\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}\mu_E(\mathbb{B}_n(\rho)). \tag{1.7}$$

In particular, (X,d,μ) has the n-dimensional volume growth

$$C_0^{-1}\left(C^{-1}\Phi\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}} w_n\rho^n \leq \mu(B(x_0,\rho)) \leq C_0 w_n \rho^n$$

for all $\rho > 0$, where w_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 1.3. (1). When $q = \frac{(n-1)p}{n-p}$, then $\theta = 1$, $r = \frac{np}{n-p}$, and correspondingly, the Gagliardo-Nirenberg inequality (1.4) degenerates into the following Sobolev inequality

$$\left(\int_X |u(x)|^{\frac{np}{n-p}} d\mu(x)\right)^{\frac{n-p}{np}} \leq C\left(\int_X |\nabla u|^p(x) d\mu(x)\right)^{\frac{1}{p}}$$

for $u \in \mathrm{Lip}_0(X)$.

(2). The non-compactness of (X,d) can be assured by the validity of (1.4). In fact, if (X,d) is bounded, then one can choose $q = \frac{(n-1)p}{n-p}$, then $\theta = 1$ and $r = \frac{np}{n-p}$, which lets (1.4) become the Sobolev inequality mentioned above, and in this setting, $u + \ell$ with $\ell \to \infty$ clearly violates the

validity of (1.4).
(3). By (1.5), we have

$$\frac{\mu(B(x_0,R))}{w_nR^n} \le C_0\frac{\mu(B(x_0,r))}{w_nr^n} = C_0\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}$$

for $x_0 \in X$ and $0 < r < R$. Fixing R and letting r tends to zero, by the volume condition (1.6) which describes the volume behavior near x_0, we can obtain

$$\frac{\mu(B(x_0,R))}{w_nR^n} \le C_0 \cdot \liminf_{r\to 0}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))} = 1,$$

which implies that $\mu(B(x_0,R)) \le C_0w_nR^n$ for any $R > 0$. So, one can get the n-dimensional volume growth, i.e., the last assertion of Theorem 1.1, directly provided (1.7) is proven.
(4). As pointed out in [17, Remark 1.2 (b)], if (X,d,μ) satisfies the *volume doubling condition*

$$\mu(B(x,2r)) \le \Lambda\mu(B(x,r)), \qquad \text{for some } \Lambda \ge 1 \text{ and all } x \in X,\ r > 0,$$

then it is easy to get that the volume condition (1.5) is satisfied with, e.g., $n \ge \log_2\Lambda$ and $C_0 = 1$. Therefore, (1.5) can be comprehended as the volume doubling condition with the explicit exponent n. Besides, one can regard the volume condition (1.6) as a generalization of the classical Bishop-Gromov volume comparison for complete manifolds with non-negative Ricci curvature.
(5). The assertion of having n-dimensional volume growth implies that, for instance, the cylinder $\mathbb{S}^{n-1}\times\mathbb{R}$ does not satisfy (1.4) for any $x \in X$ and C. The volume doubling condition (1.5) implies that the Hausdorff dimension $\dim_H X$ of (X,d) is at most n. Besides, as in (3), by the volume conditions (1.5) and (1.6), we have

$$\frac{\mu(B(x_0,R))}{\mu_E(\mathbb{B}_n(R))} \le C_0\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}$$

for $x_0 \in X$ and $0 < r < R$, which implies that

$$\limsup_{R\to 0}\frac{\mu(B(x_0,R))}{\mu_E(\mathbb{B}_n(R))} \le \liminf_{r\to 0} C_0\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))} = C_0.$$

Therefore, we know that the *Ahlfors n-regularity* at x_0 in the sense that $\eta^{-1}r^n \le \mu(B(x_0,r)) \le \eta r^n$ for some $\eta \ge 1$ and small $r > 0$, which means that $\dim_H X = n$. The volume doubling condition and the Ahlfors regularity are important in analysis on metric measure spaces. For this fact, see, e.g., [15] for the details. Note that the choice of the constant 1 chosen at the right hand side of (1.6) is only for simplicity. In fact, by (1.5), we know that $\eta_{x_0} := \liminf_{r\to 0}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}$ is positive. So, one can normalize μ so as to satisfy (1.6) once η_{x_0} is bounded.

By Theorem 2.3 (equivalently, see also Shen [27] or Ohta [25]), we know that for Finsler manifolds with non-negative n-Ricci curvature (for this notion, see Definition 2.1 for the precise statement), the volume doubling condition (1.5) holds with $C_0 = 1$. For complete Finsler manifolds with non-negative n-Ricci curvature, when the Gagliardo-Nirenberg inequality (1.4) is satisfied with the best constant (i.e., $C = \Phi$), by applying Theorems 1.2 and 2.3, we can prove the following rigidity theorem.

Corollary 1.4. *Let (X,F) be a complete n-dimensional Finsler manifold. Let p, q, r, θ, n be as in Theorem 1.1, $x_0 \in X$, and $C_0 \geq 1$. Fix a positive smooth measure μ on X and assume that the n-Ricci curvature* Ric_n *of (X,F,μ) is nonnegative. If the Gagliardo-Nirenberg inequality (1.4) is satisfied with the best constant (i.e., $C = \Phi$) and $\lim_{r\to 0}\frac{\mu(B(x_0,r))}{w_n r^n} = 1$, then under the volume doubling condition (1.5), we have the flag curvature of (X,F) is identically zero.*

Remark 1.5. Finsler manifolds are special metric measure spaces with prescribed Finsler structures. See Subsection 2.2 for a brief introduction to Finsler manifolds.

