# Love of Variety*

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**Abstract**

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# 1 Introduction

### Literature review

# 2 Theoretical framework

## 2.1 Economic environment

**Preferences.** Consider an economy populated by an infinitely-lived representative household of measure $N_t$ with logarithmic preferences over consumption:

$$U = \int_0^\infty e^{-(\rho-n)t} \ln(c_t) \mathrm{d}t \quad \text{where} \quad c_t \equiv \frac{C_t}{N_t} \tag{1}$$

and where $\rho > 0$ is the rate of time preference. In particular, aggregate consumption $C_t$ is a Cobb-Douglas aggregate of sectoral consumption bundles from $S \in \mathbb{N}$ sectors indexed by $s \in \{1, \ldots, S\}$:

$$C_t = \prod_{s=1}^{S} C_{st}^{\beta_s} \quad \text{where} \quad \sum_{s=1}^{S} \beta_s = 1. \tag{2}$$

Within each sector, the household consumes a bundle of products indexed by $i$:

$$C_{st} = \left( M_{st}^{-1/\theta} \int_0^{M_{st}} C_{ist}^{\frac{\theta-1}{\theta}} \mathrm{d}i \right)^{\frac{\theta}{\theta-1}} \tag{3}$$

where $C_{ist}$ is the consumed quantity of product $i$ from sector $s$, $M_{st}$ is the measure of products in that sector and $\theta > 1$ is the elasticity of substitution between those products. The representative household inelastically supplies $L_t^P \equiv \mathcal{S}^P \cdot N_t$ units of production labor and $L_t^E \equiv \mathcal{S}^E \cdot N_t$ units of entry labor where $\mathcal{S}^P + \mathcal{S}^E = 1$ and $\mathcal{S}^P, \mathcal{S}^E > 0$.

**Production technology.** Each sector $s$ is composed of $M_{st}$ firms producing a single product $i$ using production labor $l_{ist}$ and a bundle $x_{ist}$ of products from other sectors:

$$\begin{aligned} y_{ist} = x_{ist} l_{ist}^{\alpha_s} \quad \text{where} \quad x_{ist} = \prod_{k=1}^{S} x_{ist}(k)^{\omega_{sk}} \\ \text{and} \quad x_{ist}(k) = \left( M_{kt}^{\psi_{sk}-1/\theta} \int_0^{M_{kt}} x_{ist}(j,k)^{\frac{\theta-1}{\theta}} \mathrm{d}j \right)^{\frac{\theta}{\theta-1}}. \end{aligned} \tag{4}$$

Here, $y_{ist}$ is the output of firm $i$ from sector $s$ and $x_{ist}(j,k)$ is the quantity of product $j$ from sector $k$ demanded by firm $i$ from sector $s$. The parameter $\alpha_s \in (0,1)$ is the output elasticity of production labor in sector $s$ and $\omega_{sk}$ measures the importance of intermediate inputs from sector $k$ in the production technology of sector $s$, which are collected in the matrix $\mathbf{\Omega}$. In particular, we have:

$$\sum_{k=1}^{S} \omega_{sk} = 1 - \alpha_s.$$

Finally, $\psi_{sk} > 0$ measures the strength of the taste for variety by sector $s$'s firms for products from sector $k$, which are collected in the matrix $\mathbf{\Psi}$. This formulation developed by Benassy (1996) isolates the "taste for variety" from the elasticity of substitution between products.

**Entry technology.** In every point in time, a unit measure of potential entrants in each sector attempt to introduce new products. Specifically, these entrants can direct $1/\epsilon_s$ units of labor to entry in order to create a unit flow of these new products. Hence, the evolution of the measure of products in sector $s$ is given by:

$$\dot{M}_{st} = \epsilon_s L^E_{st}. \tag{5}$$

**Resource constraints.** The resource constraints for the products of each sector are given by:

$$C_{ist} + \sum_{k=1}^{S} \int_0^{M_{kt}} x_{jkt}(i,s)\mathrm{d}j \leq y_{ist} \quad \forall i \in M_{st} \quad \forall s \in \{1,\ldots,S\}, \tag{6}$$

the resource constraint for production labor is given by:

$$\sum_{s=1}^{S} \int_0^{M_{st}} l_{ist}\mathrm{d}i \leq \mathcal{S}^P N_t, \tag{7}$$

and the resource constraint for entry labor is given by:

$$\sum_{s=1}^{S} L^E_{st} \leq \mathcal{S}^E N_t. \tag{8}$$

Finally, the population grows at constant rate $n > 0$:

$$\dot{N}_t = n \cdot N_t. \tag{9}$$

The economic environment is summarized in Table 1.

