Article

ϕ-informational measures: Calculations for the Gamma distribution

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Appendix A. Inverse maximum entropy problem and associated inequalities: some examples

Appendix A.1. The gamma distribution and (partial) p-order moment(s)

As a very special case, consider here the gamma distribution expressed as

$$f_X(x) = \frac{(\Gamma(q)\,x)^{q-1}\exp\left(-\frac{\Gamma(q)}{r}\,x\right)}{r^q} \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}_+. \tag{A1}$$

Parameter $q > 0$ is known as shape parameter of the law, while $\sigma = \frac{r}{\Gamma(q)} > 0$ is a scaling parameter. This distribution also appears in various applications, as described for instance in [1].

Let us concentrate on the case $q > 1$ for which the distribution is non-monotonous, unimodal, where the mode is located at $x = \frac{r\,(q-1)}{\Gamma(q)}$, and $f_X(\mathbb{R}_+) = \left[0\,;\,\frac{(q-1)^{q-1}\,e^{1-q}}{r}\right]$.

Here again it cannot be a maximizer of a ϕ-entropy constraint subject to a moment of order $p > 0$. Here, we can again consider partial moments as constraints,

$$T_{k,1}(x) = x^p\,\mathbb{1}_{\mathcal{X}_k}(x), \qquad k \in \{0,-1\} \qquad \text{where}$$
$$\mathcal{X}_0 = \left[0\,;\,\frac{r\,(q-1)}{\Gamma(q)}\right) \qquad \text{and} \qquad \mathcal{X}_{-1} = \left[\frac{r\,(q-1)}{\Gamma(q)}\,;\,+\infty\right),$$

or interpret f_X as a critical point of an ϕ-like entropy by constraining the moment

$$T_1(x) = x^p \quad \text{over} \quad \mathcal{X} = \mathbb{R}_+. \tag{A2}$$

Inverting $y = f_X(x)$ leads to the equation

$$-\frac{\Gamma(q)\,x}{r\,(q-1)}\exp\left(-\frac{\Gamma(q)\,x}{r\,(q-1)}\right) = -\frac{(ry)^{\frac{1}{q-1}}}{q-1}$$

to be solved. As expected, this equation has two solutions. These solutions can be expressed via the multivalued Lambert-W function W defined by $z = \mathrm{W}(z)\exp(\mathrm{W}(z))$, i.e., W is the inverse function of $u \mapsto u\exp(u)$[2, § 1], leading to the inverse functions

$$f_{X,k}^{-1}(y) = -\frac{r\,(q-1)}{\Gamma(q)}\,\mathrm{W}_k\left(-\frac{(ry)^{\frac{1}{q-1}}}{q-1}\right), \qquad ry \in \left[0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right], \tag{A3}$$

where k denotes the branch of the Lambert-W function. $k = 0$ gives the principal branch and here it is related to the entropy part on $\mathcal{X}_0$, while $k = -1$ gives the secondary branch, related to $\mathcal{X}_{-1}$ here.

Applying (**??**) to obtain the branches of the functionals of the multiform entropy, one has thus to integrate the functions

$$\phi_k'(y) = \lambda_0 + \lambda_{k,1} \left[-\frac{r\,(q-1)}{\Gamma(q)}\ \mathrm{W}_k\left(-\frac{(ry)^{\frac{1}{q-1}}}{q-1} \right) \right]^p$$

where, to insure the convexity of the ϕ_k,

$$(-1)^k \lambda_{k,1} > 0$$

The same approach allows to design $\widetilde{\phi}_k$, with a unique λ_1 instead of the $\lambda_{k,1}$s and without restriction on λ_1.

First, let us reparametrize the λ_is so as to include the factor $r/\Gamma(q)$ inside $\lambda_{k,1}$ so that one can write formally

$$\phi_k(y) = \phi_{k,\mathrm{u}}(ry) \qquad \text{with} \tag{A4}$$

$$\phi_{k,\mathrm{u}}(u) = \beta\, u + \gamma_k + \alpha_k \int \left[(1-q)\, \mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^p du, \qquad (-1)^k \alpha_k \geq 0$$

