SECTION 2

Chapitre 3

Approximation polynomiale

Taylor Young

Introduction

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + f^{n}(0)\frac{x^n}{n!} + o(x^n)$$

Pourquoi le factoriel ?

Dérivation répétée d'une puissance

On prend la fonction la plus simple : $f(x) = x^n$

$$f'(x) = n x^{n-1}$$

$$f''(x) = n(n-1)x^{n-2}$$

A chaque dérivation, on multiplie par un facteur de décroissance.

$$f'''(x) = n(n-1)(n-2)x^{n-3}$$

$$f^{4}(x) = n(n-1)(n-2)(n-3)x^{n-4}$$

$$f^{n}(x) = n(n-1)(n-2)\dots 3 \times 2 \times 1$$

$$f^{n}(x) = n!$$

Exercices

7 Entrainement

Exercice 1 : Application de la formule de Taylor Young

En appliquant la formule de Taylor Young, calculer le DL en 0 à l'ordre 3 des fonctions :

a) $x \mapsto \ln(1+x)$

b) $x \mapsto e^x$

c) $x \mapsto \frac{1}{1+x}$

d) $x \mapsto \sqrt{1-x}$

09/03/23

a) $x \mapsto \ln(1+x)$

① $f'(x) = \frac{d}{dx} f(x)$:

$\ln(1+x) = \ln(u) \Rightarrow$ $\boxed{\ln(u) : \frac{d}{dx} \ln(u) = \frac{u'}{u}}$

$u = 1+x$

$u' = 1$

$\frac{d}{dx} \ln(1+x) = \boxed{f'(x) = \frac{1}{1+x}}$

$f'(0) = \frac{1}{1+0} = 1$

② $f''(x) = \frac{d}{dx} f'(x)$

$\frac{d}{dx}\left(\frac{1}{1+x}\right)$:

$\hookrightarrow \frac{1}{u} = u^{-1} \rightarrow$ $\boxed{\frac{d}{dx}(u^n) = n \cdot u^{n-1} \cdot u'}$

$\frac{d}{dx}(u^{-1}) = -1 \cdot u^{-1-1} \cdot u' = -1 \cdot u^{-2} \cdot u' = -u^{-2} \cdot u'$

$= -\frac{u'}{u^2}$

$\boxed{\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{u'}{u^2}}$

$\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}$

$\boxed{f''(x) = -\frac{1}{(1+x)^2}} \rightarrow f''(0) = -1$

③ $f'''(x) = \frac{d}{dx} f''(x)$:

$\frac{d}{dx}\left(-\frac{1}{(1+x)^2}\right) \rightarrow -\frac{1}{u^2} = -u^{-2} \rightarrow$ $\boxed{\frac{d}{dx} u^n = n \cdot u^{n-1} \cdot u'}$

$\hookrightarrow u(x) = 1+x$

$n = -2$

$u'(x) = 1$

$f'''(x) = (-2) \cdot (1+x)^{-2-1} \cdot 1$

$\boxed{f'''(x) = \frac{2}{(1+x)^3}} \rightarrow f'''(0) = 2$

① $f'(0) = 1$ (et $f(0) = 0$)

② $f''(0) = -1$

③ $f'''(0) = 2$

$\Rightarrow$ <u>Approx de Taylor Young</u> :

$$\boxed{f(x) = f(0) + f'(0) \cdot (x-0) + f''(0) \cdot \frac{(x-0)^2}{2!} + f'''(0) \cdot \frac{(x-0)^3}{3!} + o\left((x-0)^3\right)}$$

$\ln(1+x) = 0 + 1 \cdot (x-0) + (-1) \cdot \frac{(x-0)^2}{2!} + 2 \cdot \frac{(x-0)^3}{3!} + o\left((x-0)^3\right)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$

[Rappel : $3! = 3 \times 2 \times 1 = 6$

b) $x \mapsto e^x$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

① $f'(x) = \frac{d}{dx} e^x$:

$$\boxed{f'(x) = e^x} \to f'(0) = e^0 = 1$$

② $f''(x) = \frac{d}{dx} f'(x)$:

$$\boxed{f''(x) = e^x} \to f''(0) = 1$$

③ $f'''(x) = \frac{d}{dx} f''(x)$:

$$\boxed{f'''(x) = e^x} \to f'''(0) = 1$$

⇒ Approx de Taylor Young :

$$e^x = 1 + 1 \cdot (x-0) + 1 \cdot \frac{(x-0)^2}{2!} + 1 \cdot \frac{(x-0)^3}{3!} + o\left((x-0)^3\right)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$$

c) $x \longmapsto \dfrac{1}{1+x}$

① $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{1+x}\right)$

$\hookrightarrow \dfrac{1}{u} = u^{-1} \longrightarrow \boxed{\dfrac{d}{dx} u^n = n \cdot u^{n-1} \cdot u'}$

$\hookrightarrow \dfrac{d}{dx}\left(u^{-1}\right) = -1 \cdot u^{-2} \cdot u' = -\dfrac{u'}{u^2}$

$\boxed{\dfrac{d}{dx}\left(\dfrac{1}{u}\right) = -\dfrac{u'}{u^2}}$ $\longleftarrow$ avec : $u = 1+x$, $u' = 1$

$\boxed{f'(x) = -\dfrac{1}{(1+x)^2}} \longrightarrow f'(0) = -1$

② $f''(x) = \dfrac{d}{dx} f'(x)$

$f'(x) = -\dfrac{1}{(1+x)^2} \longrightarrow -\dfrac{1}{u^2} = -u^{-2}$

$f'(x) = -(1+x)^{-2} \longleftarrow \boxed{\dfrac{d}{dx} u^n = n \cdot u^{n-1} \cdot u'}$