Table 1: The economic environment

| | | |
|---|---|---|
| (1) | $U = \int_0^\infty e^{-(\rho-n)t} \ln(C_t/N_t) \mathrm{d}t$ | Lifetime utility |
| (2) | $C_t = \prod_{s=1}^S C_{st}^{\beta_s}$ | Aggregate consumption |
| (3) | $C_{st} = (M_{st}^{-1/\theta} \int_0^{M_{st}} C_{ist}^{\frac{\theta-1}{\theta}} \mathrm{d}i)^{\frac{\theta}{\theta-1}}$ | Sectoral consumption |
| (4) | $y_{ist} = \prod_{k=1}^S (M_{kt}^{\psi_{sk}-1/\theta} \int_0^{M_{kt}} x_{ist}(j,k)^{\frac{\theta-1}{\theta}} \mathrm{d}j)^{\frac{\omega_{sk}\theta}{\theta-1}} l_{ist}^{P\ \alpha_s}$ | Production technology |
| (5) | $\dot{M}_{st} = \epsilon_s L_{st}^E$ | Entry technology |
| (6) | $C_{ist} + \sum_{k=1}^S \int_0^{M_{kt}} x_{jkt}(i,s) \mathrm{d}j \leq y_{ist}$ | Product resources |
| (7) | $\sum_{s=1}^S \int_0^{M_{st}} \ell_{ist} \mathrm{d}i \leq \mathcal{S}^P N_t$ | Production labor resources |
| (8) | $\sum_{s=1}^S L_{st}^E \leq \mathcal{S}^E N_t$ | Entry labor resources |
| (9) | $\dot{N}_t = n \cdot N_t$ | Population growth |

## 2.2 The market equilibrium allocation

## 2.3 The optimal allocation

## References

**Benassy, Jean-Pascal**, "Taste for Variety and Optimum Production Patterns in Monopolistic Competition," *Economics Letters*, 1996, *52* (1), 41–47.

# Appendix

# A Theoretical appendix

## A.1 The optimal allocation

By symmetry across firms within sectors, we have:

$$C_{ist} = \frac{C_{st}}{M_{st}}, \quad l_{ist} = \frac{L^P_{st}}{M_{st}}, \quad \text{and} \quad x_{ist}(j,k) = \frac{x_{st}(k)}{M_{st}M_{kt}}.$$

Substituting these into the equations of Table 1, we have:

$$C_t = \prod_{s=1}^{S} C_{st}^{\beta_s} \quad \text{where} \quad C_{st} = \prod_{k=1}^{S} \left(x_{st}(k) M_{kt}^{\psi_{sk}}\right)^{\omega_{sk}} {L^P_{st}}^{\alpha_s} - \sum_{k=1}^{S} x_{kt}(s).$$

Therefore, the planner's static allocation problem is equivalent to:

$$\begin{aligned}
&\max_{C_{st}, x_{st}(k), L^P_{st}} \sum_{s=1}^{S} \beta_s \ln(C_{st}) \\
&\text{s.t.} \quad C_{st} = \prod_{k=1}^{S} \left(x_{st}(k) M_{kt}^{\psi_{sk}}\right)^{\omega_{sk}} {L^P_{st}}^{\alpha_s} - \sum_{k=1}^{S} x_{kt}(s) \\
&\text{s.t.} \quad \sum_{s=1}^{S} L^P_{st} = \mathcal{S}^P N_t.
\end{aligned}$$