A smooth metric measure space, which is also known as the weighted measure space, is actually a Riemannian manifold equipped with some measure (which is conformal to the usual Riemannian measure). More precisely, for a given complete n-dimensional Riemannian manifold (M,g) with the metric g, the triple $(M,g,e^{-f}dv_g)$ is called a smooth metric measure space, with f a *smooth real-valued* function on M and dv_g the Riemannian volume element related to g (sometimes, we also call dv_g the volume density). For a geodesic ball $B(x,r)$, we can define its *weighted (or f-)volume* $\mathrm{Vol}_f[B(x,r)]$ as follows

$$\mathrm{Vol}_f[B(x,r)] = \int_{B(x,r)} e^{-f} dv_g. \tag{1.8}$$

On a smooth metric measure space $(M,g,e^{-f}dv_g)$, the so called *∞-Bakry-Émery Ricci tensor* Ric_f is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f,$$

which is also called the *weighted Ricci curvature*. Bakry and Émery [4, 5] introduced firstly and investigated extensively the generalized Ricci tensor above and its relationship with diffusion processes. In 2014, Mao [23] studied complete non-compact smooth measure metric spaces with nonnegative weighted Ricci curvature on which some Gagliardo-Nirenberg type inequality is satisfied, and proved the following result.

Theorem 1.6. *([23]) Let p, q, r, θ, n be as in Theorem 1.1, and let $\left(M,g,e^{-f}dv_g\right)$ be an n-dimensional ($n \geq 2$) complete noncompact smooth metric measure space with non-negative weighted Ricci curvature. For a point $x_0 \in M$ at which $f(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t f$ satisfies $\partial_t f \geq 0$ along all minimal geodesic segments from x_0, with $t := d(x_o,\cdot)$ the distance to x_0. Furthermore, for any $u \in C_0^\infty(M)$ and some constant $C > 0$, if the following Gagliardo-Nirenberg type inequality*

$$\left(\int_M |u(x)|^r e^{-f} dv_g\right)^{\frac{1}{r}} \leq C\left(\int_M |\nabla u|^p(x) e^{-f} dv_g\right)^{\frac{\theta}{p}}\left(\int_M |u(x)|^q e^{-f} dv_g\right)^{\frac{1-\theta}{q}} \tag{1.9}$$

is satisfied, then we have

$$\mathrm{Vol}_f[B(x_0,R)] \geq e^{-f(x_0)}\left(C^{-1}\Phi\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}} V_0(R), \qquad \forall R > 0, \tag{1.10}$$

where $V_0(R)$ denotes the volume of an R-ball in $\mathbb{R}^n$.

Remark 1.7. (1) By applying Theorem 1.2, we can give an *alternative* proof to Theorem 1.6 for smooth metric measure spaces of dimension $n \geq 2$ – see Subsection 3.3 for the details.
(2). If the Gagliardo-Nirenberg type inequality (1.9) is satisfied with the best constant (i.e., $C = \Phi$), then by Theorems 1.6 and 3.1, and together with generalized Bishop-type volume comparisons (cf. [11, Theorem 3.3, Corollary 3.5 and Theorem 4.2]) for complete manifolds with *radial* curvature bounded, a rigidity conclusion, (M,g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ with $g_{\mathbb{R}^n}$ being the usual Euclidean metric, can be obtained (cf. [23, Corollary 1.5] for the precise statement).

It is interesting to know *under what kind of conditions* a complete open n-manifold ($n \geqslant 2$) is isometric to $\mathbb{R}^n$ or has finite topological type, which in essence has relation with the splittingness of the prescribed manifold. This is a classical topic in the global differential geometry, which has been investigated intensively (see, e.g., [7, 20, 26]).

2 Proofs of Theorem 1.2 and Corollary 1.4

2.1 Proof of Theorem 1.2

Proof. As pointed out in Remark 1.3 (3), if we want to get the n-dimensional volume growth assertion in Theorem 1.2, we only need to show (1.7). Now, in the rest of this subsection, we would like to give the details of the proof of (1.7) as follows.

First, we introduce two auxiliary functions $F, G : (0, +\infty) \to \mathbb{R}$ defined by

$$F(\lambda) := \int_X \frac{1}{\left(\lambda + d(x_0, x)^{\frac{p}{p-1}}\right)^{\frac{(p-1)q}{q-p}}} d\mu(x)$$

and

$$G(\lambda) := \int_{\mathbb{R}^n} \frac{1}{\left(\lambda + |x|^{\frac{p}{p-1}}\right)^{\frac{(p-1)q}{q-p}}} d\mu_E(x)$$

respectively, which are well defined and of class C^1.

By the layer cake representation of functions, one has

$$F(\lambda) = \int_0^{+\infty} \mu \left\{ x \in X : \frac{1}{\left(\lambda + d(x_0, x)^{\frac{p}{p-1}}\right)^{\frac{(p-1)q}{q-p}}} > s \right\} ds.$$

By taking into account that $\operatorname{diam}(X) = \infty$ and making the variable change

$$s = \frac{1}{\left(\lambda + \rho^{\frac{p}{p-1}}\right)^{\frac{(p-1)q}{q-p}}},$$

then

$$F(\lambda) = \frac{pq}{q-p} \int_0^{+\infty} \mu(B(x_0, \rho)) f(\lambda, \rho) d\rho, \tag{2.1}$$

where

$$f(\lambda,\rho)=\frac{\rho^{\frac{1}{p-1}}}{\left(\lambda+\rho^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}.$$

Similar to the above process, we can also get

$$G(\lambda)=\frac{pq}{q-p}\int_0^{+\infty}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho. \tag{2.2}$$

