A General Nonlinear Fokker-Planck Equation and its Associated Entropy

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Abstract

A recently introduced nonlinear Fokker-Planck equation, derived directly from a master equation, comes out as a very general tool to describe phenomenologically systems presenting complex behavior, like anomalous diffusion, in the presence of external forces. Such an equation is characterized by a nonlinear diffusion term that may present, in general, two distinct powers of the probability distribution. Herein, we calculate the stationary-state distributions of this equation in some special cases, and introduce associated classes of generalized entropies in order to satisfy the H-theorem. Within this approach, the parameters associated with the transition rates of the original master-equation are related to such generalized entropies, and are shown to obey some restrictions. Some particular cases are discussed.

Keywords: Nonlinear Fokker-Planck Equation, Generalized Entropies, H-Theorem, Nonextensive Thermostatistics.

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I. INTRODUCTION

The standard statistical-mechanics formalism, as proposed originally by Boltzmann and Gibbs (BG), is considered as one of the most successful theories of physics, and it has enabled physicists to propose theoretical models in order to derive thermodynamical properties for real systems, by approaching the problem from the microscopic scale. Such a prescription has led to an adequate description of a large diversity of physical systems, essentially those represented by linear equations and characterized by short-range interactions and/or short-time memories. Although BG statistical mechanics is well formulated (under certain restrictions) for systems at equilibrium, the same is not true for out-of-equilibrium systems, in such a way that most of this theory applies only near equilibrium [1, 2, 3]. One of the most important phenomenological equations of nonequilibrium statistical mechanics is the linear Fokker-Planck equation (FPE), that rules the time evolution of the probability distribution associated with a given physical system, in the presence of an external force field [4], provided that the states of the system can be expressed by a continuum. This equation deals satisfactorily with many physical situations, e.g., those associated with normal diffusion, and is essentially associated with the BG formalism, in the sense that the Boltzmann distribution, which is usually obtained through the maximization of the BG entropy under certain constraints (the so-called MaxEnt principle), also appears as the stationary solution of the linear FPE [4, 5].

Nevertheless, restrictions to the applicability of the BG statistical-mechanics formalism have been found in many systems, including for instance, those characterized by nonlinearities, long-range interactions and/or long-time memories, which may present several types of anomalous behavior, e.g., stationary states far from equilibrium [6, 7, 8]. These anomalous behaviors suggest that a more general theory is required; as a consequence of that, many attempts have been made, essentially by proposing generalizations of the BG entropy [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Among these, the most successful proposal, so far, appears to be the one suggested by Tsallis [9], through the introduction of a generalized entropy, characterized by an index q, in such a way that the BG entropy is recovered in the limit $q \to 1$. The usual extensivity property of some thermodynamic quantities holds only for $q = 1$, and if $q \neq 1$ such quantities do not increase linearly with the size of the system; this has led to the so-called nonextensive statistical-mechanics formalism [6, 7, 8].

Among many systems that present unusual behavior, one should mention those characterized by anomalous diffusion, e.g., particle transport in disordered media. A possible alternative for describing anomalous-transport processes consists in introducing modifications in the standard FPE. Within the most common procedure, one considers nonlinear FPEs [19], that in most of the cases come out as simple phenomenological generalizations of the usual linear FPE [20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. In these nonlinear systems, interesting new aspects appear, leading to a wide range of open problems, in such a way that their investigation has led to a new research area in physics, with a lot of applications in natural systems. It is very common to find the power-law-like probability distribution that maximizes Tsallis' entropy as solutions of some nonlinear FPEs [19, 20, 21, 23, 24, 30]. It seems that the nonextensive statistical mechanics formalism appears to be intimately related to nonlinear FPEs, motivating an investigation for a better understanding of possible connections between generalized entropies and nonlinear FPEs [19, 20, 21, 25, 29, 31, 32, 33, 34].

