

SOME WEIGHTED GAGLIARDO-NIRENBERG INEQUALITIES AND APPLICATIONS

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ABSTRACT. We obtain conditions on the measure μ so that the $L^2(\mu)$ -norm of a function is controlled by the L^2 -norms of the function and its gradient. Applications to eigenvalues of the Schrödinger operator and to other inequalities are also given.

1. INTRODUCTION

We will be interested in inequalities of the type

$$(1.1) \quad \left(\int_{\mathbf{R}^d} |f(x)|^2 d\mu(x) \right)^{1/2} \leq C \|f\|_2^\theta \|\nabla f\|_2^{1-\theta}.$$

We would like to give sufficient conditions on μ so that (1.1) holds and to study the best constant in the inequality if possible. Inequalities of this type with an unweighted L^q -norm on the left-hand side are known as Gagliardo-Nirenberg inequalities. In this paper we will focus attention on the case $\theta = 1/2$. Other values of θ and other norms different from L^2 -norms are also of interest but will not be considered here.

We consider the Sobolev space $H^1(\mathbf{R}^d)$ of complex-valued functions $f \in L^2(\mathbf{R}^d)$ such that $|\nabla f| \in L^2(\mathbf{R}^d)$.

The basic theorem for $d \geq 2$ and $d\mu(x) = w(x) dx$ is the following.

Theorem 1.1. *Let $d \geq 2$. Let f be a function in $H^1(\mathbf{R}^d)$ and $w(x) \geq 0$. Then*

$$(1.2) \quad \int_{\mathbf{R}^d} w(x) |f(x)|^2 dx \leq 2K(w) \|f\|_2 \|\nabla f\|_2,$$

where

$$K(w) = \inf_{a \in \mathbf{R}^d} \sup_{x \in \mathbf{R}^d} |x| \int_0^1 w(tx + a) t^{d-1} dt.$$

Equality holds when $w(x)$ is a multiple of $|x - b|^{-1}$ for some $b \in \mathbf{R}^d$. In that case, $f(x)$ must be a multiple of $e^{-c|x-b|}$ with $c > 0$, and $K(|x - b|^{-1}) = (d - 1)^{-1}$.

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In the one-dimensional case the basic result is the following pointwise inequality. It can be found in [5, Theorem 8.5].

Theorem 1.2. *Let f be a function in $H^1(\mathbf{R})$. Then the pointwise inequality*

$$(1.3) \quad |f(x)|^2 \leq \|f\|_2 \|f'\|_2$$

holds. It becomes an equality if and only if $f(t) = c_1 e^{-c_2|t-x|}$ with $c_1 \in \mathbf{C}$ and $c_2 > 0$.

The proof of both theorems is deduced from the fundamental theorem of calculus and is presented in the next section together with some comments and extensions. In the last section we will show applications to the eigenvalues of the Schrödinger operator and to a trace theorem. We include also an unweighted Gagliardo-Nirenberg inequality in a Lorentz space and some variants of the original inequality.

Weighted inequalities of Gagliardo-Nirenberg type with power weights on both sides of the inequality and not restricted to L^2 -norms are in [3]. See also [6], where higher derivatives are considered.

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2. PROOFS, COMMENTS AND EXTENSIONS

Proof of Theorem 1.1. We assume that f is in the Schwartz class and extend the inequality to H^1 by a density argument. From the fundamental theorem of calculus we have

$$\begin{aligned} |f(x)|^2 &= - \int_1^\infty \frac{d}{dt} f(a + t(x-a)) \bar{f}(a + t(x-a)) dt \\ &= -2\Re \int_1^\infty f(a + t(x-a)) ((x-a) \cdot \nabla \bar{f}(a + t(x-a))) dt. \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbf{R}^d} w(x) |f(x)|^2 dx &= -2\Re \int_{\mathbf{R}^d} \int_1^\infty w(x) f(a + t(x-a)) ((x-a) \cdot \nabla \bar{f}(a + t(x-a))) dt dx \\ &= -2\Re \int_{\mathbf{R}^d} \int_1^\infty w(a + \frac{x}{t}) \frac{dt}{t^{d+1}} f(a+x) (x \cdot \nabla \bar{f}(a+x)) dx \\ &= -2\Re \int_{\mathbf{R}^d} \int_0^1 w(a+tx) t^{d-1} dt f(a+x) (x \cdot \nabla \bar{f}(a+x)) dx. \end{aligned}$$

Denoting

$$(2.1) \quad K(w, a) = \sup_{x \in \mathbf{R}^d} |x| \int_0^1 w(tx+a) t^{d-1} dt$$

and using the Cauchy-Schwarz inequality we deduce

$$(2.2) \quad \int_{\mathbf{R}^d} w(x) |f(x)|^2 dx \leq 2K(w, a) \|f\|_2 \|\nabla f\|_2.$$

Now take the infimum in a at the right-hand side.