On the other hand, since the inequalities (1.5) and (1.6) hold on (X,d,μ), we have

$$\mu(B(x_0,\rho))\le C_0\mu_E(\mathbb{B}_n(\rho)). \tag{2.3}$$

Then, it follows from (2.1)–(2.3) that

$$0\le F(\lambda)\le C_0G(\lambda). \tag{2.4}$$

Since $q<\frac{np}{n-p}$, we have

$$n+\frac{1}{p-1}-\frac{p^2(q-1)}{(p-1)(q-p)}=n+\frac{1}{p-1}-\frac{p^2}{p-1}-\frac{p^2}{q-p}<-1,$$

and from which we know that $0\le F(\lambda)<\infty$, for any $\lambda>0$, and $F(\lambda)$ is differentiable. Also, we have

$$F'(\lambda)=-\frac{(p-1)q}{q-p}\int_X\frac{d\mu(x)}{\left(\lambda+d(x_0,x)^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}. \tag{2.5}$$

For each $\lambda>0$, consider the sequence of functions

$$u_{\lambda,k}(x):=\max\{0,\min\{0,k-d(x_0,x)\}+1\}\left(\lambda+\max\{d(x_0,x),k^{-1}\}^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

Since (X,d) is proper, $\mathrm{supp}(u_{\lambda,k}):=\{x\in X: d(x_0,x)\le k+1\}$ is compact. Therefore, we have $u_{\lambda,k}\in\mathrm{Lip}_0(X)$ for every $\lambda>0$ and $k\in\mathbb{N}$. Set

$$u_\lambda(x):=\lim_{k\to\infty}u_{\lambda,k}(x)=\left(\lambda+d(x_0,x)^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

Since the functions $u_{\lambda,k}$ verify the Gagliardo-Nirenberg inequality (1.4), by an approximation based on (2.4), we know that u_λ verifies the Gagliardo-Nirenberg inequality (1.4) also. Together with the fact that $x\to d(x_0,x)$ is 1-Lipschitz (i.e., $|Dd(x_0,\cdot)|(x)\le 1$ for all x), we can obtain

$$\begin{aligned}&\left(\int_X\frac{d\mu(x)}{\left(\lambda+d(x_0,x)^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}\right)^{\frac{1}{r}}\\ \le\ & C\left(\frac{p}{q-p}\right)^{\theta}\left(\int_X\frac{d(x_0,x)^{\frac{p}{p-1}}d\mu(x)}{\left(\lambda+d(x_0,x)^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}\right)^{\frac{\theta}{p}}\left(\int_X\frac{d\mu(x)}{\left(\lambda+d(x_0,x)^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}\right)^{\frac{1-\theta}{q}}\end{aligned} \tag{2.6}$$

by using a chain rule for the local Lipschitz constant. By the definition of $F(\lambda)$ and (2.5), the above equality can be rewritten as follows

$$\left(-F'(\lambda)\right)^{\frac{p}{\theta r}} \le l\left(F(\lambda)+\frac{q-p}{(p-1)q}\lambda F'(\lambda)\right)F(\lambda)^{\frac{(1-\theta)p}{\theta q}}, \tag{2.7}$$

where

$$l = C^{\frac{p}{\theta}}\left(\frac{p}{q-p}\right)^{p}\left(\frac{(p-1)q}{q-p}\right)^{\frac{p}{\theta r}}.$$

Since $v_\lambda(x)=\left(\lambda+|x|^{\frac{p}{q-p}}\right)^{-\frac{p-1}{q-p}}$ is a minimizer of the Gagliardo-Nirenberg inequality in $\mathbb{R}^n$, then for every $\lambda>0$, the following equality

$$\left(\int_{\mathbb{R}^n}|v_\lambda(x)|^r d\mu_E(x)\right)^{\frac{1}{r}} = \Phi\left(\int_{\mathbb{R}^n}|\nabla v_\lambda|^p(x)d\mu_E(x)\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}^n}|v_\lambda(x)|^q d\mu_E(x)\right)^{\frac{1-\theta}{q}}$$

holds. By the definition of $G(\lambda)$ and a similar argument as (2.7), the above equality can be rewritten as follows

$$\left(-G'(\lambda)\right)^{\frac{p}{\theta r}} = \widetilde{l}\left(G(\lambda)+\frac{q-p}{(p-1)q}\lambda G'(\lambda)\right)G(\lambda)^{\frac{(1-\theta)p}{\theta q}}, \tag{2.8}$$

where

$$\widetilde{l} = \Phi^{\frac{p}{\theta}}\left(\frac{p}{q-p}\right)^{p}\left(\frac{(p-1)q}{q-p}\right)^{\frac{p}{\theta r}}.$$

Substituting $G(\lambda)=G(1)\lambda^{(p-1)\left(\frac{n}{p}-\frac{q}{q-p}\right)}$ into (2.8), we have

$$\left(1-\frac{n(q-p)}{pq}\right)^{\frac{p}{\theta r}} = \Phi^{\frac{p}{\theta}}\left(\frac{p}{q-p}\right)^{p}\left(\frac{(q-p)^n}{pq}\right)G(1)^{\frac{p}{\theta}\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)}. \tag{2.9}$$

Consider the constant A given by

$$\left(1-\frac{n(q-p)}{pq}\right)^{\frac{p}{\theta r}} = C^{\frac{p}{\theta}}\left(\frac{p}{q-p}\right)^{p}\left(\frac{(q-p)^n}{pq}\right)A^{\frac{p}{\theta}\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)}. \tag{2.10}$$

It is easy to check that the function

$$H_0(\lambda) = A\lambda^{(p-1)\left(\frac{n}{p}-\frac{q}{q-p}\right)}, \qquad \lambda\in(0,+\infty),$$

satisfies the differential equation

$$\left(-q'(\lambda)\right)^{\frac{p}{\theta r}} = l\left(q(\lambda)+\frac{q-p}{(p-1)q}\lambda q'(\lambda)\right)q(\lambda)^{\frac{(1-\theta)p}{\theta q}}. \tag{2.11}$$

