Article

ϕ-informational measures: Calculations for the Gamma distribution

Steeve Zozor [1,*] and Jean-François Bercher [2]

Citation: . *Entropy* **2021**, *1*, 0.
https://doi.org/

Received:
Accepted:
Published:

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1 Univ. Grenoble Alpes, CNRS, Grenoble INP*, GIPSA-Lab, 38000 Grenoble, France
*Institute of Engineering Univ. Grenoble Alpes
steeve.zozor@cnrs.fr
2 LIGM, Univ Gustave Eiffel, CNRS, F-77454 Marne-la-Vallée, France
jean-francois.bercher@esiee.fr
* Correspondence: steeve.zozor@cnrs.fr

Appendix A. Inverse maximum entropy problem and associated inequalities: some examples

Appendix A.1. The gamma distribution and (partial) p-order moment(s)

As a very special case, consider here the gamma distribution expressed as

$$f_X(x) = \frac{(\Gamma(q)x)^{q-1} \exp\left(-\frac{\Gamma(q)}{r}\, x\right)}{r^q} \qquad \text{on} \qquad \mathcal{X} = \mathbb{R}_+. \tag{A1}$$

Parameter $q > 0$ is known as the shape parameter of the law, while $\sigma = \frac{r}{\Gamma(q)} > 0$ is a scaling parameter. This distribution also appears in various applications, as described for instance in [1].

Let us concentrate on the case $q > 1$ for which the distribution is non-monotonous, unimodal, where the mode is located at $x = \frac{r\,(q-1)}{\Gamma(q)}$, and $f_X(\mathbb{R}_+) = \left[0\,;\, \frac{(q-1)^{q-1}\, e^{1-q}}{r}\right]$.

Here again it cannot be a maximizer of a ϕ-entropy constraint subject to a moment of order $p > 0$. Here, we can again consider partial moments as constraints,

$$T_{k,1}(x) = x^p\, \mathbb{1}_{\mathcal{X}_k}(x), \qquad k \in \{0, -1\} \qquad \text{where}$$
$$\mathcal{X}_0 = \left[0\,;\, \frac{r\,(q-1)}{\Gamma(q)}\right) \qquad \text{and} \qquad \mathcal{X}_{-1} = \left[\frac{r\,(q-1)}{\Gamma(q)}\,;\, +\infty\right),$$

or interpret f_X as a critical point of an ϕ-like entropy by constraining the moment

$$T_1(x) = x^p \quad \text{over} \quad \mathcal{X} = \mathbb{R}_+. \tag{A2}$$

Inverting $y = f_X(x)$ leads to the equation

$$-\frac{\Gamma(q)\,x}{r\,(q-1)}\exp\left(-\frac{\Gamma(q)\,x}{r\,(q-1)}\right) = -\frac{(ry)^{\frac{1}{q-1}}}{q-1}$$

to be solved. As expected, this equation has two solutions. These solutions can be expressed via the multivalued Lambert-W function W defined by $z = \mathrm{W}(z)\exp(\mathrm{W}(z))$, i.e., W is the inverse function of $u \mapsto u\exp(u)$ [2, § 1], leading to the inverse functions

$$f_{X,k}^{-1}(y) = -\frac{r\,(q-1)}{\Gamma(q)}\,\mathrm{W}_k\left(-\frac{(ry)^{\frac{1}{q-1}}}{q-1}\right), \qquad ry \in \left[0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right], \tag{A3}$$

where k denotes the branch of the Lambert-W function. $k = 0$ gives the principal branch and here it is related to the entropy part on $\mathcal{X}_0$, while $k = -1$ gives the secondary branch, related to $\mathcal{X}_{-1}$ here.

Applying (**??**) to obtain the branches of the functionals of the multiform entropy, one has thus to integrate the functions

$$\phi_k'(y) = \lambda_0 + \lambda_{k,1}\left[-\frac{r\,(q-1)}{\Gamma(q)}\,\mathrm{W}_k\left(-\frac{(ry)^{\frac{1}{q-1}}}{q-1}\right)\right]^p$$

where, to ensure the convexity of the ϕ_k,

$$(-1)^k\lambda_{k,1} > 0$$

The same approach allows to design $\widetilde{\phi}_k$, with a unique λ_1 instead of the $\lambda_{k,1}$s and without restriction on λ_1.