Equality holds in (2.2) if and only if

$$|x| \int_0^1 w(tx+a)t^{d-1} dt = K(w, a),$$

where $f(a+x)(\frac{x}{|x|} \cdot \nabla \bar{f}(a+x))$ does not vanish, and $f(a+x)$ and $(\frac{x}{|x|} \cdot \nabla \bar{f}(a+x))$ are proportional. This gives $w(x) = C|x-a|^{-1}$ and $f(x) = c_1 e^{-c_2|x-a|}$ with $c_2 > 0$. \square

The proof itself offers the possibility of obtaining other inequalities. Take $a = 0$, for instance, and denote

$$W(x) = |x| \int_0^1 w(tx)t^{d-1} dt.$$

We get

$$\begin{aligned} \int_{\mathbf{R}^d} w(x)|f(x)|^2 dx &\leq 2 \int_{\mathbf{R}^d} W(x)|f(x)| |\nabla f(x)| dx \\ &\leq \sup_x (Wu^{-1/2}v^{-1/2})(x) \left(\int_{\mathbf{R}^d} |f|^2 u \right)^{1/2} \left(\int_{\mathbf{R}^d} |\nabla f|^2 v \right)^{1/2}. \end{aligned}$$

When w , u , and v are powers of $|x|$ we get inequalities similar to those in [3]. We will not pursue this method further.

We next state a theorem for measures. Let μ be a positive Borel measure. If φ is a function define the translation $\tau_a \varphi(x) = \varphi(x+a)$ for $a \in \mathbf{R}^d$, and the dilation $\delta_t \varphi(x) = \varphi(tx)$ for $t > 0$. These operators can be defined for the measure μ by duality as

$$\langle \tau_a \mu, \varphi \rangle = \langle \mu, \tau_{-a} \varphi \rangle \quad \text{and} \quad \langle \delta_t \mu, \varphi \rangle = t^{-d} \langle \mu, \delta_{1/t} \varphi \rangle,$$

where φ is a compactly supported continuous function. With small modifications Theorem 1.1 can be stated and proved for measures.

Theorem 2.1. *Let $d \geq 2$. Let f be a function in $C_0^1(\mathbf{R}^d)$ and μ a positive Borel measure. Then*

$$(2.3) \quad \int_{\mathbf{R}^d} |f(x)|^2 d\mu(x) \leq 2K(\mu) \|f\|_2 \|\nabla f\|_2,$$

where

$$K(\mu) = \inf_{a \in \mathbf{R}^d} \sup_{x \in \mathbf{R}^d} |x| \int_0^1 \delta_t(\tau_a \mu) t^{d-1} dt.$$

This means that the measure $\int_0^1 \delta_t(\tau_a \mu) t^{d-1} dt$ is actually a function for some a , and the product of this function by $|x|$ is bounded. Inequality (2.3) holds for $f \in H^1(\mathbf{R}^d)$ if its left-hand side is defined through a density argument.

Remark 2.2. The infimum in Theorems 1.1 and 2.1 is not an essential infimum. For instance, if $w(x) = |x|^{-1}$, the constant $K(w, a)$ in (2.1) is finite only for $a = 0$.

Let $w_i(x) = |x - a_i|^{-1}$ for $i = 1, 2$, and $w = w_1 + w_2$. If $a_1 \neq a_2$, $K(w) = +\infty$. Nevertheless, it is clear that (1.2) holds with constant $K(w_1) + K(w_2)$ instead of $K(w)$. This example shows that in some cases it can be convenient to decompose the function w into a sum of functions and apply Theorem 1.1 to each one of them. Of course, the same remark is valid for measures.

Proof of Theorem 1.2. This proof appears in [5, Theorem 8.5], and we include it here for completeness.

