Love of Variety*

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Abstract

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1 Introduction

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2.1 Economic environment

Preferences. Consider an economy populated by an infinitely-lived representative household of measure N_t with logarithmic preferences over consumption:

$$U = \int_0^\infty e^{-(\rho - n)t} \ln(c_t) \mathrm{d}t \quad \text{where} \quad c_t \equiv \frac{C_t}{N_t} \tag{1}$$

and where $\rho > 0$ is the rate of time preference. In particular, aggregate consumption C_t is a Cobb-Douglas aggregate of sectoral consumption bundles from $S \in \mathbb{N}$ sectors indexed by $s \in \{1, \ldots, S\}$:

$$C_t = \prod_{s=1}^{S} C_{st}^{\beta_s} \quad \text{where} \quad \sum_{s=1}^{S} \beta_s = 1. \tag{2}$$

Within each sector, the household consumes a bundle of products indexed by i:

$$C_{st} = \left(M_{st}^{-1/\theta} \int_0^{M_{st}} C_{ist}^{\frac{\theta-1}{\theta}} \mathrm{d}i \right)^{\frac{\theta}{\theta-1}} \tag{3}$$

where C_{ist} is the consumed quantity of product i from sector s, M_{st} is the measure of products in that sector and $\theta > 1$ is the elasticity of substitution between those products. The representative household inelastically supplies $L_t^P \equiv \mathcal{S}^P \cdot N_t$ units of production labor and $L_t^E \equiv \mathcal{S}^E \cdot N_t$ units of entry labor where $\mathcal{S}^P + \mathcal{S}^E = 1$ and $\mathcal{S}^P, \mathcal{S}^E > 0$.

Production technology. Each sector s is composed of M_{st} firms producing a single product i using production labor l_{ist} and a bundle x_{ist} of products from other sectors:

$$\begin{aligned} y_{ist} = x_{ist} l_{ist}^{\alpha_s} \quad \text{where} \quad x_{ist} = \prod_{k=1}^{S} x_{ist}(k)^{\omega_{sk}} \\ \text{and} \quad x_{ist}(k) = M_{kt}^{\psi_{sk}} \left(M_{kt}^{-1/\theta} \int_0^{M_{kt}} x_{ist}(j,k)^{\frac{\theta-1}{\theta}} \mathrm{d}j \right)^{\frac{\theta}{\theta-1}}. \end{aligned} \tag{4}$$

Here, y_{ist} is the output of firm i from sector s and $x_{ist}(j,k)$ is the quantity of product j from sector k demanded by firm i from sector s. The parameter $\alpha_s \in (0,1)$ is the output elasticity of production labor in sector s and ω_{sk} measures the importance of intermediate inputs from sector k in the production technology of sector s, which are collected in the matrix $\mathbf{\Omega}$. In particular, we have:

$$\sum_{k=1}^{S} \omega_{sk} = 1 - \alpha_s.$$

Finally, $\psi_{sk} > 0$ measures the strength of the taste for variety by sector s's firms for products from sector k, which are collected in the matrix $\mathbf{\Psi}$. This formulation developed by Benassy (1996) isolates the "taste for variety" from the elasticity of substitution between products.

Entry technology. In every point in time, a unit measure of potential entrants in each sector attempt to introduce new products. Specifically, these entrants can direct $1/\epsilon_s$ units of labor to entry in order to create a unit flow of these new products. Hence, the evolution of the measure of products in sector s is given by:

$$\dot{M}_{st} = \epsilon_s L_{st}^E. \tag{5}$$

Resource constraints. The resource constraints for the products of each sector are given by:

$$C_{ist} + \sum_{k=1}^{S} \int_0^{M_{kt}} x_{jkt}(i,s)\mathrm{d}j \leq y_{ist} \quad \forall i \in M_{st} \quad \forall s \in \{1,\ldots,S\}, \tag{6}$$

the resource constraint for production labor is given by:

$$\sum_{s=1}^{S} \int_0^{M_{st}} l_{ist}\mathrm{d}i \leq \mathcal{S}^P N_t, \tag{7}$$

and the resource constraint for entry labor is given by:

$$\sum_{s=1}^{S} L_{st}^E \leq \mathcal{S}^E N_t. \tag{8}$$

Finally, the population grows at constant rate $n > 0$:

$$\dot{N}_t = n \cdot N_t. \tag{9}$$

The economic environment is summarized in Table 1.

