Formule explicite de X_t (processus de Vašíček)

$$dX_t = -a(X_t - \mu)\,dt + \sigma\, dW_t$$

$$dX_t = -aX_t\,dt + a\mu\,dt + \sigma\,dW_t$$ On développe

$$dX_t + aX_t\,dt = a\mu\,dt + \sigma\,dW_t$$ On regroupe X_t

Rappel : $d(f(t)X_t) = f(t)\,dX_t + f'(t)X_t\,dt$

On multiplie par e^{at} :

$$\frac{d}{dt}\left(e^{at}X_t\right) = e^{at}\,dX_t + ae^{at}X_t\,dt = e^{at}\left(dX_t + aX_t\,dt\right)$$

On multiplie toute l'équation par e^{at}

$$e^{at}\left(dX_t + aX_t\,dt\right) = a\mu e^{at}\,dt + \sigma e^{at}\,dW_t$$

calcul stochastique → $d\left(e^{at}X_t\right) = a\mu e^{at}\,dt + \sigma e^{at}\,dW_t$

On intègre :

$$e^{at}X_t - X_0 = \int_0^t a\mu e^{a\cdot s}\,ds + \int_0^t \sigma e^{a\cdot s}\,dW_s$$

$$e^{at}X_t = X_0 + \int_0^t a\mu e^{a\cdot s}\,ds + \int_0^t \sigma e^{a\cdot s}\,dW_s$$

$$\frac{\cancel{e^{at}}X_t}{\cancel{e^{at}}} = \frac{1}{e^{at}}X_0 + \frac{1}{e^{at}}\int_0^t a\mu e^{a\cdot s}\,ds + \frac{1}{e^{at}}\int_0^t \sigma e^{a\cdot s}\,dW_s$$

$$X_t = e^{-at}X_0 + e^{-at}\int_0^t a\mu e^{a\cdot s}\,ds + \sigma e^{-at}\int_0^t e^{a\cdot s}\,dW_s$$

On calcule l'intégrale déterministe :

$$\int_0^t a\mu e^{a\cdot s}\,ds = a\mu\int_0^t e^{a\cdot s}\,ds = a\cdot\mu\left[\frac{1}{a}e^{as}\right]_0^t = \mu\left(e^{at}-1\right)$$

car $e^{-at}\cdot e^{at} = e^0 = 1$

Donc : $e^{-at}\mu\left(e^{at}-1\right) = \mu\left(e^{-at}e^{at} - e^{-at}\right) = \mu\left(1-e^{-at}\right)$

$$X_t = e^{-at}X_0 + \mu\left(1-e^{-at}\right) + \sigma e^{-at}\int_0^t e^{a\cdot s}\,dW_s$$

$$e^{-at}X_0 + \mu\left(1-e^{-at}\right) = X_0e^{-at} + \mu - \mu e^{-at} = e^{-at}(X_0-\mu) + \mu$$

$$\boxed{X_t = \mu + e^{-at}(X_0-\mu) + \sigma e^{-at}\int_0^t e^{a\cdot s}\,dW_s}$$