Assume that f is smooth and tends to 0 at infinity. We have

$$|f(x)|^2 = 2\Re \int_{-\infty}^x f(t)\overline{f'}(t) dt = -2\Re \int_x^{+\infty} f(t)\overline{f'}(t) dt,$$

so that

$$\begin{aligned} |f(x)|^2 &= \Re \int_{-\infty}^x f(t)\overline{f'}(t) dt - \Re \int_x^{+\infty} f(t)\overline{f'}(t) dt \\ &= \Re \int_{-\infty}^{+\infty} f(t)\overline{f'}(t) \operatorname{sgn}(x-t) dt \leq \|f\|_2 \|f'\|_2. \end{aligned}$$

Equality holds if and only if $f(t)$ is a multiple of $\overline{f'}(t) \operatorname{sgn}(x-t)$, that is, if $f(t) = c_1 e^{-c_2|t-x|}$. The assumption $c_2 > 0$ is needed to get an L^2 -function. \square

Corollary 2.3. *Let μ be a positive Borel measure in \mathbf{R} and $f \in H^1(\mathbf{R})$. Then*

$$(2.4) \quad \int_{\mathbf{R}} f(x)^2 d\mu(x) \leq \|\mu\| \|f\|_2 \|f'\|_2,$$

where $\|\mu\|$ denotes the total variation of μ . The inequality is optimal in the sense that $\|\mu\|$ cannot be replaced with a smaller constant. Equality holds if and only if μ is a multiple of the Dirac delta at a point.

Proof. Inequality (2.4) is immediately deduced from (1.3).

To check that $\|\mu\|$ is the optimal constant take $f(x) = e^{-\alpha|x|}$, which satisfies $\|f\|_2 \|f'\|_2 = 1$. Then

$$\sup_f \frac{\int_{\mathbf{R}} f(x)^2 d\mu(x)}{\|f\|_2 \|f'\|_2} \geq \sup_{\alpha>0} \int_{\mathbf{R}} e^{-2\alpha|x|} d\mu(x) = \|\mu\|.$$

Equality holds in (2.4) if and only if (1.3) becomes an equality for all x in the support of μ . But the same f cannot produce equality at different values of x , so that equality in (2.4) requires that the support of μ contains no more than one point. \square

If \mathbf{R} is replaced with $[0, +\infty)$ the constant in the right-hand side of (1.3) is 2 and in (2.4) is $2\|\mu\|$, and these constants are optimal. But now equality can only hold in (1.3) if $x = 0$ and $f(t) = e^{-\alpha t}$; for other values of x the inequality is strict.

Remark 2.4. Let μ be a positive radial measure. Then

$$(2.5) \quad \int_{\mathbf{R}^d} |f(x)|^2 d\mu(x) \leq C \|f\|_2 \|\nabla f\|_2$$

holds for some $C > 0$ if and only if $\mu(B(0, r)) \leq A r^{d-1}$ for some constant A and all $r > 0$. Here $B(0, r)$ is the ball centered at the origin with radius r .

For simplicity we only consider the case of a function. Take (2.1) with $a = 0$ and write $x = |x|x'$. For radial w we have

$$\begin{aligned} |x| \int_0^1 w(tx) t^{d-1} dt &= \frac{1}{|x|^{d-1}} \int_0^{|x|} w(tx') t^{d-1} dt \\ &= \frac{1}{|x|^{d-1} |S^{d-1}|} \int_{B(0, |x|)} w(y) dy, \end{aligned}$$

where $|S^{d-1}|$ is the measure of the unit sphere. Together with Theorem 1.1 this gives the sufficiency.

To show the necessity, take $f(x) = 1 - |x|/(2r)$ for $|x| \leq 2r$ and $f(x) = 0$ for $|x| > 2r$. Then

$$\frac{1}{4} \int_{B(0,r)} w(x) dx \leq \int_{\mathbf{R}^d} w(x) f(x)^2 dx \leq C \|f\|_2 \|\nabla f\|_2 \leq C r^{d/2} r^{d/2-1}.$$

3. SOME APPLICATIONS

3.1. Eigenvalues of the Schrödinger equation.

Corollary 3.1. *Let $d \geq 2$ and $V \leq 0$.*

- (1) *If λ is an eigenvalue of the operator $-\Delta + V$, then $\lambda \geq -A^2$, where A is any constant for which the inequality*

$$-\int_{\mathbf{R}^d} |f(x)|^2 V(x) dx \leq 2A \|f\|_2 \|\nabla f\|_2$$

holds. If $\lambda = -A^2$ is an eigenvalue of $-\Delta + V$, the constant $2A$ is sharp and the equality is achieved for the associated eigenfunctions.