By (2.9) and (2.10), we get

$$A=\left(\frac{\Phi}{C}\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}G(1),$$

which implies

$$H_0(\lambda)=\left(\frac{\Phi}{C}\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}\lambda^{(p-1)\left(\frac{n}{p}-\frac{q}{q-p}\right)}G(1)=\left(\frac{\Phi}{C}\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}G(\lambda). \tag{2.12}$$

In the following, we will show that when $C>\Phi$, for every $\lambda>0$,

$$F(\lambda)\geq H_0(\lambda). \tag{2.13}$$

First, we *claim* that if $F(\lambda_0)<H_0(\lambda_0)$, for some $\lambda_0>0$, then $F(\lambda)<H_0(\lambda),\forall\lambda\in(0,\lambda_0]$. We prove this by contradiction. Suppose that the claim is not true. Then there exists some $\widetilde{\lambda}\in(0,\lambda_0)$ such that $F(\widetilde{\lambda})\geq H_0(\widetilde{\lambda})$. Set $\lambda_1:=\sup\{\lambda<\lambda_0:F(\lambda)=H_0(\lambda)\}$. Then for any $\lambda\in[\lambda_1,\lambda_0], 0<F(\lambda)\leq H_0(\lambda)$, and so from (2.7), we have

$$\left(-F'(\lambda)\right)^{\frac{p}{\theta r}}\leq l\left(H_0(\lambda)+\frac{q-p}{(p-1)q}\lambda F'(\lambda)\right)H_0(\lambda)^{\frac{(1-\theta)p}{\theta q}}. \tag{2.14}$$

For every $\lambda>0$, define a function $z_\lambda:(0,\infty)\to\mathbb{R}$ by $z_\lambda(\rho)=\rho^{\frac{p}{\theta r}}+\frac{l\lambda(q-p)\rho}{(p-1)q}H_0(\lambda)^{\frac{(1-\theta)p}{\theta q}}$. Clearly, z_λ is increasing. Hence, when $\lambda\in[\lambda_1,\lambda_0]$, we infer from (2.14) and (2.11) that

$$\begin{aligned} z_\lambda(-F'(\lambda)) &= (-F'(\lambda))^{\frac{p}{\theta r}}+\frac{l\lambda(q-p)}{(p-1)q}(-F'(\lambda))H_0(\lambda)^{\frac{(1-\theta)p}{\theta q}} \\ &\leq lH_0(\lambda)^{1+\frac{(1-\theta)p}{\theta q}}=z_\lambda(-H_0'(\lambda)), \end{aligned}$$

which means $F'(\lambda)\geq H_0'(\lambda),\forall\lambda\in[\lambda_1,\lambda_0]$. Thus, we know that the function $F-H_0$ is increasing on $[\lambda_1,\lambda_0]$, which implies that

$$0=(F-H_0)(\lambda_1)\leq(F-H_0)(\lambda_0)<0.$$

This is a contradiction. Hence, the above *claim* is true.

By (1.6), we know that for every $\varepsilon>0$, there exists $\delta>0$ such that $\mu(B(x_0,\rho))\geq(1-\varepsilon)\mu_E(\mathbb{B}(\rho))$ for all $0\leq\rho\leq\delta$. Therefore, by (2.7) and making a variable change $\rho=\lambda^{\frac{1}{2-ap}}t$, we can get

$$\begin{aligned} F(\lambda) &\geq \frac{pq}{p-q}(1-\varepsilon)\int_0^\delta \mu_E(\mathbb{B}_n(\rho)f(\lambda,\rho))d\rho \\ &= \frac{1-a}{n-1+a}(1-\varepsilon)\lambda^{\frac{(p-1)n}{p}+1-\frac{p(q-1)}{q-p}}\int_0^{\delta\lambda^{\frac{1-p}{p}}}\mu_E(\mathbb{B}_n(t)f(1,t))dt. \end{aligned}$$

On the other hand, we have

$$G(\lambda)=\frac{1-a}{n-1+a}\lambda^{\frac{(p-1)n}{p}+1-\frac{p(q-1)}{q-p}}\int_0^\infty\mu_E(\mathbb{B}_n(t)f(1,t))dt.$$

Therefore, we have

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{G(\lambda)}\geq 1-\varepsilon.$$

Letting $\varepsilon\to 0$ yields

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{G(\lambda)}\geq 1.$$

When $C>\Phi$, we infer from the above inequality and (2.12) that

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{H_0(\lambda)}=\left(\frac{C}{\Phi}\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}\liminf_{\lambda\to 0}\frac{F(\lambda)}{G(\lambda)}\geq\left(\frac{C}{\Phi}\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}>1.$$

Then, together with the previous *claim*, we can get $F(\lambda)\geq H_0(\lambda),\forall\lambda>0.$ Thus, for any $\lambda>0$, we can get from (2.1), (2.2), (2.12) that

$$\int_0^{+\infty}\{\mu(B(x_0,\rho))-b\mu_E(\mathbb{B}_n(\rho))\}f(\lambda,\rho)d\rho\geq 0, \tag{2.15}$$

where $b=\left(C^{-1}\Phi\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}$. By (1.5), for a fixed $\rho>0$, we have

$$C_0\frac{\mu(B(x_0,\rho))}{\mu_E(\mathbb{B}_n(\rho))}\geq\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}$$

for any $r>\rho\geq 0$. We can assume

$$b_0:=\limsup_{r\to\infty}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}.$$

In order to prove (1.7) in the case that $C>\Phi$, it suffices to show that $b_0\geq b$. We will prove this by contradiction. By the definition of b_0, we know that for some $\rho_0>0$, there exists $\varepsilon_0>0$ such that

$$\frac{\mu(B(x_0,\rho))}{\mu_E(\mathbb{B}_n(\rho))}\leq b-\varepsilon_0,\qquad\forall\rho\geq\rho_0. \tag{2.16}$$