Recently, a general nonlinear FPE has been derived directly from a standard master equation, by introducing nonlinear contributions in the associated transition probabilities, leading to [35, 36]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial (A(x)P(x,t))}{\partial x} + \frac{\partial}{\partial x}\left(\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right) \ ; \quad \Omega[P] = a\mu P^{\mu-1} + b(2-\nu)P^{\nu-1} \ , \tag{1}$$

where a and b are constants, whereas μ and ν are real exponents [38]. The system is in the presence of an external potential $\phi(x)$, associated with a dimensionless force $A(x) = -d\phi(x)/dx$ [$\phi(x) = -\int_{-\infty}^{x} A(x')dx'$]; herein, we assume analyticity of the potential $\phi(x)$ and integrability of the force $A(x)$ in all space.

In what concerns the functional $\Omega[P(x,t)]$, we are assuming its differentiability and integrability with respect to the probability distribution $P(x,t)$, in such a way that at least its first derivative exists, i.e., that it should be at least $\Omega[P] \in C^1$. Furthermore, this functional should be a positive finite quantity, as expected for a proper diffusion-like term; this property will be verified later on, as a direct consequence of the H-theorem.

As usual, we assume that the probability distribution, together with its first derivative, as well as the product $A(x)P(x,t)$, should all be zero at infinity,

$$P(x,t)|_{x\to\pm\infty} = 0 \ ; \quad \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0 \ ; \quad A(x)P(x,t)|_{x\to\pm\infty} = 0 \ , \quad (\forall t) \ . \tag{2}$$

The conditions above guarantee the preservation of the normalization for the probability distribution, i.e., if for a given time t_0 one has that $\int_{-\infty}^{\infty} dx \ P(x,t_0) = 1$, then a simple integration of Eq. (1) with respect to the variable x yields,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx \ P(x,t) = -\left[A(x)P(x,t)\right]_{-\infty}^{\infty} + \left(\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right)_{-\infty}^{\infty} = 0 \ , \tag{3}$$

and so,

$$\int_{-\infty}^{\infty} dx \ P(x,t) = \int_{-\infty}^{\infty} dx \ P(x,t_0) = 1 \qquad (\forall t) \ . \tag{4}$$

In the present work we investigate further properties of the nonlinear FPE of Eq. (1), finding stationary solutions in several particular cases, and discussing its associated entropies, that were introduced in order to satisfy the H-theorem. In the next section we present stationary solutions of this equation; in section III we prove the H-theorem by using Eq. (1), and show that the validity of this theorem can be directly related to the definition of a general entropic form associated with this nonlinear FPE. In section IV we discuss particular cases of this general entropic form and their associated nonlinear FPEs. Finally, in section V we present our conclusions.

II. STATIONARY STATE

The nonlinear FPE of Eq. (1) is very general and covers several particular cases, e.g., the one related to Tsallis' thermostatistics [20, 21, 30]. In this section we will restrict ourselves to a stationary state, and will derive the corresponding solutions for particular values of the parameters associated with this equation.

Let us then rewrite Eq. (1) in the form of a continuity equation,

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0 \ ; \quad j(x,t) = A(x)P(x,t) - \Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x} \ , \tag{5}$$

in such a way that a stationary solution of Eq. (1), $P_{st}(x)$, is associated with a stationary probability flux, $j_{st}(x)$ = constant, which becomes $j_{st}(x) = 0$, when one uses Eq. (2). Therefore, using the functional $\Omega[P]$ of Eq. (1), the stationary-state solution satisfies,

$$A(x) = \left[a\mu P_{st}^{\mu-2}(x) + b(2-\nu)P_{st}^{\nu-2}(x)\right] \frac{\partial P_{st}(x)}{\partial x} \ , \tag{6}$$

which, after integration, becomes

$$\phi_0 - \phi(x) = a\frac{\mu}{\mu-1}P_{st}^{\mu-1}(x) + b\frac{2-\nu}{\nu-1}P_{st}^{\nu-1}(x) \ , \tag{7}$$