Table 1: The economic environment

(1)	$U = \int_0^\infty e^{-(\rho-n)t} \ln(C_t/N_t) \mathrm{d}t$	Lifetime utility
(2)	$C_t = \prod_{s=1}^S C_{st}^{\beta_s}$	Aggregate consumption
(3)	$C_{st} = (M_{st}^{-1/\theta} \int_0^{M_{st}} C_{ist}^{\frac{\theta-1}{\theta}} \mathrm{d}i)^{\frac{\theta}{\theta-1}}$	Sectoral consumption
(4)	$y_{ist} = \prod_{k=1}^S M_{kt}^{\omega_{sk}\psi_{sk}} (M_{kt}^{-1/\theta} \int_0^{M_{kt}} x_{ist}(j,k)^{\frac{\theta-1}{\theta}} \mathrm{d}j)^{\frac{\omega_{sk}\theta}{\theta-1}} l_{ist}^{P\ \alpha_s}$	Production technology
(5)	$\dot{M}_{st} = \epsilon_s L_{st}^E$	Entry technology
(6)	$C_{ist} + \sum_{k=1}^S \int_0^{M_{kt}} x_{jkt}(i,s) \mathrm{d}j \leq y_{ist}$	Product resources
(7)	$\sum_{s=1}^S \int_0^{M_{st}} \ell_{ist} \mathrm{d}i \leq \mathcal{S}^P N_t$	Production labor resources
(8)	$\sum_{s=1}^S L_{st}^E \leq \mathcal{S}^E N_t$	Entry labor resources
(9)	$\dot{N}_t = n \cdot N_t$	Population growth

2.2 The market equilibrium allocation

2.3 The optimal allocation

References

Benassy, Jean-Pascal, "Taste for Variety and Optimum Production Patterns in Monopolistic Competition," *Economics Letters*, 1996, *52* (1), 41–47.

Appendix

A Theoretical appendix

A.1 The optimal allocation

By symmetry across firms within sectors, we have:

$$C_{ist} = \frac{C_{st}}{M_{st}}, \quad l_{ist} = \frac{L^P_{st}}{M_{st}}, \quad \text{and} \quad x_{ist}(j,k) = \frac{x_{st}(k)}{M_{st}M_{kt}}.$$

Substituting these into the equations of Table 1, we have:

$$C_t = \prod_{s=1}^{S} C_{st}^{\beta_s} \quad \text{where} \quad C_{st} = \prod_{k=1}^{S} \left(x_{st}(k) M_{kt}^{\psi_{sk}} \right)^{\omega_{sk}} {L^P_{st}}^{\alpha_s} - \sum_{k=1}^{S} x_{kt}(s).$$

Therefore, the planner's static allocation problem is equivalent to:

$$\begin{aligned}
&\max_{C_{st}, x_{st}(k), L^P_{st}} \sum_{s=1}^{S} \beta_s \ln(C_{st}) \\
&\text{s.t.} \quad C_{st} = \prod_{k=1}^{S} \left(x_{st}(k) M_{kt}^{\psi_{sk}} \right)^{\omega_{sk}} {L^P_{st}}^{\alpha_s} - \sum_{k=1}^{S} x_{kt}(s) \\
&\text{s.t.} \quad \sum_{s=1}^{S} L^P_{st} = \mathcal{S}^P N_t.
\end{aligned}$$

The first-order conditions of this problem imply that:

$$x_{st}(k) = \frac{\beta_s \omega_{sk} Y_{st} C_{kt}}{\beta_k C_{st}} \quad \text{and} \quad \frac{\alpha_s Y_{st}}{\lambda_t L^P_{st}} = \frac{C_{st}}{\beta_s}$$

where λ_t denotes the Lagrange multiplier on the production labor resource constraint and aggregate output in sector s is defined as:

$$Y_{st} \equiv \prod_{k=1}^{S} \left(x_{st}(k) M_{kt}^{\psi_{sk}} \right)^{\omega_{sk}} {L^P_{st}}^{\alpha_s}. \tag{A.1}$$

Using the problem's optimality conditions, we obtain an expression for the quantity of intermediate inputs from sector k used by sector s:

$$x_{st}(k) = \omega_{sk} Y_{kt} \cdot \frac{\alpha_k L^P_{st}}{\alpha_s L^P_{kt}}. \tag{A.2}$$

