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IMPROVED GAGLIARDO-NIRENBERG INEQUALITIES ON HEISENBERG TYPE GROUPS*

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Abstract Motivated by the idea of M. Ledoux who brings out the connection between Sobolev embeddings and heat kernel bounds, we prove an analogous result for Kohn's sub-Laplacian on the Heisenberg type groups. The main result includes features of an inequality of either Sobolev or Galiardo-Nirenberg type.

Key words Heisenberg type group; heat kernel; Sobolev inequality; Galiardo-Nirenberg inequality

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1 Introduction

The classical Sobolev inequalities states that, for every function f in the Sobolev space $W^{1,p}(R^n)$, there holds

$$\|f\|_p \leq C(n,p)\|\nabla f\|_q, \quad q = np/(n-p), \quad 1 \leq p < n. \tag{1.1}$$

When $n = p$, (1.1) does not hold for $q = \infty$. In [4] and [11], the following improvement of the Sobolev inequality was derived: for $1 \leq p < q < \infty$,

$$\|f\|_{L^q(R^n)} \leq C'(n,p,q)\|\nabla f\|_{L^p(R^n)}^{p/q}\|f\|_B^{1-p/q}. \tag{1.2}$$

The space B is a Besov space defined in terms of the heat kernel semigroup $\mathrm{e}^{t\Delta}$. This includes, in particular, the Sobolev and Galiardo-Nirenberg inequalities, and also has important features not possessed by (1.1) (see [4] and [11] for details).

It was known since the work of Folland [8] and Varopoulos [16], that the following version of the Sobolev inequality holds on Heisenberg type groups, a remarkable class of stratified groups of step two introduced by Kaplan [10] (see Section 2 for definitions and properties)

$$\|f\|_{L^p(G)} \leq C(n,p)\|\nabla_G f\|_{L^{p^*}(G)}, \quad p^* = Qp/(Q-p), \tag{1.3}$$

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provided that $1 \le p < Q$, where $|\nabla_G f|$ stands for the norm of the horizontal gradient of a function $f \in C_0^\infty(G)$ and Q is the homogeneous dimension.

In this paper, we shall derive an improved version of (1.3) which is analogous to (1.2). In fact, we have the following theorem.

Theorem 1 For every $1 \le p < q < \infty$ and every function f, $f \in B^{\theta/(\theta-1)}_{\infty,\infty}$ with $\nabla_G f \in L^p(G)$, there holds

$$\|f\|_{L^q(G)} \le C(n,p,q)\|\nabla_G f\|^{\theta}_{L^p(G)}\|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}, \tag{1.4}$$

where $\theta = p/q$ and $B^{\theta/(\theta-1)}_{\infty,\infty}$ is the homogeneous Besov space defined in terms of the heat semigroup $\mathrm{e}^{h\Delta_G}$ (see Section 2).

By the heat kernel embedding (see Lemma 2), one can easily recover from Theorem 1 the Sobolev inequalities (1.3) as well as the Galiardo-Nirenberg inequalities

$$\|f\|_{L^q(G)} \le C(n,p,q)\|\nabla_G f\|^{\theta}_{L^p(G)}\|f\|^{1-\theta}_{L^r(G)}, \quad \theta = \frac{p}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{r}{qQ}. \tag{1.5}$$

2 Notation and Preliminaries

We summarize in this section the main properties of the Heisenberg type groups that we use in the present paper. For more details, we refer the reader to [3, 6–9, 12, 17] and the references therein. A H-type group G is a Carnot group of step two with the following properties: the Lie algebra $\mathfrak{g}$ of G is endowed with an inner product $\langle , \rangle$ such that , if $\mathfrak{z}$ is the center of $\mathfrak{g}$, then $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$ and moreover, for every fixed $z \in \mathfrak{z}$, the map $J_z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_z(v), \omega\rangle = \langle z, [v,\omega]\rangle, \quad \forall \omega \in \mathfrak{z}^\perp$$