The first-order conditions of this problem imply that:

$$x_{st}(k) = \frac{\beta_s \omega_{sk} Y_{st} C_{kt}}{\beta_k C_{st}} \quad \text{and} \quad \frac{\alpha_s Y_{st}}{\lambda_t L^P_{st}} = \frac{C_{st}}{\beta_s}$$

where $\lambda_t$ denotes the Lagrange multiplier on the production labor resource constraint and aggregate output in sector $s$ is defined as:

$$Y_{st} \equiv \prod_{k=1}^{S} \left(x_{st}(k) M_{kt}^{\psi_{sk}}\right)^{\omega_{sk}} {L^P_{st}}^{\alpha_s}. \tag{A.1}$$

Using the problem's optimality conditions, we obtain an expression for the quantity of intermediate inputs from sector $k$ used by sector $s$:

$$x_{st}(k) = \omega_{sk} Y_{kt} \cdot \frac{\alpha_k L^P_{st}}{\alpha_s L^P_{kt}}. \tag{A.2}$$

Substituting this and the optimality condition for the production labor choice into the product resource constraint of sector $s$, and rearranging, we have:

$$\frac{\beta_s}{\lambda_t} = \frac{L^P_{st}}{\alpha_s} - \sum_{k=1}^{S} \frac{\omega_{ks} L^P_{kt}}{\alpha_k}.$$

This expression can be rewritten in vector notation as:

$$\lambda_t \mathbf{L}^P_t = \text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}.$$

The multiplier $\lambda_t$ can be solved for by using the production labor resource constraint:

$$\lambda_t L^P = \mathbf{1}^\top \text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}$$

Therefore, the share of production labor allocated to each sector is:

$$\frac{\mathbf{L}^P_t}{L^P_t} = \frac{\text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}{\mathbf{1}^\top \text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}.$$

Substituting equation (A.2) in equation (A.1), we obtain an expression for the marginal product of production labor in sector $s$ as a function of the marginal product of production labor of its suppliers:

$$\frac{\alpha_s Y_{st}}{L^P_{st}} = \alpha_s^{\alpha_s} \prod_{k=1}^{S} \left( \omega_{sk} M_{kt}^{\psi_{sk}} \cdot \frac{\alpha_k Y_{kt}}{L^P_{kt}} \right)^{\omega_{sk}}.$$

Denoting these marginal products by $Y^L_{st}$ and taking logarithms, we can rewrite:

$$\ln(Y^L_{st}) = \alpha_s \ln(\alpha_s) + \sum_{k=1}^{S} \omega_{sk} \ln(\omega_{sk}) + \sum_{k=1}^{S} \omega_{sk}\psi_{sk} \ln(M_{kt}) + \sum_{k=1}^{S} \omega_{sk} \ln(Y^L_{kt}).$$

Defining $\hat{\mathbf{X}}$ as the element-wise logarithm of an arbitrary vector or matrix $\mathbf{X}$, we can rewrite in log-vector notation:

$$\hat{\mathbf{Y}}^L_t = (\mathbf{I} - \boldsymbol{\Omega})^{-1}[\text{diag}(\boldsymbol{\alpha})\hat{\boldsymbol{\alpha}} + (\boldsymbol{\Omega} \circ \hat{\boldsymbol{\Omega}})\mathbf{1} + (\boldsymbol{\Omega} \circ \Psi)\hat{\mathbf{M}}_t]$$

where $\circ$ denotes the Hadamard product. Using the definition of $Y^L_{st}$, we have:

$$\hat{\mathbf{Y}}_t = (\mathbf{I} - \boldsymbol{\Omega})^{-1}[\text{diag}(\boldsymbol{\alpha})\hat{\boldsymbol{\alpha}} + (\boldsymbol{\Omega} \circ \hat{\boldsymbol{\Omega}})\mathbf{1} + (\boldsymbol{\Omega} \circ \Psi)\hat{\mathbf{M}}_t] + \hat{\mathbf{L}}^P_t - \hat{\boldsymbol{\alpha}}.$$

Substituting equation (A.2) in the sectoral resource constraints, we have:

$$C_{st} = Y_{st}\left(1 - \frac{\alpha_s}{L^P_{st}} \times \sum_{k=1}^{S} \frac{\omega_{ks} L^P_{kt}}{\alpha_k}\right).$$