Substituting this and the optimality condition for the production labor choice into the product resource constraint of sector s, and rearranging, we have:

$$\frac{\beta_s}{\lambda_t} = \frac{L^P_{st}}{\alpha_s} - \sum_{k=1}^{S} \frac{\omega_{ks} L^P_{kt}}{\alpha_k}.$$

This expression can be rewritten in vector notation as:

$$\lambda_t \mathbf{L}^P_t = \text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1} \boldsymbol{\beta}.$$

The multiplier λ_t can be solved for by using the production labor resource constraint:

$$\lambda_t L^P = \mathbf{1}^\top \text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1} \boldsymbol{\beta}$$

Therefore, the share of production labor allocated to each sector is:

$$\frac{\mathbf{L}^P_t}{L^P_t} = \frac{\text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1} \boldsymbol{\beta}}{\mathbf{1}^\top \text{diag}(\boldsymbol{\alpha})(\mathbf{I} - \boldsymbol{\Omega}^\top)^{-1} \boldsymbol{\beta}}.$$

Substituting equation (A.2) in equation (A.1), we obtain an expression for the marginal product of production labor in sector s as a function of the marginal product of production labor of its suppliers:

$$\frac{\alpha_s Y_{st}}{L^P_{st}} = \alpha_s^{\alpha_s} \prod_{k=1}^{S} \left(\omega_{sk} M_{kt}^{\psi_{sk}} \cdot \frac{\alpha_k Y_{kt}}{L^P_{kt}} \right)^{\omega_{sk}}.$$

Denoting these marginal products by Y^L_{st} and taking logarithms, we can rewrite:

$$\ln(Y^L_{st}) = \alpha_s \ln(\alpha_s) + \sum_{k=1}^{S} \omega_{sk} \ln(\omega_{sk}) + \sum_{k=1}^{S} \omega_{sk} \psi_{sk} \ln(M_{kt}) + \sum_{k=1}^{S} \omega_{sk} \ln(Y^L_{kt}).$$

Defining $\hat{\mathbf{X}}$ as the element-wise logarithm of an arbitrary vector or matrix $\mathbf{X}$, we can rewrite in log-vector notation:

$$\hat{\mathbf{Y}}^L_t = (\mathbf{I} - \boldsymbol{\Omega})^{-1} [\text{diag}(\boldsymbol{\alpha}) \hat{\boldsymbol{\alpha}} + (\boldsymbol{\Omega} \circ \hat{\boldsymbol{\Omega}}) \mathbf{1} + (\boldsymbol{\Omega} \circ \Psi) \hat{\mathbf{M}}_t]$$

where $\circ$ denotes the Hadamard product. Using the definition of Y^L_{st}, we have:

$$\hat{\mathbf{Y}}_t = (\mathbf{I} - \boldsymbol{\Omega})^{-1} [\text{diag}(\boldsymbol{\alpha}) \hat{\boldsymbol{\alpha}} + (\boldsymbol{\Omega} \circ \hat{\boldsymbol{\Omega}}) \mathbf{1} + (\boldsymbol{\Omega} \circ \Psi) \hat{\mathbf{M}}_t] + \hat{\mathbf{L}}^P_t - \hat{\boldsymbol{\alpha}}.$$

Substituting equation (A.2) in the sectoral resource constraints, we have:

$$C_{st} = Y_{st}\left(1 - \frac{\alpha_s}{L_{st}^P} \times \sum_{k=1}^{S} \frac{\omega_{ks} L_{kt}^P}{\alpha_k}\right).$$

In log-vector notation, we can analogously rewrite:

$$\hat{\mathbf{C}}_t = \hat{\mathbf{Y}}_t + \hat{\mathbf{\Gamma}}_t \quad \text{where} \quad \mathbf{\Gamma}_t \equiv \mathbf{1} - \text{diag}(\boldsymbol{\alpha})\text{diag}(\mathbf{L}_t^P)^{-1}\mathbf{\Omega}^\top \text{diag}(\boldsymbol{\alpha})^{-1}\mathbf{L}_t^P.$$