is an orthogonal map whenever $\langle z, z\rangle = 1$. Set $m = \dim \mathfrak{z}^\perp$ and $n = \dim \mathfrak{z}$. Since G has step two and since the stratification of the Lie algebra $\mathfrak{g}$ is evidently $\mathfrak{z}^\perp \oplus \mathfrak{z}$, in the sequel we shall fix on G a system of coordinates (x,t) and that the group law has the form (see [3])

$$(x,t) \circ (x',t') = \begin{pmatrix} x_i + x'_i, \quad i = 1,2,\cdots,m \\ t_j + t'_j + \frac{1}{2}\langle x, U^{(j)}x'\rangle, \quad j = 1,2,\cdots,n \end{pmatrix},$$

where the matrices $U^{(1)}, U^{(2)}, \cdots, U^{(n)}$ have the following properties.

(1) $U^{(j)}$ is a $m \times m$ skew symmetric and orthogonal matrix, for every $j = 1,2,\cdots,n$;

(2) $U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0$ for every $i,j \in \{1,2,\cdots,n\}$ with $i \ne j$.

A easy computation shows that the vector field in the algebra $\mathfrak{g}$ of $G = (\mathbb{R}^{m+n}, \circ)$ that agrees at the origin with $\frac{\partial}{\partial x_j}(j = 1,\cdots,m)$ is given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m} U^{(k)}_{i,j} x_i\right)\frac{\partial}{\partial t_k},$$

and that $\mathfrak{g}$ is spanned by the left-invariant vector fields

$$X_1, \cdots, X_m, \quad T_1 = \frac{\partial}{\partial t_1}, \cdots, T_n = \frac{\partial}{\partial t_n}.$$

The Kohn's sub-Laplacian on the Heisenberg type group G is given by (see [3])

$$\begin{aligned}\Delta &= \sum_{j=1}^{m} X_j^2 = \sum_{j=1}^{m}\left(\frac{\partial}{\partial x_j}+\frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m}U_{j,i}^{(k)}x_i\right)\frac{\partial}{\partial t_k}\right)^2\\ &= \Delta_x+\frac{1}{4}|x|^2\Delta_t+\sum_{k=1}^{n}\langle U^{(k)}x,\nabla_x\rangle\frac{\partial}{\partial t_k},\end{aligned}$$

where

$$\Delta_x=\sum_{j=1}^{m}\left(\frac{\partial}{\partial x_j}\right)^2,\ \ \Delta_t=\sum_{k=1}^{n}\left(\frac{\partial}{\partial t_k}\right)^2.$$

The corresponding horizontal gradient is the m-dimensional vector given by $\nabla_G=(X_1,\cdots,X_m)$.

We call a curve $\gamma:[a,b]\to G$ a horizontal curve connecting two points $\xi,\eta\in G$ if $\gamma(a)=\xi$, $\gamma(b)=\eta$ and $\dot\gamma(s)\in\operatorname{span}\{X_1,\cdots,X_m\}$ for all s. The Carnot-Carathéodory distance between ξ,η is defined as

$$d(\xi,\eta)=\inf_{\gamma}\int_a^b\sqrt{\langle\dot\gamma(s),\dot\gamma(s)\rangle}\mathrm{d}s,$$

where the infimum is taken over all horizontal curves γ connecting ξ and η. It is known that any two points ξ,η on G can be joined by a horizontal curve of finite length and then d is a metric on G. An important feature of this distance function is that the distance is left-invariant. With this norm, we can define the metric ball centered at origin e and with radius ρ by

$$B(e,\rho)=\{\eta:d(e,\eta)<\rho\}$$

and the unit sphere $\Sigma=\partial B(e,1)$. For simplicity, we write $d(\xi)=d(e,\xi)$.