- (2) *Let $K(|V|)$ be the constant defined in Theorem 1.1. $-K(|V|)^2$ is an eigenvalue of $-\Delta f + Vf$ only when $V(x) = c|x-b|^{-1}$ for some $c < 0$. The eigenfunctions are multiples of $e^{-K(|V|)|x-b|}$.*

Proof. If λ is an eigenvalue of $-\Delta f + Vf$,

$$\int (|\nabla f|^2 - \lambda |f|^2) = \int -V |f|^2 \leq 2A \|f\|_2 \|\nabla f\|_2.$$

The inequality is only possible for $f \neq 0$ if $\lambda \geq -A^2$.

If $-A^2$ is an eigenvalue of the operator, the corresponding eigenfunction satisfies $\|\nabla f\|_2 = A \|f\|_2$ and the inequality becomes an equality.

The second part is derived from the first part and Theorem 1.1. \square

The corresponding one-dimensional result, deduced from Corollary 2.3, is the following.

Corollary 3.2. *Let μ be a positive Borel measure in \mathbf{R} .*

- (1) *If λ is an eigenvalue of the operator $-\frac{d^2}{dx^2} - \mu$, then $\lambda \geq -\|\mu\|^2/4$.*
- (2) *$-\|\mu\|^2/4$ is an eigenvalue of $-\frac{d^2}{dx^2} - \mu$ only when μ is a multiple of the Dirac delta at some $b \in \mathbf{R}$. The eigenfunctions are multiples of $e^{-2\|\mu\||x-b|}$.*

3.2. A trace theorem.

Corollary 3.3. *Let A be a measurable map from $B \subset \mathbf{R}^{d-1}$ to \mathbf{R} . Then*

$$(3.1) \quad \int_B |f(\bar{x}, A(\bar{x}))|^2 d\bar{x} \leq \|f\|_2 \left\| \frac{\partial f}{\partial x_d} \right\|_2,$$

where the L^2 -norms are taken on $B \times \mathbf{R}$. Equality holds if $f(x) = c_1(\bar{x})e^{-c_2|x_d - A(\bar{x})|}$ with $c_1 \in L^2(B)$ and $c_2 > 0$.

Proof. Using Theorem 1.2 we have

$$|f(\bar{x}, A(\bar{x}))|^2 \leq \left(\int_{\mathbf{R}} f(\bar{x}, t)^2 dt \right)^{1/2} \left(\int_{\mathbf{R}} \frac{\partial f}{\partial x_d}(\bar{x}, t)^2 dt \right)^{1/2}.$$

Integrate in \bar{x} on B and use the Cauchy-Schwarz inequality to complete the proof. \square

Remark 3.4. Inequality (3.1) implies obviously a similar one with $\|\nabla f\|$ instead of the derivative with respect to x_d . In such a situation equality requires f to be independent of \bar{x} . If B has infinite measure, both sides of the inequality would be infinite. If B has finite measure, equality only holds when A is constant, $A(\bar{x}) = c_3$, and $f(x) = c_1 e^{-c_2|x_d - c_3|}$.

3.3. An unweighted Gagliardo-Nirenberg inequality in Lorentz spaces.

The sharp inequality

$$(3.2) \quad \int_{\mathbf{R}^d} \frac{|f(x)|^2}{|x|} dx \leq \frac{2}{d-1} \|f\|_2 \|\nabla f\|_2$$

can be turned into a sharp inequality in a Lorentz space.

Given a measurable function f in \mathbf{R}^d such that the measure of its level sets $\{x : |f(x)| > t\}$ is finite for all $t > 0$ we can associate to f two decreasing rearrangements. The first one defines a radial nonnegative function in \mathbf{R}^d , f_d^* , decreasing on rays from the origin, and the second one defines a decreasing nonnegative function in $(0, +\infty)$, f_1^* . Both definitions have in common that the measure of the level sets is preserved; that is, the d -dimensional measure of $\{x : |f(x)| > t\}$ and $\{x : |f_d^*(x)| > t\}$ and the one-dimensional measure of $\{x : |f_1^*(x)| > t\}$ are the same for all $t > 0$. See [5, Section 3.3] for the definition and properties of f_d^* , called there a symmetric-decreasing rearrangement, and see [2, Chapter 2, Section 1] for f_1^* .