where ϕ_0 represents a constant. The equation above may be solved easily in some particular cases, e.g., $\nu = \mu$, $\nu = 2$, and $\mu = 0$,

$$P_{st}(x) = \frac{1}{Z^{(1)}}\left[1 - \frac{\phi(x)}{\phi_0}\right]_+^{\frac{1}{\alpha-1}} \quad ; \quad Z^{(1)} = \int\limits_{-\infty}^{\infty} dx \left[1 - \frac{\phi(x)}{\phi_0}\right]_+^{\frac{1}{\alpha-1}} \ , \tag{8}$$

where $\alpha = \mu$, in the cases $\nu = \mu$ and $\nu = 2$, whereas $\alpha = \nu$, if $\mu = 0$. In the equation above, $[y]_+ = y$, for $y > 0$, and zero otherwise. Another type of solution applies for $\mu = 2\nu - 1$, or $\nu = 2\mu - 1$,

$$P_{st}(x) = \frac{1}{Z^{(2)}}\left[1 \pm \sqrt{1 + K\left(\phi(x) - \phi_0\right)}\right]_+^{\frac{1}{\alpha-1}} \quad ; \quad Z^{(2)} = \int\limits_{-\infty}^{\infty} dx \left[1 \pm \sqrt{1 + K\left(\phi(x) - \phi_0\right)}\right]_+^{\frac{1}{\alpha-1}} \ , \tag{9}$$

where

$$K = \begin{cases} \dfrac{2a}{b^2}\,\dfrac{(2\nu-1)(\nu-1)}{(2-\nu)^2} \ , & \text{if} \quad \mu = 2\nu - 1 \quad (\alpha = \nu) \ , \\ \\ \dfrac{2b}{a^2}\,\dfrac{(3-2\mu)(\mu-1)}{\mu^2} \ , & \text{if} \quad \nu = 2\mu - 1 \quad (\alpha = \mu) \ , \end{cases} \tag{10}$$

and we are assuming that $[1 + K(\phi(x) - \phi_0)] \geq 0$. Some well-known particular cases of the stationary solutions presented above come out easily, e.g., from Eq. (8) one obtains, in all three situations that yielded this equation, the exponential solution associated with the linear FPE in the limit $\alpha \to 1$ [with $\phi_0 \propto (\alpha - 1)^{-1}$], as well as the generalized exponential solution related to Tsallis thermostatistics, for $\alpha = 2 - q$, where q denotes Tsallis' entropic index.

III. H–THEOREM AND THE ASSOCIATED ENTROPY

In this section we will demonstrate the H-theorem by making use of Eq. (1), and for that purpose, an entropic form related to this equation will be introduced. Let us therefore suppose a general entropic form satisfying the following conditions,

$$S = \int_{-\infty}^{\infty} dx \; g[P(x)] \; ; \;\; g[0] = 0 \; ; \;\; g[1] = 0 \; ; \;\; \frac{d^2 g}{dP^2} \leq 0 \; , \tag{10}$$

where one should have $g[P(x,t)]$ at least as $g[P(x,t)] \in C^2$; in addition to that, let us also define the free-energy functional,

$$F = U - \frac{1}{\beta} \, S \; ; \qquad U = \int_{-\infty}^{\infty} dx \; \phi(x) P(x,t) \; , \tag{11}$$

where β represents a Lagrange multiplier, restricted to $\beta \geq 0$. Furthermore, we will show that this free-energy functional is bounded from below; this condition, together with the H-theorem $[(\partial F/\partial t) \leq 0)$, leads, after a long time, the system towards a stationary state.