$\dfrac{d}{dx} (1+x)^{-2} = (-2) \cdot (1+x)^{-3} \cdot u'$ ← $u' = 1$

$f''(x) = -(-2) \cdot (1+x)^{-3} \cdot 1$

$f''(x) = 2(1+x)^{-3}$

$\boxed{f''(x) = \dfrac{2}{(1+x)^3}} \longrightarrow f''(0) = 2$

③ $f'''(x) = \dfrac{d}{dx} f''(x)$:

$f''(x) = \dfrac{2}{(1+x)^3} = 2 \cdot (1+x)^{-3} \longleftarrow \boxed{\dfrac{d}{dx} u^n = n \cdot u^{n-1} \cdot u'}$

$\hookrightarrow \dfrac{d}{dx} (1+x)^{-3} = -3 \cdot u^{-4} \cdot u' = -3 \cdot (1+x)^{-4} \cdot 1$

avec : $u = (1+3)$, $u' = 1$, $n = -3$

$f'''(x) = 2 \cdot (-3)(1+x)^{-4} \cdot 1$

$f'''(x) = -6(1+x)^{-4}$

$\boxed{f'''(x) = -\dfrac{6}{(1+x)^4}} \longrightarrow f'''(0) = -6$

$f(0) = 1$

$f'(0) = -1$

$f''(0) = 2$

$f'''(0) = -6$

$\Rightarrow$ Approx de Taylor Young :

$\dfrac{1}{1+x} = 1 + (-1)(x-0) + 2 \cdot \dfrac{(x-0)^2}{2!} + (-6) \cdot \dfrac{(x-0)^3}{3!} + O\left((x-0)^3\right)$

$\dfrac{1}{1+x} = 1 - x + \dfrac{2x^2}{2!} + \dfrac{-6x^3}{3} + O(x^3)$

$\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + O(x^3)$

$6_{bis})$ $x \longmapsto \frac{1}{1-x}$

① $f'(x) = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = n(1-x)^{n-1}\cdot(-1)$

$= (-1)(1-x)^{-2}\cdot(-1) = (1-x)^{-2}$

$\boxed{f'(x) = (1-x)^{-2} = \frac{1}{(1-x)^2}}$ $\rightarrow$ $f'(0) = 1$ $\qquad f(0) = \sqrt{1-1} = 1$

② $f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}(1-x)^{-2} = -2(1-x)^{-3}\cdot(-1)$

$= 2(1-x)^{-3}$

$\boxed{f''(x) = 2(1-x)^{-3} = \frac{2}{(1-x)^3}}$ $\rightarrow$ $f''(0) = 2$

③ $f'''(x) = \frac{d}{dx}f''(x) = \frac{d}{dx}2(1-x)^{-3} = 2\cdot(-3)\cdot(1-x)^{-4}\cdot(-1)$

$= 6(1-x)^{-4}$

$\boxed{f'''(x) = 6(1-x)^{-4} = \frac{6}{(1-x)^4}}$ $\rightarrow$ $f'''(0) = 6$

Approx de Taylor Young:

$\sqrt{1-x} = 1 + 1\cdot(x-0)^1 + 2\cdot\frac{(x-0)^2}{2!} + 6\cdot\frac{(x-0)^3}{3!} + o\left((x-0)^3\right)$

$\sqrt{1-x} = 1 + x + x^2 + x^3 + o(x^3)$

d) $x \mapsto \sqrt{1-x}$

(1) $f'(x) = \frac{d}{dx}\sqrt{1-x} = \frac{d}{dx}(1-x)^{1/2} \to \frac{d}{dx}(1-x)^n = n(1-x)^{n-1}\cdot(-1)$

avec $n = \frac{1}{2}$ $\hookrightarrow f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

$= -\frac{1}{2\sqrt{1-x}}$

$\boxed{f'(x) = -\frac{1}{2\sqrt{1-x}} = -\frac{1}{2}(1-x)^{1/2}} \to f'(0) = -\frac{1}{2}$ $\qquad f(0) = 1$

(2) $f''(x) = \frac{d}{dx} - \frac{1}{2\sqrt{1-x}} = \frac{d}{dx} - \frac{1}{2}(1-x)^{-\frac{1}{2}}$

$\hookrightarrow f''(x) = -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)(1-x)^{-\frac{3}{2}}\cdot(-1)$

$= -\frac{1}{2}\cdot\frac{1}{2}(1-x)^{-\frac{3}{2}}$

$= -\frac{1}{4}(1-x)^{-\frac{3}{2}}$

$\boxed{f''(x) = -\frac{1}{4}(1-x)^{-\frac{3}{2}} = -\frac{1}{4(1-x)^{3/2}}} \to f''(0) = -\frac{1}{4}$

(3) $f^3(x) = \frac{d}{dx} - \frac{1}{4}(1-x)^{-\frac{3}{2}}$

$\hookrightarrow f^3(x) = -\frac{1}{4}\cdot\left(-\frac{3}{2}\right)(1-x)^{-\frac{5}{2}}\cdot(-1)$

$= -\frac{1}{4}\cdot\left(\frac{3}{2}\right)(1-x)^{-\frac{5}{2}} \qquad = -\frac{3}{8}(1-x)^{-\frac{5}{2}}$

$\boxed{f^3(x) = -\frac{3}{8}(1-x)^{-\frac{5}{2}} = -\frac{3}{8(1-x)^{5/2}}} \to f^3(0) = -\frac{3}{8}$

Approx de Taylor young :

$$\sqrt{1-x} = 1 + \left(-\frac{1}{2}\right)(x-0) + \left(-\frac{1}{4}\right)\cdot\frac{(x-0)^2}{2!} + \left(-\frac{3}{8}\right)\frac{(x-0)^3}{3!} + O(x^3)$$

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^3)$$