First, let us reparametrize the λ_is so as to include the factor $r/\Gamma(q)$ inside $\lambda_{k,1}$ so that one can write formally

$$\phi_k(y) = \phi_{k,\mathrm{u}}(ry) \qquad \text{with} \tag{A4}$$

$$\phi_{k,\mathrm{u}}(u) = \gamma_k + \beta\,u + (-1)^k\alpha_k\int\left[(1-q)\,\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right)\right]^p du, \qquad \alpha_k \geq 0$$

Obtaining a closed forme expression for the integral term is not an easy task. But relation $z\,(1+\mathrm{W}_k(z))\;\mathrm{W}_k'(z) = \mathrm{W}_k(z)$ [2, Eq. 3.2] suggests that a way to make the integration is to search for

$$\Phi_k(u) = \int\left[(1-q)\,\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right)\right]^p du \tag{A5}$$

under the form of a series

$$\Phi_k(u) = u\sum_{l\geq 0} a_l\left[(1-q)\,\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right)\right]^{l+p}$$

identifying the coefficients a_l. This gives, by derivation and omitting the argument of W_k by sake of simplicity

$$\Big[(1-q)\,\mathrm{W}_k\Big]^p = \sum_{l\geq 0} a_l\Big[(1-q)\,\mathrm{W}_k\Big]^{l+p} + \frac{u^{\frac{1}{q-1}}}{q-1}\;\mathrm{W}_k'\sum_{l\geq 0}(l+p)\,a_l\Big[(1-q)\,\mathrm{W}_k\Big]^{l+p-1}$$

Now with $z = -\frac{u^{\frac{1}{q-1}}}{q-1}$ one has $\frac{u^{\frac{1}{q-1}}}{q-1}\;\mathrm{W}_k' = -\frac{\mathrm{W}_k}{1+\mathrm{W}_k}$ so that

$$\Big[(1-q)\,\mathrm{W}_k\Big]^p = \sum_{l\geq 0} a_l\Big[(1-q)\,\mathrm{W}_k\Big]^{l+p} + \sum_{l\geq 0}\frac{(l+p)\,a_l}{q-1}\,\frac{\Big[(1-q)\,\mathrm{W}_k\Big]^{l+p}}{1+\mathrm{W}_k}$$

that is, simplifying both sides by $\Big[(1-q)\,\mathrm{W}_k\Big]^p$ and multiplying both sides by $1+\mathrm{W}_k$,

$$1+\mathrm{W}_k = \sum_{l\ge 0} a_l \Big[(1-q)\,\mathrm{W}_k\Big]^l + \sum_{l\ge 0} \frac{a_l}{1-q} \Big[(1-q)\,\mathrm{W}_k\Big]^{l+1} + \sum_{l\ge 0} \frac{(l+p)\,a_l}{q-1} \Big[(1-q)\,\mathrm{W}_k\Big]^l$$

i.e.,

$$1+\mathrm{W}_k = \frac{(p+q-1)\,a_0}{q-1} + \sum_{l\ge 1} \frac{a_{l-1}-(p+q+l-1)\,a_l}{1-q} \Big[(1-q)\,\mathrm{W}_k\Big]^l$$

As a consequence

$$a_0 = \frac{q-1}{p+q-1}$$

$1 = a_0 - (p+q)\,a_1$ so that $a_1 = \frac{a_0-1}{p+q}$, i.e.,

$$a_1 = -\frac{p}{(p+q)(p+q-1)}$$

For $l \ge 2$, $a_{l-1}-(p+q+l-1)a_l = 0$ i.e., $a_l = \frac{1}{p+q+l-1} a_{l-1}$

$$\forall\, l \ge 2, \quad a_l = \frac{1}{(p+q+l-1)\cdots(p+q+1)}\, a_1$$

For the Pochhammer symbol $(a)_l = a\cdots(a+l-1)$ for $l \ge 1$ and $(a)_0 = 1$ one has

$$\forall\, l \ge 1, \quad a_l = -\frac{p}{(p+q-1)(p+q)_l}$$

(given for $l \ge 2$, but one can see that it remains valid for $l = 1$). Therefore

$$\Phi_k(u) = u \Big[(1-q)\,\mathrm{W}_k\Big]^p \left(\frac{q-1}{p+q-1} - \frac{p}{p+q-1} \sum_{l\ge 1} \frac{1}{(p+q)_l} \Big[(1-q)\,\mathrm{W}_k\Big]^l \right)$$