Therefore, the current-value Hamiltonian that corresponds to the planner's dynamic allocation problem is:

$$\mathcal{H}_t = \boldsymbol{\beta}^\top(\hat{\mathbf{Y}}_t + \hat{\mathbf{\Gamma}}_t) - \ln(N_t) + \sum_{s=1}^{S-1} \mu_{st}\epsilon_s L_{st}^E + \mu_{St}\left(L_t^E - \sum_{s=1}^{S-1} L_{st}^E\right)$$

where μ_{st} are costate variables. The optimality conditions of this problem are:

$$\begin{aligned}
\frac{\partial \mathcal{H}_t}{\partial L_{st}^E} &= \mu_{st}\epsilon_s - \mu_{St} = 0, \\
\frac{\partial \mathcal{H}_t}{\partial M_{st}} &= \frac{\boldsymbol{\beta}^\top(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{\Omega} \circ \mathbf{\Psi})\mathbf{e}_s}{M_{st}} = (\rho - n)\mu_{st} - \dot{\mu}_{st}.
\end{aligned}$$

On a balanced growth path, the measure of varieties within any sector must grow at rate n, which implies:

$$M_{st} = \frac{\epsilon_s L_{st}^E}{n}.$$

Using this result and the problem's optimality conditions, and rearranging, we have:

$$(\rho/n)\mu_{St} L_{st}^E = \boldsymbol{\beta}^\top(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{\Omega} \circ \mathbf{\Psi})\mathbf{e}_s.$$

The costate variable μ_{St} is solved for by using the research labor resource constraint:

$$(\rho/n)\mu_{St} L_t^E = \mathbf{1}^\top \boldsymbol{\beta}^\top(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{\Omega} \circ \mathbf{\Psi})\mathbf{e}_s.$$

Therefore, the share of entry labor allocated to each sector is:

$$\frac{\mathbf{L}_t^E}{L_t^E} = \frac{\boldsymbol{\beta}^\top(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{\Omega} \circ \mathbf{\Psi})}{\mathbf{1}^\top \boldsymbol{\beta}^\top(\mathbf{I} - \mathbf{\Omega})^{-1}(\mathbf{\Omega} \circ \mathbf{\Psi})}.$$

A.2 The market equilibrium allocation

The household's problem. Taking prices and the measures of varieties as given, the representative household's problem is to choose its consumption C_{ist} of each variety to maximize lifetime utility:

$$U = \max_{C_{ist}} \int_0^\infty e^{-(\rho-n)t} \ln(C_t) \mathrm{d}t \quad \text{where} \quad C_t = \prod_{s=1}^{S} \left(M_{st}^{-1/\theta} \int_0^{M_{st}} C_{ist}^{\frac{\theta-1}{\theta}} \mathrm{d}i \right)^{\frac{\beta_s \theta}{\theta-1}}$$

subject to the flow budget constraint:

$$\dot{A}_t = r_t A_t + w_t^P L_t^P + w_t^E L_t^E - \sum_{s=1}^{S} \int_0^{M_{st}} p_{ist} C_{ist} \mathrm{d}i$$

where A_t is financial wealth at time t, w_t^P and w_t^E are the wages paid to production and entry labor, respectively, and p_{ist} is the price of product i from sector s. Choosing aggregate consumption as the numéraire, this problem delivers the usual Euler equation and final demand functions:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho + n \quad \text{and} \quad C_{ist} = \frac{\beta_s C_t P_{st}^{\theta-1}}{M_{st} p_{ist}^{\theta}}, \quad \forall i \in [0, M_{st}],\ \forall s \in \{1, \dots, S\}$$

where $c_t \equiv C_t / N_t$ and P_{st} is the ideal price index of final consumption from sector s:

$$P_{st} \equiv \left(M_{st}^{-1} \int_0^{M_{st}} p_{ist}^{1-\theta} \mathrm{d}i \right)^{\frac{1}{1-\theta}} \quad \forall s \in \{1, \dots, S\}.$$

Similarly, the ideal price index of aggregate consumption, denoted as P_t, is normalized to unity and is defined as:

$$P_t \equiv \prod_{s=1}^{S} \left(\frac{P_{st}}{\beta_s} \right)^{\beta_s}.$$

The firm's problem. After entry, a firm engages in monopolistic competition on the output market and perfect competition on the markets for inputs. That is, it chooses a price as well as intermediate inputs and production labor to maximize profits π_{ist} while taking as given the demand for its product, the price p_{jkt} of intermediate inputs, and the production wage:

$$\pi_{ist} = \max_{p_{ist}, x_{ist}(j,k), l_{ist}} \{ p_{ist} y_{ist} - \sum_{k=1}^{S} \int_0^{M_{kt}} p_{jkt} x_{ist}(j,k) \mathrm{d}j - w_t^P l_{ist} \}.$$