A. H–Theorem

The H-theorem for a system that exchanges energy with its surrounding, herein represented by the potential $\phi(x)$, corresponds to a well-defined sign for the time derivative of the free-energy functional defined in Eq. (11). Using the definitions above,

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} dx\ \phi(x)P(x,t) - \frac{1}{\beta}\int_{-\infty}^{\infty} dx\ g[P] \right) \\
&= \int_{-\infty}^{\infty} dx\ \left(\phi(x) - \frac{1}{\beta}\frac{\partial g[P]}{\partial P} \right) \frac{\partial P}{\partial t}\ . \qquad (12)
\end{aligned}
$$

Now, one may use the FPE of Eq. (1) for the time derivative of the probability distribution; carrying out an integration by parts, and using the conditions of Eq. (2), one obtains,

$$
\frac{\partial F}{\partial t} = -\int_{-\infty}^{\infty} dx\ \left[\frac{d\phi(x)}{dx} P(x,t) + \Omega[P]\frac{\partial P}{\partial x} \right] \left[\frac{d\phi(x)}{dx} - \frac{1}{\beta}\frac{\partial^2 g[P]}{\partial P^2}\frac{\partial P}{\partial x} \right]\ . \qquad (13)
$$

Usually, one is interested in verifying the H-theorem from a well-defined FPE, together with a particular entropic form, in such a way that the quantities $\Omega[P]$ and $\partial^2 g[P]/\partial P^2$ are previously defined (see, e.g., Refs. [19, 37]). Herein, we follow a different approach, by assuming that the general Eqs. (1) and (10) should be satisfied; then, we impose the condition,

$$
\frac{\partial^2 g[P]}{\partial P^2} = -\beta \frac{\Omega[P]}{P(x,t)}\ , \qquad (14)
$$

in such way that,

$$
\frac{\partial F}{\partial t} = -\int_{-\infty}^{\infty} dx\ P(x,t) \left[\frac{d\phi(x)}{dx} + \frac{\Omega[P]}{P(x,t)}\frac{\partial P}{\partial x} \right]^2 \leq 0\ . \qquad (15)
$$

It should be noticed that Eq. (14), introduced in such a way to provide a well-defined sign for the time derivative of the free-energy functional, yields two important conditions, as described next.

(i) $\Omega[P] \geq 0$ [cf. Eq. (10)], which is expected for an appropriate diffusion-like term.

(ii) It expresses a relation involving the FPE of Eq. (1) and an associated entropic form, allowing for the calculation of such an entropic form, given the FPE, and vice-versa. Since the FPE is a phenomenological equation that specifies the dynamical evolution associated with a given physical system, Eq. (14) may be useful in the identification of the entropic form associated with such a system. In particular, the present approach makes it possible to identify entropic forms associated with some anomalous systems, exhibiting unusual behavior, that are appropriately described by nonlinear FPEs, like the one of Eq. (1). As an illustration of this point, let us consider the simple case of a linear FPE, that describes the dynamical evolution of many physical systems, essentially those characterized by normal diffusion. This equation may be recovered from Eq. (1) by choosing $\mu = \nu = 1$, in such a way that $\Omega[P] = a + b = D$, where D represents a positive constant diffusion coefficient with units (time)$^{-1}$. One may now set the Lagrange multiplier $\beta = k_B/D$, where k_B represents the Boltzmann constant; integrating Eq. (14), and using the conditions of Eq. (10), one gets the well-known BG entropic form,

$$g[P] = -k_B P \ln P \ . \tag{16}$$

In the next section we will explore further the relation of Eq. (14), by analyzing other particular cases.

The simplest situation for which condition (i) above is satisfied may be obtained by imposing both terms of the functional $\Omega[P]$ to be positive, which leads to

$$a \geq 0 \ ; \quad \mu \geq 0 \ , \qquad \text{or} \qquad a < 0 \ ; \quad \mu < 0 \ , \tag{17}$$

and

$$b \geq 0 \ ; \quad \nu \leq 2 \ , \qquad \text{or} \qquad b < 0 \ ; \quad \nu > 2 \ . \tag{18}$$

It should be stressed that it is possible to have $\Omega[P] \geq 0$ with less restrictive ranges for the parameters above. However, an additional property for the free-energy functional of Eq. (11) to be discussed next, namely, the boundness from below, requires the conditions of Eqs. (17) and (18), with the additional restriction $\nu > 0$.