Adding and removing a term in $l = 0$ in the sum, and noting that $l! = (1)_l$, one finally obtains

$$\Phi_k(u) = u \Big[(1-q)\,\mathrm{W}_k\Big]^p \left(1 - \frac{p}{p+q-1} \sum_{l\ge 0} \frac{(1)_l}{(p+q)_l\, l!} \Big[(1-q)\,\mathrm{W}_k\Big]^l \right)$$

One finally recognizes in the sum the confluent hypergeometric (or Kummer's) function ${}_1F_1(1; p+q; \cdot)$ [3, § 13] or [4, § 9.2], so that, we achieve to

$$\phi_{k,\mathrm{u}}(u) = \gamma_k + \beta\, u + (-1)^k \alpha_k\, u \left[(1-q)\ \mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right]^p \times$$
$$\left[1 - \frac{p}{p+q-1}\, {}_1F_1\left(1\,;\, p+q\,;\, (1-q)\,\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1} \right) \right) \right] \mathbb{1}_{\left(0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right)}(u), \qquad \alpha_k > 0 \tag{A6}$$

Again, p, q, r are additional parameters for this family of entropies.

Verification a posteriori

We first write (omitting the arguments for sake of simplicity)

$$\begin{aligned}\Phi_k' \quad = \quad & [(1-q)\,\mathrm{W}_k]^p \left[1 - \frac{p}{p+q-1}\,{}_1F_1\right]\\ & + p\,[(1-q)\,\mathrm{W}_k]^{p-1}\left[u\Big((1-q)\,\mathrm{W}_k\Big)'\right]\left[1-\frac{p}{p+q-1}\,{}_1F_1\right]\\ & - \frac{p}{p+q-1}[(1-q)\,\mathrm{W}_k]^p\left[u\Big((1-q)\,\mathrm{W}_k\Big)'\right]{}_1F_1'\end{aligned}$$

We note now that

$$u\Big((1-q)\,\mathrm{W}_k\Big)' = u\,(1-q)\,\frac{-\frac{1}{q-1}\,u^{\frac{1}{q-1}-1}}{q-1}\,\mathrm{W}_k' = \frac{u^{\frac{1}{q-1}}}{q-1}\,\mathrm{W}_k'$$

which is, from [2, Eq. 3.2] $z\,\mathrm{W}_k'(z) = \frac{\mathrm{W}_k(z)}{1+\mathrm{W}_k(z)}$

$$u\Big((1-q)\,\mathrm{W}_k\Big)' = -\frac{\mathrm{W}_k}{1+\mathrm{W}_k}$$

This gives, grouping the terms in the hypergeometric function

$$\Phi_k' = \frac{[(1-q)\,\mathrm{W}_k]^p}{1+\mathrm{W}_k}\left[(1+\mathrm{W}_k)\left(1-\frac{p}{p+q-1}{}_1F_1\right) - \frac{p}{1-q}\left(1-\frac{p}{p+q-1}{}_1F_1\right) + \frac{p\,\mathrm{W}_k}{p+q-1}\,{}_1F_1'\right]$$

Hence, grouping the termes in the hypergeometric function,

$$\Phi_k' = \frac{[(1-q)\,\mathrm{W}_k]^p}{1+\mathrm{W}_k}\left[1+\mathrm{W}_k - \frac{p}{1-q} + \frac{p}{(p+q-1)(1-q)}\big(((p+q-1)-(1-q)\,\mathrm{W}_k){}_1F_1 + (1-q)\,\mathrm{W}_k\,{}_1F_1'\big)\right]$$