The L^p -norms of f , f_d^* , and f_1^* are the same due to the equimeasurability of their level sets. Moreover, for real-valued f the symmetric-decreasing rearrangement does not increase the L^2 -norm of the gradient ([5, Lemma 7.17]). On the other hand, it is easy to see that f_d^* and f_1^* are related by

$$(3.3) \quad f_d^*(x) = f_1^*(\omega_d |x|^d),$$

where ω_d is the measure of the unit ball of \mathbf{R}^d .

Decreasing rearrangements are used to define Lorentz spaces. Following [2, Chapter 4, Section 4] we say that f is in the Lorentz space $L^{p,q}$ for some $0 < p, q < \infty$ if

$$(3.4) \quad \|f\|_{p,q} := \left(\int_0^\infty [t^{1/p} f_1^*(t)]^q \frac{dt}{t} \right)^{1/q}$$

is finite, with the usual modification if $q = \infty$. Due to the relation (3.3), this definition can be adapted to f_d^* . In particular, for $p = 2d/(d-1)$ and $q = 2$ the left-hand side of (3.2) appears, namely,

$$(3.5) \quad \|f\|_{2d/(d-1),2}^2 = \frac{1}{\omega_d^{1/d}} \int_{\mathbf{R}^d} \frac{f_d^*(x)^2}{|x|} dx.$$

Combining (3.5), (3.2), and the inequality $\|\nabla f_d^*\|_2 \leq \|\nabla f\|_2$ we get immediately the following.

Corollary 3.5. *Let f be a real-valued function in $H^1(\mathbf{R}^d)$ for $d \geq 2$. Then f is in the Lorentz space $L^{2d/(d-1),2}$ and*

$$(3.6) \quad \|f\|_{2d/(d-1),2}^2 \leq \frac{2}{\omega_d^{1/d}(d-1)} \|f\|_2 \|\nabla f\|_2.$$

Equality holds if f is a multiple of $e^{-c|x-b|}$ with $c > 0$ and $b \in \mathbf{R}^d$.

If we do not care about the constant, (3.6) can be obtained from Pitt's inequality

$$(3.7) \quad \int_{\mathbf{R}^d} \frac{|f(x)|^2}{|x|} dx \leq C \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 |\xi| d\xi = C \|D^{1/2} f\|_2^2$$

and the Cauchy-Schwarz inequality. But the sharp constant in (3.7) is greater than in (3.2), and it has no extremals (see [1]). The added value of (3.6) is its sharpness and the existence of extremal functions.

3.4. Some other inequalities.

Corollary 3.6. *Let $d \geq 2$, $0 \leq a < d-1$, and $b \geq 0$. Then*

$$\begin{aligned} & \int_{\mathbf{R}^d} \frac{|f(x)|^2}{|x|} dx \\ & \leq \frac{b}{d-1-a} \int_{\mathbf{R}^d} |f(x)|^2 dx + \frac{1}{d-1-a} \int_{\mathbf{R}^d} |\nabla f(x)|^2 \frac{|x|}{a+b|x|} dx. \end{aligned}$$

Equality holds for nontrivial f if and only if $a < (d-1)/2$, $b > 0$, and f is a multiple of $|x|^{-a} e^{-b|x|}$.

Proof. Take $w(|x|) = |x|^{-1}$. Following the proof of Theorem 1.1 with $a = 0$ we get

$$\int_{\mathbf{R}^d} \frac{|f(x)|^2}{|x|} dx = \frac{-2}{d-1} \Re \int_{\mathbf{R}^d} f(x) D_r \bar{f}(x) dx,$$

where $D_r f$ is the radial derivative of f . Bound the integral of the right-hand side as

$$(3.8) \quad \left| \int_{\mathbf{R}^d} f D_r f \right| \leq \frac{1}{2} \int_{\mathbf{R}^d} |f(x)|^2 \frac{a+b|x|}{|x|} dx + \frac{1}{2} \int_{\mathbf{R}^d} |\nabla f|^2 \frac{|x|}{a+b|x|} dx$$

and proceed.

Equality in (3.8) requires f radial and

$$f(x) \frac{a+b|x|}{|x|} = -D_r f(x).$$

The solutions to this equation are $f(x) = c|x|^{-a} e^{-b|x|}$. If $c \neq 0$, the integrals involved in the equality are finite only if $a < (d-1)/2$ and $b > 0$. \square

The particular case $b = 0, a = (d-1)/2$ is the classical Hardy's inequality; for $d = 3, a = 1 - \sqrt{1 - \nu^2}$, and $b = \nu$, we obtain for the gradient the inequality given in [4, p. 222] for the Dirac operator.

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