In log-vector notation, we can analogously rewrite:

$$\hat{\mathbf{C}}_t = \hat{\mathbf{Y}}_t + \hat{\boldsymbol{\Gamma}}_t \quad \text{where} \quad \boldsymbol{\Gamma}_t \equiv \mathbf{1} - \text{diag}(\boldsymbol{\alpha})\text{diag}(\mathbf{L}^P_t)^{-1}\boldsymbol{\Omega}^\top \text{diag}(\boldsymbol{\alpha})^{-1}\mathbf{L}^P_t.$$

Therefore, the current-value Hamiltonian that corresponds to the planner's dynamic allocation problem is:

$$\mathcal{H}_t = \boldsymbol{\beta}^\top(\hat{\mathbf{Y}}_t + \hat{\boldsymbol{\Gamma}}_t) - \ln(N_t) + \sum_{s=1}^{S-1} \mu_{st}\epsilon_s L^E_{st} + \mu_{St}\left(L^E_t - \sum_{s=1}^{S-1} L^E_{st}\right)$$

where $\mu_{st}$ are costate variables. The optimality conditions of this problem are:

$$\begin{aligned}
\frac{\partial \mathcal{H}_t}{\partial L^E_{st}} &= \mu_{st}\epsilon_s - \mu_{St} = 0, \\
\frac{\partial \mathcal{H}_t}{\partial M_{st}} &= \frac{\boldsymbol{\beta}^\top(\mathbf{I} - \boldsymbol{\Omega})^{-1}(\boldsymbol{\Omega} \circ \boldsymbol{\Psi})\mathbf{e}_s}{M_{st}} = (\rho - n)\mu_{st} - \dot{\mu}_{st}.
\end{aligned}$$

On a balanced growth path, the measure of varieties within any sector must grow at rate $n$, which implies:

$$M_{st} = \frac{\epsilon_s L^E_{st}}{n}.$$

Using this result and the problem's optimality conditions, and rearranging, we have:

$$(\rho/n)\mu_{St} L^E_{st} = \boldsymbol{\beta}^\top(\mathbf{I} - \boldsymbol{\Omega})^{-1}(\boldsymbol{\Omega} \circ \boldsymbol{\Psi})\mathbf{e}_s.$$

The costate variable $\mu_{St}$ is solved for by using the research labor resource constraint:

$$(\rho/n)\mu_{St} L^E_t = \mathbf{1}^\top\boldsymbol{\beta}^\top(\mathbf{I} - \boldsymbol{\Omega})^{-1}(\boldsymbol{\Omega} \circ \boldsymbol{\Psi})\mathbf{e}_s.$$

Therefore, the share of entry labor allocated to each sector is:

$$\frac{\mathbf{L}^E_t}{L^E_t} = \frac{\boldsymbol{\beta}^\top(\mathbf{I} - \boldsymbol{\Omega})^{-1}(\boldsymbol{\Omega} \circ \boldsymbol{\Psi})}{\mathbf{1}^\top\boldsymbol{\beta}^\top(\mathbf{I} - \boldsymbol{\Omega})^{-1}(\boldsymbol{\Omega} \circ \boldsymbol{\Psi})}.$$

## A.2 The market equilibrium allocation

### A.2.1 The Household's Problem

Taking prices and the measures of varieties as given, the representative household's problem is to choose its consumption $C_{ist}$ of each variety to maximize lifetime utility:

$$U = \max_{C_{ist}} \int_0^\infty e^{-(\rho-n)t} \ln(C_t) \mathrm{d}t \quad \text{where} \quad C_t = \prod_{s=1}^{S} \left( \int_0^{A_{st}} C_{ist}^{\frac{\theta_s-1}{\theta_s}} \mathrm{d}i \right)^{\frac{\beta_s \theta_s}{\theta_s-1}}$$

subject to the flow budget constraint:

