Démonstration Drift d'un actif étranger

On connaît : μ_S sous $\mathbb{Q}^d$

$$dX_t = \mu_X X_t\, dt + \sigma_X X_t\, dW_t^S \qquad dS_t^d = \mu_S S_t^d\, dt + \sigma_S S_t^d\, dW_t^f$$

1) Dév. produit stochastique :

$$d(S_t^d X_t) = \underbrace{S_t^d\, dX_t}_{1} + \underbrace{X_t\, dS_t^d}_{2} + \underset{\underbrace{d\langle S_t^d, X_t\rangle}_{3}}{d(S_t^d X_t)}$$

2) Dév. de chaque terme :

1. $S_t^d\, dX_t = S_t^d(\mu_X X_t\, dt + \sigma_X X_t\, dW_t^x) = X_t S_t^d(\mu_X\, dt + \sigma_X\, dW_t^x)$

2. $X_t\, dS_t^d = X_t(\mu_S S_t^d\, dt + \sigma_S S_t^d\, dW_t^f) = X_t S_t^d(\mu_S\, dt + \sigma_S\, dW_t^f)$

3. $d(S_t^d X_t) = (\sigma_S S_t^d\, dW_t)(\sigma_X X_t\, dW_t^x) = \sigma_S \sigma_X X_t S_t^d \underbrace{dW_t\, dW_t^x}_{\rho}$

$$= \rho\, \sigma_S S_t^d\, \sigma_X X_t\, dt$$

Règle de calc. sto : $d\langle A, B_t\rangle$ = produits des termes dW $\rho \cdot dt$

3) Consolidation :

$$d(S_t^d X_t) = X_t S_t^d(\mu_X\, dt + \sigma_X\, dW_t^x) + X_t S_t^d(\mu_S\, dt + \sigma_S\, dW_t^f) + \rho\, \sigma_S S_t^d\, \sigma_X X_t\, dt$$

$$d(S_t^d X_t) = X_t S_t^d\Big[\underbrace{(\mu_X + \mu_S + \rho\, \sigma_S \sigma_X)\, dt}_{\text{drift}} + \underbrace{\sigma_X\, dW_t^x + \sigma_S\, dW_t^S}_{\text{aléatoire}}\Big]$$

le modèle impose un drift de r_d sous la mesure-risque neutre.

4) Passage sous Q :

$$d(S_t^d X_t) = X_t S_t^d \left[\left(\underbrace{r_d\, dt}_{\text{drift sous } Q} + \underbrace{\sigma_X\, dW_t^X + \sigma_S\, dW_t^S}_{\text{aléatoire}} \right) \right]$$

Donc on a : $r_d = \underbrace{\mu_X}_{\text{On sait que } \mu_X = (r_d - r_f)} + \mu_S + \rho \sigma_S \sigma_X$

$$r_d = (r_d - r_f) + \mu_S + \rho \sigma_S \sigma_X$$

$$-\mu_S = \cancel{r_d} - r_f \cancel{- r_d} + \rho \sigma_S \sigma_X$$

$$-\mu_S = -r_f + \rho \sigma_S \sigma_X$$

$$\mu_S = r_f - \rho \sigma_S \sigma_X$$