Obtaining a closed forme expression for the integral term is not an easy task. But relation $z\,(1 + \mathrm{W}_k(z))\ \mathrm{W}_k'(z) = \mathrm{W}_k(z)$ [2, Eq. 3.2] suggests that a way to make the integration is to search for

$$\Phi_k(u) = \int \left[(1-q)\, \mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^p du \tag{A5}$$

under the form of a series

$$\Phi_k(u) = u \sum_{l \geq 0} a_l \left[(1-q)\, \mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^{l+p}$$

identifying the coefficients a_l. This gives, by derivation and omitting the argument of W_k by sake of simplicity

$$\Big[(1-q)\, \mathrm{W}_k \Big]^p = \sum_{l \geq 0} a_l \Big[(1-q)\, \mathrm{W}_k \Big]^{l+p} + \frac{u^{\frac{1}{q-1}}}{q-1}\ \mathrm{W}_k' \sum_{l \geq 0} (l+p)\, a_l \Big[(1-q)\, \mathrm{W}_k \Big]^{l+p-1}$$

Now with $z = -\frac{u^{\frac{1}{q-1}}}{q-1}$ one has $\frac{u^{\frac{1}{q-1}}}{q-1}\ \mathrm{W}_k' = -\frac{\mathrm{W}_k}{1+\mathrm{W}_k}$ so that

$$\Big[(1-q)\, \mathrm{W}_k \Big]^p = \sum_{l \geq 0} a_l \Big[(1-q)\, \mathrm{W}_k \Big]^{l+p} + \sum_{l \geq 0} \frac{(l+p)\, a_l}{q-1}\ \frac{\Big[(1-q)\, \mathrm{W}_k \Big]^{l+p}}{1 + \mathrm{W}_k}$$

that is, simplifying both sides by $\Big[(1-q)\, \mathrm{W}_k \Big]^p$ and multiplying both sides by $1 + \mathrm{W}_k$,

$$1 + \mathrm{W}_k = \sum_{l \geq 0} a_l \Big[(1-q)\, \mathrm{W}_k \Big]^l + \sum_{l \geq 0} \frac{a_l}{1-q} \Big[(1-q)\, \mathrm{W}_k \Big]^{l+1} + \sum_{l \geq 0} \frac{(l+p)\, a_l}{q-1} \Big[(1-q)\, \mathrm{W}_k \Big]^l$$

i.e.,

$$1 + \mathrm{W}_k = \frac{(p+q-1)\, a_0}{q-1} \; + \sum_{l \geq 1} \frac{a_{l-1} - (p+q+l-1)\, a_l}{1-q} \left[(1-q)\, \mathrm{W}_k \right]^l$$

As a consequence

$$a_0 = \frac{q-1}{p+q-1}$$

$1 = a_0 - (p+q)\, a_1$ so that $a_1 = \frac{a_0 - 1}{p+q}$, i.e.,

$$a_1 = -\, \frac{p}{(p+q)(p+q-1)}$$

For $l \geq 2$, $a_{l-1} - (p+q+l-1) a_l = 0$ i.e., $a_l = \frac{1}{p+q+l-1} a_{l-1}$

$$\forall\, l \geq 2, \quad a_l = \frac{1}{(p+q+l-1) \cdots (p+q+1)}\, a_1$$

For the Pochhammer symbol $(a)_l = a \cdots (a+l-1)$ for $l \geq 1$ and $(a)_0 = 1$ one has

$$\forall\, l \geq 1, \quad a_l = -\frac{p}{(p+q-1)(p+q)_l}$$

(given for $l \geq 2$, but one can see that it remains valid for $l = 1$). Therefore

$$\Phi_k(u) = u \left[(1-q)\, \mathrm{W}_k \right]^p \left(\frac{q-1}{p+q-1} - \frac{p}{p+q-1} \sum_{l \geq 1} \frac{1}{(p+q)_l} \left[(1-q)\, \mathrm{W}_k \right]^l \right)$$

Adding and removing a term in $l = 0$ in the sum, and noting that $l! = (1)_l$, one finally obtains

$$\Phi_k(u) = u \left[(1-q)\, \mathrm{W}_k \right]^p \left(1 - \frac{p}{p+q-1} \sum_{l \geq 0} \frac{(1)_l}{(p+q)_l\, l!} \left[(1-q)\, \mathrm{W}_k \right]^l \right)$$

One finally recognizes in the sum the confluent hypergeometric (or Kummer's) function [3, § 13] or [4, § 9.2] ${}_1F_1(1; p+q; \cdot)$, so that, we achieve to

$$\phi_{k,\mathrm{u}}(u) = \beta\, u \, + \, \gamma_k \, + \, \alpha_k\, u \left[(1-q)\; \mathrm{W}_k \left(- \frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^p \times$$

$$\left[1 - \frac{p}{p+q-1}\, {}_1F_1 \left(1\, ;\, p+q\, ;\, (1-q)\, \mathrm{W}_k \left(- \frac{u^{\frac{1}{q-1}}}{q-1} \right) \right) \right] \mathbb{1}_{\left(0\, ;\, \left(\frac{q-1}{e} \right)^{q-1} \right)}(u) \qquad \text{(A6)}$$

$$\text{with} \qquad (-1)^k \alpha_k > 0$$

Again, p, q, r are additional parameters for this family of entropies.