$$\dot{B}_t = r_t B_t + w_t^L L_t + w_t^R R_t - \sum_{s=1}^{S} \int_0^{A_{st}} p_{ist} C_{ist} \mathrm{d}i - \sum_{s=1}^{S} \int_0^{A_{st}} \tau_{st}^Y p_{ist} y_{ist} \mathrm{d}i$$

where $B_t$ is financial wealth at time $t$, $w_t^L$ and $w_t^R$ are the wages paid to workers and researchers, respectively, $p_{ist}$ and $y_{ist}$ are the price and quantity produced of variety $i$ from sector $s$, respectively, and $\tau_{st}^Y$ is the revenue subsidy to that firm. Choosing aggregate consumption as the numéraire, this problem delivers the usual Euler equation and final demand functions:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho + n \quad \text{and} \quad C_{ist} = \beta_s C_t P_{st}^{\theta_s - 1} / p_{ist}^{\theta_s} \quad \forall i \in [0, A_{st}], \forall s \in \{1, \ldots, S\}$$

where $c_t \equiv C_t / N_t$ and $P_{st}$ is the ideal price index of final consumption from sector $s$:

$$P_{st} \equiv \left( \int_0^{A_{st}} p_{ist}^{1-\theta_s} \mathrm{d}i \right)^{\frac{1}{1-\theta_s}} \quad \forall s \in \{1, \ldots, S\}.$$

Similarly, the ideal price index of aggregate consumption, denoted as $P_t$, is normalized to unity and is defined as:

$$P_t \equiv \prod_{s=1}^{S} \left( \frac{P_{st}}{\beta_s} \right)^{\beta_s}.$$

### A.2.2 The Firm's Problem

To hold a claim on a variety's perpetual profits, a firm must first purchase its patent from the R&D industry in its sector through free-entry. Once that firm holds a patent, it engages in monopolistic competition on the output market and perfect competition on the markets for inputs. That is, it chooses a price as well as intermediate inputs and production labor to maximize profits $\pi_{ist}$ while taking as given the demand for its variety, the price $p_{jkt}$ of intermediate inputs, the wage paid to production workers $w_t^L$ and the revenue subsidy $\tau_{st}^Y$:

$$\pi_{ist} = \max_{p_{ist}, x_{ist}(j,k), \ell_{ist}} \{(1+\tau_{st}^Y) p_{ist} y_{ist} - \sum_{k=1}^{S} \int_0^{A_{kt}} p_{jkt} x_{ist}(j,k) \mathrm{d}j - w_t^L \ell_{ist}\}.$$

This can be broken down into several sub-problems. The first being the following cost minimization problem for each supplying sector $k$:

$$\min_{x_{ist}(j,k)} \int_0^{A_{kt}} p_{jkt} x_{ist}(j,k) \mathrm{d}j \quad \text{s.t.} \quad \left( \int_0^{A_{kt}} x_{ist}(j,k)^{\frac{\theta_k - 1}{\theta_k}} \mathrm{d}j \right)^{\frac{\theta_k}{\theta_k - 1}} \geq x_{ist}(k).$$

The first-order conditions deliver the following demand functions:

$$x_{ist}(j,k) = (P_{kt} / p_{jkt})^{\theta_k} x_{ist}(k).$$

The second sub-problem is the following cost minimization problem:

$$\min_{x_{ist}(k)} \sum_{k=1}^{S} P_{kt} x_{ist}(k) \quad \text{s.t.} \quad \prod_{k=1}^{S} x_{ist}(k)^{\omega_{sk}} \geq x_{ist}.$$

The first-order conditions deliver the following demand functions:

$$x_{ist}(k) = \omega_{sk} (P_{st}^x x_{ist})^{\frac{1}{1-\alpha_s}} / P_{kt} \quad \text{where} \quad P_{st}^x \equiv \prod_{k=1}^{S} \left( \frac{P_{kt}}{\omega_{sk}} \right)^{\omega_{sk}}.$$

The third sub-problem is the following cost minimization problem:

$$\min_{x_{ist}, \ell_{ist}} \{(1-\alpha_s)(P_{st}^x x_{ist})^{\frac{1}{1-\alpha_s}} + w_t^L \ell_{ist}\} \quad \text{s.t.} \quad x_{ist} \ell_{ist}^{\alpha_s} \geq y_{ist}.$$