Finally, from [3, 13.4.1] one have

$$((p+q-1)-z){}_1F_1(1,p+q,z) + z{}_1F_1'(1,p+q,z) = (p+q-1){}_1F_1(0,p+q,z) = p+q-1$$

Then, from the domain of definition of the inverse of f_X, u is restricted to $\left(0\,;\,\left(\frac{q-1}{e}\right)^{q-1}\right)$, which can be compensated for by playing with parameter r. At the opposite, noting that $\mathrm{W}_k\big(-e^{-1}\big) = -1$, to extend the entropic functionals to C^1 functions on $\mathbb{R}_+$, one would have to impose $\beta + (-1)^k \alpha_k = 0$ to vanish the derivatives at $u = e^{1-a}$. This is impossible because from $\alpha_k > 0$ one cannot impose $\beta = \alpha_{-1} = -\alpha_0$. Moreover, even a convex extension relaxing the C^1 condition is impossible since we would have to impose $\beta + \alpha_k \le \beta$ to insure the increase of the ϕ_ks on $\mathbb{R}_+$.

We can however choose the γ_k such that the ϕ_k coincide at $u = 0$ for instance (e.g., to vanish them at 0 to insure the existence of the ϕ-entropy). One can also wish to impose the value(s) of the $\phi_{k,\mathrm{u}}$ at $u = \left(\frac{q-1}{e}\right)^{q-1}$.

Values at the bound of the domain of definition

From [2, Eq. 3.1] we have $\mathrm{W}_0(0) = 0$ and from [3, Eq. 13.1.2] ${}_1F_1(1;p+q;0) = 1$, so that

$$\phi_{0,\mathrm{u}}(0) = \gamma_0 \qquad \text{and} \qquad \phi_{0,\mathrm{u}}'(0) = \beta \tag{A7}$$

Then $\lim\limits_{x\to 0^-} \mathrm{W}_{-1}(x) = -\infty$ (see [2, Fig. 1 or Eq. 4.18]). From the asymptotics [3, Eq. 13.1.4] of the confluent hypergeometric function,

$${}_1F_1(1;p+q;(1-q)\,\mathrm{W}_{-1}) = \Gamma(p+q)\,e^{(1-q)\,\mathrm{W}_{-1}}\left[(1-q)\,\mathrm{W}_{-1}\right]^{1-p-q}\left(1+O\Big(\Big|\mathrm{W}_{-1}\Big|^{1-p-q}\Big)\right)$$

and thus

$$\Phi_{-1}(u) = u\Big[(1-q)\,\mathrm{W}_{-1}\Big]^p - p\,\Gamma(p+q-1)\,u\,\Big[(1-q)\,\mathrm{W}_{-1}\,e^{\mathrm{W}_{-1}}\Big]^{1-q}\left(1+O\left(\Big|\,\mathrm{W}_{-1}\Big|^{1-q}\right)\right)$$

This gives, from $\mathrm{W}(z)e^{\mathrm{W}(z)} = z$, i.e., $\mathrm{W}_{-1}\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right)\exp\left(\mathrm{W}_{-1}\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right)\right) = -\frac{u^{\frac{1}{q-1}}}{q-1}$,

$$\Phi_{-1}(u) = u\Big[(1-q)\,\mathrm{W}_{-1}\Big]^p - p\,\Gamma(p+q-1)\left(1+O\left(\Big|\,\mathrm{W}_{-1}\Big|^{1-q}\right)\right)$$

Finally, noting that, because $1-q<0$ we have $\lim\limits_{u\to 0^-}\Big|\,\mathrm{W}_{-1}\Big|^{1-q} = 0$, and from [2, Eq. 4.6 & lines that follow] $\lim\limits_{u\to 0^-} u\Big[(1-q)\,\mathrm{W}_{-1}\Big]^p = 0$ so that, finally, at the limit

$$\phi_{-1,\mathrm{u}}(0) = \gamma_{-1} + p\,\Gamma(p+q-1)\,\alpha_{-1} \qquad \text{and} \qquad \lim_{u\to 0^-}\phi'_{-1,\mathrm{u}}(u) = -\infty \tag{A8}$$