Substituting (2.16) into (2.15), and together with (2.3), for every $\lambda>0$, we have

$$\begin{aligned}
0 &\leq \int_0^{+\infty}\{\mu(B(x_0,\rho))-b\mu_E(\mathbb{B}_n(\rho))\}f(\lambda,\rho)d\rho\\
&\leq \int_0^{\rho_0}\mu(B(x_0,\rho))f(\lambda,\rho)d\rho+(b-\varepsilon_0)\int_{\rho_0}^{+\infty}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho\\
&\quad -b\int_0^{+\infty}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho\\
&\leq C_0\int_0^{\rho_0}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho-b\int_0^{\rho_0}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho\\
&\quad -\varepsilon_0\int_{\rho_0}^{+\infty}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho\\
&= (C_0-b+\varepsilon_0)\int_0^{\rho_0}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho-\varepsilon_0\int_0^{+\infty}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho\\
&= (C_0-b+\varepsilon_0)\int_0^{\rho_0}\mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho-\varepsilon_0\frac{q-p}{pq}\lambda^{(p-1)\left(\frac{n}{p}-\frac{q}{q-p}\right)}G(1).
\end{aligned}$$

Since $f(\lambda,\rho)=\frac{\rho^{\frac{1}{p-1}}}{\left(\lambda+\rho^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}$, one has

$$\int_0^{\rho_0}\rho^n f(\lambda,\rho)d\rho=\int_0^{\rho_0}\frac{\rho^{n+\frac{1}{p-1}}}{\left(\lambda+\rho^{\frac{p}{p-1}}\right)^{\frac{(q-1)p}{q-p}}}d\rho\leq\lambda^{\frac{-p(q-1)}{q-p}}\frac{\rho_0^{n+1+\frac{1}{p-1}}}{\left(n+1+\frac{1}{p-1}\right)}.$$

By the above two inequalities, we can get the inequality of type

$$M_1\lambda^{(p-1)\left(\frac{n}{p}-\frac{q}{q-p}\right)}\leq M_2\lambda^{\frac{-p(q-1)}{q-p}},\qquad\forall\lambda>0,$$

where $M_1,M_2>0$ are constants independent of λ. Observing $\frac{-p(q-1)}{q-p}-(p-1)\left(\frac{n}{p}-\frac{q}{q-p}\right)=\frac{(1-p)n}{p}-1<0$, letting $\lambda\to+\infty$ in the above inequality, one can obtain a contradiction. This means that (1.7) holds in the case that $C>\Phi$.

When $C=\Phi$, we can also get (1.7). In fact, in this case, for any fixed $\delta>0$, we have

$$\left(\int_X|u(x)|^r d\mu(x)\right)^{\frac{1}{r}}\leq(\Phi+\delta)\left(\int_X|\nabla u|^p(x)d\mu(x)\right)^{\frac{\theta}{p}}\left(\int_X|u(x)|^q d\mu(x)\right)^{\frac{1-\theta}{q}}.$$

Therefore, for any $x\in X$, by the previous argument, we have

$$\mu(B(x,\rho))\geq C_0^{-1}\left(\frac{\Phi}{\Phi+\delta}\right)^{\left(\frac{\theta}{p}+\frac{1-\theta}{q}-\frac{1}{r}\right)^{-1}}\mu_E(\mathbb{B}_n(\rho)),\qquad\forall\rho>0.$$

Letting $\delta\to0$, we can obtain

$$\mu(B(x,\rho))\geq C_0^{-1}\mu_E(\mathbb{B}_n(\rho)),\qquad\forall\rho>0,$$

which implies (1.7) holds in the case that $C=\Phi$.

This completes the proof of Theorem 1.2. □

2.2 Preliminary notions and a Bishop-Gromov type volume comparison theorem in Finsler geometry

Before applying Theorem 1.2 to prove Corollary 1.4, we briefly recall some concepts in Finsler geometry. We refer to [6] for a fundamental but overall introduction about Finsler geometry.

Definition 2.1. Let X be a connected n-dimensional smooth manifold and $TX=\bigcup_{x\in X}T_xX$ be its tangent bundle. The pair (X,F) is called a *Finsler manifold* if a continuous function $F:TX\to[0,\infty)$ satisfies the following conditions
(1) $F\in C^\infty(TX\backslash\{0\})$;
(2) $F(x,tv)=|t|F(x,v)$ for all $t\in\mathbb{R}$ and $(x,v)\in TX$;
(3) The $n\times n$ matrix

$$g_{ij}(x,v):=\frac{1}{2}\frac{\partial^2\left(F^2\right)}{\partial v^i\partial v^j},\qquad\text{where } v=\sum_{i=1}^n v^i\frac{\partial}{\partial x^i},\tag{2.17}$$

is positive definite for all $(x,v)\in TX\backslash\{0\}$. F is called the Finsler structure of (X,F).

We will denote by $\langle,\rangle_v$ the inner product on T_xX induced by (2.17). We know that (X,F) becomes a Riemannian manifold if and only if $g_{ij}(x,v)$ is independent of v in each $T_xX\setminus\{0\}$. For a smooth curve $\sigma:[0,l)\to X$, one can define its *integral length* $L_F\sigma$ by $L_F\sigma=\int_0^l F(\sigma,\dot{\sigma})dt$. Based on this, the *distance function* $d_F:X\times X\to[0,\infty)$ can be defined by $d_F(x_1,x_2):=\inf_\sigma L_F\sigma$, where σ runs over all smooth curves from x_1 to x_2. A smooth curve $\sigma:[0,l)\to X$ is called a *geodesic* if it locally minimizes d_F and has a constant speed (i.e., $F(\sigma,\dot{\sigma})$ is constant). The geodesic (Euler-Lagrange) equation can be written down in terms of covariant derivative along σ (see [6] for the details). The Finsler manifold (X,F) is complete if any geodesic $\sigma:[0,l)\to X$ can be extended to a geodesic $\sigma:\mathbb{R}\to X$.