Then, from the domain of definition of the inverse of f_X, u is restricted to $\left(0\, ;\, \left(\frac{q-1}{e} \right)^{q-1} \right)$, which can be compensated for by playing with parameter r. At the opposite, noting that $\mathrm{W}_k\left(-e^{-1}\right) = -1$, to extend the entropic functionals to C^1 functions on $\mathbb{R}_+$, one would have to impose $\beta + \alpha_k = 0$ to vanish the derivatives at $u = e^{1-a}$. This is impossible because from $(-1)\alpha_k > 0$ one cannot impose $\alpha_k = -\beta$. Moreover, even a convex extension relaxing the C^1 condition is impossible since we would have to impose $\beta + \alpha_k \leq \beta$ to insure the increase of the ϕ_ks.

We can however choose the γ_k such that the ϕ_k coincide at $u = 0$ for instance (e.g., to vanish them at 0 to insure the existence of the ϕ-entropy). One can also wish to impose the value(s) of the $\phi_{k,\mathrm{u}}$ at $u = \left(\frac{q-1}{e}\right)^{q-1}$.

Values at the bound of the domain of definition

From [2, Eq. 3.1] we have $\mathrm{W}_0(0) = 0$ and from [3, Eq. 13.1.2] ${}_1F_1(1; p+q; 0) = 1$, so that

$$\phi_{0,\mathrm{u}}(0) = \gamma_0 \qquad \text{and} \qquad \phi_{0,\mathrm{u}}'(0) = \beta \tag{A7}$$

Then $\lim\limits_{x \to 0^-} \mathrm{W}_{-1}(x) = -\infty$ (see [2, Fig. 1 or Eq. 4.18]). From the asymptotics [3, Eq. 13.1.4] of the confluent hypergeometric function,

$${}_1F_1(1; p+q; (1-q)\,\mathrm{W}_{-1}) = \Gamma(p+q)\, e^{(1-q)\,\mathrm{W}_{-1}} \left[(1-q)\,\mathrm{W}_{-1}\right]^{1-p-q} \left(1 + O\left(\left|\,\mathrm{W}_{-1}\,\right|^{1-p-q}\right)\right)$$

and thus

$$\Phi_{-1} = u\left[(1-q)\,\mathrm{W}_{-1}\right]^{p} - p\,\Gamma(p+q-1)\,u\left[(1-q)\,\mathrm{W}_{-1}\,e^{\mathrm{W}_{-1}}\right]^{1-q} \left(1 + O\left(\left|\,\mathrm{W}_{-1}\,\right|^{1-q}\right)\right)$$

This gives, from $\mathrm{W}(z)e^{\mathrm{W}(z)} = z$, i.e., $\mathrm{W}_{-1}\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right) \exp\left(\mathrm{W}_{-1}\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right)\right) = -\frac{u^{\frac{1}{q-1}}}{q-1}$,

$$\Phi_{-1} = u\left[(1-q)\,\mathrm{W}_{-1}\right]^{p} - p\,\Gamma(p+q-1)\left(1 + O\left(\left|\,\mathrm{W}_{-1}\,\right|^{1-q}\right)\right)$$

Finally, noting that, because $1-q<0$ we have $\lim\limits_{u \to 0^-} \left|\,\mathrm{W}_{-1}\,\right|^{1-q} = 0$, and from [2, Eq. 4.6 & lines that follow] $\lim\limits_{u \to 0^-} u\left[(1-q)\,\mathrm{W}_{-1}\right]^{p} = 0$ so that, finally, at the limit

$$\phi_{-1,\mathrm{u}}(0) = \gamma_{-1} - p\,\Gamma(p+q-1)\,\alpha_{-1} \qquad \text{and} \qquad \lim_{u \to 0^-} \phi_{0,\mathrm{u}}'(u) = -\infty \tag{A8}$$

($\alpha_{-1} < 0$).