The first-order conditions deliver the following demand functions:

$$x_{ist} = (m_{st} y_{ist})^{1-\alpha_s} / P_{st}^x \quad \text{and} \quad \ell_{ist} = \alpha_s m_{st} y_{ist} / w_t^L$$

where $m_{st}$ denotes the marginal cost of firms in sector $s$:

$$m_{st} \equiv P_{st}^x \left( \frac{w_t^L}{\alpha_s} \right)^{\alpha_s}.$$

The last sub-problem is the following profit maximization problem:

$$\max_{p_{ist}} \{ [(1+\tau_{st}^{Y})p_{ist} - m_{st}](P_{st}^{\theta_s - 1}/p_{ist}^{\theta_s})[\beta_s C_t + \sum_{k=1}^{S} \omega_{ks} m_{kt} Y_{kt}] \}.$$

The first-order condition implies that firms set their price to a constant markup $\mu_s$ above marginal cost, net of the revenue subsidy:

$$p_{ist} = \frac{\mu_s m_{st}}{1+\tau_{st}^{Y}} \quad \text{where} \quad \mu_s \equiv \frac{\theta_s}{\theta_s - 1}.$$

Therefore, we can rewrite the different price indices as:

$$P_{st} = \frac{\mu_s m_{st} A_{st}^{\frac{1}{1-\theta_s}}}{1+\tau_{st}^{Y}} \quad \text{and} \quad P_{st}^{x} = \prod_{k=1}^{S} \left[ \frac{\mu_k m_{kt} A_{kt}^{\frac{1}{1-\theta_k}}}{\omega_{sk}(1+\tau_{kt}^{Y})} \right]^{\omega_{sk}}.$$

Therefore, firm profits in sector $s$ are given by:

$$\pi_{ist} = [(1+\tau_{st}^{Y})p_{ist} - m_{st}]y_{ist} = \frac{(1+\tau_{st}^{Y})p_{ist}y_{ist}}{\theta_s} = \frac{m_{st}y_{ist}}{\theta_s - 1}.$$

Substituting the demand function for intermediate inputs in the firm's production function, we have:

$$y_{ist} = \frac{(m_{st}y_{ist})^{1-\alpha_s}\ell_{ist}^{\alpha_s}}{P_{st}^{x}}.$$

Solving for the output of firm $i$ in sector $s$, we have:

$$y_{ist} = \left( \frac{m_{st}^{1-\alpha_s}}{P_{st}^{x}} \right)^{\frac{1}{\alpha_s}} \ell_{ist}.$$

Using the symmetry of production labor demand functions as well as the definition of the firm's marginal cost, we can rewrite:

$$y_{ist} = \frac{(w_t^{L}/\alpha_s)^{1-\alpha_s} L_{st}}{A_{st} P_{st}^{x}}.$$

Therefore, firm profits in sector $s$ are given by:

$$\pi_{ist} = \frac{w_t^L L_{st}}{\alpha_s(\theta_s - 1)A_{st}} \quad \forall i \in [0, A_{st}], \forall s \in \{1, \dots, S\}.$$

### A.2.3 The Research Industry's Problem

The research industry of each sector is assumed to be monopolistically competitive on the market for patents and perfectly competitive on the research labor market. Hence, it chooses a patent price $q_{st}$ as well as research labor to maximize profits, while taking as given the wage $w_t^R$ paid to researchers and the sector-specific R&D taxes/subsidies $\tau_{st}^R$ set by the government:

$$\max_{q_{st}, R_{st}} \{q_{st}\eta_s R_{st} - (1 - \tau_{st}^R) w_t^R R_{st}\} \quad \forall s \in \{1, \dots, S\}.$$

With free-entry among patent buyers, the research industry sets the price of a patent as to extract all possible rents from the commercialization of an invention, which corresponds to the present value of a variety's stream of future profits. Together with perfect competition on the research labor market, this implies:

$$(1 - \tau_{st}^R) w_t^R = \eta_s q_{st} \quad \text{where} \quad q_{st} = \int_t^\infty e^{-\int_t^{t'} r_\tau \mathrm{d}\tau} \pi_{st'} \mathrm{d}t' \quad \forall s \in \{1, \dots, S\}.$$