Like the Riemannian case, we can also do the geodesic variation in the Finsler case. In fact, let $\sigma:(-\varepsilon,\varepsilon)\times[0,l]\to X$ be a smooth geodesic variation (i.e., $t\to\sigma(s,t)$ is geodesic for each s), and set $\eta(t)=\sigma(0,s)$. Then the variational vector field $J(t):=\frac{\partial\sigma}{\partial s}(0,t)$ satisfies the following Jacobi equation

$$D^{\dot{\eta}}_{\dot{\eta}}D^{\dot{\eta}}_{\dot{\eta}}J+R^{\dot{\eta}}(J,\dot{\eta})\dot{\eta}=0, \tag{2.18}$$

where $D^{\dot{\eta}}$ is the covariant derivative w.r.t. the vector $\dot{\eta}$, and $R^{\dot{\eta}}$ is the curvature tensor (see [6] for the details). For vectors $v,w\in T_xX$, which are linearly independent, and $\mathscr{S}=\mathrm{span}\{v,w\}$, the *flag curvature* of the *flag* $(\mathscr{S};v)$ can be defined as follows

$$K(\mathscr{S},v):=\frac{\langle R^v(w,v)v,w\rangle_v}{F(v)^2\langle w,w\rangle_v-\langle v,w\rangle_v^2}.$$

If (X,F) is a Riemannian manifold, then the flag curvature degenerates into the sectional curvature which only depends on $\mathscr{S}$ (not on the choice of $v\in\mathscr{S}$). Choose $v\in T_xX$ with $F(x,v)=1$, and let $\{e_i\}_{i=1}^n$ with $e_n=v$ be an orthonormal basis of $(T_xX,\langle,\rangle_v)$ with $\langle,\rangle_v$ induced from (2.17). Set $\mathscr{S}_i=\mathrm{span}\{e_i,v\}$ for $i=1,2,\ldots,n-1$. The Ricci curvature of v is defined by $\mathrm{Ric}(v):=\sum_{i=1}^{n-1}K(\mathscr{S}_i;v)$. We also set $\mathrm{Ric}(cv):=c^2\mathrm{Ric}(v)$ for $c\geq 0$.

For those Finsler curvatures mentioned above, Shen has explained them from the Riemannian viewpoint (see [28, Section 6.2 of Chapter 6]). Fixing $v\in T_xX\setminus\{0\}$ and extending it to a smooth vector field V around x such that all integral curves of V are geodesics, then the flag curvature $K(\mathscr{S},v)$ is the same as the sectional curvature of $\mathscr{S}$ w.r.t. the Riemannian structure $\langle,\rangle_V$, and correspondingly, $\mathrm{Ric}(v)$ is the same as the Ricci curvature of v w.r.t. $\langle,\rangle_V$. This fact leads to the following definition of N-Ricci curvature associated with an arbitrary measure on X (see also, e.g., [17, 25] for this notion).

Definition 2.2. Let μ be a positive smooth measure on X. Given $v\in T_xX\setminus\{0\}$, let $\sigma:(-\varepsilon,\varepsilon)\to X$ be the geodesic with $\dot{\sigma}=v$ and decompose μ along σ as $\mu=e^{-\psi}\mathrm{Vol}_{\dot{\sigma}}$, where $\mathrm{Vol}_{\dot{\sigma}}$ is the volume element of the Riemannian structure $\langle,\rangle_{\dot{\sigma}}$. Then, for $N\in[n,\infty]$, the *N-Ricci curvature* Ric_N is defined by

$$\mathrm{Ric}_N(v)=\mathrm{Ric}(v)+(\psi\circ\sigma)''(0)-\frac{(\psi\circ\sigma)'(0)^2}{N-n},$$

where the third term is understood as 0 if $N=\infty$ or if $N=n$ with $(\psi\circ\sigma)'(0)=0$, and as $-\infty$ if $N=n$ with $(\psi\circ\sigma)'(0)\neq 0$.

By applying the concept of the N-Ricci curvature Ric_N, Ohta [25] proved the following Bishop-Gromov type volume comparison result in the Finsler case.

Theorem 2.3. *([25, Theorem 7.3]) Let (X,F,μ) be a complete n-dimensional Finsler manifold with nonnegative N-Ricci curvature. Then we have*

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq \left(\frac{R}{r}\right)^N, \quad \text{for every } x \in X, \text{ and } 0 < r < R.$$

Moreover, if equality holds with $N = n$ for all $x \in X$ and $0 < r < R$, then any Jacobi field J along a geodesic σ has the form $J(t) = tP(t)$, where P is a parallel vector field along σ (i.e., $D^{\dot{\sigma}}_{\dot{\sigma}} P \equiv 0$).