Now, from $\mathrm{W}_k(-e^{-1}) = -1$ we immediately have

$$\phi_{k,\mathrm{u}}\left(\left(\frac{q-1}{e}\right)^{q-1}\right) = \gamma_k + \left(\frac{q-1}{e}\right)^{q-1}\left(\beta + (-1)^k\alpha_k\,(q-1)^p\left[1-\frac{p}{p+q-1}\,{}_1F_1(1;p+q;q-1)\right]\right) \tag{A9}$$

and

$$\phi'_{k,\mathrm{u}}\left(\left(\frac{q-1}{e}\right)^{q-1}\right) = \beta + (-1)^k\,\alpha_k\,(q-1)^p \tag{A10}$$

Limit $q \to 1^+$

When $q\to 1^+$ one has

$$\lim_{q\to 1^+} f_X(x) = \text{ exponential law,} \qquad \lim_{q\to 1^+}\mathcal{X}_0 = \emptyset, \qquad \lim_{q\to 1^+}\mathcal{X}_{-1} = \mathbb{R}_+ = \mathcal{X}$$

Hence, in accordance

- The constraints degenerate to a single uniform constraint $T_1(x) = x^p$;
- In this limit, conditions **??** and **??** are both satisfied.
- The entropic functional become state-independent (uniform), where only the branch ϕ_{-1} remains.

The study lies on [5, Th. 3.2] that states

$$\Big|\mathrm{W}_{-1}\left(-e^{-(t+1)}\right) + \log(t+1) + (t+1)\Big| \le 1-\log(e-1)$$

We apply this theorem to the positive real t given by

$$e^{-(t+1)} = \frac{u^{\frac{1}{q-1}}}{q-1} \qquad \text{i.e.,} \qquad t = -\frac{1}{q-1}\log u + \log(q-1) - 1$$

(see domain where u lives), which thus gives, from $q>1$

$$\left|(1-q)\,\mathrm{W}_{-1}\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right) + \log u - (q-1)\log\Big((q-1)\log(q-1)-\log u\Big)\right| \le (q-1)(1-\log(e-1))$$

As a consequence, the left handside tends uniformly to 0 when $q \to 1^+$. Finally, $(q-1)\log\Big((q-1)\log(q-1) - \log u\Big)$ goes also uniformy to 0 as $q \to 1^+$, which allows to obtain that

$$\lim_{q\to 1^+} (1-q)\,\mathrm{W}_k\left(-\frac{u^{\frac{1}{q-1}}}{q-1}\right) = -\log(u) \quad \text{uniformly}$$

As a conslusion, from the continuity of ${}_1F_1$ both w.r.t., its parameters and its variable, we have

$$\lim_{q\to 1^+} \phi_{1,\mathrm{u}}(u) = \gamma_{-1} + \beta u - \alpha_{-1}\, u\,(-\log u)^p \Big(1 - {}_1F_1(1;p+1;-\log u)\Big)$$

$u \in (0\,;\,1)$ but the domain can be expanded to $\mathbb{R}_+$.

Finally, for $p = 1$, due to [3, 13.6.14] stating that ${}_1F_1(1;2;x) = \frac{e^x-1}{x}$, we obtain after simple algebra

$$\lim_{q\to 1^+, p=1} \phi_{-1,\mathrm{u}} = \alpha_{-1}\, u\, \log u \;+\; (\beta - \alpha_{-1})\, u \;+\; \gamma_{-1} + \alpha_{-1}$$

which is nothing more than the Shannon entropic functional: we recover here that the exponential distribution is the maximal Shannon entropy distribution subject to the first order moment constraint.