Now, integrating Eq. (14) for the general functional $\Omega[P]$ of Eq. (1), and using the standard conditions for $g[P]$ defined in Eq. (10), one gets that

$$g[P] = -\beta \left[\frac{a}{\mu - 1} P^{\mu} + b \frac{2-\nu}{\nu(\nu-1)} P^{\nu} + \frac{a\nu(1-\nu) + b(2-\nu)(1-\mu)}{(1-\mu)(1-\nu)\nu} P \right] . \tag{19}$$

This entropic form recovers, as particular cases, the BG entropy (e.g., when $\mu, \nu \to 1$) and several generalized entropies defined previously in the literature, like those introduced by Tsallis [9], Abe [10], Borges-Roditi [12], and Kaniadakis [16, 17]. Such particular cases, as well as their associated FPEs, will be discussed in the next section.

For the simpler situation of an isolated system, i.e., $\phi(x)$ = constant, the H-theorem should be expressed in terms of the time derivative of the entropy, in such a way that Eq. (13) should be replaced by

$$\frac{\partial S[P]}{\partial t} = -\int_{-\infty}^{\infty} dx \ \left(\Omega[P] \frac{\partial P}{\partial x} \right) \left(\frac{\partial^2 g[P]}{\partial P^2} \frac{\partial P}{\partial x} \right) = -\int_{-\infty}^{\infty} dx \ \Omega[P] \ \frac{\partial^2 g[P]}{\partial P^2} \left(\frac{\partial P}{\partial x} \right)^2 \geq 0 \ . \tag{20}$$

In this case all that one needs is the standard condition associated with the FPE [same condition (i) above], i.e., $\Omega[P] \geq 0$, and the general restrictions of Eq. (10) for the entropy. A similar result may also be obtained by proving the H-theorem using the master equation from which Eq. (1) was derived, with the transition probabilities introduced in Ref. [35, 36] (see the Appendix).

B. Boundness from Below

Above, we have proven that the free-energy functional decreases in time, and so, for the existence of a stationary state at long times of an evolution process, characterized by a probability distribution $P_{st}(x)$, one should have that

$$F(P(x,t)) \geq F(P_{st}(x)) \qquad (\forall t). \tag{21}$$

In what follows, we will show this inequality and find the conditions for its validity. Therefore, using Eqs. (7), (10), and (11), we can write,

$$F(P(x,t)) = \int\limits_{-\infty}^{\infty} P(x,t) \left(\phi_0 - a \frac{\mu}{\mu - 1} P_{st}^{\mu-1}(x) - b \frac{2-\nu}{\nu-1} P_{st}^{\nu-1}(x) \right) dx - \frac{1}{\beta} \int\limits_{-\infty}^{\infty} g[P] dx \ , \quad (22)$$

and so,

$$F(P_{st}) - F(P) = \int\limits_{-\infty}^{\infty} (P - P_{st}) \left(a \frac{\mu}{\mu - 1} P_{st}^{\mu-1} + b \frac{2-\nu}{\nu-1} P_{st}^{\nu-1} \right) dx + \frac{1}{\beta} \int\limits_{-\infty}^{\infty} \left(g[P] - g[P_{st}] \right) dx \ , \quad (23)$$

where we have used the normalization condition for the probabilities. Now, we insert the entropic form of Eq. (19) in the equation above to obtain,

$$F(P_{st}) - F(P) = \int\limits_{-\infty}^{\infty} \left[\frac{a}{\mu - 1} P_{st}^{\mu} \Gamma_{\mu}[P/P_{st}] + \frac{b(2-\nu)}{\nu(\nu - 1)} P_{st}^{\nu} \Gamma_{\nu}[P/P_{st}] \right] dx \ , \quad (24)$$

where,

$$\Gamma_{\alpha}[z] = 1 - \alpha + \alpha z - z^{\alpha} \qquad (\alpha = \mu, \nu) \ . \quad (25)$$