2.3 Proof of Corollary 1.4

Proof. Since (X,F) is complete, by applying the Hopf-Rinow theorem, we know that (X,d_F,μ) is a *proper* metric measure space. Since the n-Ricci curvature Ric_n is nonnegative, by Theorem 2.3, we can obtain (1.5) with $C_0 = 1$. As pointed out in Remark 1.3 (5), one can normalize the fixed positive measure μ such that (1.6) is satisfied. Then by these two facts, similar to Remark 1.3 (3), we can easily get

$$\mu(B(x,\rho)) \leq \mu_E(\mathbb{B}_n(\rho)), \qquad \text{for all } \rho > 0, \ x \in X.$$

However, since the Gagliardo-Nirenberg inequality (1.4) is satisfied with the best constant (i.e., $C = \Phi$), by Theorem 1.2, we have

$$\mu(B(x,\rho)) \geq \mu_E(\mathbb{B}_n(\rho)), \qquad \text{for all } \rho > 0, \ x \in X.$$

Therefore, $\mu(B(x,\rho)) = \mu_E(\mathbb{B}_n(\rho))$ for all $\rho > 0$ and $x \in X$. By applying Theorem 2.3 directly, we know that every Jacobi field J along a geodesic σ has the form $J(t) = tP(t)$, where P is a parallel vector field along σ. Together with the Jacobi equation (2.18), it follows that $R^{\dot{\sigma}}(J,\dot{\sigma})\dot{\sigma} \equiv 0$. Then $K(\mathscr{S};\dot{\sigma}) \equiv 0$ with $\mathscr{S} = \mathrm{span}(\dot{\sigma},P)$. Since σ and J are arbitrary, we know $K \equiv 0$, which equivalently says that the flag curvature of (X,F) is identically zero. □

3 Proof of Theorem 1.6

In this section, as mentioned before, we would like to give an alternative proof to Theorem 1.6. However, before that, we need to introduce some notions. For more details, we refer readers to [11, 21, 22, 23, 24].

3.1 Some basic notions

Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. Given an n-dimensional ($n \geqslant 2$) complete Riemannian manifold (M,g) with the metric g, for a point $x \in M$, let S^{n-1}_x be the unit sphere with center x in the tangent space T_xM, and let $Cut(x)$ be the cut-locus of x, which is a closed set of zero n-Hausdorff measure. Clearly,

$$\mathbb{D}_x = \left\{ t\xi | 0 \leqslant t < d_\xi, \xi \in S^{n-1}_x \right\}$$

is a star-shaped open set of T_xM, and through which the exponential map $\exp_x : \mathbb{D}_x \to M\backslash Cut(x)$ gives a diffeomorphism from $\mathbb{D}_x$ to the open set $M\backslash Cut(x)$, where d_ξ is defined by

$$d_\xi = d_\xi(x) := \sup\{t>0|\ \gamma_\xi(s) := \exp_x(s\xi) \text{ is the unique minimal geodesic joining } x \text{ and } \gamma_\xi(t)\}.$$

We can introduce two important maps used to construct the geodesic spherical coordinate chart at a prescribed point on a Riemannian manifold. For a fixed vector $\xi \in T_xM$, $|\xi| = 1$, let $\xi^\perp$ be the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_xM, and let $\tau_t : T_xM \to T_{\exp_x(t\xi)}M$ be the parallel translation along $\gamma_\xi(t)$. The path of linear transformations $\mathbb{A}(t,\xi) : \xi^\perp \to \xi^\perp$ is defined by

$$\mathbb{A}(t,\xi)\eta = (\tau_t)^{-1}Y_\eta(t),$$

where $Y_\eta(t) = d(\exp_x)_{(t\xi)}(t\eta)$ is the Jacobi field along $\gamma_\xi(t)$ satisfying $Y_\eta(0) = 0$, and $(\nabla_t Y_\eta)(0) = \eta$. Moreover, for $\eta \in \xi^\perp$, set $\mathscr{R}(t)\eta = (\tau_t)^{-1}R(\gamma'_\xi(t), \tau_t\eta)\gamma'_\xi(t)$, where the curvature tensor $R(X,Y)Z$ is defined by $R(X,Y)Z = -[\nabla_X,\ \nabla_Y]Z + \nabla_{[X,Y]}Z$. Then $\mathscr{R}(t)$ is a self-adjoint operator on $\xi^\perp$, whose trace is the radial Ricci tensor $\mathrm{Ric}_{\gamma_\xi(t)}\left(\gamma'_\xi(t), \gamma'_\xi(t)\right)$. Clearly, the map $\mathbb{A}(t,\xi)$ satisfies the Jacobi equation $\mathbb{A}'' + \mathscr{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0,\xi) = 0$, $\mathbb{A}'(0,\xi) = I$. By Gauss's lemma, the Riemannian metric of $M\backslash Cut(x)$ in the geodesic spherical coordinate chart can be expressed by

$$ds^2(\exp_x(t\xi)) = dt^2 + |\mathbb{A}(t,\xi)d\xi|^2, \qquad \forall t\xi \in \mathbb{D}_x. \tag{3.1}$$

We consider the metric components $g_{ij}(t,\xi)$, $i,j \geq 1$, in a coordinate system $\{t, \xi_a\}$ formed by fixing an orthonormal basis $\{\eta_a, a \geq 2\}$ of $\xi^\perp = T_\xi S_x^{n-1}$, and then extending it to a local frame $\{\xi_a, a \geq 2\}$ of S_x^{n-1}. Define a function $J > 0$ on $\mathbb{D}_x\backslash\{x\}$ by

$$J^{n-1} = \sqrt{|g|} := \sqrt{\det[g_{ij}]}. \tag{3.2}$$

Since $\tau_t : S_x^{n-1} \to S_{\gamma_\xi(t)}^{n-1}$ is an isometry, we have

$$\langle d(\exp_x)_{t\xi}(t\eta_a), d(\exp_x)_{t\xi}(t\eta_b)\rangle_g = \langle \mathbb{A}(t,\xi)(\eta_a), \mathbb{A}(t,\xi)(\eta_b)\rangle_g,$$

and then $\sqrt{|g|} = \det \mathbb{A}(t,\xi)$. So, by applying (3.1) and (3.2), the volume $\mathrm{vol}(B(x,r))$ of a geodesic ball $B(x,r)$, with radius r and center x, on M is given by