Now, from $\mathrm{W}_k(-e^{-1}) = -1$ we immediately have

$$\phi_{k,\mathrm{u}}\left(\left(\frac{q-1}{e}\right)^{q-1}\right) = \gamma_k + \left(\frac{q-1}{e}\right)^{q-1}\left(\beta + \alpha_k (q-1)^p \left[1 - \frac{p}{p+q-1}\,{}_1F_1(1; p+q; q-1)\right]\right) \tag{A9}$$

and

$$\phi_{k,\mathrm{u}}'\left(\left(\frac{q-1}{e}\right)^{q-1}\right) = \beta + \alpha_k (q-1)^p \tag{A10}$$

The special case $p = 2-q$

From [3, 13.6.14], ${}_1F_1(1; 2; x) = \frac{e^x - 1}{x}$, so that

$$\Phi_k(u) = u\left[(1-q)\,\mathrm{W}_k\right]^{2-q}\left[1 - (2-q)\,\frac{e^{(1-q)\,\mathrm{W}_k} - 1}{(1-q)\,\mathrm{W}_k}\right]$$

that is

$$\Phi_k(u) = u\Big[(1-q)\,\mathrm{W}_k\Big]^{1-q}\Big[(1-q)\,\mathrm{W}_k + 2 - q\Big] - \frac{(2-q)\,u}{(q-1)^{q-1}}\left(-\,\mathrm{W}_k\,e^{\mathrm{W}_k}\right)^{1-q}$$

Again, from $\mathrm{W}_k(z)e^{\mathrm{W}_k(z)} = z$ we have $\left(-\,\mathrm{W}_k\,e^{\mathrm{W}_k}\right)^{1-q} = \left(\frac{u^{\frac{1}{q-1}}}{q-1}\right)^{1-q} = u^{-1}(q-1)^{q-1}$ so that

$$\Phi_k(u) = u\Big[(1-q)\,\mathrm{W}_k\Big]^{1-q}\Big[(1-q)\,\mathrm{W}_k + 2 - q\Big] - (2-q)$$

Interestingly, when $a \to 1$, the gamma law tends to the exponential distribution and, at the same time, $\mathcal{X}_0 \to \emptyset$, $\mathcal{X}_{-1} \to \mathbb{R}_+$. to finish

The multivalued function ϕ_{u} in the concave context is represented figure A1 for $p = 2, q = 2$ and $q = 5$, and with the choice $\alpha_0 = 1$, $\alpha_{-1} = -0.05$, $\beta = -\alpha_{-1}$, $\gamma_0 = 0$, $\gamma_{-1} = \frac{p\Gamma(p+q-1)}{(q-1)^p}\,\alpha_{-1}$.

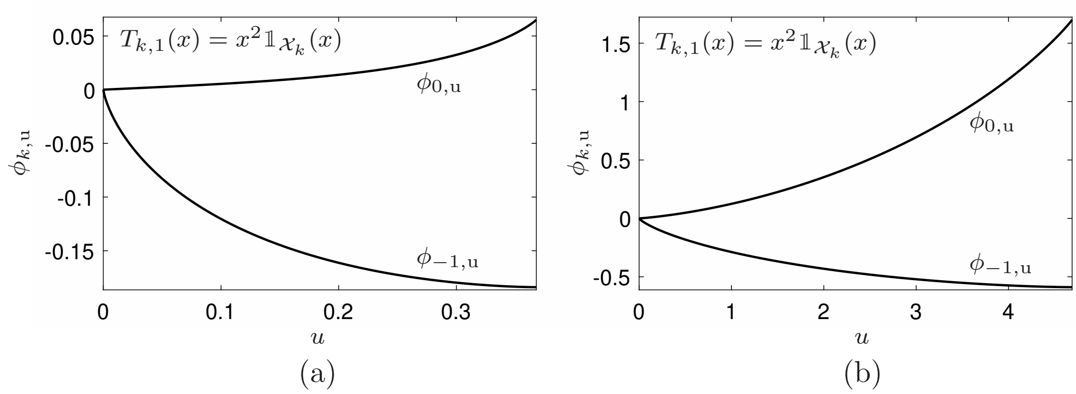

Figure A1. Multiform entropy functional ϕ_{u} derived from the gamma distribution with the partial moment constraints $T_{k,1}(x) = x^2 \mathbb{1}_{\mathcal{X}_k}(x)$, $k \in \{0, -1\}$. (a): $q = 2$; (b): $q = 5$.

- **Ponctuer titres et titre des sections?**
- **T_is ou T_i's ? De même pour les λ_is.**
- **Integrable ou summable?**
- **Limite de la Gamma vers l'exponentielle et moment d'ordre 1: on doit trouver Shannon (en paramétrisant correctement).**

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