This can be broken down into several sub-problems. The first being the following cost minimization problem for each supplying sector k:

$$\min_{x_{ist}(j,k)} \int_0^{M_{kt}} p_{jkt} x_{ist}(j,k) \mathrm{d}j \quad \text{s.t.} \quad M_{kt}^{\psi_{sk}} \left(M_{kt}^{-1/\theta} \int_0^{M_{kt}} x_{ist}(j,k)^{\frac{\theta-1}{\theta}} \mathrm{d}j \right)^{\frac{\theta}{\theta-1}} \geq x_{ist}(k).$$

The first-order conditions deliver the following demand functions:

$$x_{ist}(j,k) = (P^X_{st}(k)/p_{jkt})^{\theta} x_{ist}(k) M_{kt}^{\theta\psi_{sk}-1} \quad \text{where} \quad P^X_{st}(k) \equiv P_{kt} M_{kt}^{\frac{\theta\psi_{sk}}{1-\theta}}.$$

The second sub-problem is the following cost minimization problem:

$$\min_{x_{ist}(k)} \sum_{k=1}^{S} P^X_{st}(k) x_{ist}(k) \quad \text{s.t.} \quad \prod_{k=1}^{S} x_{ist}(k)^{\omega_{sk}} \geq x_{ist}.$$

The first-order conditions deliver the following demand functions:

$$x_{ist}(k) = \omega_{sk}(P^X_{st} x_{ist})^{\frac{1}{1-\alpha_s}} / P^X_{st}(k) \quad \text{where} \quad P^X_{st} \equiv \prod_{k=1}^{S} \left(\frac{P^X_{st}(k)}{\omega_{sk}} \right)^{\omega_{sk}}.$$

The third sub-problem is the following cost minimization problem:

$$\min_{x_{ist}, l_{ist}} \{(1-\alpha_s)(P^X_{st} x_{ist})^{\frac{1}{1-\alpha_s}} + w^P_t l_{ist}\} \quad \text{s.t.} \quad x_{ist} l_{ist}^{\alpha_s} \geq y_{ist}.$$

The first-order conditions deliver the following demand functions:

$$x_{ist} = (m_{st} y_{ist})^{1-\alpha_s} / P^X_{st} \quad \text{and} \quad \ell_{ist} = \alpha_s m_{st} y_{ist} / w^P_t$$

where m_{st} denotes the marginal cost of firms in sector s:

$$m_{st} \equiv P^X_{st} \left(\frac{w^P_t}{\alpha_s} \right)^{\alpha_s}.$$

The last sub-problem is the firm's optimal pricing problem. It is straightforward to show that firms set their price to a constant markup μ above marginal cost:

$$p_{ist} = \mu \cdot m_{st} \quad \text{where} \quad \mu \equiv \frac{\theta}{\theta - 1}.$$

Therefore, firm profits in sector s are given by:

$$\pi_{ist} = \frac{m_{st} y_{ist}}{\theta - 1}.$$

Substituting the demand function for intermediate inputs in the firm's production function, we have:

$$y_{ist} = \frac{(m_{st} y_{ist})^{1-\alpha_s} \ell_{ist}^{\alpha_s}}{P^X_{st}}.$$

Solving for the output of firm i in sector s, we have:

$$y_{ist} = \left(\frac{m_{st}^{1-\alpha_s}}{P_{st}^X} \right)^{\frac{1}{\alpha_s}} \ell_{ist}.$$

Using the symmetry of production labor demand functions as well as the definition of the firm's marginal cost, we can rewrite:

$$y_{ist} = \frac{(w_t^P/\alpha_s)^{1-\alpha_s} L_{st}}{M_{st} P_{st}^X}.$$

Therefore, firm profits in sector s are given by:

$$\pi_{ist} = \frac{w_t^P L_{st}}{\alpha_s(\theta - 1) M_{st}}, \quad \forall i \in [0, M_{st}], \ \forall s \in \{1, \ldots, S\}.$$

The value of such a firm is given by the present value of its future profits:

$$V_{st} = \int_t^\infty e^{-\int_t^{t'} r_\tau \mathrm{d}\tau} \pi_{st'} \mathrm{d}t' \quad \forall s \in \{1, \ldots, S\}.$$