For each real number $\lambda>0$, there is a dilation naturally associated with the group structure which is usually denoted as $\delta_\lambda(x,t)=(\lambda x,\lambda^2t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q=m+2n$ is the homogenous dimension of G. For simplicity, we use the notation $\lambda(x,t)=(\lambda x,\lambda^2t)$. The Carnot-Carathéodory distance d satisfies

$$d(\lambda(x,t))=\lambda d(x,t),\ \ \lambda>0.$$

Given any $\xi=(x,t)\in G\setminus\{e\}$, set $x^*=\frac{x}{d(x,t)}$, $t^*=\frac{t}{d(x,t)^2}$ and $\xi^*=(x^*,t^*)$. The polar coordinates on G associated with d is the following (see [14]):

$$\int_G f(x,t)\mathrm{d}x\mathrm{d}t=\int_0^\infty\int_\Sigma f(\lambda(x^*,t^*))\lambda^{Q-1}\mathrm{d}\sigma\mathrm{d}\lambda,\ \ f\in L^1(G).$$

Let P_h $(h>0)$ denote the heat kernel (that is, the integral kernel of $\mathrm{e}^{h\Delta_G}$) on G. For convenience, we set $P_h(x,t)=P_h((x,t);e)$ and $P(x,t)=P_1(x,t)$. It is well known that P_h has the form (see [9, 15, 17])

$$P_h(x,t)=\frac{1}{(2\pi)^n(4\pi h)^{\frac{m}{2}}}\int_{\mathbb{R}^n}\exp\left(i\langle\lambda,t\rangle-\frac{|x|^2|\lambda|\coth|\lambda|}{4h}\right)\left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}}\mathrm{d}\lambda. \tag{2.1}$$

We note that $P_h(x,t)=h^{-\frac{Q}{2}}P(x/\sqrt{h},t/h)$ for all $h>0$. The following optimal global estimates for $P(x,t)$, namely for P_h, can be found in [6, 7, 12, 13]: there exist constants A_1 and A_2 such that

$$P(x,t)\le A_1\frac{1+d(x,t)^{m-n-1}}{1+[|x|d(x,t)]^{\frac{m}{2}}}\mathrm{e}^{-d(x,t)^2/4} \tag{2.2}$$

and

$$|\nabla_G P(x,t)| \le A_2(1+d(x,t))P(x,t). \tag{2.3}$$

We obtain, by (2.2), for $0<\epsilon<1/4$,

$$P(x,t) \le A_1(1+d(x,t)^{m-n-1})\mathrm{e}^{-d(x,t)^2/4} \le C(n,\epsilon)\mathrm{e}^{-(1/4-\epsilon)d(x,t)^2}. \tag{2.4}$$

Here we use the fact

$$1+d(x,t)^{m-n-1} \le C(n,\epsilon)\mathrm{e}^{\epsilon d(x,t)^2}.$$

Similarly,

$$|\nabla_G P(x,t)| \le A_1(1+d(x,t))(1+d(x,t)^{m-n-1})\mathrm{e}^{-d(x,t)^2/4} \le C'(n,\epsilon)\mathrm{e}^{-(1/4-\epsilon)d(x,t)^2}. \tag{2.5}$$

Fix ϵ_0 such that $0<\epsilon_0<1/4$. We obtain, by (2.4) and (2.5), that there exist constants $C_1>0$, $C_2>0$, such that

$$P(x,t) \le C_1\mathrm{e}^{-C_2 d(x,t)^2}; \quad |\nabla_G P|(x,t) \le C_1\mathrm{e}^{-C_2 d(x,t)^2}.$$

Hence

$$P_h(x,t) \le C_1 h^{-n-1}\mathrm{e}^{-C_2\frac{d(x,t)^2}{h}}; \quad |\nabla_G P_h|(x,t) \le C_1 h^{-n-2}\mathrm{e}^{-C_2\frac{d(x,t)^2}{h}}. \tag{2.6}$$

For $\alpha<0$, we define $B^\alpha_{\infty,\infty}$ to be the space of all tempered distributions f on G for which the norm

$$\|f\|_{B^\alpha_{\infty,\infty}} = \sup_{h>0}\{h^{-\frac{\alpha}{2}}\|P_h f\|_{L^\infty(G)}\}.$$

To prove the theorem we first need some preliminary results on $P_h := \mathrm{e}^{h\Delta_G}$.