In particular, R&D taxes/subsidies are budget neutral, which implies:

$$\sum_{s=1}^{S} \tau_{st}^R w_t^R R_{st} = 0.$$

Differentiating $q_{st}$ with respect to time, we obtain the law of motion for the price of a patent in sector $s$:

$$\dot{q}_{st} = r_t q_{st} - \frac{w_t^L L_{st}}{\alpha_s(\theta_s - 1)A_{st}} \quad \forall s \in \{1, \dots, S\}.$$

### A.2.4 Balanced Growth Path Solution

The asset market clearing condition is:

$$B_t = \sum_{s=1}^{S} q_{st} A_{st}.$$

Differentiating with respect to time delivers:

$$\dot{B}_t = \sum_{s=1}^{S} \dot{q}_{st} A_{st} + \sum_{s=1}^{S} q_{st} \dot{A}_{st} = r_t B_t - w_t^L \sum_{s=1}^{S} \frac{L_{st}}{\alpha_s(\theta_s - 1)} + w_t^R R_t.$$

Combining this result with the household's flow budget constraint, we have:

$$C_t = w_t^L L_t + w_t^L \sum_{s=1}^{S} \frac{L_{st}}{\alpha_s(\theta_s - 1)} - \sum_{s=1}^{S} \int_0^{A_{st}} \tau_{st}^Y p_{ist} y_{ist} \mathrm{d}i.$$

Substituting in the expression for firm revenues, we have:

$$C_t = w_t^L L_t + w_t^L \sum_{s=1}^{S} \frac{\mu_s[1 - (\theta_s - 1)\tau_{st}^Y] L_{st}}{\alpha_s \theta_s (1 + \tau_{st}^Y)}.$$

Taking the logarithm of the ideal price index of aggregate consumption, we have:

$$\sum_{s=1}^{S} \beta_s [\ln(P_{st}) - \ln(\beta_s)] = 0.$$

Substituting in the expressions for $P_{st}$ and using log-vector notation, we have:

$$\boldsymbol{\beta}^\top \{\hat{\boldsymbol{\mu}}_t' + \hat{\mathbf{P}}_t^x + \mathrm{diag}(\boldsymbol{\alpha})[\ln(w_t^L)\mathbf{1} - \hat{\boldsymbol{\alpha}}] - \mathrm{diag}(\boldsymbol{\theta} - 1)^{-1} \hat{\mathbf{A}}_t - \hat{\boldsymbol{\beta}}\} = 0$$

where element $s$ of the $S \times 1$ vector $\hat{\boldsymbol{\mu}}_t'$ is equal to $\ln(\mu_s) - \ln(1 + \tau_{st}^Y)$. Using the definition of $P_{st}^x$ together with log-vector notation, we have:

$$\hat{\mathbf{P}}_t^x = (\mathbf{I} - \boldsymbol{\Omega})^{-1} \{\boldsymbol{\Omega}[\hat{\boldsymbol{\mu}}_t' + \mathrm{diag}(\boldsymbol{\alpha})[\ln(w_t^L)\mathbf{1} - \hat{\boldsymbol{\alpha}}] - \mathrm{diag}(\boldsymbol{\theta} - 1)^{-1} \hat{\mathbf{A}}_t] - \mathrm{diag}(\boldsymbol{\Omega}\hat{\boldsymbol{\Omega}}^\top)\}.$$