Differentiating V_{st} with respect to time, we obtain the law of motion for the value of a firm in sector s:

$$\dot{V}_{st} = r_t V_{st} - \frac{w_t^P L_{st}}{\alpha_s(\theta - 1) M_{st}} \quad \forall s \in \{1, \ldots, S\}.$$

The entrant's problem. The entrant's problem is to choose entry labor to maximize the expected present discounted value of introducing a new product:

$$\max_{L_{st}^E} \{ V_{st} \epsilon_s L_{st}^E - w_t^E L_{st}^E \} \quad \forall s \in \{1, \ldots, S\}.$$

With free-entry among potential entrants, this implies:

$$V_{st} = w_t^E / \epsilon_s \quad \forall s \in \{1, \ldots, S\}.$$

A.2.1 Balanced Growth Path Solution

The asset market clearing condition is:

$$B_t = \sum_{s=1}^{S} q_{st} A_{st}.$$

Differentiating with respect to time delivers:

$$\dot{B}_t = \sum_{s=1}^{S} \dot{q}_{st} A_{st} + \sum_{s=1}^{S} q_{st} \dot{A}_{st} = r_t B_t - w_t^L \sum_{s=1}^{S} \frac{L_{st}}{\alpha_s(\theta - 1)} + w_t^R R_t.$$

Combining this result with the household's flow budget constraint, we have:

$$C_t = w_t^L L_t + w_t^L \sum_{s=1}^{S} \frac{L_{st}}{\alpha_s(\theta - 1)} - \sum_{s=1}^{S} \int_0^{A_{st}} \tau_{st}^Y p_{ist} y_{ist} \mathrm{d}i.$$

Substituting in the expression for firm revenues, we have:

$$C_t = w_t^L L_t + w_t^L \sum_{s=1}^{S} \frac{\mu_s [1 - (\theta - 1)\tau_{st}^Y] L_{st}}{\alpha_s \theta (1 + \tau_{st}^Y)}.$$

Taking the logarithm of the ideal price index of aggregate consumption, we have:

$$\sum_{s=1}^{S} \beta_s [\ln(P_{st}) - \ln(\beta_s)] = 0.$$

Substituting in the expressions for P_{st} and using log-vector notation, we have:

$$\boldsymbol{\beta}^\top \{\hat{\boldsymbol{\mu}}_t' + \hat{\mathbf{P}}_t^x + \mathrm{diag}(\boldsymbol{\alpha})[\ln(w_t^L)\mathbf{1} - \hat{\boldsymbol{\alpha}}] - \mathrm{diag}(\boldsymbol{\theta} - 1)^{-1} \hat{\mathbf{A}}_t - \hat{\boldsymbol{\beta}}\} = 0$$

where element s of the $S \times 1$ vector $\hat{\boldsymbol{\mu}}_t'$ is equal to $\ln(\mu_s) - \ln(1 + \tau_{st}^Y)$. Using the definition of P_{st}^x together with log-vector notation, we have:

$$\hat{\mathbf{P}}_t^x = (\mathbf{I} - \boldsymbol{\Omega})^{-1} \{\boldsymbol{\Omega}[\hat{\boldsymbol{\mu}}_t' + \mathrm{diag}(\boldsymbol{\alpha})[\ln(w_t^L)\mathbf{1} - \hat{\boldsymbol{\alpha}}] - \mathrm{diag}(\boldsymbol{\theta} - 1)^{-1} \hat{\mathbf{A}}_t] - \mathrm{diag}(\boldsymbol{\Omega}\hat{\boldsymbol{\Omega}}^\top)\}.$$