Lemma 2 For all $h>0$, there exists a constant $C>0$ such that

$$\|P_h f\|_{L^\infty(G)} \le Ch^{-\frac{Q}{2p}}\|f\|_{L^p(G)}. \tag{2.7}$$

Proof From (2.6), we have by Hölder's inequality and polar coordinates that, for $\xi\in G$,

$$\begin{aligned}
|P_h f|(\xi) &= \left|\int_G f(\eta)P_h(\eta^{-1}\cdot\xi)\mathrm{d}\eta\right| \\
&\le \|f\|_{L^p(G)}\cdot\left(\int_G P_h^{p'}(\eta^{-1}\cdot\xi)\mathrm{d}\eta\right)^{\frac{1}{p'}} \\
&= \|f\|_{L^p(G)}\cdot\left(\int_G P_h^{p'}(\eta^{-1})\mathrm{d}\eta\right)^{\frac{1}{p'}} \\
&= \|f\|_{L^p(G)}\cdot\left(\int_G P_h^{p'}(\eta)\mathrm{d}\eta\right)^{\frac{1}{p'}} \\
&\le C_1 h^{-\frac{Q}{2}}\|f\|_{L^p(G)}\cdot\left(\int_G \mathrm{e}^{-C_2 p'\frac{d(\eta)^2}{h}}\mathrm{d}\eta\right)^{\frac{1}{p'}} \\
&\le Ch^{-\frac{Q}{2p}}\|f\|_{L^p(G)},
\end{aligned}$$

since $P_h(\xi^{-1})=P_h(\xi)$, where $\xi^{-1}=-\xi$ denote the inversion of ξ and $p'=p/(p-1)$.

Lemma 3 For all $h > 0$, there exists a constant $C > 0$ such that

$$\|\nabla_G P_h f\|_{L^p(G)} \le C h^{-\frac{1}{2}} \|f\|_{L^p(G)}. \tag{2.8}$$

Proof Note that

$$\begin{aligned} |\nabla_G P_h f|(\xi) &= \left| \int_G f(\eta) \nabla_G P_h(\eta^{-1} \cdot \xi) \mathrm{d}\eta \right| \\ &= \left| \int_G f(\eta) (\nabla_G P_h)(\eta^{-1} \cdot \xi) \mathrm{d}\eta \right| \end{aligned}$$

since ∇_G is left invariant. By (2.6) and Young's inequality for convolutions, we have

$$\begin{aligned} \|\nabla_G P_h f\|_{L^p(G)} &\le \|f\|_{L^p(G)} \cdot \left| \int_G \nabla_G P_h(\xi) \mathrm{d}\xi \right| \\ &\le C_1 \|f\|_{L^p(G)} \cdot h^{-\frac{Q}{2}-1} \left| \int_G \mathrm{e}^{-C_2 \frac{d(\xi)^2}{h}} \mathrm{d}\xi \right| \\ &\le C h^{-\frac{1}{2}} \|f\|_{L^p(G)}. \end{aligned}$$

Lemma 4 (pseudo-Poincaré inequality) For all $h > 0$ and every function f such that f, $\nabla_G f \in L^p(G) (p > 1)$, there exists a constant $C > 0$ such that

$$\|P_h f - f\|_{L^p(G)} \le C h^{\frac{1}{2}} \|\nabla_G f\|_{L^p(G)}.$$

Proof Let $g \in C_0^\infty(G)$, we have, by Lemma 2.2,

$$\begin{aligned} \int_G g(\xi)(P_h f - f) \mathrm{d}\xi &= \int_0^h \int_G g(\xi) \Delta_G P_s f \mathrm{d}\xi \mathrm{d}s \\ &= -\int_0^h \int_G (\nabla_G P_s g(\xi), \nabla_G f) \mathrm{d}\xi \mathrm{d}s \\ &\le \int_0^h \|\nabla_G P_s g(\xi)\|_{L^{p'}(G)} \mathrm{d}s \cdot \|\nabla_G f\|_{L^p(G)} \\ &\le C\sqrt{h} \|g(\xi)\|_{L^{p'}(G)} \cdot \|\nabla_G f\|_{L^p(G)}, \end{aligned}$$

where $p' = p/(p-1)$. Since $C_0^\infty(G)$ is dense in $L^{p'}(G)$, we obtain the pseudo-Poincaré inequality

$$\|P_h f - f\|_{L^p(G)} \le C\sqrt{h} \|\nabla_G f\|_{L^p(G)}.$$

Remark 5 In [1], Bakry, Baudoin, Bonnefont, and Chafai proved similar inequalities (2.7) and (2.8) on the 3-dimension Heisenberg group. Rencently, the results were generated to the context of Heisenberg type groups by Eldredge using the same method due to [1].