By analyzing the extrema of the functional $\Gamma_{\alpha}[P/P_{st}]$, one can see that $\Gamma_{\alpha}[P/P_{st}] \geq 0$ for $0 < \alpha < 1$ and $\Gamma_{\alpha}[P/P_{st}] \leq 0$ for $\alpha > 1$ and $\alpha < 0$. Therefore, the inequality of Eq. (21) is satisfied for the range of parameters specified by Eqs. (17) and (18), if one considers additionally, $\nu > 0$.

For the case of an isolated system, the stationary solution turns out to be the equilibrium state, which is the one that maximizes the entropy. Therefore, one may use the concavity property of the entropy [cf. Eq. (10)] in order to get,

$$S[P_{eq}(x)] - S[P(x,t)] = \int\limits_{-\infty}^{\infty} \left(g[P_{eq}] - g[P] \right) dx \geq 0 \qquad (\forall t). \quad (26)$$

IV. SOME PARTICULAR CASES

In this section we analyze some particular cases of the entropic form of Eq. (19) and using Eq. (14), we find for each of them, the corresponding functional $\Omega[P]$ of the associated FPE. In the examples that follow, we will set the Lagrange multiplier $\beta = k/D$, with k and D representing, respectively, a constant with dimensions of entropy and a constant diffusion coefficient.

(a) Tsallis entropy [9]: This represents the most well-known generalization of the BG entropy, which has led to the development of the area of nonextensive statistical mechanics [6, 7, 8]. One may find easily that Eq. (19) recovers Tsallis entropy in several particular cases, e.g., $\{b = 0, a = D, \mu = q\}$, $\{a = 0, b = D\nu/(2-\nu), \nu = q\}$, and $\{a = D/2, b = D\nu/[2(2-\nu)], \mu = \nu = q\}$. For all these cases one may use Eq. (14), in order to get the corresponding functional $\Omega[P]$,

$$g[P] = k\frac{P^q - P}{1-q}, \quad \Omega[P] = qDP^{q-1}\ . \tag{27}$$

With the functional $\Omega[P]$ above, one identifies the nonlinear FPE that presents the well-known q-exponential, or Tsallis distribution (replacing $q \to 2-q$), as a time-dependent solution [20, 21, 25].

(b) Abe entropy [10]: This proposal was inspired in the area of quantum groups, where certain quantities, usually called q-deformed quantities, are submitted to deformations and are often required to possess the invariance $q \leftrightarrow q^{-1}$. The Abe entropy may be obtained from Eq. (19) in the particular case $\{a = D(q-1)/(q-q^{-1}), b = -Dq(q+1)/[(q-q^{-1})(q+2)], \mu = -\nu = q\}$, for which

$$g[P] = -k\frac{P^q - P^{-q}}{q-q^{-1}}, \;\; \Omega[P] = D\left(\frac{q(q-1)}{q-q^{-1}}P^{q-1} - \frac{q(q+1)}{q-q^{-1}}P^{-q-1}\right)\ . \tag{28}$$

(c) Borges-Roditi entropy [12]: This consists in another generalization of Tsallis entropy, where now one has two distinct entropic indices, q and q', with a more general invariance

$q \leftrightarrow q'$; this case may be obtained from Eq. (19) by choosing $\{a = D(q-1)/(q'-q), b = Dq'(q'-1)/[(q-q')(2-q')], \mu = q, \nu = q'\}$. One gets,

$$g[P] = -k\frac{P^q - P^{q'}}{q-q'}, \;\; \Omega[P] = D\frac{1}{q-q'}\left(q(q-1)P^{q-1} - q'(q'-1)P^{q'-1}\right) \; . \tag{29}$$