Substituting this expression in the previous equation delivers an expression for the logarithm of the wage paid to workers:

$$\ln(w_t^L) = \frac{\boldsymbol{\Lambda}^\top [\mathrm{diag}(\boldsymbol{\alpha})\hat{\boldsymbol{\alpha}} + \mathrm{diag}(\boldsymbol{\theta} - 1)^{-1} \hat{\mathbf{A}}_t] + \boldsymbol{\beta}^\top [\mathrm{diag}(\boldsymbol{\Omega}\hat{\boldsymbol{\Omega}}^\top) + \hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\mu}}_t']}{\boldsymbol{\Lambda}^\top \boldsymbol{\alpha}}$$

where $\mathbf{\Lambda}$ is defined as in Appendix **??**. On a balanced growth path, it must be that the measure of varieties within each sector grows at the same rate as the population, which implies:

$$A_{st} = \frac{\eta_s R_{st}}{n} \quad \forall s \in \{1, \dots, S\}.$$

Similarly, the growth rate of the price of a patent must be equal to the growth rate of consumption per person, which implies:

$$R_{st} = \frac{n w_t^L L_{st}}{\alpha_s(\theta-1)(\rho-n)\eta_s q_{st}} \quad \forall s \in \{1,\ldots,S\}.$$

Therefore, the research labor market clearing condition deliver an expression for the research labor demand in each sector:

$$R_{st} = \frac{n w_t^L L_{st}}{\alpha_s(\theta-1)(\rho-n)(1-\tau_{st}^R) w_t^R} \quad \forall s \in \{1,\ldots,S\}.$$

Imposing the research labor resource constraint, we obtain an expression for the wage paid to researchers:

$$w_t^R = \frac{n w_t^L}{(\rho-n)R_t} \times \sum_{s=1}^{S} \frac{L_{st}}{\alpha_s(\theta-1)(1-\tau_{st}^R)}.$$

Substituting this back into the previous equation, we obtain an expression for the share of research labor in each sector in vector notation:

$$\frac{\mathbf{R}_t}{R_t} = \frac{\operatorname{diag}(\mathbf{1}-\boldsymbol{\tau}^R)^{-1}\operatorname{diag}(\boldsymbol{\theta}-\mathbf{1})^{-1}\operatorname{diag}(\boldsymbol{\alpha})^{-1}\mathbf{L}_t}{\mathbf{1}^\top\operatorname{diag}(\mathbf{1}-\boldsymbol{\tau}^R)^{-1}\operatorname{diag}(\boldsymbol{\theta}-\mathbf{1})^{-1}\operatorname{diag}(\boldsymbol{\alpha})^{-1}\mathbf{L}_t}.$$

The variety resource constraints are:

$$C_{ist} = y_{ist} - \sum_{k=1}^{S} \int_0^{A_{kt}} x_{jkt}(i,s)\mathrm{d}j \quad \forall i \in [0, A_{st}], \forall s \in \{1,\ldots,S\}.$$

Multiplying both sides of this equation by p_{ist} and substituting in the final and intermediate demand functions, as well as the expression for the output of firm i in sector s, we have:

$$\beta_s C_t = \frac{\mu_s w_t^L L_{st}}{\alpha_s(1+\tau_{st}^Y)} - \sum_{k=1}^{S} \frac{\omega_{ks} w_t^L L_{kt}}{\alpha_k} \quad \forall s \in \{1,\ldots,S\}.$$

This equation can be rewritten in vector notation as:

$$\frac{w_t^L \mathbf{L}_t}{C_t} = \operatorname{diag}(\boldsymbol{\alpha})(\operatorname{diag}(\boldsymbol{\mu})\operatorname{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}.$$

Imposing the production labor resource constraint, we obtain an expression for the payments to production labor as a share of aggregate consumption:

$$\frac{w_t^L L_t}{C_t} = \mathbf{1}^\top \mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}.$$

Substituting this back into the previous equation, we obtain an expression for the share of production labor in each sector:

$$\frac{\mathbf{L}_t}{L_t} = \frac{\mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}{\mathbf{1}^\top \mathrm{diag}(\boldsymbol{\alpha})(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}.$$

Substituting this back into the expression for the share of research labor in each sector, we have the following:

$$\frac{\mathbf{R}_t}{R_t} = \frac{\mathrm{diag}(\mathbf{1}-\boldsymbol{\tau}^R)^{-1}\mathrm{diag}(\boldsymbol{\theta}-\mathbf{1})^{-1}(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}{\mathbf{1}^\top \mathrm{diag}(\mathbf{1}-\boldsymbol{\tau}^R)^{-1}\mathrm{diag}(\boldsymbol{\theta}-\mathbf{1})^{-1}(\mathrm{diag}(\boldsymbol{\mu})\mathrm{diag}(\mathbf{1}+\boldsymbol{\tau}^Y)^{-1} - \boldsymbol{\Omega}^\top)^{-1}\boldsymbol{\beta}}.$$