$$\mathrm{Vol}(B(x,r)) = \int_{S_x^{n-1}} \int_0^{\min\{r,d_\xi\}} \sqrt{|g|}\,dt\,d\sigma = \int_{S_x^{n-1}} \left(\int_0^{\min\{r,d_\xi\}} \det(\mathbb{A}(t,\xi))dt \right) d\sigma, \tag{3.3}$$

where $d\sigma$ denotes the $(n-1)$-dimensional volume element on $\mathbb{S}^{n-1} \equiv S_x^{n-1} \subseteq T_xM$. As in Section 1, let $r(z) = d(x,z)$ be the intrinsic distance to the point $x \in M$. Since for any $\xi \in S_x^{n-1}$ and $t_0 > 0$, we have $\nabla r(\gamma_\xi(t_0)) = \gamma'_\xi(t_0)$ when the point $\gamma_\xi(t_0) = \exp_x(t_0\xi)$ is away from the cut locus of x (cf. [12]), then, by the definition of a non-zero tangent vector "*radial*" to a prescribed point on a manifold given in the first page of [16], we know that for $z \in M\backslash(Cut(x) \cup x)$ the unit vector field

$$v_z := \nabla r(z)$$

is the radial unit tangent vector at z. We also need the following fact about $r(z)$ (cf. Prop. 39 on p. 266 of [26]),

$$\partial_r \Delta r + \frac{(\Delta r)^2}{n-1} \leq \partial_r \Delta r + |\mathrm{Hess} r|^2 = -\mathrm{Ric}(\partial_r, \partial_r), \qquad \text{with } \Delta r = \partial_r \ln(\sqrt{|g|}),$$

with $\partial_r = \nabla r$ as a differentiable vector (cf. Prop. 7 on p. 47 of [26] for the differentiation of ∂_r), where Δ is the Laplace operator on M and Hess$r$ is the Hessian of $r(z)$. Then, together with (3.2), we have

$$\begin{aligned} & J'' + \frac{1}{(n-1)} \mathrm{Ric}\left(\gamma'_\xi(t), \gamma'_\xi(t)\right) J \leq 0, \\ & J(t,\xi) = t + O(t^2), \quad J'(t,\xi) = 1 + O(t). \end{aligned} \tag{3.4}$$

3.2 A volume comparison theorem in smooth metric measure spaces

We also need the following volume comparison theorem proven by Wei and Wylie (cf. [30, Theorem 1.2]) which is the key point to prove Theorem 1.6.

Theorem 3.1. *([30]) Let $(M, g, e^{-f} dv_g)$ be n-dimensional ($n \geqslant 2$) complete smooth metric measure space with $\mathrm{Ric}_f \geqslant (n-1)H$. Fix $x_0 \in M$. If $\partial_t f \geqslant -a$ along all minimal geodesic segments from x_0 then for $R \geqslant r > 0$ (assume $R \leqslant \pi/2\sqrt{H}$ if $H > 0$),*

$$\frac{\mathrm{Vol}_f[B(x_0,R)]}{\mathrm{Vol}_f[B(x_0,r)]} \leqslant e^{aR} \frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)},$$

where $\mathrm{Vol}_H^n(\cdot)$ is the volume of the geodesic ball with the prescribed radius in the space n-form with constant sectional curvature H, and, as before, $\mathrm{vol}_f(\cdot)$ denotes the weighted (or f-)volume of the given geodesic ball on M. Moreover, equality in the above inequality holds if and only if the radial sectional curvatures are equal to H and $\partial_t f \equiv -a$. In particular, if $\partial_t f \geqslant 0$ and $\mathrm{Ric} \geqslant 0$, then M has f-volume growth of degree at most n.

3.3 Proof of Theorem 1.6

Proof. For complete and *non-compact* smooth metric measure n-space $(M, g, e^{-f} dv_g)$, if $\partial_t f \geqslant 0$ (along all minimal geodesic segments from x_0) and $\mathrm{Ric}_f \geqslant 0$, then by Theorem 3.1 we have

$$\frac{\mathrm{Vol}_f[B(x_0,R)]}{\mathrm{Vol}_f[B(x_0,r)]} \leqslant e^{0 \cdot R} \cdot \frac{V_0(R)}{V_0(r)} = \left(\frac{R}{r}\right)^n,$$

with, as before, $V_0(\cdot)$ denotes the volume of the ball with the prescribed radius in $\mathbb{R}^n$. Clearly, here the volume doubling condition (1.5) is satisfied with $C_0 = 1$.

For $(M, g, e^{-f} dv_g)$, in order to apply Theorem 1.2 to prove Theorem 1.6, we need to normalize the original measure $e^{-f} dv_g$ such that the volume condition (1.6) can be satisfied. In fact, we need to choose the positive measure μ to be $\mu = e^{f(x_0) - f} dv_g$. Then by applying (1.8), (3.2), (3.3) and

(3.4), we can get

$$\begin{aligned}\lim_{r\to 0}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}=\lim_{r\to 0}\frac{e^{f(x_0)}\cdot \mathrm{Vol}_f[B(x_0,r)]}{V_0(r)} &= \lim_{r\to 0}\frac{\int\limits_{\mathbb{S}^{n-1}}\left(\int\limits_0^{\min\{r,d_\xi\}} J^{n-1}(t,\xi)\cdot e^{-f}dt\right)d\sigma}{e^{-f(x_0)}\int\limits_{\mathbb{S}^{n-1}}\int\limits_0^r t^{n-1}dtd\sigma}\\ &= \frac{J'(0,\xi)\cdot e^{-f(x_0)}}{e^{-f(x_0)}}=1\end{aligned}$$

by applying L'Hôpital's rule n-times, which implies (1.6) is satisfied. Therefore, if in addition the Gagliardo-Nirenberg type inequality (1.9) is satisfied, then by applying Theorem 1.2, we can get (1.10) directly. This completes the proof of Theorem 1.6. □

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