(d) Kaniadakis entropy [16, 17]: This is also a two-exponent entropic form, but slightly different from those presented in examples (b) and (c) above; it may be reproduced from Eq. (19) by choosing $\{a = b = D/[2(1+q)], \mu = 1+q, \nu = 1-q\}$, in such a way that

$$g[P] = -\frac{k}{2q}\left(\frac{1}{1+q}P^{1+q} - \frac{1}{1-q}P^{1-q}\right), \;\; \Omega[P] = \frac{D}{2}(P^q + P^{-q}) \; . \tag{30}$$

Except for the well-known example (a), i.e., Tsallis entropy and its corresponding FPE, the other three particular cases presented herein were much less explored in the literature. Their associated FPEs, defined in terms of their respective functionals $\Omega[P]$ above are, to our knowledge, presented herein for the first time. These equations, whose nonlinear terms depend essentially in two different powers of the probability distribution, may be appropriated for anomalous-diffusion phenomena where a crossover between two different diffusion regimes occurs [24].

V. CONCLUSION

We have analyzed important aspects associated with a recently introduced nonlinear Fokker-Planck equation, that was derived directly from a master equation by setting non-linear effects on its transition rates. Such equation is characterized by a nonlinear diffusion term that may present two distinct powers of the probability distribution; for this reason, it may reproduce, as particular cases, a large range of nonlinear FPEs of the literature. We have obtained stationary solutions for this equation in several cases, and some of them recover the well-known Tsallis distribution. We have proven the H-theorem, and for that, an important relation involving the parameters of the FPE and an entropic form was introduced. Since the FPE is a phenomenological equation that specifies the dynamical evolution associated with a given physical system, such a relation may be useful for identifying entropic forms associated with real systems and, in particular, anomalous systems that exhibit unusual behavior and are appropriately described by nonlinear FPEs. It is shown that, in the simple case of a linear FPE, the Boltzmann-Gibbs entropy comes out straightforwardly from this relation. Considering the above-mentioned nonlinear diffusion term, this relation yields a very general entropic form, which similarly to its corresponding FPE, depends on two distinct powers of the probability distribution. Apart from Tsallis' entropy, other entropic forms introduced in the literature are recovered as particular cases of the present one, essentially those characterized by two entropic indices. Nonlinear FPEs (as well as their associated entropic forms) whose nonlinear terms depend essentially on two different powers of the probability distribution, like the ones discussed in the present paper, are good candidates for describing anomalous-diffusion phenomena where a crossover between two different diffusion regimes may take place. As a typical example, one could have a particle transport in a system composed by two types of disordered media, characterized respectively, by significantly different grains (e.g., different average sizes), arranged in such a way that the diffusion process is dominated by one medium, at high densities, and by the other one, at low densities.

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[38] Even though one could use a simpler notation for the functional $\Omega[P]$, e.g., $\Omega[P] = a'P^{\mu'} + b'P^{\nu'}$, herein we will keep the notation of Eq. (1), as appeared naturally in the derivation of

the above FPE, for a consistency with previous publications [35, 36].

Appendix

In this appendix we will prove the H-theorem directly from the master equation, for an isolated system (i.e., no external forces). Let us consider a system described in terms of W discrete stochastic variables; then, $P_n(t)$ represents the probability of finding this system in a state characterized by the variable n at time t. This probability evolves in time according to a master equation,

$$\frac{\partial P_n}{\partial t} = \sum_{m=1}^{W} (P_m w_{m,n} - P_n w_{n,m}) \ , \tag{31}$$

where $w_{k,l}(t)$ represents the transition probability rate from state k to state l. The nonlinear effects were introduced through the transition rates [35],

$$w_{k,l}(\Delta) = \frac{1}{\Delta^2}(\delta_{k,l+1} + \delta_{k,l-1})[aP_k^{\mu-1}(t) + bP_l^{\nu-1}(t)] \ , \tag{32}$$

where Δ represents the size of the step of the random walk. Herein, we shall consider a random walk characterized by $\Delta = 1$; substituting such a transition rate in the master equation one gets,