In passing, because W_0 is bounded on the considered domain, one has immediately

$$\lim_{q\to 1^+} \phi_{0,\mathrm{u}}(u) = \gamma_0 + \beta u$$

The special case $p = 2 - q$

From [3, 13.6.14], ${}_1F_1(1;2;x) = \frac{e^x-1}{x}$, so that

$$\Phi_k(u) = u\Big[(1-q)\,\mathrm{W}_k\Big]^{2-q}\left[1 - (2-q)\,\frac{e^{(1-q)\,\mathrm{W}_k} - 1}{(1-q)\,\mathrm{W}_k}\right]$$

that is

$$\Phi_k(u) = u\Big[(1-q)\,\mathrm{W}_k\Big]^{1-q}\Big[(1-q)\,\mathrm{W}_k + 2 - q\Big] - \frac{(2-q)\,u}{(q-1)^{q-1}}\Big(-\mathrm{W}_k\, e^{\mathrm{W}_k}\Big)^{1-q}$$

Again, from $\mathrm{W}_k(z)e^{\mathrm{W}_k(z)} = z$ we have $\big(-\mathrm{W}_k\, e^{\mathrm{W}_k}\big)^{1-q} = \left(\frac{u^{\frac{1}{q-1}}}{q-1}\right)^{1-q} = u^{-1}(q-1)^{q-1}$ so that

$$\Phi_k(u) = u\Big[(1-q)\,\mathrm{W}_k\Big]^{1-q}\Big[(1-q)\,\mathrm{W}_k + 2 - q\Big] + q - 2$$

The multivalued function ϕ_{u} in the concave context is represented figure A1 for $p = 2, q = 2$ and $q = 5$, and with the choice $\alpha_0 = 1,\ \alpha_{-1} = -0.05,\ \beta = -\alpha_{-1},\ \gamma_0 = 0,\ \gamma_{-1} = \frac{p\Gamma(p+q-1)}{(q-1)^p}\,\alpha_{-1}$.

References

1. Johnson, N.L.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*, 2nd ed.; Vol. 1, John Wiley & Sons: New-York, 1995.
2. Corless, R.M.; Gonnet, G.H.; Hare, D.E.G.; Jeffrey, D.J.; Knuth, D.E. On the Lambert W Function. *Advances in Computational Mathematics* **1996**, *5*, 329–359. doi:10.1007/BF02124750.
3. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; 9th printing, Dover: New-York, 1970.
4. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 8th ed.; Academic Press: San Diego, 2015.
5. Alzahrani, F.; Salem, A. Sharp bounds for the Lambert W function. *Integral Transforms and Special Functions* **2018**, *29*, 971–978. doi:10.1080/10652469.2018.1528247.

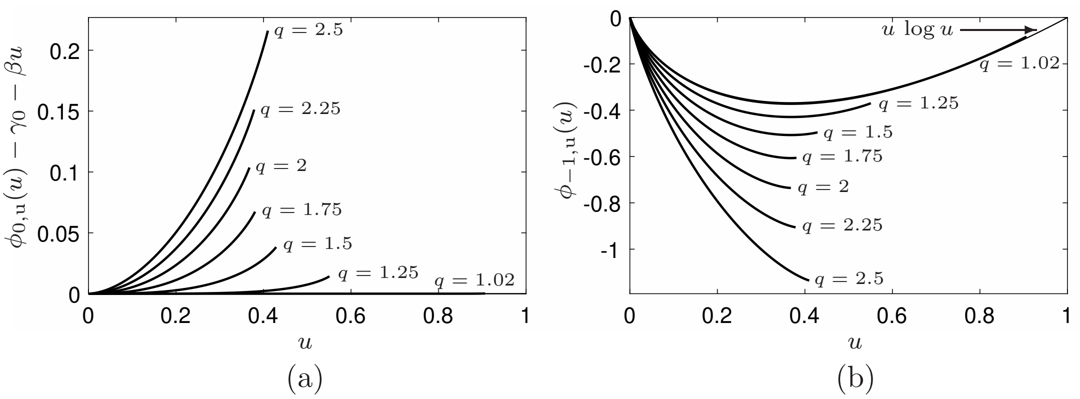

Figure A1. Multiform entropy functional ϕ_u derived from the gamma distribution with the partial moment constraints $T_{k,1}(x) = x^2 \mathbb{1}_{\mathcal{X}_k}(x)$, $k \in \{0, -1\}$. (a): $q = 2$; (b): $q = 5$.