3 Proof

We are now ready to prove Theorem 1. Our proof is inspired by that of Theorem 1 in [11].

Proof of Theorem 1 **Step 1** By homogeneity we may assume that $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \le 1$, such that for all $h > 0$,

$$|P_h f| \le h^{\theta/2(\theta-1)}.$$

For all $u > 0$ define $h = h_u = u^{2(\theta-1)/\theta}$ such that

$$|P_h f| \leq u.$$

Let λ denote the Lebesgue measure on $G \simeq R^{m+n}$. With $P_h := \mathrm{e}^{h\Delta_G}$,

$$u^q \lambda\{|f| > 2u\} \leq u^q \lambda\{|f - P_h f| > u\} \leq u^{q-p} \int_G |f - P_h f|^p \mathrm{d}\lambda.$$

Thus, by the pseudo-Poincaré inequality,

$$u^q \lambda\{|f| > 2u\} \leq C u^{q-p} h_u^{p/2} \int_G |\nabla_G f|^p \mathrm{d}\lambda = \int_G |\nabla_G f|^p \mathrm{d}\lambda$$

whence

$$\|f\|_{L^{q,\infty}} \leq \int_G |\nabla_G f|^p \mathrm{d}\lambda,$$

where $L^{q,\infty}$ denotes the weak L^q norm.

Step 2 In this step, we show that the $L^{q,\infty}$ norm can be replaced by the L^q norm if we assume that $f \in L^q(G)$. We may, and shall hereafter in the proof, assume that our functions f are real-valued. Following Ledoux in [11], we write

$$5^{-q} \|f\|^q_{L^q(G)} = \int_0^{+\infty} \lambda(\{|f| \geq 5u\}) \mathrm{d}u^q, \tag{3.1}$$

and for $u > 0$ define f_u by

$$f_u = (f - u)^+ \wedge ((c-1)u) + (G + u)^- \vee (-(c-1)u), \tag{3.2}$$

where $c \geq 5$, and $\wedge, \vee$ denote the minimum and maximum, respectively. It follows that for $u \leq |f_u| \leq cu$

$$\nabla_G f_u = \nabla_G f, \tag{3.3}$$

and is zero otherwise. Also,

$$|f| \geq 5u \Longrightarrow |f_u| \geq 4u \tag{3.4}$$

and hence

$$\int_0^{+\infty} \lambda(\{|f| \geq 5u\}) \mathrm{d}u^q \leq \int_0^{+\infty} \lambda(\{|f_u| \geq 4u\}) \mathrm{d}u^q. \tag{3.5}$$

We continue to assume that $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \leq 1$ and have $h = h_u = u^{2(\theta-1)/\theta}$, $\theta = p/q$. We have, by (3.5),

$$\begin{aligned}&\int_0^{+\infty} \lambda(\{|f| \geq 5u\}) \mathrm{d}u^q \\ \leq & \int_0^{+\infty} \lambda(\{|f_u - P_{h_u}(f_u)| \geq u\}) \mathrm{d}u^q + \int_0^{+\infty} \lambda(\{P_{h_u}|f - f_u| \geq 2u\}) \mathrm{d}u^q.\end{aligned}$$