$$n = 1 : \ \frac{\partial P_1}{\partial t} = aP_2^{\mu} - aP_1^{\mu} + bP_2P_1^{\nu-1} - bP_1P_2^{\nu-1} \ , \tag{33}$$

$$n = W : \ \frac{\partial P_W}{\partial t} = aP_{W-1}^{\mu} - aP_W^{\mu} + bP_{W-1}P_W^{\nu-1} - bP_WP_{W-1}^{\nu-1} \ , \tag{34}$$

$$\begin{aligned} n = 2, ..., (W-1) : \ \frac{\partial P_n}{\partial t} &= a\left(P_{n+1}^{\mu} + P_{n-1}^{\mu}\right) - 2aP_n^{\mu} + bP_n^{\nu-1}\left(P_{n+1} + P_{n-1}\right) \\ &- bP_n\left(P_{n+1}^{\nu-1} + P_{n-1}^{\nu-1}\right) \ , \end{aligned} \tag{35}$$

where we have treated the borders of the spectrum ($n = 1$ and $n = W$) separately from the rest. Let us now consider the entropy, $S = \sum_{n=1}^{W} g[P_n]$, satisfying the same conditions specified in the text [see Eq. (10)]; the H-theorem, to be proven below, is expressed in terms of its time derivative,

$$\frac{\partial S}{\partial t} = \sum_{n=1}^{W} \frac{dg[P_n]}{dP_n}\frac{dP_n}{dt} \geq 0 \ . \tag{36}$$

Then, using Eqs. (33)–(35) one can write this time derivative as,

$$\begin{aligned}\frac{\partial S}{\partial t} = &\sum_{n=1}^{W-1} \frac{dg[P_n]}{dP_n}\left(aP_{n+1}^{\mu} - aP_n^{\mu} - bP_nP_{n+1}^{\nu-1} + bP_{n+1}P_n^{\nu-1}\right) \\ &+ \sum_{n=2}^{W} \frac{dg[P_n]}{dP_n}\left(aP_{n-1}^{\mu} - aP_n^{\mu} + bP_{n-1}P_n^{\nu-1} - bP_nP_{n-1}^{\nu-1}\right) \ .\end{aligned} \tag{37}$$

The sum indices can be rearranged to yield all summations in the range $\sum\limits_{n=1}^{W-1}$, and thus the time derivative of the entropy becomes,

$$\frac{\partial S}{\partial t} = \sum_{n=1}^{W-1}\left(\frac{dg[P_n]}{dP_n} - \frac{dg[P_{n+1}]}{dP_{n+1}}\right)\left[a\left(P_{n+1}^{\mu} - P_n^{\mu}\right) - bP_nP_{n+1}\left(P_{n+1}^{\nu-2} - P_n^{\nu-2}\right)\right] \ . \tag{38}$$

The negative curvature of the entropic function [cf. Eq. (10)] implies that its first derivative decays monotonically with P_n. Hence, for $P_{n+1} > P_n$, the condition of Eq. (36) is satisfied for

$$a\frac{P_{n+1}^{\mu} - P_n^{\mu}}{P_{n+1} - P_n} - bP_nP_{n+1}\frac{P_{n+1}^{\nu-2} - P_n^{\nu-2}}{P_{n+1} - P_n} \geq 0 \ , \tag{39}$$

where we divided the term inside the brackets in Eq.(38) by the difference $\Delta P = P_{n+1} - P_n$. Now we consider the limit $\Delta P \to 0$ and obtain,

$$a\mu P^{\mu-1} + b(2-\nu)P^{\nu-1} \geq 0 \ , \tag{40}$$

which corresponds to the condition $\Omega[P] \geq 0$ found in the text, when proving the H-theorem by making use of the FPE of Eq. (1). It should be mentioned that the procedure above works also for $P_n > P_{n+1}$; therefore, in this appendix we have proven that the H-theorem holds for an arbitrary state n and all times t of an isolated system.