Substituting this expression in the previous equation delivers an expression for the logarithm of the wage paid to workers:

$$\ln(w_t^L) = \frac{\boldsymbol{\Lambda}^\top [\mathrm{diag}(\boldsymbol{\alpha})\hat{\boldsymbol{\alpha}} + \mathrm{diag}(\boldsymbol{\theta} - 1)^{-1} \hat{\mathbf{A}}_t] + \boldsymbol{\beta}^\top [\mathrm{diag}(\boldsymbol{\Omega}\hat{\boldsymbol{\Omega}}^\top) + \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\mu}}_t']}{\boldsymbol{\Lambda}^\top \boldsymbol{\alpha}}$$

where $\boldsymbol{\Lambda}$ is defined as in Appendix **??**. On a balanced growth path, it must be that the measure of varieties within each sector grows at the same rate as the population, which implies:

$$A_{st} = \frac{\eta_s R_{st}}{n} \quad \forall s \in \{1, \dots, S\}.$$

Similarly, the growth rate of the price of a patent must be equal to the growth rate of consumption per person, which implies:

$$R_{st} = \frac{n w_t^L L_{st}}{\alpha_s(\theta_s - 1)(\rho - n)\eta_s q_{st}} \quad \forall s \in \{1, \ldots, S\}.$$

Therefore, the research labor market clearing condition deliver an expression for the research labor demand in each sector:

$$R_{st} = \frac{n w_t^L L_{st}}{\alpha_s(\theta_s - 1)(\rho - n)(1 - \tau_{st}^R) w_t^R} \quad \forall s \in \{1, \ldots, S\}.$$

Imposing the research labor resource constraint, we obtain an expression for the wage paid to researchers:

$$w_t^R = \frac{n w_t^L}{(\rho - n) R_t} \times \sum_{s=1}^{S} \frac{L_{st}}{\alpha_s(\theta_s - 1)(1 - \tau_{st}^R)}.$$

Substituting this back into the previous equation, we obtain an expression for the share of research labor in each sector in vector notation:

$$\frac{\mathbf{R}_t}{R_t} = \frac{\mathrm{diag}(\mathbf{1} - \boldsymbol{\tau}^R)^{-1}\mathrm{diag}(\boldsymbol{\theta} - \mathbf{1})^{-1}\mathrm{diag}(\boldsymbol{\alpha})^{-1}\mathbf{L}_t}{\mathbf{1}^\top \mathrm{diag}(\mathbf{1} - \boldsymbol{\tau}^R)^{-1}\mathrm{diag}(\boldsymbol{\theta} - \mathbf{1})^{-1}\mathrm{diag}(\boldsymbol{\alpha})^{-1}\mathbf{L}_t}.$$

The variety resource constraints are:

$$C_{ist} = y_{ist} - \sum_{k=1}^{S} \int_0^{A_{kt}} x_{jkt}(i, s)\mathrm{d}j \quad \forall i \in [0, A_{st}], \forall s \in \{1, \ldots, S\}.$$

Multiplying both sides of this equation by $p_{ist}$ and substituting in the final and intermediate demand functions, as well as the expression for the output of firm $i$ in sector $s$, we have:

$$\beta_s C_t = \frac{\mu_s w_t^L L_{st}}{\alpha_s(1 + \tau_{st}^Y)} - \sum_{k=1}^{S} \frac{\omega_{ks} w_t^L L_{kt}}{\alpha_k} \quad \forall s \in \{1, \ldots, S\}.$$

This equation can be rewritten in vector notation as:

$$\frac{w_t^L \mathbf{L}_t}{C_t} = \mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1} + \boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}.$$

Imposing the production labor resource constraint, we obtain an expression for the payments to production labor as a share of aggregate consumption:

$$\frac{w_t^L L_t}{C_t} = \mathbf{1}^\top \mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}.$$

Substituting this back into the previous equation, we obtain an expression for the share of production labor in each sector:

$$\frac{\mathbf{L}_t}{L_t} = \frac{\mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}{\mathbf{1}^\top \mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}.$$

Substituting this back into the expression for the share of research labor in each sector, we have the following:

$$\frac{\mathbf{R}_t}{R_t} = \frac{\mathrm{diag}(\mathbf{1}-\boldsymbol{\tau}^R)^{-1}\mathrm{diag}(\boldsymbol{\theta}-\mathbf{1})^{-1}(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}{\mathbf{1}^\top \mathrm{diag}(\mathbf{1}-\boldsymbol{\tau}^R)^{-1}\mathrm{diag}(\boldsymbol{\theta}-\mathbf{1})^{-1}(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}.$$