From the pseudo-Poincaré inequality and recalling that $h_u = u^{2(\theta-1)/\theta}$, so that $u^{-p}h_u^{p/2} = u^{-q}$, we have

$$\begin{aligned}\lambda(\{|f_u - P_{h_u}(f_u)| &\le u^{-p}\int_G |f_u - P_{h_u}(f_u)|^p \mathrm{d}\lambda \\ &\le Cu^{-p}h_u^{p/2}\int_G |\nabla_G f_u|^p \mathrm{d}\lambda \\ &\le Cu^{-q}\int_{\{u\le|f|\le cu\}} |\nabla_G f|^p \mathrm{d}\lambda.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^\infty \lambda(\{|f_u - P_{h_u}(f_u)|\mathrm{d}u^q &\le C\int_0^\infty \left\{u^{-q}\int_{\{u\le|f|\le cu\}} |\nabla_G f|^p \mathrm{d}\lambda\right\}\mathrm{d}u^q \\ &= C\int_G |\nabla f|^p \left(\int_{|f|/c}^{|f|} \frac{\mathrm{d}u^q}{u^q}\right)\mathrm{d}\lambda \\ &= Cq\ln c\int_G |\nabla f|^p \mathrm{d}\lambda.\end{aligned}$$

Next, we consider $\lambda(\{P_{h_u}|f - f_u| \ge 2u\})$. In fact, we have

$$|f - f_u| = |f - f_u|\chi_{\{|f|\le cu\}} + |f - f_u|\chi_{|f|>cu} \le u + |f - f_u|\chi_{|f|>cu},$$

where χ_I denotes the characteristic function of the set I. This gives

$$\begin{aligned}\int_0^{+\infty} \lambda(\{P_{h_u}|f - f_u|\})\mathrm{d}u^q &\ge \int_0^{+\infty} \lambda(\{P_{h_u}|f|\chi_{\{|f|>cu\}} \ge u\})\mathrm{d}u^q \\ &\le \int_0^{+\infty} \frac{1}{u}\left(\int_G |f|\chi_{\{|f|>cu\}}\mathrm{d}\lambda\right)\mathrm{d}u^q \\ &= \frac{q}{q-1}\int_G \left(\int_0^\infty \chi_{\{|f|>cu\}}\mathrm{d}u^{q-1}\right)\mathrm{d}\lambda \\ &= \frac{q}{q-1}\cdot\frac{1}{c^{q-1}}\|f\|_{L^q(G)}^q.\end{aligned}$$

Therefore we have shown that

$$5^{-q}\|f\|_{L^q(G)}^q \le Cq\ln c\int_G |\nabla f|^p \mathrm{d}\lambda + \frac{q}{q-1}\cdot\frac{1}{c^{q-1}}\|f\|_{L^q(G)}^q$$

which by choosing c large enough yields (1.4) under the additional assumption $f \in L^q(G)$.

Step 3 The final step is to remove the assumption $f \in L^q(G)$ in Step 2. We again follow Ledoux's approach and define

$$N_\varepsilon(f) = \int_\varepsilon^{1/\varepsilon} \lambda(|f| \ge 5u)\mathrm{d}u^q < \infty.$$

It is seen that, by Step 2,

$$N_\varepsilon(f) \le Cq\ln c\int_G |\nabla f|^p \mathrm{d}\lambda + \int_\varepsilon^{1/\varepsilon} \frac{1}{u}\left(\int_G |f|\chi_{\{|f|>cu\}}\mathrm{d}\lambda\right)\mathrm{d}u^q.$$

It was proved in [11] that there holds

$$\begin{aligned}&\int_{\varepsilon}^{1/\varepsilon}\frac{1}{u}\left(\int_G |f|\chi_{\{|f|>cu\}}\mathrm{d}\lambda\right)\mathrm{d}u^q\\ \le\ &\frac{q}{q-1}\cdot\frac{5^q}{c^{q-1}}N_\varepsilon(f)+\frac{q}{q-1}\cdot\frac{1}{c^{q-1}}\|f\|_{L^{q,\infty}(G)}\left(q\ln\left(\frac{c}{5}\right)+\frac{1}{q-1}\right).\end{aligned}$$

By choosing c large enough it follows that $\sup\limits_{\varepsilon>0} N_\varepsilon(f)$ and so $f\in L^q(G)$. The proof is therefore completed.

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